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A Corrector Result for H-Converging Parabolic Problems with Time-Dependent Coefficients

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In memory of Ennio De Giorgi,
of his exceptional mathematical ideas,
and of his equally exceptional human qualities.

Abstract. In this paper we consider a sequence of linear parabolic problems with coefficients which may depend on time, namely

$$\begin{cases} \partial_t u^\varepsilon - \operatorname{div}\left(A^\varepsilon(t,x)\,\nabla u^\varepsilon\right) = f \quad \text{in } \mathcal{D}'(Q)\,, \\ u^\varepsilon \in L^2\left(0,T; H_0^1(\Omega)\right) \cap H^1\left(0,T; H^{-1}(\Omega)\right)\,, \\ u^\varepsilon\Big|_{t=0} = a\,. \end{cases}$$

Assuming that the sequence of matrices $\{A^\varepsilon(x,t)\}$ H-converges, we prove that there exists a matrix p^ε with entries in $L^2(Q)$ such that

$$\nabla u^\varepsilon - p^\varepsilon \nabla u^0 \to 0 \quad \text{strongly in } L^1\left(Q; \mathbb{R}^N\right);$$

this is a corrector result for the spatial gradient. We also prove a corrector result for the time derivative $\partial_t u^\varepsilon$.

1. - Introduction

Convergence problems in the Calculus of Variations and in the related equations were for a long time one of Ennio De Giorgi's favorite topics. His ideas deeply influenced the school he developed around him and the entire mathematical community working in this field.

In order to study these problems in the Calculus of Variations, Ennio De Giorgi introduced the notion of Γ-convergence, a very fruitful concept as can be seen by the many results it produced. This concept has in some sense its roots in the paper [8], where De Giorgi and Spagnolo used the variational viewpoint to study the G-convergence, a notion introduced by Spagnolo in [13] and [14] in order to study the convergence of solutions of elliptic problems.

In the present paper we will study the corrector problem for a sequence of G-converging parabolic operators, or more exactly, following the terminology used by Tartar and Murat in order to stress the fact that the matrices are not assumed to be symmetric (see [18]), of a sequence of H-converging matrices. More precisely we will consider a sequence of linear parabolic problems with coefficients which may depend on time, of the form

$$(0^\varepsilon) \qquad \begin{cases} \partial_t u^\varepsilon - \operatorname{div}\big(A^\varepsilon(t,x)\,\nabla u^\varepsilon\big) = f & \text{in } \mathcal{D}'(Q), \\ u^\varepsilon \in L^2(0,T;H_0^1(\Omega)) \cap H^1\big(0,T;H^{-1}(\Omega)\big), \\ u^\varepsilon\Big|_{t=0} = a, \end{cases}$$

in a cylinder $Q = (0,T)\times\Omega$, where $\{A^\varepsilon\}$ is a sequence of measurable, uniformly bounded, uniformly elliptic matrices on Q, which H-converges to a matrix A^0 of the same type. We will prove a corrector result for this problem.

By H-convergence of A^ε (see Definition 2.1 below) we mean that for every choice of the data $f \in L^2\big(0,T;H^{-1}(\Omega)\big)$ and $a \in L^2(\Omega)$, the (unique) solution u^ε of (0^ε) converges weakly in the energy space $L^2\big(0,T;H_0^1(\Omega)\big) \cap H^1\big(0,T;H^{-1}(\Omega)\big)$ to the unique solution u^0 of (0^0) (the problem corresponding to A^0), together with the weak convergence in $L^2(Q;\mathbb{R}^N)$ of $A^\varepsilon(t,x)\nabla u^\varepsilon$ to $A^0(t,x)\nabla u^0$. This notion was introduced, under the names of G-convergence and PG-convergence, by Spagnolo, who proved that from every sequence of measurable, uniformly bounded, uniformly elliptic matrices on Q, one can extract a subsequence which H-converges to some matrix A^0, see the papers [13], [14], [15] by Spagnolo and [6] by Colombini and Spagnolo (see also the surveys [19] and [20] by Zhikov, Kozlov and Oleinik, and the paper [16] by Svanstedt for the nonlinear monotone case).

This problem was also studied by Bensoussan, Lions and Papanicolaou, who in their book [3] considered the particular case of periodic coefficients (see also the book [12] by Sanchez-Palencia). In this case the matrices A^ε are of the form

$$(1) \qquad A^\varepsilon(t,x) = A\left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right),$$

where $k \geq 0$ is a fixed real number, and the matrix A is defined on $\mathbb{R}\times\mathbb{R}^N$, elliptic, bounded, periodic with period $(0,T_0)\times\prod_{i=1}^N(0,Y_i)$ (T_0 and Y_i, $i = 1,\dots,N$, are positive real numbers). In this case there are "explicit" formulas which allow one to compute the limit matrix A^0. These formulas involve solving N auxiliary special problems on the period cell $(0,T_0)\times\prod_{i=1}^N(0,Y_i)$, the solutions of which we will denote by $\Phi_i(\tau,y)$ in the present paper. In addition to the weak convergence of the solution u^ε of (0^ε) to the solution u^0 of (0^0), which can be expressed as

$$(2) \qquad u^\varepsilon - u^0 \rightharpoonup 0 \quad \text{weakly in } L^2\big(0,T;H_0^1(\Omega)\big),$$

Bensoussan, Lions and Papanicolaou proved a corrector result, namely that, when $k = 1$, 2 or 3, and A is smooth enough,

$$(3)\quad u^\varepsilon - \left[u^0 - \varepsilon \sum_{i=1}^N \Phi_i\left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right) \partial_{x_i} u^0(t,x)\right] \to 0 \text{ strongly in } L^2\big(0,T; H_0^1(\Omega)\big).$$

The last result can be seen as an improvement of (2) since it provides an approximation of ∇u^ε in the strong topology of $L^2(Q;\mathbb{R}^N)$, namely

$$(4)\quad \nabla u^\varepsilon - \left[\nabla u^0 - \sum_{i=1}^N \nabla\Phi_i\left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right) \partial_{x_i} u^0(t,x)\right] \to 0 \text{ strongly in } L^2\big(Q;\mathbb{R}^N\big).$$

Formula (3) can be interpreted as the beginning of the asymptotic expansion of the solution of (0^ε).

In the present paper, our first aim is to obtain for the spatial gradient a corrector result similar to (3), or more exactly to (4), in the general case where one only requires that $\{A^\varepsilon(t,x)\}$ is a sequence of measurable, uniformly bounded and uniformly elliptic matrices, which H-converge to some matrix A^0.

In the elliptic case, the corresponding corrector result, in the case of matrices which are measurable, uniformly bounded and uniformly elliptic on Ω, has been proved by Tartar (see [18]). We will follow the same method, that consists in using "special test functions" w_i^ε which play a role similar to the functions Φ_i introduced by Bensoussan, Lions and Papanicolaou, in the sense that in the periodic case (1) studied in [3] one can take, when $k = 1$, 2 or 3,

$$w_i^\varepsilon(t,x) = x_i - \varepsilon\, \Phi_i\left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right), \qquad i = 1, \dots, N.$$

We define the $N \times N$ matrix $p^\varepsilon(t,x)$, whose entries are in $L^2(Q)$, by

$$p^\varepsilon e_i = \nabla w_i^\varepsilon$$

(where $e_i = \nabla x_i$ is the i-th vector of the canonical basis of $\mathbb{R}^N$). Our main result (see Theorem 3.8 below) then states that the solution u^ε of (0^ε) satisfies

$$(5)\qquad \nabla u^\varepsilon - p^\varepsilon \nabla u^0 \to 0 \quad \text{strongly in } L^2\big(Q;\mathbb{R}^N\big),$$

which is the generalization of (4) to the nonperiodic case, since in the periodic framework

$$p^\varepsilon e_i = \nabla w_i^\varepsilon = e_i - \nabla\Phi_i\left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right),$$

which implies that (5) is equivalent to (4).

There is actually some inaccuracy in (5), since this statement does not hold true in general, but requires some regularity assumptions on u^0 (and/or the functions w_i^ε); when no regularity assumption is made, convergence (5) only takes place in the larger space $L^1(Q;\mathbb{R}^N)$, *i.e.*,

$$\nabla u^\varepsilon - p^\varepsilon \nabla u^0 \to 0 \quad \text{strongly in } L^1\big(Q;\mathbb{R}^N\big). \tag{6}$$

In addition to this result, which is concerned with the asymptotic behavior of the spatial gradient of the solution, we prove a corrector result for the time derivative of u^ε, that is, we show that

$$\partial_t u^\varepsilon - \left[\partial_t u^0 + \sum_{i=1}^N \operatorname{div}\Big(\partial_{x_i} u^0 \left(A^\varepsilon \nabla w_i^\varepsilon - A^0 e_i\right)\Big)\right] \to 0 \quad \text{strongly in } L^2\big(0,T;H^{-1}(\Omega)\big), \tag{7}$$

or more exactly in the larger space $L^1\big(0,T;W^{-1,1}(\Omega)\big)$ in the case where no regularity assumptions are met by u^0 and the functions w_i^ε (see again Theorem 3.8 below). Statement (7), which provides a representation of $\partial_t u^\varepsilon$ similar to the representation given by (5) (or (6)), seems to be new.

As far as we know, very few corrector results were available in the parabolic case. As far as the spatial gradient is concerned, the only results we are aware of were obtained either in the periodic case (1) (see Bensoussan, Lions and Papanicolaou [3] and Svanstedt [17] in the nonlinear case), or in the case where the coefficients do not depend on time (see Brahim-Otsmane, Francfort and Murat [5]). For the time derivative, the only result we know is due to Colombini and Spagnolo [6], who proved that in the case where the coefficients satisfy an equi-continuity condition with respect to time, that is, if

$$\lim_{\tau\to 0}\left\{\sup_\varepsilon \int_0^{T-h}\int_{\Omega_0} |A_{ij}^\varepsilon(t+\tau,x) - A_{ij}^\varepsilon(t,x)|\,dx\,dt\right\} = 0 \tag{8}$$

for every $i,j = 1,\dots,N$, for every set Ω_0 compactly contained in Ω, and for every $h > 0$, one has

$$\partial_t u^\varepsilon \to \partial_t u^0 \quad \text{strongly in } L^2\big(0,T;H^{-1}(\Omega)\big),$$

which in this case coincides with the corrector result (7).

As already mentioned above, our method of proof follows the ideas introduced by Tartar in the elliptic case, and consists first in analyzing the case where u^0 is smooth (Proposition 3.13, in this case the $L^2(Q;\mathbb{R}^N)$ convergence (5) is obtained) and then in approximating u^0 with smooth functions (which leads to a loss of regularity, that is, the convergence in $L^2(Q;\mathbb{R}^N)$ becomes a convergence in $L^1(Q;\mathbb{R}^N)$). This was indeed the method used in [18].

Besides its own interest, the corrector result is very useful in the study of homogenization of quasilinear parabolic problems of the type

$$\partial_t u^\varepsilon - \operatorname{div}\big(A^\varepsilon(t,x)\nabla u^\varepsilon\big) = f + H^\varepsilon\big(t,x,\nabla u^\varepsilon\big) \quad \text{in } \mathcal{D}'(Q), \tag{9}$$

where the perturbation H^ε of the linear problem is assumed to be uniformly (in ε) Lipschitz-continuous with respect to ∇u^ε. Then it is relatively easy to prove that the corrector result (6) still holds true, and that

$$H^\varepsilon\big(t,x,\nabla u^\varepsilon\big) = H^\varepsilon\big(t,x,p^\varepsilon\,\nabla u^0 + R^\varepsilon\big) = H^\varepsilon\big(t,x,p^\varepsilon \nabla u^0\big) + \hat{R}^\varepsilon\,,$$

where R^ε and $\hat{R}^\varepsilon$ tend to zero strongly in $L^1(Q;\mathbb{R}^N)$ and $L^1(Q)$, respectively. This allows one to prove that the solution u^ε of the nonlinear problem (9) converges to a solution of

$$\partial_t u^0 - \operatorname{div}\big(A^0(t,x)\,\nabla u^0\big) = f + H^0\big(t,x,\nabla u^0\big) \quad \text{in } \mathcal{D}'(Q),$$

where H^0 is defined by

$$H^\varepsilon(t,x,p^\varepsilon\,\xi) \rightharpoonup H^0(t,x,\xi) \quad \text{in } \mathcal{D}'(Q), \qquad \text{for every } \xi \in \mathbb{R}^N$$

(subsequences have to be extracted in order to make these statements accurate). Similar problems were studied in [4], [2] and [1] in the elliptic case; we present in Section 6 an analysis of a simple case in the parabolic framework (see also [7] for the case where the matrices A^ε do not depend on time).

The plan of the paper is as follows. In Section 2 we recall the definition of H-convergence (Definition 2.1), and some of its properties; we also fix some notation. In Section 3 we introduce the special test functions w_i^ε (Definition 3.2) and the corrector matrices p^ε (Definition 3.6), and we state our main result (Theorem 3.8). Remark 3.18 and Definition 3.19 are devoted to the presentation of a possible extension of the definition of correctors, which actually proves to be equivalent, up to a change of notation, to Definition 3.2. In Remark 3.16, by introducing an additional special test function w_0^ε (one can actually choose $w_0^\varepsilon(t,x) = t$), we show that both corrector statements (5) for ∇u^ε and (7) for $\partial_t u^\varepsilon$ can be rewritten in a unified way. Section 4 is devoted to the periodic case $A^\varepsilon(t,x) = A(t/\varepsilon^k, x/\varepsilon)$; we prove there that our result allows one to recover the result obtained in the periodic framework by Bensoussan, Lions and Papanicolaou. Section 5 is devoted to the proofs of the main results. In Section 6 we analyze an example of quasilinear problems like (9), as described above. Finally, in Section 7 we give the proof of some auxiliary results.

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2. – Definition of H-convergence

In this section we give the definition of H-convergence as well as the notation we will use throughout the paper.

Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 1$ (no smoothness is assumed on the boundary $\partial\Omega$ of Ω), T be a strictly positive real number and Q be the cylinder $(0,T)\times\Omega$. Let α and β be given strictly positive real numbers; we will denote by $\mathcal{M}(\alpha,\beta;Q)$ the set of all Lebesgue-measurable $N\times N$ matrix-valued functions $A=\{A_{ij}\}_{i,j=1,\dots,N}$ defined on Q, such that:

$$(10)\qquad \big(A(t,x)\,\xi\big)\cdot\xi \geq \alpha|\xi|^2\,, \qquad \big(A^{-1}(t,x)\,\xi\big)\cdot\xi \geq \beta^{-1}|\xi|^2\,,$$

for almost every $(t,x)\in Q$, for every $\xi\in\mathbb{R}^N$; here $A^{-1}(t,x)$ denotes the inverse matrix of $A(t,x)$, which exists in view of the first part of (10). Taking $\xi = A(t,x)\,\eta$, the last inequality of (10) is equivalent to $\eta\cdot(A(t,x)\,\eta) \geq \beta^{-1}\,|A(t,x)\,\eta|^2$ and hence implies that

$$(11)\qquad |A(t,x)\,\eta| \leq \beta\,|\eta|\,.$$

Following the notation traditionally used for homogenization problems, in the whole of this paper we will denote by ε an index which takes its values in a sequence E of strictly positive real numbers which converge to 0.

We will consider a sequence $\{A^\varepsilon\}$ of matrices in $\mathcal{M}(\alpha,\beta;Q)$. A special case of this setting is the case of periodic homogenization, where one considers matrices of the form (1) (for an extensive study of this problem, see [3]; see also Section 4 below).

Let us recall the following well known result (see for instance [11], Theorem 3.1)

$$(12)\qquad \begin{cases} \text{If } v\in L^2\big(0,T;H^1_0(\Omega)\big)\cap H^1\big(0,T;H^{-1}(\Omega)\big), \\ \text{then } v\in C^0\big([0,T];L^2(\Omega)\big), \text{ and} \\ \|v\|^2_{C^0\left([0,T];L^2(\Omega)\right)} = \sup\limits_{t\in[0,T]}\displaystyle\int_\Omega |v(t)|^2\,dx \\ \qquad\qquad \leq c(Q)\left(\|v\|^2_{L^2\left(0,T;H^1_0(\Omega)\right)} + \|\partial_t v\|^2_{L^2\left(0,T;H^{-1}(\Omega)\right)}\right). \end{cases}$$

Here and in the rest of this paper we choose (recall that Ω is assumed to be bounded)

$$\|v\|_{H^1_0(\Omega)} = \|\nabla v\|_{L^2(\Omega;\mathbb{R}^N)}$$

as norm in $H^1_0(\Omega)$, where $\nabla v = (\partial_{x_1}v,\dots,\partial_{x_N}v)$ denotes the spatial gradient of v.

Moreover we will often use the following well known compactness property for evolution spaces (see for instance [10], Chapitre 1, Théorème 5.1)

$$\text{(13)}\qquad \begin{cases} \text{If } \{v^\varepsilon\} \text{ is bounded in } L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \\ \text{then } \{v^\varepsilon\} \text{ is relatively compact in } L^2(Q). \end{cases}$$

The result (13) is also true if the space $H_0^1(\Omega)$ is replaced by $H^1(\Omega)$, provided the boundary of Ω is sufficiently smooth (for instance of class C^1).

Given f and a such that

$$\text{(14)}\qquad f \in L^2(0,T;H^{-1}(\Omega)), \qquad a \in L^2(\Omega),$$

there exists, for every fixed ε, a unique solution u^ε of problem (0^ε) (see, for instance, [10]). Note that, by (12), the initial condition $u^\varepsilon(0) = a$ makes sense. Moreover, using u^ε as a test function in (0^ε), the following well known *a priori* estimates are easily obtained

$$\text{(15)}\qquad \begin{aligned} &\|u^\varepsilon\|^2_{C^0([0,T];L^2(\Omega))} + \|u^\varepsilon\|^2_{L^2(0,T;H_0^1(\Omega))} \\ &\qquad \le c(\alpha)\left(\|f\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|a\|^2_{L^2(\Omega)}\right), \end{aligned}$$

$$\text{(16)}\qquad \|\partial_t u^\varepsilon\|_{L^2(0,T;H^{-1}(\Omega))} \le c(\alpha,\beta)\left(\|f\|_{L^2(0,T;H^{-1}(\Omega))} + \|a\|_{L^2(\Omega)}\right).$$

Let us recall the definition of parabolic H-convergence, introduced with minor variations by Spagnolo and Colombini, Spagnolo under the names of G- or PG-convergence (see [14], [6], [15]), and which generalizes to the parabolic case the definition given in the elliptic one (see [14], [18]).

DEFINITION 2.1. *We will say that a sequence $\{A^\varepsilon\}$ of matrices in $\mathcal{M}(\alpha,\beta;Q)$* H-converges *to a matrix $A^0 \in \mathcal{M}(\alpha,\beta;Q)$, and we will write*

$$A^\varepsilon \overset{H}{\rightharpoonup} A^0$$

if, for every f and for every a satisfying (14), the sequence $\{u^\varepsilon\}$ of the (unique) solutions of the parabolic Cauchy-Dirichlet problems (0^ε) satisfies

$$\text{(17)}\qquad u^\varepsilon \rightharpoonup u^0 \qquad \textit{weakly in } L^2(0,T;H_0^1(\Omega)),$$

$$\text{(18)}\qquad A^\varepsilon\nabla u^\varepsilon \rightharpoonup A^0\nabla u^0 \quad \textit{weakly in } L^2(Q;\mathbb{R}^N),$$

where u^0 is the unique solution of the problem

$$\text{(19)}\qquad \begin{cases} \partial_t u^0 - \operatorname{div}(A^0\nabla u^0) = f \qquad \textit{in } \mathcal{D}'(Q), \\ u^0 \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \\ u^0\big|_{t=0} = a. \end{cases}$$

The following fundamental result proves the interest of the notion of H-convergence.

THEOREM 2.2 (see [15]). *The class $\mathcal{M}(\alpha, \beta; Q)$ is compact with respect to H-convergence: in other words, from any sequence $\{A^\varepsilon\}$ of matrices in $\mathcal{M}(\alpha, \beta; Q)$ one can extract a subsequence which H-converges to some $A^0 \in \mathcal{M}(\alpha, \beta; Q)$.*

The next proposition shows that the H-convergence is a local property, and does not depend on the boundary conditions satisfied by the solutions of the parabolic problems (0^ε).

PROPOSITION 2.3 (see [15]). *Assume that $A^\varepsilon \overset{H}{\rightharpoonup} A^0$ in Q. Let Ω_0 be an open subset of Ω, and define $Q_0 = (0, T) \times \Omega_0$. Let $\{u^\varepsilon\}$ be a sequence of functions satisfying*

$$\begin{gathered} u^\varepsilon \in L^2\big(0, T; H^1(\Omega_0)\big) \cap H^1\big(0, T; H^{-1}(\Omega_0)\big)\,, \\ \partial_t u^\varepsilon - \operatorname{div}\big(A^\varepsilon\, \nabla u^\varepsilon\big) = f^\varepsilon \quad \textit{in } \mathcal{D}'(Q_0)\,, \\ u^\varepsilon \rightharpoonup u^0 \quad \textit{weakly in } L^2\big(0, T; H^1(\Omega_0)\big)\,, \\ f^\varepsilon \to f^0 \quad \textit{strongly in } L^2\big(0, T; H^{-1}(\Omega_0)\big)\,. \end{gathered}$$

Then

$$A^\varepsilon\, \nabla u^\varepsilon \rightharpoonup A^0\, \nabla u^0 \quad \textit{weakly in } L^2\big(Q_0; \mathbb{R}^N\big),$$

and therefore u^0 satisfies

$$\partial_t u^0 - \operatorname{div}\big(A^0\, \nabla u^0\big) = f^0 \quad \textit{in } \mathcal{D}'(Q_0)\,.$$

In the next sections we will denote by e_i, for $i = 1, \dots, N$, the vector in $\mathbb{R}^N$ whose entries are all zero except the i-th, which is 1. Moreover we will denote by $\langle \cdot, \cdot \rangle$ the duality pairing between the spaces $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, and by $\langle\langle \cdot\,, \cdot \rangle\rangle$ the duality pairing between $L^2\big(0, T; H^{-1}(\Omega)\big)$ and $L^2\big(0, T; H_0^1(\Omega)\big)$.

3. – Definition of the correctors, statement of the main result and comments

From now on, we assume that $\{A^\varepsilon\}$ is a sequence of matrices in $\mathcal{M}(\alpha, \beta; Q)$ which H-converges to A^0.

We first extend the matrices A^ε to a cylinder $\widetilde{Q} = (0, T) \times \widetilde{\Omega}$, where $\widetilde{\Omega}$ is a bounded open set in $\mathbb{R}^N$ with smooth boundary (say a large ball) such that $\Omega \subset\subset \widetilde{\Omega}$, *i.e.*, the closure of Ω is contained in $\widetilde{\Omega}$. To do this, we define $\tilde{A}^\varepsilon$, $\tilde{A}^0 \in \mathcal{M}(\alpha, \beta; \widetilde{Q})$ by

$$\tag{20} \tilde{A}^\varepsilon = \begin{cases} A^\varepsilon & \text{on } Q\,, \\ \alpha I & \text{on } \widetilde{Q} \setminus Q\,. \end{cases} \qquad \tilde{A}^0 = \begin{cases} A^0 & \text{on } Q\,, \\ \alpha I & \text{on } \widetilde{Q} \setminus Q\,. \end{cases}$$

Then we have (see Section 7 for the proof)

PROPOSITION 3.1. *The sequence* $\{\tilde{A}^\varepsilon\}$ *H-converges to* $\tilde{A}^0$ *in* $\widetilde{Q}$.

The next definition (see also Definition 3.19 below) generalizes to the parabolic case the notion of correctors introduced in the elliptic case by Tartar in [18].

DEFINITION 3.2. For $i = 1, \dots, N$, we will say that the sequence $\{w_i^\varepsilon\}$ is a sequence of *i-th special test functions* for the matrices $\{A^\varepsilon\}$ if it satisfies

$$w_i^\varepsilon \in L^2\big(0, T; H^1(\widetilde{\Omega})\big) \cap H^1\big(0, T; H^{-1}(\widetilde{\Omega})\big)\,, \tag{21}$$

$$\partial_t w_i^\varepsilon - \operatorname{div}\big(\tilde{A}^\varepsilon\, \nabla w_i^\varepsilon\big) = f_i^\varepsilon + g_i^\varepsilon \quad \text{in } \mathcal{D}'(\widetilde{Q})\,, \tag{22}$$

$$w_i^\varepsilon \rightharpoonup x_i \quad \text{weakly in } L^2\big(0, T; H^1(\widetilde{\Omega})\big)\,, \tag{23}$$

$$w_i^\varepsilon(0) \to x_i \quad \text{strongly in } L^2(\widetilde{\Omega})\,, \tag{24}$$

where

$$f_i^\varepsilon \in L^2\big(0, T; H^{-1}(\widetilde{\Omega})\big)\,, \quad f_i^\varepsilon \to f_i^0 \quad \text{strongly in } L^2\big(0, T; H^{-1}(\widetilde{\Omega})\big)\,, \tag{25}$$

$$g_i^\varepsilon \in L^2(\widetilde{Q})\,, \qquad g_i^\varepsilon \rightharpoonup g_i^0 \quad \text{weakly in } L^2(\widetilde{Q})\,. \tag{26}$$

The next proposition gives some properties of the special test functions, which will be used in the proof of the main result. Recall that a sequence $\{v^\varepsilon\}$ of measurable functions defined on Q is said to be *equi-integrable* if for every $\eta > 0$ there exists a positive number δ such that, for every measurable set $D \subset Q$ satisfying $\operatorname{meas} D < \delta$ and for every ε, one has

$$\int_D |v^\varepsilon| < \eta\,.$$

This condition is equivalent to the relative compactness of the sequence $\{v^\varepsilon\}$ in the weak topology of $L^1(Q)$.

PROPOSITION 3.3. *Assume that* $A^\varepsilon \overset{H}{\rightharpoonup} A^0$, *and that* $\{w_i^\varepsilon\}$ *is a sequence of i-th special test functions according to Definition* 3.2. *Then*

$$\tilde{A}^\varepsilon\, w_i^\varepsilon \rightharpoonup \tilde{A}^0\, e_i \quad \textit{weakly in } L^2\big(\widetilde{Q}; \mathbb{R}^N\big)\,, \tag{27}$$

$$f_i^0 + g_i^0 = -\operatorname{div}\big(\tilde{A}^0\, e_i\big) \quad \textit{in} \quad \mathcal{D}'(\widetilde{Q})\,, \tag{28}$$

$$\textit{the functions } |\nabla w_i^\varepsilon|^2 \textit{ are equi-integrable in } Q\,, \tag{29}$$

$$w_i^\varepsilon \to x_i \quad \textit{strongly in } C^0\big([0, T]; L^2(\Omega)\big)\,. \tag{30}$$

See Section 7 for the proof. In contrast with (27) and (28), statements (29) and (30) rely on Meyers' regularity result.

REMARK 3.4. There exist special test functions which satisfy the requirements of Definition 3.2. Indeed one can consider, for instance, the solutions of the following problems ($i = 1, \dots, N$):

$$
\begin{cases}
\partial_t w_i^\varepsilon - \operatorname{div}(\tilde{A}^\varepsilon \nabla w_i^\varepsilon) = -\operatorname{div}(\tilde{A}^0 e_i) \quad \text{in } \mathcal{D}'(\tilde{Q}), \\
w_i^\varepsilon - x_i \in L^2(0,T; H_0^1(\tilde{\Omega})) \cap H^1(0,T; H^{-1}(\tilde{\Omega})), \\
w_i^\varepsilon \Big|_{t=0} = x_i ,
\end{cases}
\tag{31}
$$

which satisfy (21) to (26). □

REMARK 3.5. The sequence $\{w_i^\varepsilon\}$ of i-th special test functions is "quasi-unique" in the following sense: if $\{w_i^\varepsilon\}$ and $\{\tilde{w}_i^\varepsilon\}$ are two sequences of i-th special test functions then

$$
w_i^\varepsilon - \tilde{w}_i^\varepsilon \to 0 \qquad \text{strongly in } L^2(0,T; H^1(\Omega)). \tag{32}
$$

Indeed the difference $z^\varepsilon = w_i^\varepsilon - \tilde{w}_i^\varepsilon$ satisfies

$$
z^\varepsilon \in L^2(0,T; H^1(\tilde{\Omega})) \cap H^1(0,T; H^{-1}(\tilde{\Omega})),
$$

$$
\partial_t z^\varepsilon - \operatorname{div}(\tilde{A}^\varepsilon \nabla z^\varepsilon) = f^\varepsilon + g^\varepsilon \qquad \text{in } \mathcal{D}'(\tilde{Q}), \tag{33}
$$

$$
z^\varepsilon \rightharpoonup 0 \qquad \text{weakly in } L^2(0,T; H^1(\tilde{\Omega})), \tag{34}
$$

$$
z^\varepsilon(0) \to 0 \qquad \text{strongly in } L^2(\tilde{\Omega}), \tag{35}
$$

with

$$
f^\varepsilon + g^\varepsilon \to 0 \quad \text{in } \{L^2(0,T; H^{-1}(\tilde{\Omega})) \text{ strong}\} + \{L^2(\tilde{Q}) \text{ weak}\}, \tag{36}
$$

and, by the compactness property (13),

$$
z^\varepsilon \to 0 \qquad \text{strongly in } L^2(\tilde{Q}). \tag{37}
$$

If $\varphi = \varphi(x)$ is a function of class $C_0^\infty(\tilde{\Omega})$, such that $0 \le \varphi \le 1$ on $\tilde{\Omega}$, $\varphi \equiv 1$ on Ω, by taking $\varphi(x) z^\varepsilon(t,x)$ as test function in (33), we obtain easily

$$
\begin{aligned}
\frac{1}{2} \int_{\tilde{\Omega}} |z^\varepsilon(T)|^2 \varphi \, dx + \alpha \int_{\tilde{Q}} |\nabla z^\varepsilon|^2 \varphi \le \langle\langle f^\varepsilon , z^\varepsilon \varphi \rangle\rangle + \int_{\tilde{Q}} g^\varepsilon z^\varepsilon \varphi \\
+ \frac{1}{2} \int_{\tilde{\Omega}} |z^\varepsilon(0)|^2 \varphi \, dx + \beta \int_{\tilde{Q}} |\nabla z^\varepsilon| \, |\nabla \varphi| \, |z^\varepsilon| .
\end{aligned}
$$

Using (34), (35), (36) and (37), it is easy to see that the right hand side of the last inequality converges to zero. This implies (32). □

DEFINITION 3.6. *We will call* corrector matrices (*or simply* correctors) *the* $N \times N$ *matrices* $p^\varepsilon(t,x) = \{p^\varepsilon_{ij}(t,x)\}_{i,j=1,\dots,N}$ *defined by*

$$p^\varepsilon_{ij} = \partial_{x_i} w^\varepsilon_j \,, \tag{38}$$

or, equivalently,

$$p^\varepsilon \xi = \sum_{j=1}^{n} \xi_j \nabla w^\varepsilon_j \qquad \text{for every } \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N, \tag{39}$$

where the functions w^ε_i *are special test functions in the sense of Definition 3.2: in other words, the* j*-th column of the matrix* p^ε *is the spatial gradient of the function* w^ε_j.

The corrector matrices p^ε satisfy the properties stated in the following proposition.

PROPOSITION 3.7. *If* $A^\varepsilon \overset{H}{\rightharpoonup} A^0$, *and* p^ε *are corrector matrices in the sense of Definition* 3.6, *then, for every* $\xi \in \mathbb{R}^N$, *one has*

$$p^\varepsilon \xi \rightharpoonup \xi \qquad \text{weakly in } L^2(Q;\mathbb{R}^N), \tag{40}$$

$$A^\varepsilon p^\varepsilon \xi \rightharpoonup A^0 \xi \qquad \text{weakly in } L^2(Q;\mathbb{R}^N), \tag{41}$$

$$\left(A^\varepsilon p^\varepsilon \xi\right) \cdot (p^\varepsilon \xi) \rightharpoonup \left(A^0 \xi\right) \cdot \xi \quad \text{weakly in } L^1(Q). \tag{42}$$

PROOF. Convergences (40) and (41) follow from the definition of the correctors and from the properties (23) and (27) of the functions w^ε_i; convergence (42) follows from the convergence of energy in the sense of distributions stated in Lemma 7.3 below (see Remark 7.4), and from the equi-integrability (29) of $|\nabla w^\varepsilon_i|^2$, which implies the equi-integrability of $(A^\varepsilon p^\varepsilon \xi) \cdot (p^\varepsilon \xi)$. Note that, in contrast with (40) and (41), statement (42) relies on Meyers' regularity result. □

The main result of this paper is the following corrector result.

THEOREM 3.8. *Assume that the sequence* $\{A^\varepsilon\}$ *of matrices in* $\mathcal{M}(\alpha, \beta; Q)$ *satisfies* $A^\varepsilon \overset{H}{\rightharpoonup} A^0$ *on* Q, *and let* p^ε *be corrector matrices. Let* u^ε, u^0 *be the solutions of* (0^ε), (19) *respectively. Moreover, assume that*

$$\nabla u^0 \in L^q\left(Q;\mathbb{R}^N\right), \tag{43}$$

$$\|p^\varepsilon\|_{L^r(Q;\mathbb{R}^{N\times N})} \le c_0 \,, \tag{44}$$

for some $q, r \in [2, +\infty]$. *Then, if* s *is defined by*

$$\frac{1}{s} = \max\left\{\frac{1}{2}, \frac{1}{q} + \frac{1}{r}\right\}, \tag{45}$$

one has

$$\nabla u^\varepsilon - p^\varepsilon \nabla u^0 \to 0 \quad \textit{strongly in } L^s\big(Q; \mathbb{R}^N\big)\,. \tag{46}$$

Moreover

$$u^\varepsilon \to u^0 \quad \textit{strongly in } C^0\big([0,T]; L^2(\Omega)\big), \tag{47}$$

and

$$\partial_t u^\varepsilon - \left[\partial_t u^0 + \sum_{i=1}^N \operatorname{div}\Big(\partial_{x_i} u^0\big(A^\varepsilon \nabla w_i^\varepsilon - A^0 e_i\big)\Big)\right] \to 0 \tag{48}$$

$$\textit{strongly in } L^s\big(0,T; W^{-1,s}(\Omega)\big)\,.$$

REMARK 3.9. Corrector result (46) (and result (50) below) is a generalization to the nonperiodic case of the corrector result by Bensoussan, Lions and Papanicolaou [3] (see Section 4 below for a comparison). Result (47) has been proved by Colombini and Spagnolo in [6] under the additional equi-continuity assumption (8). Result (48) (like result (52) below), instead, seems to be new. □

REMARK 3.10. Convergence (46) says that ∇u^ε may be replaced by $p^\varepsilon \nabla u^0$ at the expense of an error which converges strongly to zero. Convergence (48) similarly provides an approximation of $\partial_t u^\varepsilon$ by

$$\partial_t u^0 + \sum_{i=1}^N \operatorname{div}\big(\partial_{x_i} u^0 (A^\varepsilon \nabla w_i^\varepsilon - A^0 e_i)\big)\,.$$

Remark that this expression does not depend only on $\partial_t u^0$, but also on the spatial gradient ∇u^0. □

REMARK 3.11. In (46) and (48), one ideally would like to have the result with $s = 2$, because $L^2(Q; \mathbb{R}^N)$ and $L^2\big(0,T; H^{-1}(\Omega)\big)$ are the natural spaces for ∇u^ε and $\partial_t u^\varepsilon$ respectively; this is not possible in general when ∇u^0 is not sufficiently smooth, and the (strong) convergences only take place in $L^s(Q; \mathbb{R}^N)$ and $L^s(0,T; W^{-1,s}(\Omega))$ respectively, with s given by (45): indeed in (46) ∇u^ε is bounded in $L^2(Q; \mathbb{R}^N)$, while $p^\varepsilon \nabla u^0$ is only bounded in $L^\sigma(Q; \mathbb{R}^N)$, with $1/\sigma = 1/q + 1/r$ (a similar remark holds for (48)).

In the case where there is no extra regularity on ∇u^0 and on the sequence $\{p^\varepsilon\}$, that is, when $q = 2$ and $r = 2$, convergence (46) takes place only in $L^1(Q; \mathbb{R}^N)$, while (48) takes place only in $L^1\big(0,T; W^{-1,1}(\Omega)\big)$. □

REMARK 3.12. In the proof of Theorem 3.8, Meyers' regularity result will only be used to prove convergence (47), and to prove the particular case $q = +\infty$, $r = 2$ for the convergences (46) and (48). This regularity result will be used on w_i^ε, through the equi-integrability (29) of $|\nabla w_i^\varepsilon|^2$. □

A special case of the above theorem, which is also the most favorable one, is the case where $u^0 \in C^\infty(\overline{Q})$ and is zero near the lateral boundary of the cylinder Q. In this case we will prove the following result.

PROPOSITION 3.13. *Assume that the hypotheses of Theorem* 3.8 *hold and moreover that*

$$u^0 \in C^\infty(\overline{Q}), \quad u^0 = 0 \quad \text{in a neighborhood of } [0,T] \times \partial\Omega. \tag{49}$$

Then

$$\nabla u^\varepsilon - p^\varepsilon \nabla u^0 \to 0 \quad \text{strongly in } L^2(Q; \mathbb{R}^N), \tag{50}$$

$$u^\varepsilon \to u^0 \quad \text{strongly in } C^0([0,T]; L^2(\Omega)), \tag{51}$$

$$\partial_t u^\varepsilon - \left[\partial_t u^0 + \sum_{i=1}^N \partial_t w_i^\varepsilon \, \partial_{x_i} u^0\right] = \eta_1^\varepsilon + \eta_2^\varepsilon, \tag{52}$$

where

$$\eta_1^\varepsilon \to 0 \quad \text{strongly in } L^2(0,T; H^{-1}(\Omega)), \qquad \eta_2^\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(Q).$$

REMARK 3.14. Proposition 3.13 is a consequence of Theorem 5.1 below, and will be proved in the First Step of the Proof of Theorem 3.8, in Section 5 below (see also Remark 5.2).

Similarly to Theorem 3.8, statements (50) and (52) do not rely on Meyers' regularity result, in contrast with (51), the proof of which uses (30) and therefore this regularity result. □

REMARK 3.15. Observe that the term $\partial_t w_i^\varepsilon \, \partial_{x_i} u^0$ has a meaning as an element of $L^2(0,T; H^{-1}(\Omega))$, since it is the product of an element $h = \partial_t w_i^\varepsilon \in L^2(0,T; H^{-1}(\Omega))$ by a function $\varphi = \partial_{x_i} u^0 \in C^\infty(\overline{Q})$, product which is defined by

$$\langle\!\langle \varphi h, v \rangle\!\rangle = \langle\!\langle h, \varphi v \rangle\!\rangle \qquad \text{for every } v \in L^2(0,T; H_0^1(\Omega)).$$

In the statement of Proposition 3.13, convergences (50) and (51) are a mere repetition of (46) and (47) with $s = 2$, while (52) seems to differ from (48); actually (52) is a different writing of (48) when u^0 is smooth. Indeed for every $\psi \in C^\infty(\overline{Q})$ one has

$$\begin{aligned}
\sum_{i=1}^N \partial_t w_i^\varepsilon \, \partial_{x_i} \psi &= \sum_{i=1}^N \left[\operatorname{div}(A^\varepsilon \nabla w_i^\varepsilon - A^0 e_i)\right] \partial_{x_i} \psi \\
&\quad + \sum_{i=1}^N \left[f_i^\varepsilon + g_i^\varepsilon + \operatorname{div}(A^0 e_i)\right] \partial_{x_i} \psi \\
&= \sum_{i=1}^N \operatorname{div}\left(\partial_{x_i} \psi \, (A^\varepsilon \nabla w_i^\varepsilon - A^0 e_i)\right) \\
&\quad - \sum_{i=1}^N \left(A^\varepsilon \nabla w_i^\varepsilon - A^0 e_i\right) \cdot \nabla \partial_{x_i} \psi \\
&\quad + \sum_{i=1}^N \left[f_i^\varepsilon + g_i^\varepsilon + \operatorname{div}(A^0 e_i)\right] \partial_{x_i} \psi .
\end{aligned} \tag{53}$$

In view of (25), (26), (27) and (28) the last two sums of the right hand side of (53) tend to zero in $\{L^2(0,T;H^{-1}(\Omega))$ strong$\}+\{L^2(Q)$ weak$\}$. Therefore, by taking $\psi = u^0$, convergence (52) follows immediately from (48) when u^0 satisfies (49).

In the case where u^0 is not smooth, and only belongs to the natural space $L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$, statement (48) of Theorem 3.8 still makes sense, while (52) has no longer a meaning. □

REMARK 3.16. It would be logical to introduce a further special test function w_0^ε which converges to t, *i.e.*, which satisfies

$$(54)\qquad \begin{cases} w_0^\varepsilon \in L^2(0,T;H^1(\widetilde{\Omega})) \cap H^1(0,T;H^{-1}(\widetilde{\Omega}))\,, \\ \partial_t w_0^\varepsilon - \operatorname{div}(A^\varepsilon \nabla w_0^\varepsilon) = f_0^\varepsilon + g_0^\varepsilon \text{ in } \mathcal{D}'(\widetilde{Q}), \\ w_0^\varepsilon \rightharpoonup t \text{ weakly in } L^2(0,T;H^1(\widetilde{\Omega}))\,, \\ w_0^\varepsilon(0) \to 0 \text{ strongly in } L^2(\widetilde{\Omega})\,, \end{cases}$$

with $f_0^\varepsilon \to f_0^0$ strongly in $L^2(0,T;H^{-1}(\widetilde{\Omega})$ and $g_0^\varepsilon \rightharpoonup g_0^0$ weakly in $L^2(\widetilde{Q})$. It is clear that $w_0^\varepsilon(x,t) \equiv t$ is such a special test function, and we will make this choice here. In this setting it makes sense to define a $(N+1)\times(N+1)$ matrix Π^ε whose entries are

$$\Pi_{ij}^\varepsilon = \partial_{x_i} w_j^\varepsilon\,, \qquad i,j = 0,1,\dots,N$$

(where $x_0 = t$ and $w_0^\varepsilon = t$).

If we define $Dv = (\partial_t v, \partial_{x_1} v, \dots, \partial_{x_N} v)$ as the "full" (that is, with respect to t and x) gradient of a function v, convergences (50) and (52) of Proposition 3.13 actually state that when u^0 is smooth, $\Pi^\varepsilon Du^0$ is a good approximation of Du^ε, *i.e.*, that

$$Du^\varepsilon - \Pi^\varepsilon Du^0 \to 0 \quad \text{strongly},$$

in the sense that

$$(55)\qquad \begin{cases} (Du^\varepsilon - \Pi^\varepsilon Du^0)_0 \to 0 \text{ in } \{L^2(0,T;H^{-1}(\Omega)) \text{ strong}\} + \{L^2(Q) \text{ weak}\}, \\ (Du^\varepsilon - \Pi^\varepsilon Du^0)_i \to 0 \text{ strongly in } L^2(Q), \qquad \text{for } i = 1,\dots,N. \end{cases}$$

Note that convergence (55) takes place in the "natural spaces" where $Du^\varepsilon = (\partial_t u^\varepsilon, \nabla u^\varepsilon)$ is bounded. □

REMARK 3.17. When u^0 is smooth, an easy computation shows that convergences (50) and (52) (or, equivalently, (55)) can be written as

$$(56)\qquad u^\varepsilon - \left[u^0 + \sum_{i=1}^N (w_i^\varepsilon - x_i)\,\partial_{x_i} u^0\right] \to 0 \text{ strongly in } L^2(0,T;H_0^1(\Omega)),$$

$$(57)\qquad \partial_t u^\varepsilon - \partial_t\left[u^0 + \sum_{i=1}^N (w_i^\varepsilon - x_i)\,\partial_{x_i} u^0\right] \to 0$$
$$\text{in } \Big\{L^2(0,T;H^{-1}(\Omega)) \text{ strong}\Big\} + \Big\{L^2(Q) \text{ weak}\Big\}.$$

Convergence (56) is very similar to the corrector result obtained in the periodic case by Bensoussan, Lions and Papanicolaou (see [3] and Section 4 below, where the relationship between their results and ours is studied). □

REMARK 3.18. In place of Definition 3.2, we could have considered an apparently more general definition of special test functions, where a term which converges strongly in $H^{-1}(0,T;H^1(\Omega))$ is added to the right hand side of (22) as follows.

DEFINITION 3.19. For $i=1,\dots,N$, we will say that $\{\hat{w}_i^\varepsilon\}$ is a sequence of *i-th special test functions* if it satisfies

$$\text{(58)}\qquad \hat{w}_i^\varepsilon \in L^2(0,T;H^1(\widetilde{\Omega})) \cap H^1(0,T;H^{-1}(\widetilde{\Omega}))\,,$$

$$\text{(59)}\qquad \partial_t \hat{w}_i^\varepsilon - \operatorname{div}(\tilde{A}^\varepsilon\, \nabla \hat{w}_i^\varepsilon) = \hat{f}_i^\varepsilon + \hat{g}_i^\varepsilon + \partial_t \hat{h}_i^\varepsilon \quad \text{in } \mathcal{D}'(Q)\,,$$

$$\text{(60)}\qquad \hat{w}_i^\varepsilon \rightharpoonup x_i \quad \text{weakly in } L^2(0,T;H^1(\widetilde{\Omega}))\,,$$

$$\text{(61)}\qquad \hat{w}_i^\varepsilon(0) \to x_i \quad \text{strongly in } L^2(\widetilde{\Omega})\,,$$

where

$$\text{(62)}\qquad \hat{f}_i^\varepsilon \in L^2(0,T;H^{-1}(\widetilde{\Omega}))\,, \quad \hat{f}_i^\varepsilon \to \hat{f}_i^0 \quad \text{strongly in } L^2(0,T;H^{-1}(\widetilde{\Omega}))\,,$$

$$\text{(63)}\qquad \hat{g}_i^\varepsilon \in L^2(\widetilde{Q})\,, \quad \hat{g}_i^\varepsilon \rightharpoonup \hat{g}_i^0 \quad \text{weakly in } L^2(\widetilde{Q})\,,$$

$$\text{(64)}\begin{cases} \hat{h}_i^\varepsilon \in L^2(0,T;H^1(\widetilde{\Omega}))\,, \quad \partial_t \hat{h}_i^\varepsilon \text{ bounded in } L^2(0,T;H^{-1}(\widetilde{\Omega}))\,, \\ \hat{h}_i^\varepsilon \to \hat{h}_i^0 \text{ strongly in } L^2(0,T;H^1(\widetilde{\Omega})), \hat{h}_i^\varepsilon(0) \to \hat{h}_i^0(0) \text{ strongly}^{(1)} \text{ in } L^2(\widetilde{\Omega}). \end{cases}$$

The interest of Definition 3.19 lies in the fact that, in the case of periodic homogenization with fast oscillations with respect to time, the corrector result proved in [3] can be rewritten in terms of special test functions in the sense of Definition 3.19. This will be shown in Section 4, case $k>2$.

However the generalization introduced by Definition 3.19 is only apparent. Indeed let $\hat{w}_i^\varepsilon$ be a special test function in the sense of Definition 3.19, and define

$$\text{(65)}\qquad w_i^\varepsilon = \hat{w}_i^\varepsilon - (\hat{h}_i^\varepsilon - \hat{h}_i^0)\,.$$

We claim that w_i^ε is a special test function in the sense of Definition 3.2, and that Theorem 3.8 holds for $r=2$ with w_i^ε replaced by $\hat{w}_i^\varepsilon$ (and p^ε replaced by $\hat{p}^\varepsilon$, defined by $\hat{p}^\varepsilon e_i = \nabla \hat{w}_i^\varepsilon$ for $i=1,\dots,N$) when $\hat{w}_i^\varepsilon$ is a special test function in the sense of Definition 3.19.

(1) It can actually be shown that the three first assertions of (64) imply that $\hat{h}_i^\varepsilon(0) \to \hat{h}_i^0(0)$ strongly in $L^2_{\mathrm{loc}}(\widetilde{\Omega})$.

Indeed, when $\hat{w}_i^\varepsilon$ is a special test function in the sense of Definition 3.19, then the function w_i^ε defined by (65) satisfies

$$\begin{aligned}\partial_t w_i^\varepsilon - \operatorname{div}(\tilde{A}^\varepsilon \nabla w_i^\varepsilon) = \hat{f}_i^\varepsilon + \hat{g}_i^\varepsilon + \partial_t \hat{h}_i^\varepsilon - \partial_t(\hat{h}_i^\varepsilon - \hat{h}_i^0)\\ + \operatorname{div}(\tilde{A}^\varepsilon (\nabla \hat{h}_i^\varepsilon - \nabla \hat{h}_i^0)) = f_i^\varepsilon + g_i^\varepsilon\,,\end{aligned}$$

where f_i^ε and g_i^ε, defined by

$$f_i^\varepsilon = \hat{f}_i^\varepsilon + \partial_t \hat{h}_i^0 + \operatorname{div}(\tilde{A}^\varepsilon (\nabla \hat{h}_i^\varepsilon - \nabla \hat{h}_i^0))\,, \qquad g_i^\varepsilon = \hat{g}_i^\varepsilon\,, \tag{66}$$

satisfy (25) and (26), because of (62), (63) and (64), where

$$f_i^0 = \hat{f}_i^0 + \partial_t \hat{h}_i^0\,, \qquad g_i^0 = \hat{g}_i^0\,. \tag{67}$$

On the other hand, in view of (60), (61) and (64), we have

$$\begin{gathered} w_i^\varepsilon \rightharpoonup x_i \quad \text{weakly in } L^2(0,T;H^1(\widetilde{\Omega})),\\ w_i^\varepsilon(0) = \hat{w}_i^\varepsilon(0) - (\hat{h}_i^\varepsilon(0) - \hat{h}_i^0(0)) \to x_i \quad \text{strongly in } L^2(\widetilde{\Omega})\,.\end{gathered}$$

Therefore w_i^ε defined by (65) is a special test function in the sense of Definition 3.2.

Moreover, for p^ε and $\hat{p}^\varepsilon$ defined by

$$p^\varepsilon e_i = \nabla w_i^\varepsilon\,, \qquad \hat{p}^\varepsilon e_i = \nabla \hat{w}_i^\varepsilon$$

we have $\|(p^\varepsilon - \hat{p}^\varepsilon)e_i\|_{L^2(\widetilde{Q};\mathbb{R}^N)} = \|\nabla(\hat{h}_i^\varepsilon - \hat{h}_i^0)\|_{L^2(\widetilde{Q};\mathbb{R}^N)} \to 0$, because of (64). It is therefore clear that the results of Theorem 3.8 for $r = 2$ still hold true with p^ε and w_i^ε replaced by $\hat{p}^\varepsilon$ and $\hat{w}_i^\varepsilon$, when $\hat{w}_i^\varepsilon$ is a special test function in the sense of Definition 3.19. □

4. – Comparison with the known results for periodic homogenization

In this section we will recall the corrector results proved by Bensoussan, Lions and Papanicolaou [3] in the case of periodic homogenization, and we will analyze the connection between their corrector results and ours.

In the case of periodic homogenization, one considers a sequence of matrices of the form

$$A^\varepsilon(t,x) = A\left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right), \quad k \ge 0\,, \tag{68}$$

where $A(\tau, y) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^{N\times N}$ is a $(N \times N)$-matrix in $\mathcal{M}(\alpha, \beta; \mathbb{R}^{N+1})$ which is periodic both in τ and y (with periods respectively $(0, T_0)$ and Y_0, where

T_0 is a positive real number and Y_0 is a parallelepiped of $\mathbb{R}^N$). The choice of different values of k corresponds to different speeds of oscillations in time with respect to oscillations in space.

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, and let T be a positive number. For the sake of simplicity we will assume here that A and $\partial\Omega$ are sufficiently smooth. Then the sequence $\{A^\varepsilon\}$ defined by (68) can be shown to H-converge in $Q = (0,T) \times \Omega$ to a matrix A^0 with constant coefficients, whose expression depends on A and k. Moreover a corrector result holds (see [3], Section 2.11). Let us now recall the expression of A^0 and this corrector result.

To define the limit matrix A^0, N functions $\Phi_i(\tau, y) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $i = 1, \dots, N$, are introduced. The functions Φ_i are periodic with respect to τ and y (with periods $(0, T_0)$ and Y_0 respectively), and are solutions of some partial differential equations depending on the value of k as follows.

(i) If $0 \le k < 2$ (case of slow oscillations in time), then $\Phi_i(\tau, \cdot)$ is defined, for every fixed $\tau \in \mathbb{R}$, as the unique solution of the periodic elliptic problem

$$(69)\qquad \begin{cases} -\operatorname{div}_y\big(A(\tau,y)\,(\nabla_y \Phi_i(\tau,y) - e_i)\big) = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^N), \\ \Phi_i(\tau,\cdot) \in H^1_{\mathrm{loc}}(\mathbb{R}^N), \text{ periodic with period } Y_0, \\ \displaystyle\int_{Y_0} \Phi_i(\tau,y)\,dy = 0. \end{cases}$$

It is clear that $\Phi_i(\tau, y)$ is periodic of period $(0, T_0) \times Y_0$.

(ii) If $k = 2$, then Φ_i is the unique solution of the periodic parabolic problem

$$(70)\qquad \begin{cases} \partial_\tau \Phi_i(\tau,y) - \operatorname{div}_y\big(A(\tau,y)\,(\nabla_y \Phi_i(\tau,y) - e_i)\big) = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^{N+1}), \\ \Phi_i \in L^2_{\mathrm{loc}}\big(\mathbb{R}; H^1_{\mathrm{loc}}(\mathbb{R}^N)\big) \cap H^1_{\mathrm{loc}}\big(\mathbb{R}; H^{-1}_{\mathrm{loc}}(\mathbb{R}^N)\big), \\ \Phi_i(\cdot,\cdot) \text{ periodic with period } (0,T_0) \times Y_0, \\ \displaystyle\int_0^{T_0}\!\!\int_{Y_0} \Phi_i(\tau,y)\,d\tau\,dy = 0. \end{cases}$$

(iii) If $k > 2$ (case of fast oscillations in time), then Φ_i does not depend on τ, and is the unique solution of the periodic elliptic problem

$$(71)\qquad \begin{cases} -\operatorname{div}_y\big(M_\tau(A)(y)\,(\nabla_y \Phi_i(y) - e_i)\big) = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^N), \\ \Phi_i(\cdot) \in H^1_{\mathrm{loc}}(\mathbb{R}^N), \text{ periodic with period } Y_0, \\ \displaystyle\int_{Y_0} \Phi_i(y)\,dy = 0, \end{cases}$$

where $M_\tau(A)(y)$ denotes the average of A with respect to time, defined by

$$\big(M_\tau(A)\big)_{jh}(y) = \frac{1}{T_0}\int_0^{T_0} A_{jh}(\tau,y)\,d\tau\,.$$

Note that these functions Φ_i are denoted in [3] respectively by the symbols χ^i, θ^i and ϕ^i in the three cases $k < 2$, $k = 2$ and $k > 2$. Then the results of [3], Chapter II, Theorems 2.1 and 2.3 can be rewritten as follows.

THEOREM 4.1 (see [3]). *For every $k \geq 0$, the sequence of matrices $\{A^\varepsilon\}$ defined by* (68) *H-converge to the matrix A^0 with constant coefficients defined by*

$$A^0_{ij} = M\left(A_{ij} - \sum_{k=1}^{N} A_{ik}\partial_{y_k}\Phi_j\right),$$

where, for a periodic function $v(\tau, y)$ with period $(0, T_0) \times Y_0$, we denote by $M(v)$ its mean value defined by

$$M(v) = \frac{1}{T_0}\frac{1}{\operatorname{meas} Y_0}\int_0^{T_0}\int_{Y_0} v(\tau, y)\, d\tau\, dy.$$

Moreover, if $k = 1, 2,$ or 3, and if the data a and f are smooth enough, the solution u^ε of (0^ε) *satisfies*

$$u^\varepsilon(t,x) - \left[u^0(t,x) - \varepsilon\sum_{i=1}^{N}\Phi_i\left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right)\partial_{x_i}u^0(t,x)\right] \to 0 \quad \text{strongly in } L^2\big(0,T; H_0^1(\Omega)\big), \tag{72}$$

where u^0 is the solution of (19).

Define the functions w_i^ε, $i = 1, \dots, N$, by

$$w_i^\varepsilon(t,x) = x_i - \varepsilon\,\Phi_i\left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right), \tag{73}$$

and the matrices $p^\varepsilon = p^\varepsilon(t,x)$ by

$$p^\varepsilon\,\xi = \sum_{i=1}^{N}\xi_i\,\nabla w_i^\varepsilon, \quad \text{for every } \xi \in \mathbb{R}^N, \tag{74}$$

or in other terms

$$p^\varepsilon(t,x) = P\left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right),$$

where

$$P_{ij}(\tau, y) = \delta_{ij} - \partial_{y_i}\Phi_j(\tau, y). \qquad i, j = 1, \dots, N. \tag{75}$$

Since we have assumed the data $\partial\Omega$, A, f and a to be smooth, so are u^ε, u^0 and Φ_i, $i = 1, \dots, N$. Convergence (72) is equivalent to

$$\nabla u^\varepsilon - \left[\nabla u^0 - \sum_{i=1}^{N}\nabla_y\Phi_i\left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right)\partial_{x_i}u^0(t,x)\right] + \varepsilon\sum_{i=1}^{N}\Phi_i\left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right)\nabla\partial_{x_i}u^0(t,x) \to 0 \text{ strongly in } L^2(Q;\mathbb{R}^N).$$

Since the last sum converges to zero strongly in $L^2(Q; \mathbb{R}^N)$, and in view of the definition (74) of the matrices p^ε, convergence (72) is equivalent to

$$\nabla u^\varepsilon - p^\varepsilon \nabla u^0 \to 0 \quad \text{strongly in } L^2\big(Q; \mathbb{R}^N\big). \tag{76}$$

We claim that the functions w_i^ε defined by (73) are special test functions in the sense of Definition 3.2 (or of Definition 3.19 when $k > 2$). This claim and the equivalence between (72) and (76) immediately imply that the corrector result of [3] (which we recalled in (72)) coincides with the corrector result (50) stated in Proposition 3.13 above.

REMARK 4.2. The claim, the equivalence between (72) and (76) and Theorem 3.8 (or Proposition 3.13) above also imply that the corrector result (72) (and (76)) holds true for every $k \geq 0$, and not only for $k = 1, 2$ and 3. This result seems to be new. □

In order to prove the claim, let us first observe that, since Φ_i is smooth and periodic in y, we have

$$w_i^\varepsilon \rightharpoonup x_i \quad \text{weakly in } L^2\big(0, T; H^1(\widetilde{\Omega})\big), \qquad w_i^\varepsilon(0) \to 0 \quad \text{strongly in } L^2(\widetilde{\Omega}),$$

for every bounded open set $\widetilde{\Omega}$ of $\mathbb{R}^N$. Proving that w_i^ε is a special test function therefore reduces to proving that w_i^ε satisfies the parabolic equation (22) (or (59)) with appropriate right hand side. In order to do this we will treat separately three cases, depending on the value of k.

(i) - CASE $0 \leq k < 2$. In this case, using the definition (69) of Φ_i, we obtain

$$\partial_t w_i^\varepsilon - \operatorname{div}\big(A^\varepsilon \nabla w_i^\varepsilon\big) = -\varepsilon^{1-k}\, \partial_\tau \Phi_i\left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right). \tag{77}$$

The right hand side of (77) converges to zero strongly in $L^2\big(0, T; H^{-1}(\widetilde{\Omega})\big)$: indeed, for every $\tau \in \mathbb{R}$, let us define $v_i(\tau, \cdot)$ as the unique solution of the periodic elliptic problem

$$\begin{cases} -\Delta_y v_i(\tau, y) = \partial_\tau \Phi_i(\tau, y) \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \\ v_i(\tau, \cdot) \in H^1_{\mathrm{loc}}(\mathbb{R}^N), \ \text{periodic with period } Y_0, \\ \displaystyle\int_{Y_0} v_i(\tau, y)\, dy = 0; \end{cases} \tag{78}$$

note that (78) has a solution since the last assertion of (69) implies that the integral $\int_{Y_0} \partial_\tau \Phi_i(\tau, y)\, dy$ is zero. If we define the functions $q_i^\varepsilon(t, x)$ by

$$q_i^\varepsilon(t, x) = \nabla_y v_i\left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right),$$

we deduce from (78) that

$$\partial_\tau \Phi_i \left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right) = -\varepsilon \operatorname{div} q_i^\varepsilon .$$

Therefore

$$\partial_t w_i^\varepsilon - \operatorname{div}(A^\varepsilon \nabla w_i^\varepsilon) = f_i^\varepsilon = \varepsilon^{2-k} \operatorname{div} q_i^\varepsilon .$$

Since q^ε is bounded in $L^2(\widetilde{Q}; \mathbb{R}^N)$, and since $k < 2$, w_i^ε , f_i^ε and $g_i^\varepsilon = 0$ satisfy (22), (25) and (26), which implies that w_i^ε is a i-th special test function in the sense of Definition 3.2 above.

(ii) - CASE $k = 2$. This is the easiest case, since from (70) we obtain

$$\partial_t w_i^\varepsilon - \operatorname{div}(A^\varepsilon \nabla w_i^\varepsilon) = -\frac{1}{\varepsilon} \left[\partial_\tau \Phi_i - \operatorname{div}_y\left(A\left(\nabla_y \Phi_i - e_i\right)\right)\right] \left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) = 0 .$$

Therefore the functions w_i^ε satisfy (22) with $f_i^\varepsilon = 0$, $g_i^\varepsilon = 0$, and are special test functions according to Definition 3.2.

(iii) - CASE $k > 2$. In this case the functions w_i^ε do not depend on t, and from (71) we obtain

$$\begin{aligned} \partial_t w_i^\varepsilon - \operatorname{div}(A^\varepsilon \nabla w_i^\varepsilon) &= \frac{1}{\varepsilon} \left[\operatorname{div}_y\left(A\left(\nabla_y \Phi_i - e_i\right)\right)\right] \left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right) \\ &= \frac{1}{\varepsilon} \left[\operatorname{div}_y\left(\left(A - M_\tau(A)\right)\left(\nabla_y \Phi_i - e_i\right)\right)\right] \left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right) \\ &= \frac{1}{\varepsilon} v_i \left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right) , \end{aligned} \tag{79}$$

where we have set

$$v_i(\tau, y) = \operatorname{div}_y \left(\left(A(\tau, y) - M_\tau(A)(y)\right)\left(\nabla_y \Phi_i(y) - e_i\right)\right) .$$

It is clear that $M_\tau(v_i) = T_0^{-1} \int_0^{T_0} v_i(\tau, y)\, d\tau = 0$. Therefore the function V_i defined by

$$V_i(\tau, y) = \int_0^\tau v_i(\sigma, y)\, d\sigma$$

is a periodic function with respect to τ and y, and

$$\frac{1}{\varepsilon} v_i \left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right) = \partial_t \hat{h}_i^\varepsilon(t, x) ,$$

where

$$\hat{h}_i^\varepsilon(t, x) = \varepsilon^{k-1} V_i \left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}\right) . \tag{80}$$

The function w_i^ε therefore satisfies Definition 3.19 with $\hat{f}_i^\varepsilon = 0$, $\hat{g}_i^\varepsilon = 0$ and $\hat{h}_i^\varepsilon$ given by (80): indeed equation (59) is satisfied because of (79); since

$$\nabla \hat{h}_i^\varepsilon = \varepsilon^{k-2} \nabla_y V_i \left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon} \right),$$

$\hat{h}_i^\varepsilon$ converges to zero strongly in $L^2(0,T; H^1(\widetilde{\Omega}))$. As far as the trace at $t=0$ is concerned, we have

$$\hat{h}_i^\varepsilon(0,x) = \varepsilon^{k-1} V_i \left(0, \frac{x}{\varepsilon}\right) = 0 .$$

Finally, defining q_i, q_i^ε as

$$\begin{aligned} q_i(\tau, y) &= [A(\tau, y) - M_\tau(A)(y)] \, (\nabla_y \Phi_i(y) - e_i) , \\ q_i^\varepsilon(t,x) &= q_i \left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon} \right), \end{aligned}$$

we have $v_i = \operatorname{div}_y q_i$, so that

$$\frac{1}{\varepsilon} v_i \left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon} \right) = \operatorname{div} q_i^\varepsilon ,$$

which implies that $\partial_t \hat{h}_i^\varepsilon$ is bounded in $L^2(0,T; H^{-1}(\widetilde{\Omega}))$. Thus (64) is satisfied, and the functions w_i^ε are special test functions according to Definition 3.19, and therefore Proposition 3.13 holds with this choice of w_i^ε and p^ε (see Remark 3.18).

5. – Proof of the main results

The following theorem is the core of the proof of the corrector result.

THEOREM 5.1. *Assume that $A^\varepsilon, A^0 \in \mathcal{M}(\alpha, \beta; Q)$, $A^\varepsilon \xrightarrow{H} A^0$, and that $f \in L^2(0,T; H^{-1}(\Omega))$, $a \in L^2(\Omega)$. Let u^ε, u^0 be (respectively) the solutions of the parabolic problems* (0^ε) *and* (19). *Let ψ be a function in $C^\infty(\overline{Q})$ such that $\psi = 0$ in a neighborhood of $[0,T] \times \partial\Omega$, satisfying*[2]

$$\text{(81)} \qquad \begin{cases} \|\partial_t u^0 - \partial_t \psi\|_{L^2(0,T;H^{-1}(\Omega))} \le \delta , \\ \|u^0 - \psi\|_{L^2(0,T;H_0^1(\Omega))} \le \delta , \\ \|u^0(0) - \psi(0)\|_{L^2(\Omega)} \le \delta . \end{cases}$$

[2]The third inequality of (81) actually follows from the first two, up to a multiplicative constant $c(Q)$ (see (12)). By imposing the inequality in the form stated in (81), the constants appearing in (83), (84) and (85) do not depend on Q.

Define the functions $z^\varepsilon \in L^2\big(0, T; H_0^1(\Omega)\big)$ *by*

$$z^\varepsilon = \psi + \sum_{i=1}^{N} (w_i^\varepsilon - x_i)\, \partial_{x_i} \psi\,, \tag{82}$$

where the functions w_i^ε *are special test functions according to Definition* 3.2. *Then*

$$\alpha \limsup_{\varepsilon\to 0} \|u^\varepsilon - z^\varepsilon\|^2_{L^2\left(0,T;H_0^1(\Omega)\right)} \le c(\beta)\, \delta^2\,, \tag{83}$$

$$\frac{1}{2} \limsup_{\varepsilon\to 0} \|u^\varepsilon - z^\varepsilon\|^2_{C^0\left([0,T];L^2(\Omega)\right)} \le c(\beta)\, \delta^2\,, \tag{84}$$

$$\limsup_{\varepsilon\to 0} \left\| \partial_t u^\varepsilon - \left[\partial_t \psi + \sum_{i=1}^{N} \operatorname{div}\Big(\partial_{x_i}\psi\, (A^\varepsilon\, \nabla w_i^\varepsilon - A^0 e_i)\Big)\right] \right\|_{L^2\left(0,T;H^{-1}(\Omega)\right)} \le c(\alpha, \beta)\delta. \tag{85}$$

Remark 5.2. Using the equations satisfied by the special test functions w_i^ε and the computation (53), it is easy to see that statement (85) of Theorem 5.1 implies that the following decomposition holds

$$\partial_t u^\varepsilon - \left[\partial_t \psi + \sum_{i=1}^{N} \partial_t w_i^\varepsilon\, \partial_{x_i} \psi \right] = \eta_1^\varepsilon + \eta_2^\varepsilon\,, \tag{86}$$

where η_1^ε and η_2^ε, defined by

$$\begin{aligned} \eta_1^\varepsilon = \partial_t u^\varepsilon &- \left[\partial_t \psi + \sum_{i=1}^{N} \operatorname{div}\Big(\partial_{x_i}\psi\, (A^\varepsilon\, \nabla w_i^\varepsilon - A^0 e_i)\Big)\right] \\ &- \sum_{i=1}^{N} \Big[f_i^\varepsilon + g_i^0 + \operatorname{div}(A^0 e_i)\Big]\, \partial_{x_i}\psi\,, \\ \eta_2^\varepsilon = \sum_{i=1}^{N} &\big(A^\varepsilon\, \nabla w_i^\varepsilon - A^0 e_i\big) \cdot \nabla \partial_{x_i}\psi - \sum_{i=1}^{N} (g_i^\varepsilon - g_i^0)\, \partial_{x_i}\psi\,, \end{aligned}$$

satisfy

$$\limsup_{\varepsilon\to 0} \|\eta_1^\varepsilon\|_{L^2\left(0,T;H^{-1}(\Omega)\right)} \le c(\alpha, \beta)\, \delta\,, \qquad \eta_2^\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(Q). \tag{87}$$

In the particular case where u^0 is smooth, with $u^0 = 0$ in a neighborhood of $[0, T] \times \partial\Omega$, one can take $\psi = u^0$ and $\delta = 0$ in Theorem 5.1, so that (83), (84), (86) and (87) will prove Proposition 3.13 (see the First Step of the Proof of Theorem 3.8 below). □

REMARK 5.3. Let us explicitly emphasize that Meyers' regularity result will never be used in the proof of Theorem 5.1. □

REMARK 5.4. Observe also that in the smooth periodic case where one can take $\psi = u^0$, the function z^ε defined by (82) coincides with

$$u^0 - \varepsilon \sum_{i=1}^{N} \partial_{x_i} u^0 \, \Phi_i \left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon} \right) = u^0 + \varepsilon u^1 ,$$

which is the corrector used in [3] (see Section 4 above and in particular (73)). □

PROOF OF THEOREM 5.1.

FIRST STEP. We first observe that from the definition (82) of z^ε, from convergences (23), (24) of the special test functions w_i^ε, and from the compactness result (13) one obtains that z^ε belongs to $L^2(0,T; H_0^1(\Omega))$, and that

$$z^\varepsilon \rightharpoonup \psi \quad \text{weakly in} \quad L^2(0,T; H_0^1(\Omega)) , \tag{88}$$

$$z^\varepsilon(0) \to \psi(0) \quad \text{strongly in} \quad L^2(\Omega) , \tag{89}$$

$$z^\varepsilon \to \psi \quad \text{strongly in} \quad L^2(Q) . \tag{90}$$

Moreover, using the identity $\nabla \psi = \sum_{i=1}^N \partial_{x_i} \psi \, e_i$, one obtains

$$\begin{aligned} \nabla z^\varepsilon &= \nabla \psi + \sum_{i=1}^{N} \left(\partial_{x_i} \psi (\nabla w_i^\varepsilon - e_i) + (w_i^\varepsilon - x_i) \nabla \partial_{x_i} \psi \right) \\ &= \sum_{i=1}^{N} \partial_{x_i} \psi \, \nabla w_i^\varepsilon + \sum_{i=1}^{N} (w_i^\varepsilon - x_i) \, \nabla \partial_{x_i} \psi . \end{aligned} \tag{91}$$

Therefore

$$\begin{aligned} \operatorname{div}(A^\varepsilon \nabla z^\varepsilon) &= \sum_{i=1}^{N} \operatorname{div}\left(\partial_{x_i} \psi \, A^\varepsilon \nabla w_i^\varepsilon \right) + \sum_{i=1}^{N} \operatorname{div}\left((w_i^\varepsilon - x_i) \, A^\varepsilon \, \nabla \partial_{x_i} \psi \right) \\ &= \sum_{i=1}^{N} \partial_{x_i} \psi \operatorname{div}\left(A^\varepsilon \nabla w_i^\varepsilon \right) + \sum_{i=1}^{N} \left(A^\varepsilon \nabla w_i^\varepsilon \right) \cdot \nabla \partial_{x_i} \psi \\ &\quad + \sum_{i=1}^{N} \operatorname{div} \left((w_i^\varepsilon - x_i) \, A^\varepsilon \nabla \partial_{x_i} \psi \right) . \end{aligned} \tag{92}$$

On the other hand it is easy to see that $\partial_t z^\varepsilon \in L^2(0,T; H^{-1}(\Omega))$ and that

$$\partial_t z^\varepsilon = \partial_t \psi + \sum_{i=1}^{N} \partial_t w_i^\varepsilon \, \partial_{x_i} \psi + \sum_{i=1}^{N} (w_i^\varepsilon - x_i) \, \partial_{x_i t}^2 \psi . \tag{93}$$

Therefore from (92) and (93), recalling equation (22) satisfied by w_i^ε and equality (28), we obtain

$$\begin{aligned}
&(\partial_t u^\varepsilon - \partial_t z^\varepsilon) - \operatorname{div}\big(A^\varepsilon (\nabla u^\varepsilon - \nabla z^\varepsilon)\big) \\
&\qquad = f - \partial_t \psi - \sum_{i=1}^N \partial_{x_i}\psi\, f_i^\varepsilon - \sum_{i=1}^N \partial_{x_i}\psi\, g_i^\varepsilon - \sum_{i=1}^N (w_i^\varepsilon - x_i)\, \partial^2_{x_i t}\psi \\
&\qquad\quad + \sum_{i=1}^N \big(A^\varepsilon \nabla w_i^\varepsilon\big)\cdot \nabla\partial_{x_i}\psi + \sum_{i=1}^N \operatorname{div}\big((w_i^\varepsilon - x_i)\, A^\varepsilon \nabla\partial_{x_i}\psi\big) \\
&\qquad = f - \big(\partial_t\psi - \operatorname{div}(A^0 \nabla\psi)\big) + \sum_{i=1}^N \partial_{x_i}\psi\, (f_i^0 - f_i^\varepsilon) \\
&\qquad\quad + \sum_{i=1}^N \partial_{x_i}\psi\, (g_i^0 - g_i^\varepsilon) - \sum_{i=1}^N (w_i^\varepsilon - x_i)\, \partial^2_{x_i t}\psi \\
&\qquad\quad + \sum_{i=1}^N \big(A^\varepsilon \nabla w_i^\varepsilon - A^0 e_i\big)\cdot \nabla\partial_{x_i}\psi + \sum_{i=1}^N \operatorname{div}\big((w_i^\varepsilon - x_i)\, A^\varepsilon \nabla\partial_{x_i}\psi\big).
\end{aligned} \tag{94}$$

SECOND STEP. We now multiply (94) by $u^\varepsilon - z^\varepsilon$ and integrate on $(0,\tau)$, with $\tau \in [0,T]$. We denote by $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle_{Q_\tau}$ the duality pairing between the spaces $L^2(0,\tau; H^{-1}(\Omega))$ and $L^2(0,\tau; H_0^1(\Omega))$. Since

$$\langle\!\langle \partial_t(u^\varepsilon - z^\varepsilon)\,, u^\varepsilon - z^\varepsilon\rangle\!\rangle_{Q_\tau} = \frac12 \int_\Omega |u^\varepsilon(\tau) - z^\varepsilon(\tau)|^2\, dx - \frac12 \int_\Omega |a - z^\varepsilon(0)|^2\, dx \tag{95}$$

and since

$$\int_{Q_\tau} \big(A^\varepsilon (\nabla u^\varepsilon - \nabla z^\varepsilon)\big)\cdot\big(\nabla u^\varepsilon - \nabla z^\varepsilon\big) \ge \alpha \int_{Q_\tau} |\nabla u^\varepsilon - \nabla z^\varepsilon|^2\,, \tag{96}$$

from (94), (95) and (96) we obtain, for every $\tau\in[0,T]$,

$$\begin{aligned}
&\frac12 \int_\Omega |u^\varepsilon(\tau) - z^\varepsilon(\tau)|^2\, dx + \alpha \int_{Q_\tau} |\nabla u^\varepsilon - \nabla z^\varepsilon|^2 \\
&\le \langle\!\langle f - \big(\partial_t\psi - \operatorname{div}(A^0\nabla\psi)\big)\,, u^\varepsilon - z^\varepsilon\rangle\!\rangle_{Q_\tau} \\
&\quad + \sum_{i=1}^N \langle\!\langle \partial_{x_i}\psi\,(f_i^0 - f_i^\varepsilon)\,, u^\varepsilon - z^\varepsilon\rangle\!\rangle_{Q_\tau} \\
&\quad + \sum_{i=1}^N \int_{Q_\tau} \partial_{x_i}\psi\,(g_i^0 - g_i^\varepsilon)\,(u^\varepsilon - z^\varepsilon) - \sum_{i=1}^N \int_{Q_\tau} (w_i^\varepsilon - x_i)\,\partial^2_{x_i t}\psi\,(u^\varepsilon - z^\varepsilon) \\
&\quad + \sum_{i=1}^N \int_{Q_\tau} \big(A^\varepsilon\nabla w_i^\varepsilon - A^0 e_i\big)\cdot\nabla\partial_{x_i}\psi\,(u^\varepsilon - z^\varepsilon) \\
&\quad - \sum_{i=1}^N \int_{Q_\tau} (w_i^\varepsilon - x_i)\,\big(A^\varepsilon\nabla\partial_{x_i}\psi\big)\cdot\big(\nabla u^\varepsilon - \nabla z^\varepsilon\big) + \frac12\int_\Omega |a - z^\varepsilon(0)|^2\, dx\,.
\end{aligned} \tag{97}$$

From (97) one obtains

$$
\begin{aligned}
&\max\left\{\frac{1}{2}\,\|u^\varepsilon - z^\varepsilon\|^2_{C^0\left([0,T];L^2(\Omega)\right)}\,,\ \alpha\,\|u^\varepsilon - z^\varepsilon\|^2_{L^2\left(0,T;H^1_0(\Omega)\right)}\right\}\\
&\qquad\le \sup_{\tau\in[0,T]} \langle\!\langle f - \left(\partial_t\psi - \operatorname{div}(A^0\,\nabla\psi)\right),\ u^\varepsilon - z^\varepsilon\rangle\!\rangle_{Q_\tau}\\
&\qquad\quad + \sum_{i=1}^N \|\partial_{x_i}\psi\,(f_i^0 - f_i^\varepsilon)\|_{L^2\left(0,T;H^{-1}(\Omega)\right)}\,\|u^\varepsilon - z^\varepsilon\|_{L^2\left(0,T;H^1_0(\Omega)\right)}\\
&\qquad\quad + \sup_{\tau\in[0,T]}\sum_{i=1}^N \int_{Q_\tau} \partial_{x_i}\psi\,(g_i^0 - g_i^\varepsilon)\,(u^\varepsilon - z^\varepsilon)\\
&\qquad\quad + \sum_{i=1}^N \|w_i^\varepsilon - x_i\|_{L^2(Q)}\,\|\partial^2_{x_i t}\psi\|_{L^\infty(Q)}\,\|u^\varepsilon - z^\varepsilon\|_{L^2(Q)}\\
&\qquad\quad + \sup_{\tau\in[0,T]}\sum_{i=1}^N \int_{Q_\tau} \left(A^\varepsilon\,\nabla w_i^\varepsilon - A^0\,e_i\right)\cdot\nabla\partial_{x_i}\psi\,(u^\varepsilon - z^\varepsilon)\\
&\qquad\quad + \sum_{i=1}^N \|w_i^\varepsilon - x_i\|_{L^2(Q)}\,\|A^\varepsilon\,\nabla\partial_{x_i}\psi\|_{L^\infty(Q;\mathbb{R}^N)}\,\|u^\varepsilon - z^\varepsilon\|_{L^2\left(0,T;H^1_0(\Omega)\right)}\\
&\qquad\quad + \frac{1}{2}\int_\Omega |a - z^\varepsilon(0)|^2\,dx\,.
\end{aligned}
\tag{98}
$$

We will now pass to the limit as $\varepsilon\to 0$ in each term of the right hand side of (98). Since $u^\varepsilon - z^\varepsilon$ is bounded in $L^2\left(0,T;H^1_0(\Omega)\right)$, f_i^ε converges strongly in $L^2\left(0,T;H^{-1}(\Omega)\right)$ to f_i^0 and w_i^ε converges strongly in $L^2(Q)$ to x_i, we immediately obtain

$$
\sum_{i=1}^N \|\partial_{x_i}\psi\,(f_i^0 - f_i^\varepsilon)\|_{L^2\left(0,T;H^{-1}(\Omega)\right)}\,\|u^\varepsilon - z^\varepsilon\|_{L^2\left(0,T;H^1_0(\Omega)\right)} \to 0\,,
\tag{99}
$$

$$
\sum_{i=1}^N \|w_i^\varepsilon - x_i\|_{L^2(Q)}\,\|\partial^2_{x_i t}\psi\|_{L^\infty(Q)}\,\|u^\varepsilon - z^\varepsilon\|_{L^2(Q)} \to 0\,,
\tag{100}
$$

$$
\sum_{i=1}^N \|w_i^\varepsilon - x_i\|_{L^2(Q)}\,\|A^\varepsilon\,\nabla\partial_{x_i}\psi\|_{L^\infty(Q;\mathbb{R}^N)}\,\|u^\varepsilon - z^\varepsilon\|_{L^2\left(0,T;H^1_0(\Omega)\right)} \to 0\,.
\tag{101}
$$

Moreover, using (89), one has

$$
\frac{1}{2}\int_\Omega |a - z^\varepsilon(0)|^2\,dx \to \frac{1}{2}\int_\Omega |a - \psi(0)|^2\,dx\,.
\tag{102}
$$

Since g_i^ε converges weakly in $L^2(Q)$ to g_i^0, while $u^\varepsilon - z^\varepsilon$ converges strongly in $L^2(Q)$ to $u^0 - \psi$, we can apply Lemma 7.5 of Section 7 below, with

$\rho^\varepsilon = g_i^0 - g_i^\varepsilon$, $\sigma^\varepsilon = \partial_{x_i}\psi\,(u^\varepsilon - z^\varepsilon)$, obtaining

$$\sup_{\tau\in[0,T]} \sum_{i=1}^N \int_{Q_\tau} \partial_{x_i}\psi\,(g_i^0 - g_i^\varepsilon)\,(u^\varepsilon - z^\varepsilon) \to 0\,. \tag{103}$$

Similarly, using the weak convergence of $A^\varepsilon\,\nabla w_i^\varepsilon$ to $A^0\,e_i$ (see (27)), we obtain

$$\sup_{\tau\in[0,T]} \sum_{i=1}^N \int_{Q_\tau} \left(A^\varepsilon\,\nabla w_i^\varepsilon - A^0\,e_i\right)\cdot\nabla\partial_{x_i}\psi\,(u^\varepsilon - z^\varepsilon) \to 0\,. \tag{104}$$

It remains to study the term

$$\sup_{\tau\in[0,T]} \langle\!\langle f - \left(\partial_t\psi - \operatorname{div}(A^0\,\nabla\psi)\right),\, u^\varepsilon - z^\varepsilon\rangle\!\rangle_{Q_\tau}\,.$$

By the representation theorem for elements of $L^2\big(0,T;\,H^{-1}(\Omega)\big)$, there exists a function $F\in L^2(Q;\mathbb{R}^N)$ such that, for every v belonging to $L^2\big(0,T;\,H_0^1(\Omega)\big)$,

$$\langle\!\langle f - \left(\partial_t\psi - \operatorname{div}(A^0\,\nabla\psi)\right),\, v\rangle\!\rangle_{Q_\tau} = \int_{Q_\tau} F\cdot\nabla v\,.$$

Therefore, using again Lemma 7.5, we obtain

$$\begin{aligned}&\sup_{\tau\in[0,T]} \langle\!\langle f - \left(\partial_t\psi - \operatorname{div}(A^0\,\nabla\psi)\right),\, u^\varepsilon - z^\varepsilon\rangle\!\rangle_{Q_\tau}\\ &\qquad\to \sup_{\tau\in[0,T]} \langle\!\langle f - \left(\partial_t\psi - \operatorname{div}(A^0\,\nabla\psi)\right),\, u^0 - \psi\rangle\!\rangle_{Q_\tau}\,.\end{aligned} \tag{105}$$

Putting together the previous results, we deduce from (98) that

$$\begin{aligned}&\max\left\{\frac{1}{2}\,\|u^\varepsilon - z^\varepsilon\|^2_{C^0\left([0,T];L^2(\Omega)\right)},\ \alpha\,\|u^\varepsilon - z^\varepsilon\|^2_{L^2\left(0,T;H_0^1(\Omega)\right)}\right\}\\ &\le \|f - \left(\partial_t\psi - \operatorname{div}(A^0\nabla\psi)\right)\|_{L^2\left(0,T;H^{-1}(\Omega)\right)}\,\|u^0 - \psi\|_{L^2\left(0,T;H_0^1(\Omega)\right)}\\ &\quad + \frac{1}{2}\int_\Omega |a - \psi(0)|^2\,dx\,.\end{aligned} \tag{106}$$

Since u^0 solves problem (19), using the estimates (81) on $u^0 - \psi$ and the boundedness (11) of the matrix A^0, one has

$$\begin{aligned}&\|f - \left(\partial_t\psi - \operatorname{div}(A^0\,\nabla\psi)\right)\|_{L^2\left(0,T;H^{-1}(\Omega)\right)}\\ &\quad= \|(\partial_t u^0 - \partial_t\psi) - \operatorname{div}\left(A^0(\nabla u^0 - \nabla\psi)\right)\|_{L^2\left(0,T;H^{-1}(\Omega)\right)}\\ &\quad\le \|\partial_t u^0 - \partial_t\psi\|_{L^2\left(0,T;H^{-1}(\Omega)\right)} + \beta\|u^0 - \psi\|_{L^2\left(0,T;H_0^1(\Omega)\right)}\\ &\quad\le (1+\beta)\,\delta\,.\end{aligned} \tag{107}$$

Moreover, using the last estimate of (81), one has

$$\frac{1}{2}\int_\Omega |a-\psi(0)|^2\,dx \le \frac{\delta^2}{2}. \tag{108}$$

Statements (83) and (84), with $c(\beta)=\beta+3/2$ then follow from (106), (107) and (108).

THIRD STEP. It remains to prove (85).

We first define

$$X^\varepsilon = (\partial_t u^\varepsilon - \partial_t z^\varepsilon) - \sum_{i=1}^N \big(A^\varepsilon\,\nabla w_i^\varepsilon - A^0\,e_i\big)\cdot\nabla\partial_{x_i}\psi + \sum_{i=1}^N \partial_{x_i}\psi\,(g_i^\varepsilon - g_i^0).$$

Using equation (94) satisfied by $u^\varepsilon - z^\varepsilon$ and equation (19) satisfied by u^0, it is easy to check that

$$X^\varepsilon = \operatorname{div}\big(A^\varepsilon\,(\nabla u^\varepsilon - \nabla z^\varepsilon)\big) + (\partial_t u^0 - \partial_t\psi) - \operatorname{div}\big(A^0\,(\nabla u^0 - \nabla\psi)\big) + \rho^\varepsilon, \tag{109}$$

where

$$\begin{aligned}\rho^\varepsilon &= \sum_{i=1}^N \partial_{x_i}\psi\,(f_i^0 - f_i^\varepsilon) - \sum_{i=1}^N (w_i^\varepsilon - x_i)\,\partial^2_{x_i t}\psi\\ &\quad + \sum_{i=1}^N \operatorname{div}\big((w_i^\varepsilon - x_i)\,A^\varepsilon\,\nabla\partial_{x_i}\psi\big) \to 0 \text{ strongly in } L^2\big(0,T;H^{-1}(\Omega)\big).\end{aligned}$$

Therefore, from (81) and (83), we obtain

$$\begin{aligned}\limsup_{\varepsilon\to 0}\|X^\varepsilon\|_{L^2(0,T;H^{-1}(\Omega))} &\le \beta\,\limsup_{\varepsilon\to 0}\|u^\varepsilon - z^\varepsilon\|_{L^2(0,T;H_0^1(\Omega))}\\ &\quad + \|\partial_t u^0 - \partial_t\psi\|_{L^2(0,T;H^{-1}(\Omega))}\\ &\quad + \beta\,\|u^0-\psi\|_{L^2(0,T;H_0^1(\Omega))} \le c(\alpha,\beta)\,\delta,\end{aligned} \tag{110}$$

with $c(\alpha,\beta) = \beta\sqrt{\frac{\beta+\frac{3}{2}}{\alpha}} + 1 + \beta$.

On the other hand, recalling the definition of X^ε and (93), and then the definition of the functions w_i^ε, it is easy to check that

$$\begin{aligned}X^\varepsilon &= \partial_t u^\varepsilon - \partial_t\psi - \sum_{i=1}^N \partial_t w_i^\varepsilon\,\partial_{x_i}\psi - \sum_{i=1}^N (w_i^\varepsilon - x_i)\,\partial^2_{x_i t}\psi\\ &\quad - \sum_{i=1}^N \big(A^\varepsilon\,\nabla w_i^\varepsilon - A^0\,e_i\big)\cdot\nabla\partial_{x_i}\psi + \sum_{i=1}^N \partial_{x_i}\psi\,(g_i^\varepsilon - g_i^0)\\ &= \partial_t u^\varepsilon - \left[\partial_t\psi + \sum_{i=1}^N \operatorname{div}\big(\partial_{x_i}\psi\,(A^\varepsilon\,\nabla w_i^\varepsilon - A^0\,e_i)\big)\right] + \sigma^\varepsilon,\end{aligned} \tag{111}$$

where

$$\sigma^\varepsilon = \sum_{i=1}^{N} \partial_{x_i} \psi \, (f_i^0 - f_i^\varepsilon) - \sum_{i=1}^{N} (w_i^\varepsilon - x_i) \, \partial^2_{x_i t} \psi \to 0$$
$$\text{strongly in } L^2\big(0, T; H^{-1}(\Omega)\big).$$

Putting together (110) and (111) proves (85). □

PROOF OF THEOREM 3.8.

FIRST STEP. In the case where u^0 belongs to $C^\infty(\overline{Q})$, with $u^0 = 0$ in a neighborhood of $[0, T] \times \partial\Omega$, we can take $\psi = u^0$ and $\delta = 0$ in Theorem 5.1. We then deduce from (83) that

$$(112) \qquad u^\varepsilon - z^\varepsilon \to 0 \quad \text{strongly in } L^2\big(0, T; H_0^1(\Omega)\big).$$

Since, by the definitions of p^ε and z^ε, one has (see (91))

$$(113) \qquad \nabla z^\varepsilon = p^\varepsilon \nabla \psi + \sum_{i=1}^{N} (w_i^\varepsilon - x_i) \nabla \partial_{x_i} \psi \, ,$$

and since the last term of (113) tends to zero strongly in $L^2(Q; \mathbb{R}^N)$, we deduce from (112) that

$$\nabla u^\varepsilon - p^\varepsilon \nabla u^0 \to 0 \quad \text{strongly in } L^2\big(Q; \mathbb{R}^N\big),$$

i.e., that convergence (46) holds with $s = 2$ (note that $q = +\infty$ here).
Moreover from (84) one has

$$u^\varepsilon - z^\varepsilon \to 0 \quad \text{strongly in } C^0\big([0, T]; L^2(\Omega)\big).$$

Since, by the strong convergence (30) of w_i^ε to x_i, z^ε tends to u^0 strongly in $C^0\big([0, T]; L^2(\Omega)\big)$, (47) is proved. Finally, (85) directly gives (48) with $s = 2$. Note that the above results and Remark 5.2 prove Proposition 3.13.

SECOND STEP. In the case where u^0 does not meet the above regularity, we approximate u^0: for $\delta > 0$, let ψ be a function in $C^\infty(\overline{Q})$ which is zero in a neighborhood of $[0, T] \times \partial\Omega$ such that (81) holds. It is possible to find such a function ψ, for instance in the following way: the function u^0 is first regularized by convolution in time, obtaining a function which belongs to $C^\infty\big([0, T]; H_0^1(\Omega)\big)$; the time derivative of this function is then approximated in $L^2\big(0, T; H_0^1(\Omega)\big)$ by functions which are piecewise constant in time; these (finitely many) values in $H_0^1(\Omega)$ are next approximated by elements of $C_0^\infty(\Omega)$; finally this function is again regularized by convolution in time. It is clear that in this way one obtains an approximation of u^0 in $L^2\big(0, T; H_0^1(\Omega)\big) \cap$

$H^1\big(0,T;H^{-1}(\Omega)\big)$. Moreover, using the embedding result (12), one can always assume that the third estimate of (81) is also satisfied.

We first prove (47). Let us define z^ε as in Theorem 5.1. By the strong convergence (30) in $C^0\big([0,T];L^2(\Omega)\big)$ of the special test functions w_i^ε, z^ε satisfies

$$z^\varepsilon \to \psi \quad \text{strongly in } C^0\big([0,T];L^2(\Omega)\big).$$

Therefore, using the triangle inequality

$$\begin{aligned}\limsup_{\varepsilon\to 0}\|u^\varepsilon - u^0\|_{C^0([0,T];L^2(\Omega))} &\le \limsup_{\varepsilon\to 0}\|u^\varepsilon - z^\varepsilon\|_{C^0([0,T];L^2(\Omega))}\\ &+ \limsup_{\varepsilon\to 0}\|z^\varepsilon - \psi\|_{C^0([0,T];L^2(\Omega))} + \|\psi - u^0\|_{C^0([0,T];L^2(\Omega))}\end{aligned}$$

and recalling (84), (81) and (12), we obtain

$$\limsup_{\varepsilon\to 0}\|u^\varepsilon - u^0\|_{C^0([0,T];L^2(\Omega))} \le c\,\delta\,.$$

Since δ is arbitrary, (47) is proved.

Let us explicitly emphasize that convergence (47), whose proof relies on convergence (30) and therefore on Meyers' estimate, will never be used in the end of the proof below, except in the special case where $q=+\infty$ and $r=2$ (see the end of the third step). Therefore statements (46) and (48) do not rely on Meyers' regularity result when $(q,r)\neq(+\infty,2)$.

THIRD STEP. Let us now prove (46). We write

$$\begin{aligned}\|\nabla u^\varepsilon - p^\varepsilon\,\nabla u^0\|_{L^s(Q;\mathbb{R}^N)} &\le \|\nabla u^\varepsilon - \nabla z^\varepsilon\|_{L^s(Q;\mathbb{R}^N)}\\ &+ \|\nabla z^\varepsilon - p^\varepsilon\,\nabla\psi\|_{L^s(Q;\mathbb{R}^N)} + \|p^\varepsilon\,(\nabla\psi - \nabla u^0)\|_{L^s(Q;\mathbb{R}^N)}\,.\end{aligned} \tag{114}$$

Since $s\le 2$, applying Theorem 5.1 we obtain

$$\limsup_{\varepsilon\to 0}\|\nabla u^\varepsilon - \nabla z^\varepsilon\|_{L^s(Q;\mathbb{R}^N)} \le c\,\limsup_{\varepsilon\to 0}\|\nabla u^\varepsilon - \nabla z^\varepsilon\|_{L^2(Q;\mathbb{R}^N)} \le c\,\delta\,, \tag{115}$$

with c depending only on α, β, s and Q. Using (113), the properties of the special test functions and compactness property (13), one has

$$\nabla z^\varepsilon - p^\varepsilon\,\nabla\psi = \sum_{i=1}^N (w_i^\varepsilon - x_i)\,\nabla\partial_{x_i}\psi \longrightarrow 0 \quad \text{strongly in } L^2(Q;\mathbb{R}^N),$$

and therefore, since $s\le 2$,

$$\lim_{\varepsilon\to 0}\|\nabla z^\varepsilon - p^\varepsilon\,\nabla\psi\|_{L^s(Q;\mathbb{R}^N)} = 0\,. \tag{116}$$

It only remains to estimate the term $\|p^\varepsilon\,(\nabla\psi - \nabla u^0)\|_{L^s(Q;\mathbb{R}^N)}$.

Let us begin with the case where ∇u^0 belongs to $L^q(Q;\mathbb{R}^N)$, with $q < +\infty$. In this case, using the approximation technique described in the second step above, we can assume that, in addition to (81), we also have

$$\|\psi - u^0\|_{L^q\left(0,T;W_0^{1,q}(\Omega)\right)} \le \delta\,. \tag{117}$$

Since s satisfies (45), using Hölder's inequality, we obtain

$$\|p^\varepsilon\ (\nabla\psi - \nabla u^0)\|_{L^s(Q;\mathbb{R}^N)} \le c\, \|p^\varepsilon\|_{L^r(Q;\mathbb{R}^{N\times N})}\, \|\nabla\psi - \nabla u^0\|_{L^q(Q;\mathbb{R}^N)} \le c\,\delta\,. \tag{118}$$

Therefore from (114), (115), (116) and (118) we obtain

$$\limsup_{\varepsilon\to 0} \|\nabla u^\varepsilon - p^\varepsilon \nabla u^0\|_{L^s(Q;\mathbb{R}^N)} \le c\,\delta\,, \tag{119}$$

with c depending only on s, α, β, Q and on the bound c_0 of p^ε in L^q. Since δ is arbitrary, (46) follows in the case where $q < +\infty$.

If $q = +\infty$ (which implies $s = 2$) one cannot assume that the approximating function ψ satisfies (117). However, if $r > 2$, this is not a problem, since, in this case, we can replace $q = +\infty$ by some q finite but so large that $\frac{1}{q} + \frac{1}{r} \le \frac{1}{2}$, and proceed as before.

In the case where $q = +\infty$ and $r = 2$, this approach cannot be followed. To overcome this problem, we will use the equi-integrability of $|p^\varepsilon|^2$, which follows from Proposition 3.3 and the definition of the correctors. Therefore, for every $\delta > 0$, there exists $\eta > 0$ such that, for every measurable set $D \subset Q$, with $\operatorname{meas} D < \eta$, one has

$$\int_D |p^\varepsilon|^2 < \delta^2\,, \qquad \text{for every } \varepsilon.$$

Using Egorov's theorem, it is easy to see that one may assume that the smooth function ψ approximating u^0 obtained by the technique described in the second step satisfies, in addition to (81), the following property: there exists a set $D \subset Q$ such that

$$\|\nabla\psi - \nabla u^0\|_{L^\infty(Q\setminus D;\mathbb{R}^N)} < \delta\,, \qquad \operatorname{meas} D < \eta\,.$$

Moreover one may assume that

$$\|\nabla\psi\|_{L^\infty(Q;\mathbb{R}^N)} \le c\, \|\nabla u^0\|_{L^\infty(Q;\mathbb{R}^N)}\,.$$

Therefore

$$\begin{aligned}
\|p^\varepsilon(\nabla\psi - \nabla u^0)\|^2_{L^2(Q;\mathbb{R}^N)} &= \int_{Q\setminus D} |p^\varepsilon(\nabla\psi - \nabla u^0)|^2 + \int_D |p^\varepsilon(\nabla\psi - \nabla u^0)|^2 \\
&\le \delta^2 \int_Q |p^\varepsilon|^2 + c^2\, \|\nabla u^0\|^2_{L^\infty(Q;\mathbb{R}^N)} \int_D |p^\varepsilon|^2 \le c\,\delta^2.
\end{aligned}$$

That is, we have proved (46) when $q = +\infty$ and $r = 2$. This completes the proof of (46).

FOURTH STEP. Let us finally prove (48). We limit ourselves to the case $q < \infty$, since the case $q = \infty$ can be dealt with by arguing as above. We have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\left\|\partial_t u^\varepsilon - \left[\partial_t u^0 + \sum_{i=1}^N \operatorname{div}\left(\partial_{x_i} u^0\,(A^\varepsilon\,\nabla w_i^\varepsilon - A^0\,e_i)\right)\right]\right\|_{L^s\left(0,T;W^{-1,s}(\Omega)\right)} \\
&\quad\le \limsup_{\varepsilon\to 0}\left\|\partial_t u^\varepsilon - \left[\partial_t \psi + \sum_{i=1}^N \operatorname{div}\left(\partial_{x_i} \psi\,(A^\varepsilon\,\nabla w_i^\varepsilon - A^0\,e_i)\right)\right]\right\|_{L^s\left(0,T;W^{-1,s}(\Omega)\right)} \\
&\qquad + \left\|\partial_t \psi - \partial_t u^0\right\|_{L^s\left(0,T;W^{-1,s}(\Omega)\right)} \\
&\qquad + \limsup_{\varepsilon\to 0}\sum_{i=1}^N \left\|\operatorname{div}\left((\partial_{x_i}\psi - \partial_{x_i} u^0)\,(A^\varepsilon\,\nabla w_i^\varepsilon - A^0\,e_i)\right)\right\|_{L^s\left(0,T;W^{-1,s}(\Omega)\right)}.
\end{aligned} \tag{120}$$

Since $s \le 2$, in view of (85), the first term of the right hand side of (120) can be estimated by $c\,\delta$, with c depending only on α, β, s and Q. Similarly, by (81), we have

$$\left\|\partial_t \psi - \partial_t u^0\right\|_{L^s\left(0,T;W^{-1,s}(\Omega)\right)} \le c(s,\,Q)\,\delta\,.$$

As far as the last term of (120) is concerned, since $A^\varepsilon\,\nabla w_i^\varepsilon - A^0\,e_i$ is bounded in $L^r(Q;\mathbb{R}^N)$, using (117) we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\sum_{i=1}^N \left\|\operatorname{div}\left((\partial_{x_i}\psi - \partial_{x_i} u^0)\,(A^\varepsilon\,\nabla w_i^\varepsilon - A^0\,e_i)\right)\right\|_{L^s\left(0,T;W^{-1,s}(\Omega)\right)} \\
&\quad\le c\,\|\nabla\psi - \nabla u^0\|_{L^q(Q;\mathbb{R}^N)} \le c\,\delta\,.
\end{aligned}$$

Therefore

$$\limsup_{\varepsilon\to 0}\left\|\partial_t u^\varepsilon - \left[\partial_t u^0 + \sum_{i=1}^N \operatorname{div}\left(\partial_{x_i} u^0\,(A^\varepsilon\,\nabla w_i^\varepsilon - A^0\,e_i)\right)\right]\right\|_{L^s\left(0,T;W^{-1,s}(\Omega)\right)} \le c\,\delta\,.$$

The arbitrariness of δ completes the proof of (48). □

6. – An application to quasilinear equations

As an example, we give in this section a simple application of the corrector result to the study of a sequence of parabolic equations involving nonlinear terms depending on the gradient. More precisely, we consider the problem

$$\begin{cases} \partial_t u^\varepsilon - \operatorname{div}\bigl(A^\varepsilon \nabla u^\varepsilon\bigr) = H^\varepsilon(t,x,u^\varepsilon,\nabla u^\varepsilon) + f \quad \text{in } \mathcal{D}'(Q), \\ u^\varepsilon \in L^2\bigl(0,T; H_0^1(\Omega)\bigr) \cap H^1\bigl(0,T; H^{-1}(\Omega)\bigr), \\ u^\varepsilon\Big|_{t=0} = a. \end{cases} \tag{121}$$

We assume that the data f and a satisfy (14), and that

$$A^\varepsilon, A^0 \in \mathcal{M}(\alpha,\beta;Q), \qquad A^\varepsilon \overset{H}{\rightharpoonup} A^0; \tag{122}$$

we also assume that, for every ε, $H^\varepsilon(t,x,\sigma,\xi) : (0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying the following conditions (uniformly with respect to ε)

$$|H^\varepsilon(t,x,\sigma,\xi)| \le c_1\,(k_1(t,x) + |\sigma| + |\xi|), \tag{123}$$

$$|H^\varepsilon(t,x,\sigma,\xi) - H^\varepsilon(t,x,\sigma,\xi')| \le c_2\,(1+|\sigma|)\,|\xi-\xi'|, \tag{124}$$

$$|H^\varepsilon(t,x,\sigma,\xi) - H^\varepsilon(t,x,\sigma',\xi)| \le c_3\,(k_3(t,x) + |\sigma| + |\sigma'| + |\xi|)\,|\sigma-\sigma'| \tag{125}$$

for almost every $(t,x)\in Q$, for every $\sigma,\sigma'\in\mathbb{R}$, for every $\xi,\xi'\in\mathbb{R}^N$, where k_1 and k_3 are positive functions in $L^2(Q)$ and c_1, c_2, c_3 are positive constants.

The existence of a solution of (121) is classical; moreover, using u^ε as test function, one obtains

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |u^\varepsilon(t)|^2\,dx + \alpha\int_\Omega |\nabla u^\varepsilon(t)|^2\,dx \\ &\quad\le \frac{c_1^2}{2}\int_\Omega k_1^2(t)\,dx + \frac{1}{2}\int_\Omega |u^\varepsilon(t)|^2\,dx \\ &\qquad + c_1\int_\Omega |u^\varepsilon(t)|^2\,dx + \frac{\alpha}{4}\int_\Omega |\nabla u^\varepsilon(t)|^2\,dx + \frac{c_1^2}{\alpha}\int_\Omega |u^\varepsilon(t)|^2\,dx \\ &\qquad + \frac{1}{\alpha}\|f(t)\|^2_{H^{-1}(\Omega)} + \frac{\alpha}{4}\int_\Omega |\nabla u^\varepsilon(t)|^2\,dx, \end{aligned} \tag{126}$$

which gives an *apriori* estimate for u^ε in $L^2\bigl(0,T; H_0^1(\Omega)\bigr)$, by using Gronwall's inequality. Therefore, extracting a subsequence, u^ε converges weakly to some function u^0.

It is then reasonable to ask whether this u^0 is a solution of an equation of the same form as (121), where the matrix A^ε is replaced by A^0 and the function H^ε is replaced by a suitable function H^0, to be identified. This problem has been studied for the corresponding elliptic equations, for example in [4], [2] and [1]. The answer to this question is positive, but the new function H^0 has to be constructed by the use of the corrector, and even in the case where $H^\varepsilon = H$ does not depend on ε, the limit function H^0 is in general different from H (see Remark 6.2 below).

PROPOSITION 6.1. *Assume that* $\{H^\varepsilon\}$ *is a sequence of Carathéodory functions satisfying* (123), (124) *and* (125), *and that the matrices* A^ε, A^0 *satisfy* (122). *Let* $\{p^\varepsilon\}$ *be a sequence of corrector matrices in the sense of Definition* 3.6. *Then there exist a subsequence, still denoted by* $\{H^\varepsilon\}$, *and a Carathéodory function* $H^0(t, x, \sigma, \xi)$: $(0, T) \times \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *satisfying* (123), (124), (125) (*for different constants* c_1, c_2, c_3) *such that*

$$H^\varepsilon\big(t, x, \sigma, p^\varepsilon(t, x)\xi\big) \rightharpoonup H^0(t, x, \sigma, \xi) \quad \textit{weakly in } L^2(Q), \tag{127}$$

for every $\sigma \in \mathbb{R}$ *and for every* $\xi \in \mathbb{R}^N$.

PROOF. By (123) the sequence $\{H^\varepsilon(\cdot, \cdot, \sigma, p^\varepsilon(\cdot, \cdot)\,\xi)\}$ is bounded in $L^2(Q)$ for every fixed $\sigma \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, so that one can extract a subsequence which converges weakly in $L^2(Q)$ to a limit, which we will denote by $H^0(t, x, \sigma, \xi)$. By a diagonal argument, one can extract a subsequence such that convergence (127) holds for every rational number σ and every vector ξ with rational coordinates.

We claim that the function H^0, now defined only for rational σ and ξ, satisfies inequalities (123), (124), (125), with different constants c_1, c_2, c_3. Let us prove, for instance, (123); from now on, for the sake of brevity, we will omit the dependence of H^ε on t and x. If $\varphi \in C_0^\infty(Q)$, $\varphi \geq 0$, then

$$\begin{aligned}\int_Q |H^0(\sigma, \xi)|\,\varphi &\leq \liminf_{\varepsilon\to 0} \int_Q |H^\varepsilon(\sigma, p^\varepsilon\xi)|\,\varphi \\ &\leq c_1 \left[\int_Q k_1(t, x)\,\varphi + |\sigma| \int_Q \varphi + \liminf_{\varepsilon\to 0} \int_Q |p^\varepsilon\xi|\,\varphi\right].\end{aligned} \tag{128}$$

Using Cauchy-Schwartz's inequality and the ellipticity of the matrices A^ε one has

$$\int_Q |p^\varepsilon\xi|\,\varphi \leq \left[\int_Q |p^\varepsilon\xi|^2\,\varphi\right]^{\frac12} \left[\int_Q \varphi\right]^{\frac12} \leq \frac{1}{\sqrt{\alpha}} \left[\int_Q (A^\varepsilon\, p^\varepsilon\xi)\cdot(p^\varepsilon\xi)\,\varphi\right]^{\frac12} \left[\int_Q \varphi\right]^{\frac12}.$$

Therefore, using the convergence of energy (42) of Proposition 3.7, one obtains

$$\limsup_{\varepsilon\to 0} \int_Q |p^\varepsilon\xi|\,\varphi \leq \frac{1}{\sqrt{\alpha}} \left[\int_Q (A^0\xi)\cdot\xi\,\varphi\right]^{\frac12} \left[\int_Q \varphi\right]^{\frac12} \leq \sqrt{\frac{\beta}{\alpha}}\,|\xi| \int_Q \varphi,$$

so that, by (128),

$$\int_Q |H^0(\sigma, \xi)|\,\varphi \leq c_1 \int_Q \left(k_1(t, x) + |\sigma| + \sqrt{\frac{\beta}{\alpha}}\,|\xi|\right)\varphi.$$

Since this holds for every positive function $\varphi \in C_0^\infty(Q)$, it follows that H^0 satisfies (123), with c_1 replaced by $c_1\sqrt{\beta/\alpha}$, for all rational σ and ξ. Inequalities (124) and (125) can be proved in the same way.

Once (123), (124) and (125) are proved for rational values of $\sigma, \sigma', \xi, \xi'$, it is easy to extend $H^0(\sigma, \xi)$ to all values of σ and ξ in such a way that (127) holds for the same previously extracted subsequence (see [1] for the details if necessary). □

REMARK 6.2. An interesting and natural example of the situation under consideration is the periodic case where the matrices $A^\varepsilon(t, x)$ are of the form (68), while H^ε has the form

$$H^\varepsilon(t, x, \sigma, \xi) = H\left(\frac{t}{\varepsilon^k}, \frac{x}{\varepsilon}, \sigma, \xi\right),$$

with $H(\tau, y, \sigma, \xi)$ smooth and periodic with respect to (τ, y) (with period $(0, T_0) \times Y_0$), satisfying (123), (124) and (125). It is then easy to check that, for every σ and ξ, $H^\varepsilon(t, x, \sigma, p^\varepsilon \xi)$ converges weakly in $L^2(Q)$ to

$$H^0(\sigma, \xi) = \frac{1}{T_0 \operatorname{meas} Y_0} \int_{(0,T_0)\times Y_0} H\big(\tau, y, \sigma, P(\tau, y)\xi\big)\, d\tau\, dy,$$

where $P(\tau, y)$ is the matrix defined by (75). Taking H depending only on ξ, it is then easy to see that H^0 is in general different from H. □

Let's go back to the quasilinear problem (121). By the *a priori* estimate (126), u^ε is bounded in $L^2\big(0, T; H_0^1(\Omega)\big)$; since we have $\partial_t u^\varepsilon = \operatorname{div}(A^\varepsilon \nabla u^\varepsilon) + H^\varepsilon(t, x, u^\varepsilon, \nabla u^\varepsilon) + f$, $\partial_t u^\varepsilon$ is bounded in $L^2\big(0, T; H^{-1}(\Omega)\big)$ and therefore, in view of (13), the sequence $\{u^\varepsilon\}$ is relatively compact in $L^2(Q)$. Thus it is possible to extract a subsequence, which we will continue to denote by $\{u^\varepsilon\}$, such that

(129) $$u^\varepsilon \rightharpoonup u^0 \quad \text{weakly in } L^2\big(0, T; H_0^1(\Omega)\big),$$

(130) $$u^\varepsilon \to u^0 \quad \text{strongly in } L^2(Q), \text{ and almost everywhere in } Q.$$

The next result identifies the equation satisfied by u^0.

THEOREM 6.3. *Assume that* $a \in L^2(\Omega)$, $f \in L^2\big(0, T; H^{-1}(\Omega)\big)$, *that* $\{A^\varepsilon\}$ *is a sequence in* $\mathcal{M}(\alpha, \beta; Q)$ *which* H*-converges to* A^0, *and that* $\{H^\varepsilon\}$ *is a sequence of Carathéodory functions satisfying* (123), (124) *and* (125), *which converges to a function* H^0 *in the sense of* (127). *Assume moreover that* $\{u^\varepsilon\}$ *is a sequence of solutions of the quasilinear problems* (121) *satisfying* (129) *and* (130). *Then* u^0 *is a solution of the problem*

(131) $$\begin{cases} \partial_t u^0 - \operatorname{div}\big(A^0 \nabla u^0\big) = H^0\big(t, x, u^0, \nabla u^0\big) + f & \text{in } \mathcal{D}'(Q), \\ u^0 \in L^2\big(0, T; H_0^1(\Omega)\big) \cap H^1\big(0, T; H^{-1}(\Omega)\big), \\ u^0\big|_{t=0} = a. \end{cases}$$

PROOF. By (123), the sequence $H^\varepsilon(u^\varepsilon, \nabla u^\varepsilon)$ is bounded in $L^2(Q)$, therefore we can assume, by passing to a subsequence, that for some function $h \in L^2(Q)$

$$H^\varepsilon\big(u^\varepsilon, \nabla u^\varepsilon\big) \rightharpoonup h \quad \text{weakly in } L^2(Q).$$

Defining r^ϵ and v^ϵ by

$$(132) \qquad \begin{cases} \partial_t r^\varepsilon - \operatorname{div}(A^\varepsilon \nabla r^\varepsilon) = H^\epsilon(u^\varepsilon, \nabla u^\varepsilon) - h \quad \text{in } \mathcal{D}'(Q), \\ r^\varepsilon \in L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega)), \\ r^\varepsilon\Big|_{t=0} = 0, \end{cases}$$

$$(133) \qquad \begin{cases} \partial_t v^\varepsilon - \operatorname{div}(A^\varepsilon \nabla v^\varepsilon) = h + f \quad \text{in } \mathcal{D}'(Q), \\ v^\varepsilon \in L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega)), \\ v^\varepsilon\Big|_{t=0} = a, \end{cases}$$

we have

$$u^\varepsilon = r^\varepsilon + v^\varepsilon .$$

Lemma 7.1 below implies that

$$(134) \qquad r^\varepsilon \to 0 \quad \text{strongly in } L^2(0,T; H_0^1(\Omega)),$$

and therefore

$$v^\varepsilon \rightharpoonup u^0 \quad \text{weakly in } L^2(0,T; H_0^1(\Omega)).$$

From Definition 2.1 of H-convergence, it follows that u^0 satisfies (130).

It only remains to show that $h = H^0(u^0, \nabla u^0)$. By the corrector result of Theorem 3.8, we obtain

$$(135) \qquad \nabla v^\varepsilon - p^\varepsilon \nabla u^0 \to 0 \quad \text{strongly in } L^1(Q; \mathbb{R}^N),$$

and therefore we have

$$(136) \qquad \nabla u^\varepsilon - p^\varepsilon \nabla u^0 \to 0 \quad \text{strongly in } L^1\big(Q; \mathbb{R}^N\big).$$

For $\delta > 0$, let ϕ, ψ be two simple functions (*i.e.*, measurable functions which assume only finitely many values), with values respectively in $\mathbb{R}$ and $\mathbb{R}^N$ such that

$$(137) \qquad \|\phi - u^0\|_{L^2(Q)} \leq \delta, \qquad \|\psi - \nabla u^0\|_{L^2(Q;\mathbb{R}^N)} \leq \delta .$$

Then, if φ is an arbitrary function in $L^\infty(Q)$, we have

$$
\begin{aligned}
&\left|\int_Q \left[H^\varepsilon(u^\varepsilon, \nabla u^\varepsilon) - H^0(u^0, \nabla u^0)\right] \varphi\right| \\
&\quad \leq \|\varphi\|_{L^\infty(Q)} \int_Q \left|H^\varepsilon(u^\varepsilon, \nabla u^\varepsilon) - H^\varepsilon(u^0, \nabla u^\varepsilon)\right| \\
&\quad + \|\varphi\|_{L^\infty(Q)} \int_Q \left|H^\varepsilon(u^0, \nabla u^\varepsilon) - H^\varepsilon(\phi, \nabla u^\varepsilon)\right| \\
&\quad + \|\varphi\|_{L^\infty(Q)} \int_Q \left|H^\varepsilon(\phi, \nabla u^\varepsilon) - H^\varepsilon(\phi, p^\varepsilon \nabla u^0)\right| \\
&\quad + \|\varphi\|_{L^\infty(Q)} \int_Q \left|H^\varepsilon(\phi, p^\varepsilon \nabla u^0) - H^\varepsilon(\phi, p^\varepsilon \psi)\right| \\
&\quad + \left|\int_Q \left[H^\varepsilon(\phi, p^\varepsilon \psi) - H^0(\phi, \psi)\right] \varphi\right| \\
&\quad + \|\varphi\|_{L^\infty(Q)} \int_Q \left|H^0(\phi, \psi) - H^0(u^0, \nabla u^0)\right|.
\end{aligned}
\tag{138}
$$

Using convergences (130) and (136), and inequalities (124) and (125) satisfied by H^ε, one easily obtains that the first and third integrals of the right hand side of (138) tend to zero; similarly, it follows from (137) that the second, the fourth and the last one are bounded by $c\,\delta$, for some constant c. Finally, as a trivial consequence of (127) one obtains

$$H^\varepsilon(\phi, p^\varepsilon \psi) \rightharpoonup H^0(\phi, \psi) \quad \text{weakly in } L^2(Q),$$

and therefore the fifth integral of the right hand side of (138) tends to zero. By the arbitrariness of δ, it follows that $H^\varepsilon(u^\varepsilon, \nabla u^\varepsilon)$ tends to $H^0(u^0, \nabla u^0)$ weakly in $L^1(Q)$, and therefore that $h = H^0(u^0, \nabla u^0)$. □

7. – Some auxiliary results

In this section we prove some results which we have used in the present paper. Let us start with the proof of Proposition 3.1.

PROOF OF PROPOSITION 3.1. For $f \in L^2\big(0, T; H^{-1}(\widetilde{\Omega})\big)$ and $a \in L^2(\widetilde{\Omega})$, consider the problems

$$
\begin{cases}
\partial_t u^\varepsilon - \operatorname{div}(\tilde{A}^\varepsilon \nabla u^\varepsilon) = f & \text{in } \mathcal{D}'(\widetilde{Q}), \\
u^\varepsilon \in L^2\big(0, T; H_0^1(\widetilde{\Omega})\big) \cap H^1(0, T; H^{-1}(\widetilde{\Omega}))\,, \\
u^\varepsilon\Big|_{t=0} = a\,.
\end{cases}
$$

We must show that the whole sequence $\{u^\varepsilon\}$ satisfies

$$u^\varepsilon \rightharpoonup u^0 \qquad \text{weakly in } L^2(0,T;H_0^1(\widetilde{\Omega}))\,, \tag{139}$$

$$\tilde{A}^\varepsilon\,\nabla u^\varepsilon \rightharpoonup \tilde{A}^0\,\nabla u^0 \quad \text{weakly in } L^2(\widetilde{Q};\mathbb{R}^N)\,, \tag{140}$$

which implies that u^0 is the unique solution of problem

$$\begin{cases} \partial_t u^0 - \operatorname{div}\big(\tilde{A}^0\,\nabla u^0\big) = f \quad \text{in } \mathcal{D}'(\widetilde{Q}), \\ u^0 \in L^2(0,T;H_0^1(\widetilde{\Omega})) \cap H^1(0,T;H^{-1}(\widetilde{\Omega}))\,, \\ u^0\Big|_{t=0} = a\,. \end{cases} \tag{141}$$

By extracting a subsequence, which we will denote by the superscript ε', we can assume that

$$u^{\varepsilon'} \rightharpoonup u \quad \text{weakly in } L^2\big(0,T;H_0^1(\widetilde{\Omega})\big) \cap H^1\big(0,T;H^{-1}(\widetilde{\Omega})\big)\,,$$

(see the *a priori* estimates (15) and (16)). We only have to prove that

$$\tilde{A}^{\varepsilon'}\,\nabla u^{\varepsilon'} \rightharpoonup \tilde{A}^0\,\nabla u \quad \text{weakly in } L^2\big(\widetilde{Q};\mathbb{R}^N\big)\,; \tag{142}$$

this will imply that $u = u^0$, and therefore that the whole sequence $\{u^\varepsilon\}$ satisfies (139) and (140).

Since $\tilde{A}^\varepsilon = A^\varepsilon$, $\tilde{A}^0 = A^0$ on Q, using Proposition 2.3 we obtain that

$$\tilde{A}^{\varepsilon'}\,\nabla u^{\varepsilon'} \rightharpoonup \tilde{A}^0\,\nabla u \quad \text{weakly in } L^2\big(Q;\mathbb{R}^N\big)\,. \tag{143}$$

On the other hand, on $\widetilde{Q}\setminus Q$ one has

$$\tilde{A}^{\varepsilon'}\,\nabla u^{\varepsilon'} = \alpha\,\nabla u^{\varepsilon'} \rightharpoonup \alpha\,\nabla u = \tilde{A}^0\,\nabla u \quad \text{weakly in } L^2\big(\widetilde{Q}\setminus Q;\mathbb{R}^N\big)\,. \tag{144}$$

Putting together (143) and (144) we obtain (142), and this completes the proof. $\square$

We now turn our attention to the proof of Proposition 3.3. We begin with a simple lemma.

LEMMA 7.1. *Assume that $\{A^\varepsilon\}$ is a sequence of matrices in $\mathcal{M}(\alpha,\beta;Q)$, and that $\{r^\varepsilon\}$ is a sequence of functions satisfying*

$$\begin{cases} \partial_t r^\varepsilon - \operatorname{div}\big(A^\varepsilon\,\nabla r^\varepsilon\big) = f^\varepsilon + g^\varepsilon \quad \text{in } \mathcal{D}'(Q)\,, \\ r^\varepsilon \in L^2(0,T;H_0^1(\Omega)) \cap H^1\big(0,T;H^{-1}(\Omega)\big)\,, \\ r^\varepsilon\Big|_{t=0} = a^\varepsilon\,, \end{cases} \tag{145}$$

where

$$a^\varepsilon \to 0 \quad \textit{strongly in} \;\; L^2(\Omega)\,, \tag{146}$$

$$f^\varepsilon \in L^2\big(0, T; H^{-1}(\Omega)\big)\,, \;\; f^\varepsilon \to f^0 \; \textit{strongly in} \;\; L^2\big(0, T; H^{-1}(\Omega)\big)\,, \tag{147}$$

$$g^\varepsilon \in L^2(Q)\,, \quad g^\varepsilon \rightharpoonup g^0 \;\; \textit{weakly in} \quad L^2(Q)\,, \tag{148}$$

with

$$f^0 + g^0 = 0 \quad \textit{in} \quad \mathcal{D}'(Q)\,. \tag{149}$$

Then

$$r^\varepsilon \to 0 \quad \textit{strongly in}\; L^2\big(0, T; H_0^1(\Omega)\big) \;\textit{and in}\; C^0\big([0, T]; L^2(\Omega)\big)\,. \tag{150}$$

Proof. Taking r^ε as test function in (145), one immediately obtains

$$\|r^\varepsilon\|_{L^2\left(0,T;H_0^1(\Omega)\right)} \le c\,, \qquad \|\partial_t r^\varepsilon\|_{L^2\left(0,T;H^{-1}(\Omega)\right)} \le c\,.$$

Therefore, by extracting a subsequence and using the compactness result (13), we can assume that for some r^0

$$r^\varepsilon \rightharpoonup r^0 \quad \text{weakly in} \quad L^2\big(0, T; H_0^1(\Omega)\big)\,, \tag{151}$$

$$r^\varepsilon \to r^0 \quad \text{strongly in} \;\; L^2(Q)\,. \tag{152}$$

Taking r^ε as test function in (145) and integrating on $(0, \tau)$, with $\tau \in [0, T]$, we now obtain:

$$\frac{1}{2}\int_\Omega |r^\varepsilon(\tau)|^2\,dx + \alpha\int_{Q_\tau} |\nabla r^\varepsilon|^2 \le \frac{1}{2}\int_\Omega |a^\varepsilon|^2\,dx + \int_0^\tau \langle f^\varepsilon(t), r^\varepsilon(t)\rangle\,dt + \int_{Q_\tau} g^\varepsilon r^\varepsilon,$$

where $Q_\tau = (0, \tau) \times \Omega$.

Using convergences (146), (147), (148), (151) and (152), recalling equality (149), and using Lemma 7.5 below as in the same way as in the proof of (103) and (104) in Theorem 3.8, one easily obtains

$$\sup_{\tau\in[0,T]}\int_\Omega |r^\varepsilon(\tau)|^2\,dx \to 0\,, \qquad \int_Q |\nabla r^\varepsilon|^2 \to 0\,,$$

which imply (150) for the extracted subsequence and, *a posteriori*, for the whole sequence $\{r^\varepsilon\}$. □

PROOF OF PROPOSITION 3.3. In the case where $g_i^\varepsilon = g_i^0$, statement (27) is a mere application of Proposition 2.3. When $g_i^\varepsilon \neq g_i^0$, we define r_i^ε as the solutions of

$$\begin{cases} \partial_t r_i^\varepsilon - \operatorname{div}\big(\tilde{A}^\varepsilon \nabla r_i^\varepsilon\big) = g_i^\varepsilon - g_i^0 \quad \text{in } \mathcal{D}'(\widetilde{Q}), \\ r_i^\varepsilon \in L^2\big(0, T; H_0^1(\widetilde{\Omega})\big) \cap H^1(0, T; H^{-1}(\widetilde{\Omega})), \\ r_i^\varepsilon \Big|_{t=0} = 0. \end{cases}$$

Applying Lemma 7.1 to r_i^ε, we obtain that

$$r_i^\varepsilon \to 0 \quad \text{strongly in } L^2\big(0, T; H_0^1(\widetilde{\Omega})\big) \text{ and in } C^0\big([0, T]; L^2(\widetilde{\Omega})\big), \tag{153}$$

and (27) easily follows under the assumptions of Definition 3.2. Equality (28) follows by passing to the limit in each term of (22).

Let us now prove the equi-integrability property (29). This is where we need Ω to be embedded in a larger, smooth domain $\widetilde{\Omega}$. Let $\varphi(x)$ be a smooth function in $C_0^\infty(\widetilde{\Omega})$ such that $0 \le \varphi \le 1$ on $\widetilde{\Omega}$, $\varphi \equiv 1$ on Ω. Then, using equation (22) satisfied by the special test function w_i^ε, the function φw_i^ε satisfies

$$\begin{aligned} \partial_t(\varphi w_i^\varepsilon) - \operatorname{div}\big(\tilde{A}^\varepsilon \nabla(\varphi w_i^\varepsilon)\big) &= \varphi\, \partial_t w_i^\varepsilon - \operatorname{div}\big(\varphi\, \tilde{A}^\varepsilon \nabla w_i^\varepsilon\big) - \operatorname{div}\big(w_i^\varepsilon\, \tilde{A}^\varepsilon \nabla \varphi\big) \\ &= \varphi\, \Big(\partial_t w_i^\varepsilon - \operatorname{div}(\tilde{A}^\varepsilon \nabla w_i^\varepsilon)\Big) - \big(\tilde{A}^\varepsilon \nabla w_i^\varepsilon\big)\cdot \nabla\varphi - \operatorname{div}\big(w_i^\varepsilon\, \tilde{A}^\varepsilon \nabla\varphi\big) \\ &= \varphi f_i^\varepsilon + \varphi g_i^\varepsilon - \big(\tilde{A}^\varepsilon \nabla w_i^\varepsilon\big)\cdot\nabla\varphi - \operatorname{div}\big(w_i^\varepsilon\, \tilde{A}^\varepsilon \nabla\varphi\big). \end{aligned} \tag{154}$$

The first term of the right hand side of equality (154) converges strongly in $L^2\big(0, T; H^{-1}(\widetilde{\Omega})\big)$, the second and the third ones are bounded in $L^2(\widetilde{Q})$. Since w_i^ε is bounded in $L^2\big(0, T; H^1(\widetilde{\Omega})\big) \cap H^1(0, T; H^{-1}(\widetilde{\Omega}))$, and therefore, by (12), in $C^0\big([0, T]; L^2(\widetilde{\Omega})\big)$, the classical Gagliardo-Nirenberg embedding theorem (see for instance [9], Chapter I, Proposition 3.1) implies that w_i^ε is bounded in $L^\sigma(\widetilde{Q})$, where $\sigma = 2(N+2)/N > 2$. Therefore the last term of (154) is bounded in $L^\sigma\big(0, T; W^{-1,\sigma}(\widetilde{\Omega})\big)$ for some $\sigma > 2$. Thus using Proposition 7.2 below we obtain that $|\nabla(\varphi w_i^\varepsilon)|^2$ is equi-integrable in $\widetilde{Q}$, which implies that

$$\varphi^2 |\nabla w_i^\varepsilon|^2 \text{ is equi-integrable in } \widetilde{Q}, \tag{155}$$

which in turn implies (29).

Let us finally prove (30). Multiplying (154) by $\varphi(w_i^\varepsilon - x_i)$, and integrating on $(t_1, t_2) \times \widetilde{\Omega}$, with $0 \le t_1 < t_2 \le T$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_{\widetilde{\Omega}} \varphi^2\, |w_i^\varepsilon(t_2) - x_i|^2\, dx - \frac{1}{2}\int_{\widetilde{\Omega}} \varphi^2\, |w_i^\varepsilon(t_1) - x_i|^2\, dx \\ &= \int_{t_1}^{t_2} \langle \varphi f_i^\varepsilon(t), \varphi\,(w_i^\varepsilon(t) - x_i)\rangle\, dt + \int_{t_1}^{t_2}\int_{\widetilde{\Omega}} \varphi^2 g_i^\varepsilon\, \big(w_i^\varepsilon(t) - x_i\big) \\ &\quad - \int_{t_1}^{t_2}\int_{\widetilde{\Omega}} \varphi^2\, \big(\tilde{A}^\varepsilon \nabla w_i^\varepsilon\big)\cdot\nabla(w_i^\varepsilon - x_i) - 2\int_{t_1}^{t_2}\int_{\widetilde{\Omega}} \big(\tilde{A}^\varepsilon \nabla w_i^\varepsilon\big)\cdot\nabla\varphi\, \varphi\, (w_i^\varepsilon - x_i). \end{aligned}$$

Therefore

$$\begin{aligned}
&\left| \frac{1}{2} \int_{\widetilde{\Omega}} \varphi^2 |w_i^\varepsilon(t_2) - x_i|^2 \, dx - \frac{1}{2} \int_{\widetilde{\Omega}} \varphi^2 |w_i^\varepsilon(t_1) - x_i|^2 \, dx \right| \\
&\leq \left[\int_{t_1}^{t_2} \| f_i^\varepsilon(t) \|^2_{H^{-1}(\widetilde{\Omega})} \, dt \right]^{\frac{1}{2}} \| \varphi^2 (w_i^\varepsilon - x_i) \|_{L^2\left(0,T; H_0^1(\widetilde{\Omega})\right)} \\
&\quad + \| g_i^\varepsilon \|_{L^2(\widetilde{Q};\mathbb{R}^N)} \left[\int_{t_1}^{t_2} \int_{\widetilde{\Omega}} \varphi^2 |w_i^\varepsilon - x_i|^2 \right]^{\frac{1}{2}} \\
&\quad + \beta \int_{t_1}^{t_2} \int_{\widetilde{\Omega}} \varphi^2 \, |\nabla w_i^\varepsilon| \, |\nabla (w_i^\varepsilon - x_i)| \\
&\quad + 2\beta \int_{t_1}^{t_2} \int_{\widetilde{\Omega}} \varphi \, |w_i^\varepsilon - x_i| \, |\nabla w_i^\varepsilon| \, |\nabla \varphi|
\end{aligned} \tag{156}$$

Using (25), (26), the strong convergence of w_i^ε in $L^2(Q)$ and the equi-integrability (155), it is easy to see that the right hand side of (156) is small (uniformly with respect to ε) if $t_2 - t_1$ is small.

Therefore the functions

$$\Psi^\varepsilon(t) = \frac{1}{2} \int_{\widetilde{\Omega}} \varphi^2 \, |w_i^\varepsilon(t) - x_i|^2 \, dx$$

are equi-continuous (and equi-bounded, since, by (12), $w_i^\varepsilon - x_i$ is bounded in $C^0\big(0, T; L^2(\widetilde{\Omega})\big)$). By Ascoli's theorem, the sequence $\{\Psi^\varepsilon\}$ is relatively compact in $C^0([0, T])$. Since, by the compactness property (13), w_i^ε tends to x_i strongly in $L^2(\widetilde{Q})$, it follows that Ψ^ε tends to zero strongly in $L^2(0, T)$ and therefore in $C^0([0, T])$. This implies (30), and Proposition 3.3 is proved. □

The following equi-integrability result has been used above in the proof of Proposition 3.3 (proof of (155)).

PROPOSITION 7.2. *Assume that the boundary of $\widetilde{\Omega}$ is sufficiently smooth (for instance of class C^1), and that v^ε is solution of*

$$\begin{cases}
\partial_t v^\varepsilon - \operatorname{div}\big(\widetilde{A}^\varepsilon \, \nabla v^\varepsilon\big) = f^\varepsilon + g^\varepsilon + h^\varepsilon \quad \text{in } \mathcal{D}'(\widetilde{Q}), \\
v^\varepsilon \in L^2\big(0, T; H_0^1(\widetilde{\Omega})\big) \cap H^1\big(0, T; H^{-1}(\widetilde{\Omega})\big), \\
v^\varepsilon \Big|_{t=0} = a^\varepsilon,
\end{cases}$$

where $A^\varepsilon \in \mathcal{M}(\alpha, \beta; \widetilde{Q})$, and the data f^ε, g^ε, h^ε and a^ε satisfy

$$f^\varepsilon \in L^2\big(0, T; H^{-1}(\widetilde{\Omega})\big), \quad f^\varepsilon \to f^0 \quad \textit{strongly in } L^2\big(0, T; H^{-1}(\widetilde{\Omega})\big),$$

$$g^\varepsilon \in L^2(\widetilde{Q}), \qquad g^\varepsilon \rightharpoonup g^0 \quad \textit{weakly in } L^2(\widetilde{Q}),$$

$$h^\varepsilon \textit{ is bounded in } L^\sigma\big(0, T; W^{-1,\sigma}(\widetilde{\Omega})\big), \quad \textit{for some } \sigma > 2,$$

$$a^\varepsilon \in L^2(\widetilde{\Omega}), \qquad a^\varepsilon \to a^0 \quad \textit{strongly in } L^2(\widetilde{\Omega}).$$

Then the functions $|\nabla v^\varepsilon|^2$ are equi-integrable on $\widetilde{Q}$.

PROOF. Let z^ε be the (unique) solution of

$$
(157)\qquad \begin{cases} \partial_t z^\varepsilon - \operatorname{div}\big(\tilde{A}^\varepsilon\, \nabla z^\varepsilon\big) = f^0 + g^0 + h^\varepsilon \quad \text{in } \mathcal{D}'(\widetilde{Q}), \\ z^\varepsilon \in L^2\big(0,T;\, H_0^1(\widetilde{\Omega})\big) \cap H^1\big(0,T;\, H^{-1}(\widetilde{\Omega})\big), \\ z^\varepsilon\Big|_{t=0} = a^0 . \end{cases}
$$

By Lemma 7.1, we have

$$
\int_{\widetilde{Q}} |\nabla v^\varepsilon - \nabla z^\varepsilon|^2 \to 0 \quad \text{as } \varepsilon \to 0 .
$$

Since, for every measurable set $D \subset \widetilde{Q}$ we have

$$
\int_D |\nabla v^\varepsilon|^2 \le 2 \int_D |\nabla z^\varepsilon|^2 + 2 \int_{\widetilde{Q}} |\nabla v^\varepsilon - \nabla z^\varepsilon|^2 ,
$$

the equi-integrability of $|\nabla v^\varepsilon|^2$ follows from the equi-integrability of $|\nabla z^\varepsilon|^2$, that we will prove now.

Let $\hat{f}^0 \in L^\infty\big(0,T;\, W^{-1,\infty}(\widetilde{\Omega})\big)$, $\hat{g}^0 \in L^\infty(\widetilde{Q})$ and $\hat{a}^0 \in W_0^{1,\infty}(\widetilde{\Omega})$ be such that, for some $\delta > 0$,

$$
(158)\quad \|f^0 - \hat{f}^0\|_{L^2(0,T;H^{-1}(\widetilde{\Omega}))} < \delta , \quad \|g^0 - \hat{g}^0\|_{L^2(\widetilde{Q})} < \delta , \quad \|a^0 - \hat{a}^0\|_{L^2(\widetilde{\Omega})} < \delta ,
$$

and let $\hat{z}^\varepsilon$ be the (unique) solution of

$$
(159)\qquad \begin{cases} \partial_t \hat{z}^\varepsilon - \operatorname{div}\big(\tilde{A}^\varepsilon\, \nabla \hat{z}^\varepsilon\big) = \hat{f}^0 + \hat{g}^0 + h^\varepsilon \quad \text{in } \mathcal{D}'(\widetilde{Q}), \\ \hat{z}^\varepsilon \in L^2\big(0,T;\, H_0^1(\widetilde{\Omega})\big) \cap H^1\big(0,T;\, H^{-1}(\widetilde{\Omega})\big), \\ \hat{z}^\varepsilon\Big|_{t=0} = \hat{a}^0 . \end{cases}
$$

Using Meyers' type regularity estimates (see for instance [3], Chapter 2, Theorem 2.2), the sequence $\{\hat{z}^\varepsilon\}$ is bounded in $L^p\big(0,T;\, W_0^{1,p}(\widetilde{\Omega})\big)$, for some p (with $2 < p \le \sigma$), and the bound depends only on the norms of $\hat{f}^0 + \hat{g}^0 + h^\varepsilon$ and $\hat{a}^0$ respectively in $L^p\big(0,T;\, W^{-1,p}(\widetilde{\Omega})\big)$ and $W_0^{1,\infty}(\widetilde{\Omega})$. Therefore, for every measurable set $D \subset \widetilde{Q}$, we obtain from Hölder's inequality that

$$
\int_D |\nabla \hat{z}^\varepsilon|^2 \le (\operatorname{meas} D)^{1-\frac{2}{p}} \left[\int_D |\nabla \hat{z}^\varepsilon|^p \right]^{\frac{2}{p}} \le c\big(\hat{f}^0, \hat{g}^0, h^\varepsilon, \hat{a}^0\big)\, (\operatorname{meas} D)^{1-\frac{2}{p}} .
$$

On the other hand, using $z^\varepsilon - \hat{z}^\varepsilon$ as test function in (157) and (159), one obtains

$$
\begin{aligned} \|\nabla z^\varepsilon - \nabla \hat{z}^\varepsilon\|_{L^2(\widetilde{Q};\mathbb{R}^N)} \le c\big[&\|f^0 - \hat{f}^0\|_{L^2\left(0,T;H^{-1}(\widetilde{\Omega})\right)} \\ &+ \|g^0 - \hat{g}^0\|_{L^2(\widetilde{Q})} + \|a^0 - \hat{a}^0\|_{L^2(\widetilde{\Omega})} \big] < c\,\delta . \end{aligned}
$$

Hence

$$\int_D |\nabla z^\varepsilon|^2 \le 2\int_D |\nabla \hat{z}^\varepsilon|^2 + 2\int_D |\nabla z^\varepsilon - \nabla \hat{z}^\varepsilon|^2$$
$$\le c\big(\hat{f}^0, \hat{g}^0, h^\varepsilon, \hat{a}^0\big)\,(\operatorname{meas} D)^{1-\frac{2}{p}} + c\,\delta^2 .$$

Thus, for every fixed number $\eta > 0$, we first choose δ such that $c\,\delta^2 = \eta/2$, then $\hat{f}^0$, $\hat{g}^0$ and $\hat{a}^0$ satisfying (158); for any measurable set D with $\operatorname{meas} D$ small enough (independently of ε), we obtain

$$\int_D |\nabla z^\varepsilon|^2 < \eta .$$

Therefore the sequence $\{|\nabla z^\varepsilon|^2\}$ is equi-integrable, and the result is proved. □

We now prove a result of "convergence of energy", which is the main step in the proof of the last assertion of Proposition 3.7.

LEMMA 7.3. *Assume that $A^\varepsilon, A^0 \in \mathcal{M}(\alpha, \beta; Q)$, and that $A^\varepsilon \overset{H}{\rightharpoonup} A^0$. Moreover assume that z^ε satisfies*

$$\partial_t z^\varepsilon - \operatorname{div}\big(A^\varepsilon \nabla z^\varepsilon\big) = f^\varepsilon + g^\varepsilon \quad \textit{in } \mathcal{D}'(Q), \tag{160}$$

$$z^\varepsilon \in L^2\big(0, T; H^1(\Omega)\big) \cap H^1\big(0, T; H^{-1}(\Omega)\big)\,, \tag{161}$$

$$z^\varepsilon \rightharpoonup z^0 \quad \textit{weakly in } L^2\big(0, T; H^1(\Omega)\big), \tag{162}$$

$$f^\varepsilon \to f^0 \quad \textit{strongly in } L^2\big(0, T; H^{-1}(\Omega)\big), \tag{163}$$

$$g^\varepsilon \rightharpoonup g^0 \quad \textit{weakly in } L^2(Q). \tag{164}$$

Then

$$\big(A^\varepsilon \nabla z^\varepsilon\big)\cdot \nabla z^\varepsilon \rightharpoonup \big(A^0 \nabla z^0\big)\cdot \nabla z^0 \quad \textit{in } \mathcal{D}'(Q). \tag{165}$$

REMARK 7.4. It follows immediately from Lemma 7.3 that if w_i^ε are special test functions in the sense of Definition 3.2, and if

$$z^\varepsilon = \sum_{i=1}^N \xi_i w_i^\varepsilon\,, \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^N\,,$$

then

$$\int_Q \big(A^\varepsilon \nabla z^\varepsilon\big)\cdot \nabla z^\varepsilon\, \varphi \to \int_Q \big(A^0 \xi\big)\cdot \xi\, \varphi \quad \text{for every } \varphi \in C_0^\infty(Q).$$

This proves that assertion (42) of Proposition 3.7 holds in the sense of distributions. □

PROOF OF LEMMA 7.3. Using φz^ε as test function in (160), with $\varphi \in C_0^\infty(Q)$, we have the equality

$$-\frac{1}{2}\int_Q |z^\varepsilon|^2\,\partial_t\varphi + \int_Q \left(A^\varepsilon\,\nabla z^\varepsilon\right)\cdot\nabla z^\varepsilon\,\varphi \\ + \int_Q \left(A^\varepsilon\,\nabla z^\varepsilon\right)\cdot\nabla\varphi\, z^\varepsilon = \langle\!\langle f^\varepsilon\,,\, z^\varepsilon\varphi\rangle\!\rangle + \int_Q g^\varepsilon\,\varphi\, z^\varepsilon\,.$$

Using the strong convergence of z^ε to z^0 in $L^2(Q)$ (which results from the compactness property (13)), the local character of the H-convergence (see Proposition 2.3), and the hypotheses on the data, it is easy to pass to the limit in each term, except the second one, which is the one we want to study. We obtain

$$\lim_{\varepsilon\to 0}\int_Q \left(A^\varepsilon\,\nabla z^\varepsilon\right)\cdot\nabla z^\varepsilon\,\varphi = \frac{1}{2}\int_Q |z^0|^2\,\partial_t\varphi - \int_Q \left(A^0\,\nabla z^0\right)\cdot\nabla\varphi\, z^0 \\ + \langle\!\langle f^0\,,\, z^0\varphi\rangle\!\rangle + \int_Q g^0\,\varphi\, z^0\,,$$

which, using φz^0 as test function in the limit equation of (160), implies (165). □

In the final part of this section, we prove a lemma about uniform convergence of integrals. This result has been used in the proof of Theorem 5.1 and in the proof of Lemma 7.1.

LEMMA 7.5. *Assume that* $\{\rho^\varepsilon\}$ *and* $\{\sigma^\varepsilon\}$ *are two sequences of functions satisfying*

$$\rho^\varepsilon \rightharpoonup \rho^0 \quad \text{weakly in } L^2(Q;\mathbb{R}^k), \qquad \sigma^\varepsilon \rightharpoonup \sigma^0 \quad \text{strongly in } L^2(Q;\mathbb{R}^k),$$

with $k \geq 1$, *and denote by* Q_τ *the cylinder* $(0,\tau)\times\Omega$, *for any* $\tau \in [0,T]$.

Then

$$\sup_{\tau\in[0,T]}\left|\int_{Q_\tau}\rho^\varepsilon\cdot\sigma^\varepsilon - \int_{Q_\tau}\rho^0\cdot\sigma^0\right| \to 0\,. \tag{166}$$

PROOF. We introduce the functions

$$\Psi_\varepsilon(\tau) = \int_{Q_\tau}\left(\rho^\varepsilon\cdot\sigma^\varepsilon - \rho^0\cdot\sigma^0\right),$$

which converge to zero for every $\tau \in [0,T]$. Moreover, if $0 \leq \tau_1 < \tau_2 \leq T$, one has

$$\begin{aligned} |\Psi_\varepsilon(\tau_2) - \Psi_\varepsilon(\tau_1)| &\leq \int_{\tau_1}^{\tau_2}\int_\Omega \left(|\rho^\varepsilon\cdot\sigma^\varepsilon| + |\rho^0\cdot\sigma^0|\right) \\ &\leq \|\rho^\varepsilon\|_{L^2(Q;\mathbb{R}^k)}\left[\int_{\tau_1}^{\tau_2}\int_\Omega |\sigma^\varepsilon|^2\right]^{1/2} \\ &\quad + \|\rho^0\|_{L^2(Q;\mathbb{R}^k)}\left[\int_{\tau_1}^{\tau_2}\int_\Omega |\sigma^0|^2\right]^{1/2} \end{aligned} \tag{167}$$

Since σ^ε converges strongly in $L^2(Q;\mathbb{R}^k)$, inequality (167) implies the equicontinuity of the sequence Ψ^ε. Therefore, by Ascoli's theorem, Ψ_ε converges uniformly to zero in $[0, T]$, which is equivalent to (166). □

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