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# Deterministic Nonlinear Filtering

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Dedicated to the memory of Ennio De Giorgi

**Abstract.** A model for nonlinear filtering is considered in which errors in state dynamics and observations are modelled deterministically. Mortensen's deterministic estimator and a minimax estimator are considered. A risk sensitive stochastic filter model with small state and observation noise intensities is also considered. The minimax estimator is obtained in the zero noise intensity limit, using asymptotic properties of a pathwise interpretation of the Zakai stochastic partial differential equation.

## 1. – Introduction

In general terms, the filtering problem can be stated as follows. Let $x_T$ denote a state (or signal) at time $T$ and $y_T$ an observation at time $T$. The observation depends on both $x_T$ and certain measurement errors. Estimates $e_T$ for $x_T$ are allowed which depend on past observations $y_t$ for $0 \leq t \leq T$. The goal of filtering is to find estimates which are "best" or at least "good" according to some criterion. Moreover, to be useful in applications, filtering algorithms must be computationally implementable in real time.

In the traditional stochastic filter model, state dynamics are governed by white-noise driven stochastic differential equations, and observation errors are also modelled as white noises. The objective is to minimize an expected mean squared estimation error criterion. For linear-gaussian filter models, the Kalman-Bucy filter provides a highly successful solution. This filter involves only the solution off-line of a matrix Riccati differential equation and the solution on-line of a linear stochastic differential equation for the estimate. See for example [FR, p. 136]. For nonlinear stochastic filter models, the expected mean squared estimation error depends on the conditional distribution of the state at time $T$ given past observations for $0 \leq t \leq T$. Nonlinear filtering theory reduces

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the problem to the solution of the Zakai stochastic PDE for an unnormalized version of the conditional density, followed by integrations [Ku]. Except in the lowest state dimensions this is a computationally formidable task. However, a splitting method considered by Rozovskii and associates is useful for doing part of the computation off line rather than in real time [LMR]. In engineering practice, extended Kalman filters are frequently used instead. The extended Kalman filter approximation has been rigorously justified in some special cases, in which observation noise is of low intensity. See Picard [P].

Instead of the stochastic filter model, we consider in this paper a deterministic model described in Section 2. Errors in the state dynamics and observations are modelled as deterministic functions (called disturbances) rather than as white noises. See (2.1), (2.2). The unnormalized conditional density is replaced by a certain "information state" function, which in our notation is $-V(T,x)$, where $V(T,x)$ is defined by minimizing a least squared disturbance error criterion. See (2.5). The function $V$ evolves according to the first-order PDE (2.6) of Hamilton-Jacobi-Bellman type, and is interpreted as a viscosity sense solution to (2.6). This function $V$ is central to the analysis. In Section 3 we consider the solution operator $S_T$, which maps initial data $\phi$ for the PDE (2.6) to the solution $V(T,x)$. The solution operator is "linear" when considered in the so-called max-plus algebra [AQV] rather than with respect to ordinary addition and scalar multiplication. See (3.3), (3.4). Another important property is that the solution operator preserves semiconcavity (Lemma 2.5). These properties are useful to obtain a representation (3.7) of solutions in terms of quadratic basis functions.

The deterministic approach to filtering was pioneered by Mortensen [Mo]. Mortensen's estimator $\hat{x}_T$ minimizes $V(T,x)$ as a function of $x$. Thus, $\hat{x}_T$ maximizes the information state $-V(T,x)$. In Section 4 we consider another estimator $\hat{e}_T^0$, called a minimax estimator. It is a slight variant of the minimax estimator introduced by McEneaney [Mc2]. The minimax estimator minimizes over $e$ the maximum over $x$ of $\mu\ell(|x-e|) - V(T,x)$, where $\mu > 0$ and $\ell(r)$ represents the loss from absolute estimation error $r$. This minimax point of view in filtering is similar in spirit to the robust approach to control (also called H-infinity control theory). See [BB].

In Section 5 we return to a stochastic filter model. However, instead of the traditional expected mean square error criterion, an expected exponential-of-loss criterion is minimized. The estimator obtained in this way is called a risk sensitive filter. Hijab's thesis [H] obtained the deterministic information state via WKB —type asymptotics of the unnormalized conditional density, as a state and observation noise intensity parameter $\varepsilon$ tends to 0. See also [JB] for a proof of this kind of result using viscosity solution methods. In order to compare stochastic and deterministic filter models, so called pathwise filtering theory is needed [Da] [Mi] [FP]. Under some simplifying assumptions, we sketch a stochastic control proof of this asymptotic result (Lemma 5.1 (a)). Then we prove convergence of the risk sensitive estimate to the minimax estimate as $\varepsilon \to 0$ (Theorem 5.2).

The results presented here are part of ongoing joint research with W. M. McEneaney. The mathematical framework, and similar results under a different set of assumptions, have been announced in [FM2]. In the present paper we impose restrictive assumptions as needed to avoid various technical issues addressed in [FM2] and its more detailed version which is in preparation.

## 2. – Deterministic filter model

In this paper we are concerned with the following model. The state $x_t$ which is to be estimated evolves according to the differential equation

$$\dot{x}_t = f(x_t) + \sigma(x_t)w_t, \tag{2.1}$$

where $w_t$ is the state disturbance. The observation $y_t$ at time $t$ satisfies

$$y_t = h(x_t) + \rho v_t, \quad \rho > 0 \tag{2.2}$$

where $v_t$ is the observation disturbance. Here $x_t \in \mathbb{R}^n$, $w_t \in \mathbb{R}^m$, $y_t, v_t \in \mathbb{R}^p$. We assume throughout:

$$f, \sigma, h \in C^1(\mathbb{R}^n) \text{ with } f_x, \sigma, \sigma_x, h, h_x \text{ bounded}. \tag{A1}$$

Here $f_x$ is the matrix of partial derivatives of $f$ with $\sigma_x, h_x$ defined similarly.

An estimator is a function which assigns to each observation trajectory $y_.$ and time $T$ an estimate $e_T$ for the unknown state $x_T$. It is required to be nonanticipative, in the sense that $y_t = \tilde{y}_t$ for $0 \leq t \leq T$ imply that the corresponding estimates satisfy $e_T = \tilde{e}_T$. Following a rather common abuse of terminology, we shall also refer to $e_T$ as the estimator.

Mortensen's deterministic estimator [Mo] is described as follows. Let $\phi$ satisfy:

$$\phi \text{ is locally Lipschitz and } \phi \geq 0. \tag{A2}$$

We regard $-\phi(x_0)$ as a measure of the likelihood of unknown initial state $x_0$. (Later additional assumptions (A3)-(A5) will be made as needed.) Let

$$J = \phi(x_0) + \frac{1}{2}\int_0^T \left[|w_t|^2 + |v_t|^2\right] dt. \tag{2.3}$$

Given observations $y_t$ for $0 \leq t \leq T$, one wishes to minimize $J$ among all $x_0, w_. v_.$ consistent with the observations. Suppose that the minimum occurs at $x_0^*, w_.^*, v_.^*$, and let $x_t^*$ be the corresponding solution to (2.1). The Mortensen estimator at time $T$ is $\hat{x}_T = x_T^*$. (If the minimum is not unique, then $\hat{x}_T$ need not be unique.)

It is convenient to describe the Mortensen estimator in the following way. We fix the final state $x_T = x$ and solve (2.1) backward in time. An observation trajectory $y_.$ is fixed throughout the discussion which follows. We can rewrite (2.3) as

$$J = J(T, x; w_.) = \phi(x_0) + \frac{1}{2}\int_0^T \left[|w_t|^2 + \rho^{-2}|y_t - h(x_t)|^2\right] dt. \tag{2.3$'$}$$

For given $(T, x)$, a standard existence theorem in control theory implies that there exists $w^*_.$ which minimizes J. See [FR, Cor. III. 4.1].

Let $K \subset \mathbb{R}^{n+1}$ be any compact set. There exists a constant $B_K$ such that $\|w^*_.\|_2 \leq B_K$, where $\|\cdot\|_2$ is the norm in $L^2[0, T]$, provided that $(T, x) \in K$. This follows from (A2), (2.3′) and $0 \leq J(T, x; w^*_.) \leq J(T, x; 0)$. It then suffices to consider disturbances $w_.$ with $\|w_.\|_2 \leq B_K$. If $x_t$ is the corresponding solution to (2.1) with $x_T = x$, then $|x_t| \leq R$ for some $R$ depending only on $K$. We call this the *range of dependence property*. Let us next obtain a $L^\infty$ bound for $w_t^*$:

LEMMA 2.1. (a) *For every compact set $K \subset \mathbb{R}^{n+1}$ there exists $M_K$ such that $|w_t^*| \leq M_K$ provided $(T, x) \in K$.*

(b) *If $\phi$ is Lipschitz and $\sigma$ is constant, then $|w_t^*| \leq M$ where $M$ depends only on $T$.*

PROOF. It suffices to assume that $\phi$ is smooth, by approximating $\phi$ uniformly on compact sets by smooth functions which satisfy uniform local Lipschitz bounds. By Pontryagin's principle [FR, Sec. II. 11], $\frac{1}{2}|w|^2 + p_t \cdot \sigma(x_t^*)w$ is minimum for $w = w_t^*$, where

$$\dot{p}_t = -p_t \cdot \left[f_x(x_t^*) + \sigma_x(x_t^*)w_t^*\right] + \rho^{-2}h_x(x_t^*)(y_t - h(x_t^*)) \tag{2.4}$$

with $p_0 = -\phi_x(x_0^*)$ which is bounded by the range of dependence property. From (2.4) and (A1)

$$\left|\frac{d}{dt}\log(1 + |p_t|^2)\right| \leq C_1(1 + |w_t^*|)$$

for suitable $C_1$ depending on $K$. Since $\|w^*_.\|_2 \leq B_K$, this implies that $\log(1 + |p_t|^2)$ is bounded, and hence also $p_t$ and $w_t^*$. This proves (a). If $\phi$ is Lipschitz and $\sigma$ is constant, then $p_0 = -\phi_x(x_0^*)$ is bounded and $\sigma_x = 0$. A bound for $p_t$, and hence also a bound for $w_t^*$, which depends only on $T$ is obtained immediately from (2.4). □

Following the dynamic programming method consider the optimal cost function

$$V(T, x) = \min_{w_.} J(T, x; w_.). \tag{2.5}$$

One can interpret $-V(T, x)$ as a measure of the likelihood of state $x_T = x$ at time $T$. Sometimes $-V(T, \cdot)$ is called the *information state* at time $T$. The

Mortensen estimator $\hat{x}_T \in \operatorname{argmin}_x V(T, x)$ is in this sense a maximum likelihood estimator for $x_T$. The function $V(T, \cdot)$ satisfies a local Lipschitz condition, uniformly for $T$ in any finite interval. To see this, by the range of dependence property it suffices to consider $\phi$ Lipschitz on $\mathbb{R}^n$ and then to use Lemma 2.1 (a) and a standard argument [FS, Sec. 4.8]. Another standard argument using the dynamic programming principle gives a uniform local Lipschitz estimate for $V(\cdot, x)$. Thus, $V(\cdot, \cdot)$ is locally Lipschitz.

Moreover, $V$ satisfies in the viscosity sense the dynamic programming PDE (also called Hamilton- Jacobi-Bellman PDE)

$$V_T = -f(x) \cdot V_x - \frac{1}{2} a(x) V_x \cdot V_x + \frac{1}{2\rho^2} |y_T - h(x)|^2, \tag{2.6}$$

with initial data

$$V(0, x) = \phi(x). \tag{2.7}$$

Here $a = \sigma\sigma'$ where $'$ denotes matrix transpose.

In order to obtain uniqueness of viscosity solutions to (2.6)-(2.7), further assumptions on the initial data $\phi$ are needed. A general uniqueness result when $|\phi_x(x)|$ grows at most linearly in $|x|$ as $|x| \to \infty$ is given in [Mc1]. In the present paper, this uniqueness issue will not arise. We remark that if $\phi$ is Lipschitz and $\sigma$ is constant, then $V(T, \cdot)$ satisfies a uniform Lipschitz condition on any finite time interval. Uniqueness holds in the class of such viscosity solutions.

In addition to (A1) and (A2) let us now assume

$$\lim_{|x| \to \infty} \phi(x) = +\infty. \tag{A3}$$

LEMMA 2.2. *For each $T > 0$,*

$$\lim_{|x| \to \infty} V(T, x) = +\infty.$$

SKETCH OF PROOF. If $w^*_\cdot$ is minimizing and $x^*_t$ the corresponding solution to (2.1) with $x^*_T = x$, then by (2.3′)

$$\phi(x^*_0) + \frac{1}{2} \int_0^T |w^*_t|^2 dt \le V(T, x).$$

If $V(T, x_n)$ is bounded for a sequence $x_n$ with $|x_n| \to \infty$, then the corresponding $x^*_{0n}$ is bounded by (A3) and $w^*_{\cdot n}$ is bounded in $L^2$-norm. However, this is impossible by (2.1) and (A1). □

In particular, $V(T, \cdot)$ has a minimum at some $\hat{x}_T$ (Mortensen estimator).

REMARK 2.3. If $f(x) = Ax, \sigma =$ constant, $h(x) = Hx$, and $\phi$ is quadratic the model (2.1)-(2.2) is the deterministic counterpart of the stochastic Kalman-Bucy model. In this case, the Mortensen filter agrees with the Kalman-Bucy filter. It satisfies the linear differential equation

$$\frac{d\hat{x}_T}{dT} = A\hat{x}_T + \frac{\psi_T}{\rho}(y_T - H\hat{x}_T), \tag{2.8}$$

where $\psi_T$ can be precomputed from the solution to a matrix Riccati differential equation. For nonquadratic $\phi$, there are similar asymptotic results as $T \to \infty$. See formula (2.13) in the 1-dimensional case and discussion after it.

REMARK 2.4. In the nonlinear case there is an analogue of (2.8). For notational simplicity, let $x_t, y_t$ be scalar valued ($n = p = 1$). Suppose that $V$ is smooth (class $C^2$) in a neighborhood of $(T, \hat{x}_T)$ and that $V_{xx}(T, \hat{x}_T) > 0$. By differentiating the PDE (2.6) with respect to $x$ and the equation $V_x(T, \hat{x}_T) = 0$ with respect to $T$, one gets

$$\frac{d\hat{x}_T}{dT} = f(\hat{x}_T) + \frac{h_x(\hat{x}_T)}{\rho^2 V_{xx}(T, \hat{x}_T)}(y_T - h(\hat{x}_T)), \tag{2.9}$$

with initial data $\hat{x}_0 \in \operatorname{argmin}_x \phi(x)$. Unfortunately, $V_{xx}$ is not known without solving (2.6)-(2.7); and hence (2.9) does not provide a finite dimensional procedure for finding $\hat{x}_T$. In engineering practice, extended Kalman filters involving repeated linearizations of (2.1) and (2.2) are frequently used instead. At the end of Section 4, we will consider the special case of one to one observation function $h$ and small observation parameter $\rho$. In that case a simplified version (4.5) of equation (2.9) provides a robust filter.

SEMICONCAVITY. A function $\phi$ is called *semiconcave* if for every $R$ there exists $C_R$ such that

$$\tilde{\phi}(x) = \phi(x) - \frac{1}{2}C_R|x|^2 \tag{2.10}$$

is concave on the ball $\{|x| \leq R\}$. In addition to (A1) let us now assume:

(A4) $f, \sigma, h \in C^2(\mathbb{R}^n)$ with $f_{xx}, \sigma_{xx}, h_{xx}$ bounded.

LEMMA 2.5. *If $\phi$ is semiconcave, then $V(T, \cdot)$ is semiconcave. In fact, there exists $\Gamma_R(T)$ such that for $0 \leq \tau \leq T$, $R > 0$*

$$\tilde{V}(\tau, x) = V(\tau, x) - \frac{1}{2}\Gamma_R(T)|x|^2$$

*is concave on the ball $\{|x| \leq R\}$.*

SKETCH OF PROOF. Let

$$\tilde{J}(\tau, x; w_.) = J(\tau, x; w_.) - \frac{1}{2}\Gamma|x|^2$$

where $\Gamma = \Gamma_R(T)$ is to be chosen suitably. Since the infimum of any family of concave functions is concave, it suffices to show that $\tilde{J}(\tau, \cdot; w_.)$ is concave on $\{|x| \leq R\}$ for each $w_.$. By smoothing via convolution with approximations to the identity, we may assume that $\phi$ is smooth. Then semiconcavity is equivalent to $\phi_{xx}(x)\xi \cdot \xi \leq C_R|\xi|^2$ for all $\xi$, when $|x| \leq R$. To show concavity of $\tilde{J}$ it suffices to show that

$$J_{xx}\xi \cdot \xi \leq \Gamma \tag{2.11}$$

whenever $|x| \leq R$ and $|\xi| = 1$. The solution $x_t = x_t(x)$ to (2.1) depends smoothly on the final data $x = x_\tau$. Let $\zeta_t^1, \zeta_t^2$ denote the first and second order derivatives of $x_t$ in the direction $\xi$. Then

$$\begin{aligned} J_{xx}(\tau, x; w_.)\xi \cdot \xi &= \frac{d^2}{dh^2} J(\tau, x + h\xi; w_.)\Big|_{h=0} \\ &= \phi_x(x_0)\zeta_0^2 + \phi_{xx}(x_0)\zeta_0^1 \cdot \zeta_0^1 + \int_0^T \left[G_x\zeta_t^2 + G_{xx}\zeta_t^1 \cdot \zeta_t^1\right] dt, \end{aligned} \tag{2.12}$$

$$G(t, x, w) = \frac{1}{2}\left[|w|^2 + \rho^{-2}|y_t - h(x)|^2\right].$$

By Lemma 2.1 we may assume that $|w_t| \leq M_K$, where $K = [0, T] \times \{|x| \leq R\}$. Then $|x_0| \leq R_1$ for suitable $R_1$ and hence $\phi_{xx}(x_0)\zeta_0^1 \cdot \zeta_0^1 \leq C_{R_1}|\zeta_0^1|^2$. Moreover, from (A1), (A2) and (A4) there are bounds for $|\zeta_t^1|$, $|\zeta_t^2|$ and all other terms on the right side of (2.2). This implies (2.11), for suitable $\Gamma$. □

LARGE TIME BEHAVIOR. We recall that $-\phi(x^0)$ is interpreted as a measure of the likelihood of initial state $x_0$. The function $\phi$ is often chosen rather arbitrarily. It is an interesting question to describe conditions under which the dependence of the Mortensen filter on $\phi$ is lost asymptotically as $T \to \infty$. For the stochastic (white noise disturbance) counterpart of the model which we consider, see [OP] and references cited there for results of this kind.

Let us consider only the linear model in Remark 2.3, but with $\phi$ not necessarily quadratic. (For the stochastic model, this corresponds to linear dynamics and observations, but nongaussian initial data.) As in Remark 2.4, for simplicity we consider the 1 dimensional case. Equation (2.9) now becomes

$$\frac{d\hat{x}_T}{dT} = A\hat{x}_T + \frac{H}{\rho^2 V_{xx}(T, \hat{x}_T)}(y_T - H\hat{x}_T). \tag{2.13}$$

We assume that $H \neq 0$ and that $\phi$ is smooth (class $C^2$) with $0 < \phi_{xx} \leq C$. We will show that $V$ is also smooth with $V_{xx} > 0$, and that as $T \to \infty$, $V_{xx}(T, \hat{x}_T)$

tends at an exponential rate to a constant $K$. See formula (2.21). The constant $K$ is the same one obtained via the Kalman-Bucy filter, and does not depend on $\phi$. Since $f(x) = Ax$, $h(x) = Hx$, $\sigma$ is constant and $\phi$ is strictly convex, $J(T, x; w.)$ is strictly convex by (2.1) and (2.3′). Hence the minimizing $w_\cdot^*$ is unique. Moreover, by Pontryagin's principle $w_t^* = -\sigma p_t$ with $p_t$ as in (2.4). Then (2.1), (2.4), with $x_t = x_t^*$ become

$$(2.14) \qquad \begin{aligned} &\text{(a)} \quad \dot{x}_t = Ax_t - \sigma^2 p_t \\ &\text{(b)} \quad \dot{p}_t = -Ap_t + \rho^{-2} H(y_t - Hx_t). \end{aligned}$$

These are the characteristic equations for the PDE (2.6), for this choice of $f, h, \sigma$. Let $x_t(\alpha)$, $p_t(\alpha)$ denote the solution to (2.14) with initial data

$$(2.15) \qquad x_0(\alpha) = \alpha, \quad p_0(\alpha) = -\phi_x(\alpha).$$

Given $T, x$, we have $x_t^* = x_t(\alpha^*)$ for a unique $\alpha^*$ such that $x = x_T(\alpha^*)$.

The method of characterisitics provides a smooth solution $\overline{V}(T, x)$ to (2.6)-(2.7) on some interval $0 \le T < T_1$, and $\overline{V}(T, x) = V(T, x)$ in this interval. Let us show that $T_1 = \infty$ and that $V_{xx}(T, x) > 0$. For $0 \le t < T_1$,

$$(2.16) \qquad V_x(t, x_t(\alpha)) = -p_t(\alpha)$$

$$(2.17) \qquad V_{xx}(t, x_t(\alpha)) \frac{\partial x_t}{\partial \alpha} = -\frac{\partial p_t}{\partial \alpha}.$$

Let $\zeta_t = \partial x_t / \partial \alpha$, $\eta_t = -\partial p_t / \partial \alpha$. By (2.14)

$$(2.18) \qquad \begin{aligned} &\text{(a)} \quad \dot{\zeta}_t = A\zeta_t + \sigma^2 \eta_t \\ &\text{(b)} \quad \dot{\eta}_t = \rho^{-2} H^2 \zeta_t - A\eta_t \end{aligned}$$

with the initial data $\zeta_0 = 1$, $\eta_0 = \phi_{xx}(\alpha) > 0$. An elementary analysis shows that $\zeta_t > 0$, $\eta_t > 0$ for all $t \ge 0$. Since $\partial x_t / \partial \alpha > 0$ the mapping $\alpha \to x_t(\alpha)$ is one-to-one, and $\overline{V}(T, x) = V(T, x)$ for all $T$. Thus $T_1 = +\infty$. Moreover, by (2.17), $V_{xx} > 0$. Since $V_{xx} = \zeta_t^{-1} \eta_t$, we have by (2.17) and (2.18) for each fixed $\alpha$

$$(2.19) \qquad \frac{d}{dt} V_{xx} = \rho^{-2} H^2 - 2AV_{xx} - \sigma^2 V_{xx}^2,$$

where $V_{xx} = V_{xx}(t, x_t(\alpha))$ and the initial data are $V_{xx}(0, \alpha) = \phi_{xx}(\alpha)$. Thus, $r_t = V_{xx}(t, x_t(\alpha))$ solves the Riccati differential equation $\dot{r}_t = g(r_t)$, where

$$g(r) = \rho^{-2} H^2 - 2Ar - \sigma^2 r^2.$$

Then $g(K) = 0, g'(K) = -\lambda$, where

$$(2.20) \qquad \begin{aligned} &\text{(a)} \quad K = \sigma^{-2}\left(-A + \sqrt{A^2 + \rho^{-2} H^2 \sigma^2}\right), \\ &\text{(b)} \quad \lambda = 2\sqrt{A^2 + \rho^{-2} H^2 \sigma^2}. \end{aligned}$$

For initial data $0 < r_0 \leq C$, there exists $B$ such that $|r_T - K| \leq Be^{-\lambda T}$ for all $T$. By choosing $\alpha$ such that $x_T(\alpha) = \hat{x}_T$,

$$|V_{xx}(T, \hat{x}_T) - K| \leq Be^{-\lambda T}. \tag{2.21}$$

Thus, (2.13) is asymptotically as $T \to \infty$ the same as for quadratic initial data. From this we wish to obtain a result which says that, asymptotically as $T \to \infty$, $\hat{x}_T$ loses dependence on $\phi$.

By (2.20), (2.21) we can rewrite (2.13) as

$$\frac{d\hat{x}_T}{dT} = -\frac{\lambda}{2}\hat{x}_T + \frac{Hy_T}{\rho^2 K} + \psi_T(y_T - H\hat{x}_T), \tag{2.22}$$

$$|\psi_T| \leq \beta e^{-\lambda T} \tag{2.23}$$

for some $\beta$. Consider another initial $\tilde{\phi}$ with $0 < \tilde{\phi}(x) \leq C$. The corresponding Mortensen estimate $\hat{\hat{x}}_T$ also satisfies (2.22), with $\psi_T$ replaced by $\tilde{\psi}_T$ which also satisfies (2.23). Let $\xi_T = \hat{x}_T - \hat{\hat{x}}_T$. Then

$$\frac{d\xi_T}{dT} = -\frac{\lambda}{2}\xi_T + (\psi_T - \tilde{\psi}_T)y_T + H\tilde{\psi}_T\hat{\hat{x}}_T - H\psi_T\hat{x}_T. \tag{2.24}$$

Let us assume that the state and observation disturbances satisfy

$$\begin{aligned} &\limsup_{T\to\infty} T^{-1}\int_0^T |w_t|^2 dt < \infty \\ &\limsup_{T\to\infty} T^{-1}\int_0^T |v_t|^2 dt < \infty. \end{aligned} \tag{2.25}$$

An elementary analysis shows that $|y_t|$, $|\hat{x}_t|$ and $|\hat{\hat{x}}_t|$ are all $o(e^{\mu t})$ as $t \to \infty$ for any $\mu > A_+$, where $A_+ = \max(A, 0)$. By (2.20b), $A_+ < \lambda/2$. From (2.23), (2.24) it then follows that $\xi_T \to 0$ as $T \to \infty$. The author wishes to thank D. Hernandez-Hernandez for helpful comments concerning this argument.

## 3. – The solution operator

Let $S_T$ denote the nonlinear operator which maps the initial data $\phi$ in (2.7) to the optimal cost function $V$ defined by (2.5):

$$S_T\phi(x) = V(T, x). \tag{3.1}$$

Recall that $V$ satisfies the PDE (2.6) in the viscosity sense. Let $\mathcal{C}$ denote the cone of all functions $\phi$ such that $\phi$ is semiconcave and satisfies (A2), (A3). By results of Section 2, $S_T$ maps $\mathcal{C}$ into $\mathcal{C}$. Moreover,

$$S_T\phi \leq S_T\psi \text{ if } \phi \leq \psi; \tag{3.2}$$

$$S_T(\phi + c) = S_T\phi + c, c \text{ any constant.} \tag{3.3}$$

Moreover, the solution operator has the following important property. Consider a collection $Z$ of functions $\phi_z \in \mathcal{C}$.

LEMMA 3.1. *If* $\phi(x) = \min_{z\in Z}\phi_z(x)$ *for all* $x$, *then*

$$S_T\phi(x) = \min_{z\in Z} S_T\phi_z(x) \textit{ for all } x. \tag{3.4}$$

PROOF. Since $\phi \leq \phi_z$ for all $z$, (3.2) implies that $S_T\phi(x)$ is no more than the right side of (3.4). Given $(T, x)$, let $w^*_\cdot$ minimize $J(T, x; w_\cdot)$ with associated solution $x^*_\cdot$ to (2.1), $x^*_T = x$. Then $\phi(x^*_0) = \phi_\zeta(x^*_0)$ for some $\zeta \in Z$ and hence

$$\begin{aligned} S_T\phi(x) &= J(T, x; w^*_\cdot) = J_\zeta(T, x; w^*_\cdot) \\ &\geq S_T\phi_\zeta(x) \geq \min_{z\in Z} S_T\phi_z(x), \end{aligned}$$

where for $J_\zeta$ we replace $\phi$ by $\phi_\zeta$ in (2.3′). □

Properties (3.3), (3.4) state that the solution operator $S_T$ is "linear", not with respect to the usual addition and scalar multiplication but when considered with respect to the so-called "max-plus" algebra [AQV]. We shall return to this point in Section 5 in discussing the small noise asymptotics of linear, parabolic PDEs which arise in risk sensitive filtering. See Remark 5.3.

BASIS FUNCTIONS. It is often useful to approximate solutions to linear, time-dependent PDEs by linear combinations of solutions which have initial data chosen from a given set of basis functions. Something similar can be done for solutions $V(T, x)$ to (2.6)-(2.7) provided we work in the max-plus algebra. Let us consider "basis functions" of the form

$$\phi_z(x) = \frac{C}{2}|x - C^{-1}z|^2 \tag{3.5}$$

where for notational simplicity we suppress the dependence of $\phi_z$ on $C$.

We wish to represent a semiconcave function $\phi$ in terms of basis functions $\phi_z$. For simplicity, let us first assume that $\phi(x) - \frac{1}{2}C_1|x|^2$ is concave on $\mathbb{R}^n$ for some $C_1$. Let $C > C_1$ and as in (2.10) $\tilde{\phi}(x) = \phi(x) - \frac{1}{2}C|x|^2$. Then $\tilde{\phi}$ is concave on $\mathbb{R}^n$ and $|x|^{-1}\tilde{\phi}(x) \to -\infty$ as $|x| \to \infty$. Let $a(z)$ be the dual of the convex function $-\tilde{\phi}$:

$$a(z) = \max_{x\in\mathbb{R}^n}[z \cdot x + \tilde{\phi}(x)].$$

Then - $\tilde{\phi}(x)$ is the dual of the convex function $a(z)$ [RW, Chap. 11]:

$$-\tilde{\phi}(x) = \max_{z\in\mathbb{R}^n}[z \cdot x - a(z)],$$

or equivalently

$$\phi(x) = \min_{z\in\mathbb{R}^n}[\phi_z(x) + a_1(z)],\qquad a_1(z) = a(z) - \frac{1}{2C}|z|^2. \tag{3.6}$$

By (3.3) and (3.4)

$$S_T\phi(x) = \min_{z\in\mathbb{R}^n}[S_T\phi_z(x) + a_1(z)]. \tag{3.7}$$

To remove the global restriction above on $\tilde{\phi}$, let $\phi$ be semiconcave. Given a compact set $K$, it suffices to consider disturbances $w_.$ such that $|x_t| \leq R$ where $R$ depends only on $K$ (the range of dependence property in Section 2.) We can choose $\bar{\phi}$ such that $\bar{\phi}(x) = \phi(x)$ whenever $|x| \leq R$ and for suitable $C_R$

$$\tilde{\phi}(x) = -\frac{1}{2}C_R|x|^2 + \bar{\phi}(x)$$

is concave on $\mathbb{R}^n$ with $|x|^{-1}\tilde{\phi}(x) \to -\infty$ as $|x| \to \infty$. Then $S_T\bar{\phi}(x) = S_T\phi(x)$ for all $(T, x) \in K$ and we can replace $\bar{\phi}$ by $\phi$ in (3.7). For example one can choose

$$\bar{\phi}(x) = \phi(x)\theta(x) - B(|x|^2 - R^2)^+$$

where $B$ is sufficiently large and $\theta$ is smooth with $\theta(x) = 1$ for $|x| \leq R$ and $\theta(x) = 0$ for $|x| \geq R + 1$.

Basis representations may be useful in connection with a numerical technique which does part of the task of solving (2.6)-(2.7) "off line" rather than in real time. This will be discussed in a forthcoming paper with McEneaney. A similar approach for solving the Zakai stochastic PDE of nonlinear filtering was developed by Rozovskii and associates. See [LMR].

## 4. – Robust filters

The definition of robust filter which we will use is a slight modification of one introduced by McEneaney [Mc2]. Let $e_T$ be any estimator for the state $x_T$ at time $T$. We consider a measure $\ell(|x_T - e_T|)$ of the loss from estimation error $x_T - e_T$. In [Mc2] the quadratic loss function $\ell(r) = \frac{1}{2}|r|^2$ is used. We assume:

$$\text{(A5)} \qquad \begin{aligned} &\ell \in C^2(\mathbb{R}^1_+) \text{ with } \ell(0) = \ell'(0) = 0, \\ &\ell'(r) > 0 \text{ for } r > 0, \quad 0 < \ell''(r) \leq C \text{ for } r \geq 0. \end{aligned}$$

Let $\mu > 0$ be a parameter. We say that $e_T$ achieves robust estimation at level $\mu$ if, for all $x_0, w_., v_.$

$$\mu\ell(|x_T - e_T|) \leq \phi(x_0) + \frac{1}{2}\int_0^T [|w_t|^2 + |v_t|^2]dt. \tag{4.1}$$

If one sets $\gamma^2 = \mu^{-1}$, then $\gamma$ has the role of a familiar $H_\infty$ - bound parameter in robust control and estimation. See [BB] [Mc2]. Since $\ell''(0) > 0$, for small estimation errors the left side of (4.1) behaves nearly like a quadratic. However (A5) allows for loss functions with less than quadratic growth of $\ell(r)$ as $|r| \to \infty$. In Section 5 we will consider linearly growing $\ell(r)$, such as for example $\ell(r) = (1 + r^2)^{1/2} - 1$.

By (2.3) and (2.5), $e_T$ achieves robust estimation at level $\mu$ at time $T$ if and only if

$$\mu\ell(|x - e_T|) - V(T, x) \leq 0 \tag{4.1$'$}$$

for all $x \in \mathbb{R}^n$. Under rather general assumptions we should expect that there are many robust estimators $e_T$ if $\mu < \mu^*$, and no robust estimators if $\mu > \mu^*$, for a certain critical parameter value $\mu^*$. See [Mc2] for the case of quadratic $\ell(r)$ and quadratically growing $\phi(x)$. Let us consider the following estimator, introduced in [Mc2].

MINIMAX ESTIMATOR. Let

$$G(T, e) = \max_x[\mu\ell(|x - e)| - V(T, x)].$$

We assume that the maximum is attained (finite). By (A5) the function $\ell(|x - \cdot|)$ is strictly convex on $\mathbb{R}^n$ for each $x$ and moreover

$$\lim_{|e|\to\infty} G(T, e) = +\infty. \tag{4.2}$$

This implies that $G(T, \cdot)$ is strictly convex on $\mathbb{R}^n$ and has a minimum at a unique $\hat{e}^0_T$. We call $\hat{e}^0_T$ the *minimax estimator* for $x_T$. By (4.1$'$) the minimax estimator $\hat{e}^0_T$ is robust if and only if

$$G(T, \hat{e}^0_T) = \min_e \max_x[\mu\ell(|x - e|) - V(T, x)] \leq 0. \tag{4.3}$$

If the min and max are taken in the opposite order then

$$\max_x \min_e[\mu\ell(|x - e|) - V(T, x)] = \max_x[-V(T, x)] \leq 0$$

and the Mortensen estimator $\hat{x}_T$ attains the maximum over $x$. In general, min max $\geq$ max min and $\hat{e}_T^0$ is not the same as $\hat{x}_T$.

EXAMPLE 4.1. Take $T = 0, n = 1$ and

$$\ell(r) = \frac{1}{2}r^2, \phi(x) = (x^2 - 1)^2 + \alpha x^2,$$

with $0 < \alpha < 1$, $\mu < 2\alpha$. A calculation shows that $\hat{e}_0^0 = 0$ is robust and that the minimum of $\phi(x)$ occurs at $\pm\hat{x}_0 \neq 0$. □

On the other hand, if max min = min max, then $\hat{e}_T^0 = \hat{x}_T$. In particular, this saddle point property holds if $\mu\ell(|x - e|) - V(T, x)$ is concave in $x$, since this function is convex in $e$. For instance, this happens in the Kalman - Bucy setting (Remark 2.3) with $\phi(x)$ and $\ell(r)$ quadratic. Then $\frac{\mu}{2}|x - e|^2 - V(T, x)$ is quadratic, and concave in $x$ for $\mu < \mu^*$ where $\mu^*$ is the critical parameter value.

SMALL OBSERVATION NOISE. When $\rho$ is small in (2.2), then $h(x_T)$ can be estimated with small error. We call this the case of small observation noise. In general, the filtering problem has no easy solution for small $\rho$, or even for the case $\rho = 0$ considered in [JL]. We consider here only the following easy case in which states and observations have the same dimension $n = p = 1$ and $h$ is one-one with Lipschitz inverse:

$$|h_x(x)| \geq c_1 > 0. \tag{4.4}$$

Following an argument in Picard [P], for the corresponding stochastic small observation noise model, the following proposition provides an easy way to generate robust filters, in this special case.

PROPOSITION 4.2. *Assume* (4.4) *and let* $\psi_.$ *be any bounded function such that* $\psi_t \geq c > 0$. *Define the estimator* $e_T$ *by*

$$\frac{d}{dT}e_T = f(e_T) + \rho^{-1}\psi_T(y_T - h(e_T)), \tag{4.5}$$

*with initial data* $e_0$. *Let* $\phi(x) = k(x - e_0)^2$ *where* $k > 0$ *and* $\ell(r) = \frac{1}{2}r^2$. *Then given* $\mu > 0$, $T_0 > 0$ *there exists* $\rho_0 > 0$ *such that* $e_T$ *achieves robust estimation at level* $\mu$ *for* $T \geq T_0$ *and* $0 < \rho \leq \rho_0$.

PROOF. We may assume that $h_x > 0$. By subtracting (4.5) from (2.1), we have

$$\frac{d}{dt}|x_t - e_t|^2 = 2(x_t - e_t)\left[f(x_t) - f(e_t) - \frac{\psi_t}{\rho}(h(x_t) - h(e_t)) + \sigma(x_t)w_t - \psi_t v_t\right],$$

$$2(x_t - e_t)\psi_t(h(x_t) - h(e_t)) \geq 2cc_1(x_t - e_t)^2.$$

By using the inequality $2ab \le \mu a^2 + \mu^{-1} b^2$ and the boundedness of $\sigma(x_t)$ and $\psi_t$, we obtain for small $\rho$

$$\frac{\mu}{2}\frac{d}{dt}|x_t - e_t|^2 \le -\frac{\mu c c_1}{2\rho}|x_t - e_t|^2 + \frac{1}{2}(w_t^2 + v_t^2),$$

$$\frac{\mu}{2}|x_T - e_T|^2 \le \frac{\mu}{2}|x_0 - e_0|^2 \exp\left(\frac{-\mu c_1 c T}{2\rho}\right) + \frac{1}{2}\int_0^T [w_t^2 + v_t^2]dt.$$

The first term on the right side is no more than $\phi(x_0)$ if $T \ge T_0 > 0$ and $\rho$ is small enough ($\rho \le \rho_0$.) □

Under the special circumstances of Proposition 4.2, it seems likely that the minimax and Mortensen estimators are the same. It would be interesting to find choices of $\psi_t$ in (4.5) which give especially good approximations to the Mortensen estimator $\hat{x}_T$. (See [P] for detailed analysis of a corresponding question in the stochastic model, in which $\hat{x}_T$ is replaced by the conditional mean.) One possibility is to find $\psi_T$ from (2.9) with $V_{xx}(T, \hat{x}_T)$ replaced by an approximation obtained by linearizing $f(x_t)$ and $h(x_t)$ about $f(y_t)$ and $h(y_t)$, and a quadratic appromiation to $\phi(x_0)$ about $\phi(y_0)$. The solution to this linear-quadratic approximation suggests that $\rho V_{xx}(T, \hat{x}_T)$ is bounded below by a positive constant if $T \ge T_0 > 0$ and $\rho$ is small.

REMARK 4.2. A more general formulation of robust filtering involves estimation errors over $0 \le t \le T$ as well as at time $T$. Let $\ell(r) = \frac{1}{2}r^2$. Then the formulation in [BJP] requires that, for all $x_0, w_., v_.$

$$\mu_1|x_T - e_T|^2 + \mu_2\int_0^T |x_t - e_t|^2 dt \le 2\phi(x_0) + \int_0^T [|w_t|^2 + |^2 v_t|^2]dt, \tag{4.6}$$

where $\mu_1, \mu_2$ are nonnegative parameters (not both 0.) In (4.1) we took $\mu_1 = \mu, \mu_2 = 0$. Other authors [DBB] [Kr] took $\mu_1 = 0, \mu_2 > 0$. When $\mu_2 > 0$ the analysis becomes more complicated since the dynamics of an information state function $\bar{V}(T, x)$, analogous to our $V(T, x)$, depend on the estimation trajectory $e_.$ which is to be chosen optimally. Under restrictive assumptions there is an analogue $\bar{x}_T$ of the Mortensen estimator $\hat{x}_T$, obtained by minimizing $\bar{V}(T, \cdot)$ [Kr, Thm. 4.17]. The function $\bar{V}$ satisfies an equation (or inequality) rather similar to (2.6). However, it is not a PDE (or partial differential inequality) since a term involving $|x - \bar{x}_T|^2$ must be added to the right side of (2.6). See [Kr, formula 4.4.12]. The robust filtering problem with $\mu_2 > 0$ and $\mu_1 = 0$ is a special case of a nonlinear $H_\infty$-control problem with partial state observations. The control is the estimate $e_t$, which affects the running cost but not the state dynamics. The result in [Kr] just mentioned is an instance of a certainty equivalence principle in $H_\infty$ control. See also [JBE, Section 4.6]

## 5. – Risk sensitive filters

Instead of the deterministic model in Section 2, let us now consider a stochastic filter model for system states and observations. To simplify the presentation, let us make in this section the following assumptions:

(A6)
(a) (A1) holds and $\sigma$ = identity matrix;
(b) $\phi$ is Lipschitz, $\phi \geq 0$, $\lim_{|x|\to\infty} |x|^{-1}\phi(x) > 0$;
(c) (A5) holds and $\ell'(r)$ is bounded.

In addition, we only sketch various arguments which are well known and are given in more detail elsewhere. The development will be similar to that in [FM2] where a technically more difficult set of assumptions is made (including $\sigma$ nonconstant, $\phi(x)$ quadratically growing as $|x| \to \infty$.)

Let $X_t^\varepsilon$ denote the state process and $Y_t^\varepsilon$ an accumulated observations process. They satisfy stochastic differential equations

$$dX_t^\varepsilon = f(X_t^\varepsilon)dt + \sqrt{\varepsilon}dB_t \tag{5.1}$$

$$dY_t^\varepsilon = h(X_t^\varepsilon)dt + \sqrt{\varepsilon}\rho d\tilde{B}_t, \quad Y_0^\varepsilon = 0, \tag{5.2}$$

where $\varepsilon > 0$ is a parameter and $B_\cdot$, $\tilde{B}_\cdot$ are independent Brownian motion processes. Moreover, $X_0^\varepsilon$ is independent of $B_\cdot$, $\tilde{B}_\cdot$ and has density $k_\varepsilon \exp[-\varepsilon^{-1}\phi(x)]$ where $k_\varepsilon$ is a normalizing constant. The traditional nonlinear filtering problem is to find a minimum expected least squares estimate $e_T$ for $X_T^\varepsilon$, such that $e_T$ is measurable with respect to the $\sigma$ - algebra generated by the accumulated observations $Y_t^\varepsilon$ for $0 \leq t \leq T$. For the risk sensitive filter, instead of expected least squares, expected exponential-of-loss is to be minimized. Thus, $e_T$ is chosen to minimize

$$E\exp\left[\frac{\mu}{\varepsilon}\ell(|X_T^\varepsilon - e_T|)\right],$$

where $\mu > 0$ is a parameter.

The risk sensitive filter problem can be restated in terms of an unnormalized conditional density $q^\varepsilon(T,x)$, which satisfies the Zakai stochastic PDE

$$dq^\varepsilon = \mathcal{L}_\varepsilon^* q^\varepsilon dT + \frac{hq^\varepsilon}{\varepsilon\rho^2}dY_T^\varepsilon \tag{5.3}$$

with initial data

$$q^\varepsilon(0,x) = \exp\left[-\varepsilon^{-1}\phi(x)\right]. \tag{5.4}$$

Here $\mathcal{L}_\varepsilon$ is the generator of the Markov process $X_t^\varepsilon$:

$$\mathcal{L}_\varepsilon g = \frac{\varepsilon}{2}\Delta g + f\cdot g_x \tag{5.5}$$

and $\mathcal{L}_\varepsilon^*$ is the (formal) adjoint of $\mathcal{L}_\varepsilon$. See for example [Da] [Ku]. The risk sensitive filter problem can be reformulated as one of choosing $e = \hat{e}_T^\varepsilon$ which minimizes

$$\Phi^\varepsilon(T, e) = \int_{\mathbb{R}} \exp\left[\frac{\mu}{\varepsilon}\ell(|x - e|)\right] q^\varepsilon(T, x)dt. \tag{5.6}$$

By the estimate (5.12) below, $q^\varepsilon(T, x)$ decreases to 0 at an exponential rate as $|x| \to \infty$, which assures finiteness of the integral for small enough $\mu$.

We wish to show that the optimal stochastic risk sensitive estimator tends to the deterministic minimax estimator in Section 4 as $\varepsilon \to 0$. This cannot be done directly, since a fixed observation function $y_.$ is given in the deterministic filter model, while $Y_.^\varepsilon$ is a nowhere differentiable sample path of a stochastic process. This apparent difficulty is avoided by using pathwise nonlinear filtering theory. This provides a version of the unnormalized conditional density process $q^\varepsilon$ which "depends continuously" on the accumulated observation sample paths. See [JB] [Da] [FP]. By using pathwise filtering theory it suffices to consider any fixed smooth (class $C^1$) accumulated observation function $Y_.$ and to rewrite the Zakai equation (5.3) in Stratonovich form:

$$\frac{\partial q^\varepsilon}{\partial T} = \mathcal{L}_\varepsilon^* q^\varepsilon - \frac{1}{\varepsilon\rho^2}\left(\frac{1}{2}|h|^2 - y_T \cdot h\right) q^\varepsilon \tag{5.3$'$}$$

where $y_T = dY_T/dT$. Let us multiply $q^\varepsilon$ by a convenient factor depending only on the given observation path $y_.$, which does not affect the minimization over $e$ in (5.6). Let

$$\tilde{q}^\varepsilon(T, x) = q^\varepsilon(T, x) \exp\left[-\frac{1}{2\varepsilon\rho^2}\int_0^T |y_t|^2 dt\right].$$

Then $\tilde{q}^\varepsilon$ satisfies the linear parabolic PDE

$$\frac{\partial \tilde{q}^\varepsilon}{\partial T} = \frac{\varepsilon}{2}\Delta\tilde{q}^\varepsilon - \operatorname{div}\,(f\tilde{q}^\varepsilon) - \frac{1}{2\varepsilon\rho^2}|y_T - h|^2\tilde{q}^\varepsilon. \tag{5.7}$$

Let $V^\varepsilon = -\varepsilon \log \tilde{q}^\varepsilon$. Then

$$\frac{\partial V^\varepsilon}{\partial T} = \frac{\varepsilon}{2}\Delta_x V^\varepsilon - f \cdot V_x^\varepsilon - \frac{1}{2}|V_x^\varepsilon|^2 + \frac{1}{2\rho^2}|y_T - h|^2 + \varepsilon \operatorname{div} f \tag{5.8}$$

with initial data

$$V^\varepsilon(0, x) = \phi(x). \tag{5.9}$$

Note that, when $\varepsilon = 0$, (5.8)-(5.9) become (2.6)-(2.7) in case $\sigma$ = identity. By viscosity solution methods it can be proved that $V^\varepsilon \to V$ uniformly on compact sets as $\varepsilon \to 0$. See [JB]. However, let us sketch a direct stochastic control proof of this fact (see Lemma 5.1(a).)

There is the following stochastic control representation for $V^\varepsilon(T,x)$. Let $\xi_t^\varepsilon$ denote a controlled process, satisfying backward in time the stochastic differential equation

$$d\xi_t^\varepsilon = [f(\xi_t^\varepsilon) + w_t]dt + \sqrt{\varepsilon}d\beta_t, 0 \le t \le T, \qquad (5.10)$$
$$\xi_T^\varepsilon = x,$$

with $\beta_.$ a backward in time Brownian motion and $w_.$ any backward in time bounded progressively measurable control process. (A more precise formulation in terms of reference probability systems is given in [FS, p. 160].) Let

$$J^\varepsilon(T,x;w_.) = E\left\{\phi(\xi_0^\varepsilon) + \int_0^T \left[\varepsilon \text{ div } f(\xi_t^\varepsilon) + \frac{1}{2}|w_t|^2 + \frac{1}{2\rho^2}|y_t - h(\xi_t^\varepsilon)|^2\right] dt\right\}.$$

Since (A6) holds, a standard argument [FS p. 190] gives a bound $|J_x^\varepsilon| \le M$, where $M$ depends only on $T$. The dynamic programming PDE for the problem of minimizing $J^\varepsilon$, as a function of $w_.$, is (5.8). A verification theorem then yields

$$V^\varepsilon(T,x) = \inf_{w_.} J^\varepsilon(T,x;w_.).$$

Moreover, $|V_x^\varepsilon| \le M$, which implies that it suffices to assume that $|w_t| \le M$ since $w_t^* = V_x^\varepsilon(t, \xi_t^{\varepsilon *})$ provides an optimal (feedback) control. Here $\xi_t^{\varepsilon *}$ solves (5.10) with $w_t = w_t^*$.

LEMMA 5.1. (a) *There exists* $C_1(T)$ *such that*

$$|V^\varepsilon(T,x) - V(T,x)| \le C_1(T)\sqrt{\varepsilon}. \qquad (5.11)$$

(b) *There exist positive* $K_1(T)$, $K_2(T)$ *such that, for* $0 < \varepsilon < 1$,

$$V^\varepsilon(T,x) \ge K_1(T)|x| - K_2(T). \qquad (5.12)$$

SKETCH OF PROOF. Given any sample path $w_.$ for an admissible control process such that $|w_t| \le M(M = M(T))$, let $x_t$ be the solution of (2.1) with $\sigma$ = identity and $x_T = x$. Then

$$\|x_. - \xi_.^\varepsilon\| \le C(T)\sqrt{\varepsilon}\|\beta_.\|$$

where $\| \quad \|$ is the sup norm on $[0,T]$. By (A6) this implies for all such control processes $w_.$

$$|J^\varepsilon(T,x;w_.) - EJ(T,x;w_.)| \le C_1(T)\sqrt{\varepsilon},$$

which implies (a).

To prove (b) it suffices by (a) to show that

$$V(T,x) \ge \bar{K}_1(T)|x| - \bar{K}_2(T) \qquad (5.13)$$

for suitable $\bar{K}_1(T), \bar{K}_2(T)$. For this purpose choose $\alpha > 0$ such that

$$\alpha(|x|^2 + 1) - x \cdot (f(x) + w) \geq 0$$

for all $x$, $|w| \leq M$. Then

$$\frac{d}{dt}\left[(|x_t|^2 + 1)\exp 2\alpha(T - t)\right] \leq 0,$$
$$|x_0|^2 + 1 \geq (|x|^2 + 1)\exp(-2\alpha T).$$

By using (A6) (b), for suitable $\bar{K}_1(T), \bar{K}_2(T)$

$$J(T, x; w_.) \geq \bar{K}_1(T)|x| - \bar{K}_2(T)$$

for all $w_.$ such that $|w_t| \leq M(T)$. By (2.5) this gives (b). □

By (5.12)

$$q^\varepsilon(T, x) \leq c_\varepsilon(T)\exp[-\varepsilon^{-1}K_1(T)|x|].$$

Moreover, by (A6) (c), $\ell(r) \leq Dr$ for some constant $D$. Then

$$\mu\ell(|x - e|) - K_1(T)|x| \leq \mu D|x - e| - K_1(T)|x|, \tag{5.14}$$

which implies finiteness of the integral in (5.6) for $\mu < \mu_1$, where $\mu_1 = D^{-1}K_1(T)$. Since $\ell(|x - \cdot|)$ is strictly convex, the function $\Phi^\varepsilon(T, \cdot)$ in (5.6) is strictly convex. Moreover, $\Phi^\varepsilon(T, e) \to +\infty$ as $|e| \to \infty$. Hence, $\Phi^\varepsilon(T, e)$ has a minimum at a unique $\hat{e}_T^\varepsilon$, called the *risk sensitive* estimator for $x_T$.

The main result of this section is the following.

THEOREM 5.2. *As $\varepsilon \to 0$ the risk sensitive estimator $\hat{e}_T^\varepsilon$ tends to the minimax estimator $\hat{e}_T^0$, provided $\mu < D^{-1}K_1(T)$ with $K_1(T)$ as in* (5.12).

PROOF. Let

$$F^\varepsilon(T, x, e) = \mu\ell(|x - e|) - V^\varepsilon(T, x),$$
$$G^\varepsilon(T, e) = \varepsilon \log \int_{\mathbb{R}^n} \exp[\varepsilon^{-1}F^\varepsilon(T, x, e)]dx.$$

Then $\hat{e}_T^\varepsilon$ minimizes $G^\varepsilon(T, \cdot)$, since $G^\varepsilon(T, e)$ is a function of $T$ plus $\log \Phi^\varepsilon(T, e)$. A standard Laplace-Varadhan asymptotic principle argument together with (5.11), (5.14) shows that $G^\varepsilon(T, e) \to G(T, e)$ uniformly on compact sets (for each fixed $T$) as $\varepsilon \to 0$, provided $\mu < K_1(T)$. Since $G(T, \cdot)$ is strictly convex and is minimum at $\hat{e}_T^0$ the conclusion follows. □

REMARK 5.3. We write $\tilde{q}^\varepsilon \sim \exp[-\varepsilon^{-1}V]$ in the sense that $V^\varepsilon = -\varepsilon \log \tilde{q}^\varepsilon$ tends to $V$ uniformly on compact sets as $\varepsilon \to 0$. The PDE (5.7) for $\tilde{q}^\varepsilon$ is linear. Thus

$$\alpha^\varepsilon \sim \exp\left[-\varepsilon^{-1}c\right] \text{ implies } \alpha^\varepsilon \tilde{q}^\varepsilon \sim \exp\left[-\varepsilon^{-1}(V+c)\right].$$

Similarly, if $\tilde{q}_i^\varepsilon \sim \exp[-\varepsilon^{-1}V_i]$ is a solution to (5.7) for $i = 1, 2$, then

$$\tilde{q}_1^\varepsilon + \tilde{q}_2^\varepsilon \sim \exp\left[-\varepsilon^{-1}\min(V_1, V_2)\right].$$

These asymptotic properties make clear why properties (3.3), (3.4) of the solution operator must hold, and thus $S_T$ is a linear operator in the max plus algebra.

## REFERENCES


[AQV] M. AKIAN – J.-P. QUADRAT – M. VIOT, *Bellman Processes*, in "Lecture Notes in Control and Information Science", No. 199, Springer-Verlag, 1994, 302-311.

[BB] T. BASAR – P. BERNHARD, *"$H_\infty$-Optimal Control and Related Minimax Design Problems"*, Birkhauser, Boston, 1991.

[BJP] R. K. BOEL – M. R. JAMES – I. R. PETERSEN, *Robustness and Risk Sensitive Filtering*, submitted to IEEE Trans. Auto. Control.

[Da] M. H. A. DAVIS, "Lectures on Stochastic Control and Nonlinear Filtering", Tata Institute Lectures on Math. and Physics No. 75, Springer Verlag, 1984.

[DBB] G. DIDINSKY – T. BASAR – P. BERNHARD, *Structural properties of minimax policies of a class of differential games arising in nonlinear $H_\infty$-control and filtering*, in "Proc. 32nd IEEE Conf. on Decision and Control", San Antonio, 1993.

[FM1] W. H. FLEMING – W. M. MCENEANEY, *Risk sensitive optimal control and differential games*, in "Lecture Notes in Control and Info. Sci.", No. 184, Springer Verlag, 1992, 185-197.

[FM2] W. H. FLEMING – W. M. MCENEANEY, *Risk Sensitive and Robust Nonlinear Filtering*, in "Proc. 36-th IEEE Conf. on Decision and Control", San Diego, 1997.

[FP] W. H. FLEMING – E. PARDOUX, *Optimal control for partially observed diffusions*, SIAM J. Control and Optim. **2** (1982), 261-285.

[FR] W. H. FLEMING – R. W. RISHEL, "Deterministic and Stochastic Optimal Control", Springer-Verlag, 1975.

[FS] W. H. FLEMING – H. M. SONER, "Controlled Markov Processes and Viscosity Solutions", Springer-Verlag, 1992.

[H] O. HIJAB, "Minimum Energy Estimation", Ph.D. Dissertation, Univ. of Calif., Berkeley, 1980.

[JB] M. R. JAMES – J. S. BARAS, *Nonlinear filtering and large deviations: A PDE-control theoretic approach*, Stochastics **23** (1988), 391-412.

[JBE] M. R. JAMES – J. S. BARAS – R. J. ELLIOTT, *Risk-sensitive control and dynamic games for partially observed discrete-time nonlinear systems*, IEEE Trans. Auto. Control **39** (4) (1994), 780-792.

[JL] M. Joannides – F. LeGland, *Nonlinear filtering with continuous time perfect observations and noninformative quadratic variation*, in "Proc. 36th IEEE Conf. on Decision and Control", San Diego, 1997.

[Kr] A. J. Krener, *Necessary and sufficient conditions for nonlinear worst case ($H_\infty$) control and estimation*, J. Math. Systems, Estimation and Control **4** (1994), 485-488; erratum **5** (1995), 257.

[Ku] H. Kunita, *Stochastic partial differential equations connected with nonlinear filtering*, in "Lecture Notes in Math.", No. 972, Springer-Verlag, 1982.

[LMR] S. Lototsky – Mikulevicius – B. L. Rozovskii, *Nonlinear filtering revisited: a spectral approach*, SIAM J. Control Optim. **35** (1997), 435-461.

[Mc1] W. M. McEneaney, *Uniqueness for viscosity solutions of nonstationary HJB equations under some a priori conditions (with applications)*, SIAM J. Control and Optim **33** (1995), 1560-1576.

[Mc2] W. M. McEneaney, *Robust/$H_\infty$ Filtering for Nonlinear Systems*, to appear in Systems Control Lett.

[Mi] S. K. Mitter, *Lectures on nonlinear filtering and stochastic control*, in "Lecture Notes in Math.", No. 972, Springer-Verlag, 1982.

[Mo] R. E. Mortensen, *Maximum likelihood recursive nonlinear filtering*, J. Optim. Theory Appl. **2** (1968), 386-394.

[OP] D. Ocone – E. Pardoux, *Asymptotic stability of the optimal filter with respect to its initial condition*, SIAM J. Control Optimiz. **34** (1996), 226-242.

[P] J. Picard, *Nonlinear filtering of one-dimensional diffusions in case of a high-signal-to-noise ratio*, SIAM J. Appl. Math. **46** (1986), 1098-1125.

[RW] R. T. Rockafellar – R. J. B. Wets, "Variational Analysis", Springer Verlag, 1997.

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