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With deep sorrow for the loss of our Teacher and Friend, we dedicate this paper to Ennio De Giorgi.
We have tried to pursue our research in the direction that he had pointed out.

Abstract. We prove the existence of strong minimizers for functionals depending on free discontinuities, free gradient discontinuities and second derivatives, which are related to image segmentation.

1. – Introduction

The issue of minimizing functionals depending on both a bulk energy and a surfacic (or lineic in two dimensions) discontinuity energy has attracted the interest of many researchers (see for instance [A1], [A2], [AFP], [AV], [BZ], [C], [CL], [CLT1]-[CLT5], [Co], [DA], [DCL], [DG], [DMS], [DS], [Fo], [LS], [MS], [MSh], [T1], [T2]).

Here we focus the functional

$$F(K_0, K_1, u) = \int_{\Omega\setminus(K_0\cup K_1)} \left(|D^2u|^p + \mu|u-g|^q\right) dy + \alpha\mathcal{H}^{n-1}(K_0\cap\Omega) + \beta\mathcal{H}^{n-1}\big((K_1\setminus K_0)\cap\Omega\big) \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is an open set, $n\in\mathbb{N}$, $n\geq 2$, $\alpha,\beta,\mu,p,q \in \mathbb{R}$, with

$$p>1,\quad q\geq 1,\quad \mu>0,\quad 0<\beta\leq\alpha\leq 2\beta,\quad g\in L^{nq}_{\mathrm{loc}}(\Omega)\cap L^q(\Omega)\ , \tag{1.2}$$

are given; while K_0, $K_1\subset\mathbb{R}^n$ are Borel sets (a priori unknown) with $K_0\cup K_1$ closed, and $u\in C^2(\Omega\setminus(K_0\cup K_1))$ is approximately continuous on $\Omega\setminus K_0$.

We notice that in case $n=p=q=2$ the energy (1.1) reduces to the following one

$$\int_{\Omega\setminus(K_0\cup K_1)} \left(|D^2u|^2+\mu|u-g|^2\right) dy+\alpha\mathcal{H}^1(K_0\cap\Omega)+\beta\mathcal{H}^1\big((K_1\setminus K_0)\cap\Omega\big)\ . \tag{1.3}$$

Functional (1.3) was introduced by Blake & Zisserman (thin plate surface under tension [BZ]) as an energy to be minimized in order to achieve a segmentation of a monocromatic picture. In this context g describes the light intensity level on the screen Ω, μ is a scale parameter, α is a contrast parameter and a measure of immunity to noise, β is a gradient-contrast parameter.

The elements of a minimizing triplet (K_0, K_1, u) play respectively the role of edges, creases and smoothly varying intensity in the region $\Omega \setminus (K_0 \cup K_1)$ for the segmented image. The second-order model (1.3) was introduced to overcome the over-segmentation of steep gradients (ramp effect) and other inconvenients which occur in lower order models as in case of Mumford & Shah functional ([MSh]).

We prove the following statements.

THEOREM 1.1. *Under assumptions* (1.2) *with $n = p = 2$ there is at least one triplet among K_0, $K_1 \subset \mathbb{R}^2$ Borel sets with $K_0 \cup K_1$ closed and $u \in C^2(\Omega \setminus (K_0 \cup K_1))$ approximately continuous on $\Omega \setminus K_0$ minimizing functional* (1.1) *and having finite energy. Moreover the sets $K_0 \cap \Omega$ and $K_1 \cap \Omega$ are* $(\mathcal{H}^1, 1)$ *rectifiable.*

THEOREM 1.2. *Under assumptions* (1.2) *with $n = p = 2$ and $\alpha = \beta$ there is at least one pair among $K \subset \mathbb{R}^2$ closed set and $u \in C^2(\Omega \setminus K)$ minimizing the functional*

$$\int_{\Omega \setminus K} \left(|D^2 u|^2 + \mu |u - g|^q\right) dy + \alpha \mathcal{H}^1(K \cap \Omega)$$

and having finite energy. Moreover the set $K \cap \Omega$ is $(\mathcal{H}^1, 1)$ *rectifiable.*

We proved the existence of strong solutions for different free discontinuity problems involving second derivatives in previous papers: models of elastic-plastic plates [CLT3] and rigid-plastic slabs [CLT4], but in those cases the continuity of competing functions was built into the functional ($K_0 = \emptyset$), and this property entailed an "a priori" L^∞ estimate which is not true any more in the present case. Here the finiteness of energy (1.3) entails neither essential boundedness of u or ∇u nor local summability of ∇u (see example (1.4) of [CLT5]), and this fact develops many substantial difficulties.

In the paper [CLT5] we introduced a relaxed version of (1.3) (weak formulation of the Blake & Zisserman functional) which in case of (1.1) leads to

$$\int_\Omega \left(|\nabla^2 v|^p + \mu |v - g|^q\right) dy + \alpha \mathcal{H}^{n-1}(S_v) + \beta \mathcal{H}^{n-1}\left(S_{\nabla v} \setminus S_v\right) \tag{1.4}$$

to be minimized over $v \in GSBV(\Omega)$, with $\nabla v \in [GSBV(\Omega)]^n$ and we proved the existence of minimizers of (1.4). Here we prove that for $p = n = 2$ a weak minimizer actually provides a minimizing triplet of (1.3) by taking a representative of the function and the closure of singular sets of the function itself and of its gradient.

The proofs rely on a deep use of the coarea formula and of a new Poincaré type inequality in $GSBV(\Omega)$ (see Theorem 4.1). These tools allow the study of a normalized energy density (see (5.1)) for which we prove a suitable decay outside the singular set of weak solutions.

By a careful application of our Poincaré inequality we show that blown up sequences (v_h) of a weak minimizer are nice around points with vanishing normalized energy density: in the sense that the functions v_h, though converging only in measure, are strongly convergent in $L^p(\Omega)$ togheter with their approximate gradient, if modified on sets of small area and perimeter in a way preserving median and gradient median (see Theorem 4.3). This compactness property is the most technical and difficult point of the whole argument and allows us to show that the pointwise limit of (v_h) solves an elliptic fourth order equation (see Theorem 5.1): the decay estimates of its solutions are transferred (by a technical joining lemma) to the sequence (v_h) (see Theorem 5.4). This entails that for a weak minimizer v of (1.4), we find $\mathcal{H}^1(\Omega \cap (\overline{S_v \cup S_{\nabla v}} \setminus (S_v \cup S_{\nabla v}))) = 0$ and $\tilde{v} \in C^2(\Omega \setminus \overline{S_v \cup S_{\nabla v}})$.

We stress the fact that for the Mumford & Shah functional ([DCL]), the elastic-plastic plate functional ([CLT3]) and the rigid-plastic slab functional ([CLT4]), the minimizers u are quasi-minimizers of the main part of the functional so that the forcing term in the functional plays the role of a small perturbation. While for the Blake & Zissermann functional, as like as in the Mumford & Shah functional with unbounded datum (see [L]), the forcing term $\int_\Omega |v - g|^q dy$ has to be considered in the main part of the functional and not as a lower order perturbation.

The only point where the assumption $p = 2$ is needed is the decay estimate for solutions of a linear elliptic equation of the fourth order (see Theorem 5.2); elsewhere in the proofs the only assumption $p \geq n$ is enough.

The assumption $g \in L^{nq}_{\mathrm{loc}}(\Omega)$ is sharp in the sense that for every $\tau < nq$ there exist $g \in L^\tau(\Omega)$ such that functional (1.1) has no minimizing triplet. We refer for a counterexample to a forthcoming paper where additional properties of solutions are given: lower and upper density estimates, necessary conditions satisfied by strong minimizers, a study of the vector valued case and a more general case for p and n.

The outline of the paper is the following.

2. – Notations and functional spaces

From now on we denote by Ω an open set in $\mathbb{R}^n$, $n \geq 2$.

For a given set $U \subset \mathbb{R}^n$ we denote by ∂U its topological boundary, by $\mathcal{H}^k(U)$ its k–dimensional Hausdorff measure and by $|U|$ its Lebesgue outer measure; χ_U is the characteristic function of U. We indicate by $B_\rho(x)$ the open ball $\{y \in \mathbb{R}^n;\ |y-x| < \rho\}$, and we set $B_\rho = B_\rho(0)$, $\omega_n = |B_1|$. If Ω, Ω' are open subsets in $\mathbb{R}^n$, by $\Omega \subset\subset \Omega'$ we mean that $\overline{\Omega}$ is compact and $\overline{\Omega} \subset \Omega'$. We introduce the following notations: $s \wedge t = \min\{s,t\}$, $s \vee t = \max\{s,t\}$ for every $s, t \in \mathbb{R}$; given two vectors a, b, we set $a \cdot b = \sum_i a_i b_i$ and $(a \otimes b)_{ij} = a_i b_j$.

In the following with the same letter c we denote suitable constants which may change in different inequalities.

For any Borel function $v : \Omega \to \mathbb{R}$ the approximate upper and lower limits of v are the Borel functions $v^+, v^- : \Omega \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ defined for any $x \in \Omega$ by

$$v^+(x) = \inf\left\{t \in \overline{\mathbb{R}} : \lim_{\rho \to 0} \rho^{-n} |\{v > t\} \cap B_\rho(x)| = 0\right\},$$

$$v^-(x) = \sup\left\{t \in \overline{\mathbb{R}} : \lim_{\rho \to 0} \rho^{-n} |\{v < t\} \cap B_\rho(x)| = 0\right\}.$$

The set

$$S_v = \left\{x \in \Omega;\ \ v^-(x) < v^+(x)\right\}$$

is a Borel set, of negligible Lebesgue measure (see e.g. [F], 2.9.13); we say that v is approximately continuous on $\Omega \setminus S_v$ and we denote by $\tilde{v} : \Omega \setminus S_v \to \overline{\mathbb{R}}$ the function

$$\tilde{v}(x) = \operatorname{ap}\lim_{y \to x} v(y) = v^+(x) = v^-(x).$$

Let $x \in \Omega \setminus S_v$ be such that $\tilde{v}(x) \in \mathbb{R}$; we say that v is approximately differentiable at x if there exists a vector $\nabla v(x) \in \mathbb{R}^n$ (the approximate gradient of v at x) such that

$$\operatorname{ap}\lim_{y \to x} \frac{|v(y) - \tilde{v}(x) - \nabla v(x) \cdot (y-x)|}{|y-x|} = 0.$$

If v is a smooth function then ∇v is the classical gradient. In the following with the notation $|\nabla v|$ we mean the euclidean norm of ∇v and we set $\nabla_i v = (\mathbf{e}_i \cdot \nabla) v$, $\{\mathbf{e}_i\}$ denoting the canonical basis of $\mathbb{R}^n$. In the one dimensional case ($n = 1$) we shall use the notation v' in place of ∇v.

We recall the definition of the space of functions of bounded variation in Ω with values in $\mathbb{R}$:

$$BV(\Omega) = \left\{v \in L^1(\Omega);\ Dv \in \mathcal{M}_n(\Omega)\right\}$$

where $Dv = (D_1 v, \ldots, D_n v)$ denotes the distributional gradient of v and $\mathcal{M}_n(\Omega)$ denotes the space of vector–valued Radon measure with finite total variation. We denote by $\int_\Omega |Dv|$ the total variation of the measure Dv in Ω.

If $v = \chi_E$ is the characteristic function of a set E, then $v \in BV(\Omega)$ if and only if E is a set with finite perimeter in Ω; the perimeter of E in Ω is given by

$$P(E, \Omega) = \int_\Omega |D\chi_E|.$$

For every $v \in BV(\Omega)$ the following properties hold:

1) $v^+(x), v^-(x) \in \mathbb{R}$ for $\mathcal{H}^{n-1}$–almost all $x \in \Omega$ (see [Z], 5.9.6);
2) S_v is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable (see [F], 4.5.9(16));
3) ∇v exists a.e. in Ω and coincides with the Radon–Nikodym derivative of Dv with respect to the Lebesgue measure (see [F], 4.5.9(26));
4) for $\mathcal{H}^{n-1}$ almost all $x \in S_v$ there exists a unique $\nu = \nu_v(x) \in \partial B_1$ such that, setting $B_\rho^+ = \{y \in B_\rho(x) : (y-x)\cdot\nu > 0\}$ and $B_\rho^- = \{y \in B_\rho(x) : (y-x)\cdot\nu < 0\}$, then (see [Z], 5.14.3)

$$\lim_{\rho\to 0}\left(\fint_{B_\rho^+}|v(y)-v^+(x)|^{\frac{n}{n-1}}dy + \fint_{B_\rho^-}|v(y)-v^-(x)|^{\frac{n}{n-1}}dy\right) = 0,$$

and also (see [F], 4.5.9(15))

$$\int_\Omega |Dv| \geq \int_\Omega |\nabla v|\,dy + \int_{S_v}|v^+ - v^-|\,d\mathcal{H}^{n-1}.$$

Moreover $\nu_v(x)$ is an approximate normal vector to S_v at x (see [Z], 5.9.6).

We recall the definitions of some function spaces and we refer to [DA] and [A1] for their properties.

DEFINITION 2.1. *SBV*(Ω) denotes the class of functions $v \in BV(\Omega)$ such that

$$\int_\Omega |Dv| = \int_\Omega |\nabla v|\,dy + \int_{S_v}|v^+ - v^-|\,d\mathcal{H}^{n-1}.$$

$SBV_{\mathrm{loc}}(\Omega)$ denotes the class of functions $v \in SBV(\Omega')$ for every $\Omega' \subset\subset \Omega$.

Moreover we define

$$GBV(\Omega) = \{v : \Omega \to \mathbb{R} \text{ Borel function}; -k \vee v \wedge k \in BV_{\mathrm{loc}}(\Omega)\ \forall k \in \mathbb{N}\},$$
$$GSBV(\Omega) = \{v : \Omega \to \mathbb{R} \text{ Borel function}; -k \vee v \wedge k \in SBV_{\mathrm{loc}}(\Omega)\ \forall k \in \mathbb{N}\}.$$

LEMMA 2.2. *Let $v \in GSBV(\Omega)$. Then S_v is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable and ∇v exists a.e. in Ω.*

PROOF. See [A2], Proposition 1.3 and 1.4. □

LEMMA 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and $v \in GSBV(\Omega)$. Assume that*

$$\int_\Omega |\nabla v|\,dy < +\infty, \quad \mathcal{H}^{n-1}(S_v) < +\infty.$$

Then for every $a, b \in \mathbb{R}$, $a < b$ we have $(a \vee v \wedge b) \in SBV(\Omega)$,

$$\int_\Omega |D(a \vee v \wedge b)| = \int_a^b P(\{v > \sigma\}, \Omega)\,d\sigma,$$

and $(a \vee v \wedge b)\chi_E \in SBV(\Omega)$ for every set E with finite perimeter in Ω.

PROOF. Follows easily by Definition 2.1 and the coarea formula (see e.g. [Z], 5.4.4). □

LEMMA 2.4. *Let $a, b \in \mathbb{R}$ and $v \in GSBV(a, b)$.*

If $\int_a^b |v'(t)|dt + \mathcal{H}^0(S_v) < +\infty$ then $v \in SBV(a, b)$.
If $\int_a^b |v'(t)|^p dt < +\infty$ and $\mathcal{H}^0(S_v) = 0$ then $v \in W^{1,p}(a, b)$.

The previous lemma, when applied to any function in $GSBV^2(a, b)$ and to its approximate derivative, shows that in dimension one the strong formulation (1.1) and the weak formulation (1.4) coincide.

Now we may define the following function spaces; even if we will use only $GSBV^2(\Omega)$ in the following, we recall the definitions of various spaces related to bounded second derivatives, to avoid confusion with other authors' notations.

DEFINITION 2.5. We set

$$\begin{aligned} SBH(\Omega) &= \{v \in W^{1,1}(\Omega), \quad Dv \in [SBV(\Omega)]^n\} \\ SBV^2(\Omega) &= \{v \in SBV(\Omega), \quad \nabla v \in [SBV(\Omega)]^n\} \\ GSBV^2(\Omega) &= \{v : \Omega \to \mathbb{R} : v \in GSBV(\Omega), \quad \nabla v \in [GSBV(\Omega)]^n\}. \end{aligned}$$

Notice that $Dv = \nabla v$ in $SBH(\Omega)$, $Dv \neq \nabla v$ in $SBV^2(\Omega)$ and in $GSBV^2(\Omega)$; moreover we set

$$S_{\nabla v} = \bigcup_{i=1}^{n} S_{\nabla_i v}.$$

Eventually we introduce the weak energy functional and a space for competing functions.

DEFINITION 2.6. For $\Omega \subset \mathbb{R}^n$ open set, under the assumption (1.2), setting $X(\Omega) = GSBV^2(\Omega) \cap L^q(\Omega)$, we define $\mathcal{F} : X(\Omega) \to [0, +\infty]$ as

$$\mathcal{F}(v) = \int_\Omega \left(|\nabla^2 v|^p + \mu |v - g|^q\right) dy + \alpha \mathcal{H}^{n-1}(S_v) + \beta \mathcal{H}^{n-1}(S_{\nabla v} \setminus S_v). \tag{2.1}$$

We need a localization of the previous functional $\mathcal{F}$.

DEFINITION 2.7. For every Borel subset $A \subset \Omega$ and $v \in X(\Omega)$ we define

$$\begin{aligned} \mathcal{F}_g(v, \mu, \alpha, \beta, A) = &\int_A \left(|\nabla^2 v|^p + \mu |v - g|^q\right) dy + \alpha \mathcal{H}^{n-1}(S_v \cap A) \\ &+ \beta \mathcal{H}^{n-1}\left((S_{\nabla v} \setminus S_v) \cap A\right). \end{aligned}$$

We shall use $\mathcal{F}(v, A)$ if g, μ, α, β are clearly defined by the context.

The following two theorems have been proved in [CLT5] with the inessential restriction $p = q = 2$. In the first statement we use the following sequence of smooth truncation functions:

$$\begin{cases} \varphi_k \in C^2(\mathbb{R}), & 0 \le \varphi_k' \le 1\,, \\ \varphi_k(t) = t & \forall\, t \in [-k+1, k-1]\,, \\ |\varphi_k(t)| = k & \forall\, t : |t| > k+1\,, \end{cases} \tag{2.2}$$

for every integer $k \ge 2$. The property $v \in GSBV(\Omega)$ (see Definition 2.1) is clearly equivalent to the requirement

$$\varphi_k \circ v \in SBV_{\mathrm{loc}}(\Omega)$$

(see [A2], Section 1).

THEOREM 2.8 (Interpolation inequality). *Let $n \in \mathbb{N}$ and let $Q \subset \mathbb{R}^n$ be a cube with edges of lenght l and parallel to the coordinate axes, $p \ge 1$ and $v \in GSBV^2(Q)$. Then for every $i = 1, \ldots, n$ the following inequality holds true*

$$\int_Q |\nabla_i(\varphi_k \circ v)|\, dy \le 2k\big(\mathcal{H}^{n-1}(S_v \cup S_{\nabla v}) + l^{n-1}\big) + \frac{p\, l^{\frac{n-1}{2} + \frac{2p-1}{p}}}{2p-1} \left(\int_Q |\nabla^2 v|^p dy\right)^{\frac{1}{p}}.$$

THEOREM 2.9 (Existence of weak solutions). *Let $\Omega \subset \mathbb{R}^n$ be an open set. Assume* (1.2) *with $g \in L^q(\Omega)$. Then there is $v_0 \in X(\Omega)$ such that*

$$\mathcal{F}(v_0) \le \mathcal{F}(v) \qquad \forall\, v \in X(\Omega).$$

We recall that assumption $\beta \le \alpha \le 2\beta$ is necessary for lower semicontinuity of $\mathcal{F}$ (see [BZ], [Co]).

3. – Preliminary results

In this section we prove some technical results.

LEMMA 3.1 (Density upper bound). *Let $v \in GSBV^2(\Omega)$ be a minimizer for the functional* (2.1) *with $g \in L^\tau_{\mathrm{loc}}(\Omega)$ ($\tau \ge nq$). Then for every $0 < \rho \le 1$ and for every $x \in \Omega$ such that $\overline{B}_\rho(x) \subset \Omega$ we have*

$$\mathcal{F}_g\big(v, \mu, \alpha, \beta, \overline{B}_\rho(x)\big) \le c_0 \rho^{n-1}$$

where $c_0 = \mu \|g\|^q_{L^\tau(B_\rho(x))} \omega_n^{1-\frac{q}{\tau}} + \alpha n \omega_n$.

In particular if $p = q = n = 2$ and $g \in L^\infty(\Omega)$ then $c_0 = \pi \mu \|g\|^2_{L^\infty(\Omega)} + 2\pi\alpha$.

PROOF. By minimality of v for $\mathcal{F}$ we get $\mathcal{F}(v) \leq \mathcal{F}(w)$, where $w = v\chi_{\Omega\setminus B_\rho(x)}$. Since $S_w \cup S_{\nabla w} \subset ((S_v \cup S_{\nabla v}) \setminus B_\rho(x)) \cup \partial B_\rho(x)$ and taking into account $\beta \leq \alpha$, by subtraction we obtain

$$\begin{aligned}\int_{B_\rho(x)} \left(|\nabla^2 v|^p + \mu|v-g|^q\right) dy + \alpha\mathcal{H}^{n-1}\left(S_v \cap \overline{B}_\rho(x)\right)& \\ + \beta\mathcal{H}^{n-1}\left((S_{\nabla v} \setminus S_v) \cap \overline{B}_\rho(x)\right)& \\ \leq \mu \int_{B_\rho(x)} |g|^q\, dy + \alpha\mathcal{H}^{n-1}\left(\partial B_\rho(x)\right)& \\ \leq \mu\|g\|^q_{L^\tau(B_\rho(x))}(\omega_n\rho^n)^{1-\frac{q}{\tau}} + \alpha n\omega_n\rho^{n-1},&\end{aligned}$$

hence, by the assumptions on τ and ρ, we achieve the proof. □

LEMMA 3.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $K_0, K_1 \subset \mathbb{R}^n$ be Borel sets with $K_0 \cup K_1$ closed. Let $v \in C^2(\Omega \setminus (K_0 \cup K_1))$ and let v be approximately continuous in $\Omega \setminus K_0$. Assume that*

$$F(K_0, K_1, v) < +\infty,$$

then $v \in X(\Omega)$, $S_v \subset K_0$, $S_{\nabla v} \subset K_0 \cup K_1$ and $\mathcal{F}(v) \leq F(K_0, K_1, v)$.

PROOF. The function v is obviously in $L^q(\Omega)$ and, by Lemma 2.6 in [CLT3], we have $\nabla v \in [GSBV(\Omega)]^n$. So, by Lemma 2.3 in [DCL], the proof will be achieved as soon as we show that for every cube $Q \subset \Omega$ with edges parallel to the coordinate axes and for every integer $k \geq 2$ we have $\nabla(\varphi_k \circ v) \in \left[L^1(Q)\right]^n$ (possibly not uniformly in k, Q), where (φ_k) satisfies (2.2).

Fix $Q \subset \Omega$ as above and $i \in \{1, \dots, n\}$, then, by Theorem 2.8, we obtain the following interpolation inequality

$$\int_Q |\nabla_i(\varphi_k \circ v)|\, dy \leq 2k\left(\mathcal{H}^{n-1}(K_0 \cup K_1) + l^{n-1}\right) + \frac{p l^{\frac{n-1}{2}+\frac{2p-1}{p}}}{2p-1}\left(\int_Q |\nabla^2 v|^p dy\right)^{\frac{1}{p}}$$

where l is the edge length of Q. Hence $v \in X(\Omega)$ and the other assertions follow immediately. □

REMARK 3.3. By Theorem 2.9 and Lemma 3.2 we obtain immediately that

$$\begin{aligned}\min_{v\in X(\Omega)} \mathcal{F}(v) \leq \inf\big\{F(K_0, K_1, v) : K_0, K_1 \subset \mathbb{R}^n \text{ Borel sets, } K_0 \cup K_1 \text{ closed,}& \\ v \in C^2\left(\Omega \setminus \left(K_0 \cup K_1\right)\right),\ v \text{ approximately continuous on } \Omega \setminus K_0\big\}.&\end{aligned}$$

LEMMA 3.4 (Scaling). *Let $v \in GSBV^2(B_r(x_0))$. For $\lambda > 0$ and for every $x \in B_1$ set $v_r(x) = \dfrac{v(x_0 + rx)}{(\lambda r^{2p-1})^{1/p}}$ and $g_r(x) = \dfrac{g(x_0 + rx)}{(\lambda r^{2p-1})^{1/p}}$.*
Then $v_r \in GSBV^2(B_1)$ and

$$\mathcal{F}_g\left(v, \mu, \alpha, \beta, B_r(x_0)\right) = \lambda r^{n-1}\mathcal{F}_{g_r}\left(v_r, \mu\lambda^{\frac{q}{p}-1} r^{1+q\frac{2p-1}{p}}, \frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, B_1\right).$$

PROOF. Follows immediately by change of variables. □

DEFINITION 3.5. A function $v \in X(\Omega)$ is a local minimizer of (2.1) if, for every compact set $T \subset \Omega$,

$$\mathcal{F}_g(v,\mu,\alpha,\beta,T) = \inf_{z\in X(\Omega)} \left\{\mathcal{F}_g(z,\mu,\alpha,\beta,T) : \ z = v \text{ a.e.in } \Omega\setminus T\right\} < +\infty.$$

Actually the infimum in the Definition 3.5 is a minimum due to Theorem 2.9 and to the fact that $\{z \in X(\Omega) : \ z = v \text{ a.e.in } \Omega \setminus T\}$ is closed in $L^q(\Omega)$.

LEMMA 3.6 (Matching). *Let $z, v \in GSBV^2(\Omega)$, $\overline{B}_t(x) \subset \Omega$ and*

$$\mathcal{H}^{n-1}\big((S_z \cup S_{\nabla z}) \cap \partial B_t(x)\big) = \mathcal{H}^{n-1}\big((S_v \cup S_{\nabla v}) \cap \partial B_t(x)\big) = 0.$$

Then, by setting

$$u = \begin{cases} z & \text{in } B_t(x) \\ v & \text{in } \Omega\setminus B_t(x) \end{cases}$$

we have

$$\begin{aligned}\mathcal{F}_g(u,\mu,\alpha,\beta,\Omega) \le\ & \mathcal{F}_g\big(z,\mu,\alpha,\beta,B_t(x)\big) + \mathcal{F}_g\big(v,\mu,\alpha,\beta,\Omega\setminus\overline{B}_t(x)\big) \\ & +\alpha\mathcal{H}^{n-1}\big(\{\tilde z \ne \tilde v\}\cap\partial B_t(x)\big)+\beta\mathcal{H}^{n-1}\big(\{\widetilde{\nabla z}\ne\widetilde{\nabla v}\}\cap\partial B_t(x)\big).\end{aligned}$$

PROOF. Follows immediately by the definitions. □

LEMMA 3.7 (Joining). *Let $u, v \in GSBV^2(\Omega)$ and let $0 < s < t < 1$ be such that $\overline{B}_t(x) \subset \Omega$. Then for every $\delta \in (0,1)$ there exist $c = c(\delta, p, n) > 0$ and a cut-off function $\psi \in C_0^2(B_t(x))$, with $\psi \equiv 1$ in a neighborhoud of $\overline{B}_s(x)$, such that, by setting $U = \psi v + (1-\psi)u$, we have*

$$\begin{aligned}\mathcal{F}(U,\overline{B}_t(x)) \le\ & (1+\delta)\big(\mathcal{F}(v,\overline{B}_t(x)) + \mathcal{F}(u,\overline{B}_t(x)\setminus B_s(x))\big) \\ & + \frac{c}{(t-s)^p}\left(\int_{B_t(x)\setminus B_s(x)} |\nabla(v-u)|^p dy + \frac{c}{\delta s^p (t-s)^p}\int_{B_t(x)\setminus B_s(x)} |v-u|^p dy\right).\end{aligned}$$

PROOF. To simplify notation we consider $x = 0$. We fix $\delta \in (0,1)$ and $N = N(\delta) \in \mathbb{N}$, $N = 1 + [3^{p-1}/\delta]$. Let

$$s_j = s + j\frac{t-s}{N}, \quad j = 0, \dots, N,$$

and ψ_j $(j = 0, \dots, N-1)$ cut-off functions between B_{s_j} and $B_{s_{j+1}}$ of class C^2 such that $|D\psi_j| \le \frac{2N}{(t-s)}$ and $|D^2\psi_j| \le \frac{a_n N^2}{s(t-s)^2}$ in $C_j = B_{s_{j+1}} \setminus B_{s_j}$ (where $a_n = 6 + 2\sqrt{n-1}$), and define

$$U_j = \psi_j v + (1-\psi_j)u.$$

Then, by setting $\mathbf{a} \odot \mathbf{b} = \frac{1}{2}(\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a})$, we get, for every j,

$$\begin{aligned}\int_{B_t} |\nabla^2 U_j|^p \, dy \le & \int_{B_{s_j}} |\nabla^2 v|^p dy + \int_{B_t \setminus B_{s_j}} |\nabla^2 u|^p dy \\ & + \int_{C_j} |\psi_j \nabla^2 v + (1-\psi_j)\nabla^2 u + 2D\psi_j \odot \nabla(v-u) \\ & + D^2\psi_j \, (v-u)|^p dy \le \int_{B_t} |\nabla^2 v|^p dy + \int_{B_t \setminus B_s} |\nabla^2 u|^p dy \\ & + 3^{p-1} \int_{C_j} \big(|\nabla^2 v|^p + |\nabla^2 u|^p + |D\psi_j|^p |\nabla(v-u)|^p \\ & + |D^2 \psi_j|^p |v-u|^p \big) \, dy.\end{aligned}$$

Since $C_j \cap C_{j+1} = \emptyset$, by adding with respect to j from 0 to $N-1$, we get

$$\begin{aligned}\min_j \int_{B_t} |\nabla^2 U_j|^p dy \le & \int_{B_t} |\nabla^2 v|^p dy + \int_{B_t \setminus B_s} |\nabla^2 u|^p dy \\ & + \frac{3^{p-1}}{N} \int_{B_t \setminus B_s} \Bigg(|\nabla^2 v|^p + |\nabla^2 u|^p \\ & + \left(\frac{2N}{t-s} \right)^p |\nabla(v-u)|^p + \left(\frac{a_n N^2}{s(t-s)^2} \right)^p |v-u|^p \Bigg) \, dy.\end{aligned}$$

By choosing the index j for which the above minimum is achieved, we set $U = U_j$ so that

$$\begin{aligned}\int_{B_t} |\nabla^2 U|^p dy \le & \int_{B_t} |\nabla^2 v|^p dy + \int_{B_t \setminus B_s} |\nabla^2 u|^p dy \\ & + \delta \int_{B_t \setminus B_s} \Bigg(|\nabla^2 v|^p + |\nabla^2 u|^p + \frac{2^p (1+3^{p-1})^p}{\delta^p (t-s)^p} |\nabla(v-u)|^p \\ & + \frac{a_n^p (1+3^{p-1})^{2p}}{\delta^{2p} s^p (t-s)^{2p}} |v-u|^p \Bigg) \, dy,\end{aligned}$$

hence the thesis follows with $c = a_n^p (1+3^{p-1})^p / \delta^{p-1}$. □

LEMMA 3.8. *Let $n \ge 2$, $p \ge 1$ and let $\Omega \subset \mathbb{R}^n$ be open. Let $v \in GSBV^2(\Omega)$ such that, for every compact set $T \subset \Omega$,*

$$\mathcal{F}_g(v, \mu, \alpha, \beta, T) < +\infty.$$

Then

$$\lim_{\rho \to 0} \rho^{1-n} \mathcal{F}_g\big(v, \mu, \alpha, \beta, B_\rho(x)\big) = 0$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega \setminus (S_v \cup S_{\nabla v})$.

PROOF. Apply the same argument of Lemma 2.6 in [DCL]. □

4. – Compactness and lower semicontinuity

We prove a Poincaré type inequality in the class $GSBV$ which extends Theorem 3.1 in [DCL] allowing a more general truncation. We enphasize that $v \in GSBV^2(\Omega)$ does not even entail that either v or ∇v belongs to $L^1_{\rm loc}(\Omega)$.

Let B be an open ball in $\mathbb{R}^n$. For every measurable function $v : B \to \mathbb{R}$ we define the least median of v in B as

$$\operatorname{med}(v, B) = \inf\left\{t \in \mathbb{R};\ \ |\{v < t\} \cap B| \geq \frac{1}{2}|B|\right\}.$$

We remark that $\operatorname{med}(\cdot, B)$ is a non linear operator and in general it has no relationship with $⨍_B \cdot dy$. Obviously we have $\operatorname{med}(v\chi_{B\setminus E} + \operatorname{med}(v, B)\chi_E, B) = \operatorname{med}(v, B)$ for every $E \subset B$. For every $v \in GSBV(B)$ and $a \in \mathbb{R}$ with $(2\gamma_n\mathcal{H}^{n-1}(S_v))^{\frac{n}{n-1}} \leq a \leq \frac{1}{2}|B|$, we set

$$\tau'(v, a, B) = \inf\big\{t \in \mathbb{R};\ \ |\{v < t\}| \geq a\big\},$$
$$\tau''(v, a, B) = \inf\big\{t \in \mathbb{R};\ \ |\{v \geq t\}| \leq a\big\},$$

where γ_n is the isoperimetric constant relative to the balls of $\mathbb{R}^n$, i.e. for every Borel set E

$$\min\left\{|E \cap B|^{\frac{n-1}{n}}, |B \setminus E|^{\frac{n-1}{n}}\right\} \leq \gamma_n P(E, B).$$

For $\eta \geq 0$ we define the truncation operator

$$T(v, a, \eta) = \big(\tau'(v, a, B) - \eta\big) \vee v \wedge \big(\tau''(v, a, B) + \eta\big). \tag{4.1}$$

We get easily $T(T(v, a, \eta), a, \eta) = T(v, a, \eta)$, $\operatorname{med}(T(v, a, \eta), B) = \operatorname{med}(v, B)$ and $T(\lambda v, a, \lambda\eta) = \lambda T(v, a, \eta)$ for every $\lambda > 0$. Moreover $|\nabla T(v, a, \eta)| \leq |\nabla v|$ a.e. on B and

$$|\{v \neq T(v, a, \eta)\}| \leq 2a. \tag{4.2}$$

In case v is vector-valued the operators med and T are defined componentwise.

THEOREM 4.1 (Poincaré type inequality). *Let $B \subset \mathbb{R}^n$ be an open ball, $n \geq 2$, $p \geq 1$. Let $v \in GSBV(B)$ and $a \in \mathbb{R}$ with*

$$\big(2\gamma_n\mathcal{H}^{n-1}(S_v)\big)^{\frac{n}{n-1}} \leq a \leq \frac{1}{2}|B|, \tag{4.3}$$

let $\eta \geq 0$ and $T(v, a, \eta)$ as in (4.1). *Then*

$$\int_B |D\,T(v, a, \eta)| \leq 2|B|^{\frac{p-1}{p}}\left(\int_B |\nabla T(v, a, \eta)|^p\,dy\right)^{\frac{1}{p}} + 2\eta\mathcal{H}^{n-1}(S_v). \tag{4.4}$$

If $p < n$, setting $p^* = \frac{np}{n-p}$, we have also

$$\tag{4.5} \begin{aligned} &\int_B |T(v,a,\eta) - \mathrm{med}(v,B)|^{p^*} dy \\ &\leq \frac{1}{2}\left(\frac{4\gamma_n p(n-1)}{n-p}\right)^{p^*} \left(\int_B |\nabla T(v,a,0)|^p \, dy\right)^{\frac{p^*}{p}} + (2\eta)^{p^*} a. \end{aligned}$$

If $p \geq n$, for every $s \geq \frac{n}{n-1}$ we have also

$$\tag{4.6} \begin{aligned} &\int_B |T(v,a,\eta) - \mathrm{med}(v,B)|^{s} dy \\ &\leq \frac{1}{2}\left(\frac{4\gamma_n s(n-1)}{n}\right)^{s} \left(\int_B |\nabla T(v,a,0)|^p \, dy\right)^{\frac{s}{p}} |B|^{1+\frac{s}{n}-\frac{s}{p}} + (2\eta)^{s} a. \end{aligned}$$

PROOF. We may assume that $\mathrm{med}(v,B) = 0$ and the right hand sides are finite. If $\mathcal{H}^{n-1}(S_v) = 0$ and $a = 0$ then $v \in W^{1,p}(B)$, $T(v,0,\eta) = v$ and the inequalities are well-known. By Lemma 2.3, $T(v,a,\eta) \in SBV(B)$ and we obtain

$$\tag{4.7} \begin{aligned} \int_B |DT(v,a,\eta)| &= \int_B |\nabla T(v,a,\eta)| \, dy \\ &\quad + \int_{S_{T(v,a,\eta)}} |(T(v,a,\eta))^+ - (T(v,a,\eta))^-| \, d\mathcal{H}^{n-1} \\ &\leq \int_B |\nabla T(v,a,\eta)| \, dy + \big(\tau''(v,a,B) \\ &\quad - \tau'(v,a,B) + 2\eta\big)\mathcal{H}^{n-1}(S_v). \end{aligned}$$

By the coarea formula of Lemma 2.3 and the isoperimetric inequality we obtain

$$\tag{4.8} \begin{aligned} \int_B |DT(v,a,\eta)| &= \int_{-\infty}^{+\infty} P(\{T(v,a,\eta) < \sigma\}, B) \, d\sigma \\ &\geq \frac{1}{\gamma_n} \int_0^{\tau''} |\{T(v,a,\eta) \geq \sigma\}|^{\frac{n-1}{n}} d\sigma \\ &\quad + \frac{1}{\gamma_n} \int_{\tau'}^{0} |\{T(v,a,\eta) < \sigma\}|^{\frac{n-1}{n}} d\sigma \\ &\geq \frac{1}{\gamma_n}\big(\tau''(v,a,B) - \tau'(v,a,B)\big) a^{\frac{n-1}{n}}. \end{aligned}$$

By the assumption on a and by comparison with (4.7) in the case $\eta = 0$ we have

$$\begin{aligned} 2\big(\tau''(v,a,B) - \tau'(v,a,B)\big)\mathcal{H}^{n-1}(S_v) &\leq \frac{1}{\gamma_n}\big(\tau''(v,a,B) - \tau'(v,a,B)\big) a^{\frac{n-1}{n}} \\ \leq \int_B |\nabla T(v,a,0)| \, dy + \big(\tau''(v,a,B) &- \tau'(v,a,B)\big)\mathcal{H}^{n-1}(S_v), \end{aligned}$$

hence

$$(4.9)\qquad \big(\tau''(v,a,B)-\tau'(v,a,B)\big)\mathcal{H}^{n-1}(S_v)\le\int_B|\nabla T(v,a,0)|\,dy.$$

By substitution in (4.7), we obtain for every $\eta\ge 0$

$$(4.10)\qquad \int_B|DT(v,a,\eta)|\le 2\int_B|\nabla T(v,a,\eta)|\,dy+2\eta\mathcal{H}^{n-1}(S_v),$$

so (4.4) follows by Hölder inequality. By (4.8) and (4.10) we get also

$$(4.11)\qquad \begin{aligned}&\big(\tau''(v,a,B)-\tau'(v,a,B)\big)a^{\frac{n-1}{n}}\\ &\le 2\gamma_n\left(|B|^{\frac{p-1}{p}}\left(\int_B|\nabla T(v,a,\eta)|^p\,dy\right)^{\frac{1}{p}}+\eta\mathcal{H}^{n-1}(S_v)\right).\end{aligned}$$

By the classical Poincaré inequality (see [F], pag. 504) applied to $T(v,a,\eta)$ we get

$$(4.12)\qquad \int_B|T(v,a,\eta)|^{1^*}dy\le\left(\gamma_n\int_B|DT(v,a,\eta)|\right)^{1^*}.$$

We define

$$E=\{y\in B;\tau'(v,a,B)\le v(y)\le\tau''(v,a,B)\},$$
$$E'=\{y\in B;v(y)<\tau'(v,a,B)\},\quad E''=\{y\in B;v(y)>\tau''(v,a,B)\}.$$

Then, by taking into account (4.1), (4.2) and (4.4), we have for every $s\ge 1$

$$(4.13)\qquad \begin{aligned}\int_B|T(v,a,\eta)|^s dy&=\int_E|T(v,a,0)|^s dy+\int_{E'\cup E''}|T(v,a,\eta)|^s dy\\ &\le\int_E|T(v,a,0)|^s dy+\int_{E'}|\tau'(v,a,B)-\eta|^s dy+\int_{E''}|\tau''(v,a,B)+\eta|^s dy\\ &\le\int_E|T(v,a,0)|^s dy+2^{s-1}\left(\int_{E'\cup E''}|T(v,a,0)|^s dy+\eta^s|E'\cup E''|\right)\\ &\le 2^{s-1}\int_B|T(v,a,0)|^s dy+(2\eta)^s a.\end{aligned}$$

Hence if $p=1$, by (4.13) with $s=1^*$, (4.12) and (4.10) for $T(v,a,0)$ we get

$$\int_B|T(v,a,\eta)|^{1^*}dy\le 2^{1^*-1}\left(2\gamma_n\int_B|\nabla T(v,a,0)|\,dy\right)^{1^*}+(2\eta)^{1^*}a,$$

so that (4.5) is proved in the case $p = 1$. We focus now the case $1 < p < n$. We set

$$w = T(v, a, 0)|T(v, a, 0)|^{\frac{p^*}{1^*}-1}$$

and we notice that $w \in SBV(B)$, $\operatorname{med}(w, B) = \operatorname{med}(v, B)$ and

$$|\nabla w| = \frac{p^*}{1^*}|T(v, a, 0)|^{\frac{p^*}{1^*}-1}|\nabla T(v, a, 0)| \quad \text{a.e. on } B.$$

By plugging w in (4.12) and (4.10) with $\eta = 0$ and by Hölder inequality

$$\begin{aligned}
\left(\int_B |T(v, a, 0)|^{p^*} dy\right)^{\frac{1}{1^*}} &= \left(\int_B |T(w, a, 0)|^{1^*} dy\right)^{\frac{1}{1^*}} \\
&\le 2\gamma_n \frac{p^*}{1^*} \int_B |\nabla T(v, a, 0)||T(v, a, 0)|^{\frac{p^*}{1^*}-1} dy \\
&\le 2\gamma_n \frac{p^*}{1^*} \left(\int_B |\nabla T(v, a, 0)|^p dy\right)^{\frac{1}{p}} \left(\int_B |T(v, a, 0)|^{p^*} dy\right)^{1-\frac{1}{p}}.
\end{aligned}$$

Dividing by $\|T(v, a, 0)\|_{L^{p^*}(B)}^{p^*(1-\frac{1}{p})}$, which is finite since $T(v, a, 0)$ is bounded, we get

$$\int_B |T(v, a, 0)|^{p^*} dy \le \left(\frac{2\gamma_n p(n-1)}{n-p}\right)^{p^*} \left(\int_B |\nabla T(v, a, 0)|^p dy\right)^{\frac{p^*}{p}}. \tag{4.14}$$

By (4.13) with $s = p^*$ and (4.14) we obtain

$$\int_B |T(v, a, \eta)|^{p^*} dy \le \frac{1}{2}\left(\frac{4\gamma_n p(n-1)}{n-p}\right)^{p^*} \left(\int_B |\nabla T(v, a, 0)|^p dy\right)^{\frac{p^*}{p}} + (2\eta)^{p^*} a.$$

If $p \ge n$, fix any $s \ge \frac{n}{n-1}$ and set $r = \frac{ns}{n+s}$. Then $r < n$ and $r^* = s$, hence by (4.5) and Hölder inequality, still assuming $\operatorname{med}(v, B) = 0$ and setting $c = (4\gamma_n \frac{r(n-1)}{n-r})^s = (4\gamma_n \frac{s(n-1)}{n})^s$, we have

$$\begin{aligned}
\int_B |T(v, a, \eta)|^s dy &\le \frac{c}{2}\left(\int_B |\nabla T(v, a, 0)|^r dy\right)^{\frac{s}{r}} + (2\eta)^s a \\
&\le \frac{c}{2}|B|^{\left(1-\frac{r}{p}\right)\frac{s}{r}} \left(\int_B |\nabla T(v, a, 0)|^p dy\right)^{\frac{s}{p}} + (2\eta)^s a \\
&\le \frac{c}{2}|B|^{1+\frac{s}{n}-\frac{s}{p}} \left(\int_B |\nabla T(v, a, 0)|^p dy\right)^{\frac{s}{p}} + (2\eta)^s a
\end{aligned}$$

and the proof is completed. □

We show that, beside (4.2), also the perimeter of the set $\{v \neq T(v,a,\eta)\}$ can be estimated for many suitable $\eta \in (0,1)$.

LEMMA 4.2. *Let $B \subset \mathbb{R}^n$ be an open ball, $n \geq 2$, $s \geq 1$. Let $v \in GSBV(B)$ and $a \in \mathbb{R}$ with $\left(2\gamma_n \mathcal{H}^{n-1}(S_v)\right)^{\frac{n}{n-1}} \leq a \leq \frac{1}{2}|B|$. Then there exists $\eta \in (0,1)$ such that*

$$
\begin{aligned}
&P\big(\{v > \tau''(v,a,B)+\eta\}, B\big)\\
&\quad \leq 3a^{1-\frac{1}{s}}\left(\int_{\{v>\tau''(v,a,B)\}} |\nabla T(v,a,1)|^s dy\right)^{\frac{1}{s}} + 3\mathcal{H}^{n-1}(S_v),
\end{aligned} \tag{4.15}
$$

$$
\begin{aligned}
&P\big(\{v < \tau'(v,a,B)-\eta\}, B\big)\\
&\quad \leq 3a^{1-\frac{1}{s}}\left(\int_{\{v<\tau'(v,a,B)\}} |\nabla T(v,a,1)|^s dy\right)^{\frac{1}{s}} + 3\mathcal{H}^{n-1}(S_v).
\end{aligned} \tag{4.16}
$$

Actually $|\{\eta \in (0,1) \ :$ *both* (4.15) *and* (4.16) *hold* $\}| \geq \frac{1}{3}$.

PROOF. By coarea formula of Lemma 2.3 and by the definition of $SBV(B)$

$$
\begin{aligned}
\int_{\tau''}^{\tau''+1} P(\{v>\sigma\}, B)\, d\sigma &= \int_B |D(\tau'' \vee v \wedge (\tau''+1))|\\
&\leq \int_{\{\tau''<v<\tau''+1\}} |\nabla T(v,a,1)|\, dy + \mathcal{H}^{n-1}(S_v)\\
&\leq \int_{\{\tau''<v\}} |\nabla T(v,a,1)|\, dy + \mathcal{H}^{n-1}(S_v),
\end{aligned}
$$

and analogously

$$
\begin{aligned}
\int_{\tau'-1}^{\tau'} P(\{v<\sigma\}, B)\, d\sigma &= \int_B |D((\tau'-1) \vee v \wedge \tau')|\\
&\leq \int_{\{\tau'-1<v<\tau'\}} |\nabla T(v,a,1)|\, dy + \mathcal{H}^{n-1}(S_v)\\
&\leq \int_{\{v<\tau'\}} |\nabla T(v,a,1)|\, dy + \mathcal{H}^{n-1}(S_v).
\end{aligned}
$$

We get the thesis by Chebyshev and Hölder inequalities and by (4.2). □

For any given function in $GSBV$, we define an affine polynomial correction such that both median and gradient median vanish.

Let $B_r(x) \subset \Omega$ and $v \in GSBV(B_r(x))$; for every $y \in \mathbb{R}^n$ we set

$$
(M_{x,r}v)(y) = \operatorname{med}\big(\nabla v, B_r(x)\big) \cdot (y-x) \tag{4.17}
$$

$$
(\mathcal{P}_{x,r}v)(y) = (M_{x,r}v)(y) + \operatorname{med}\big(v - M_{x,r}v, B_r(x)\big). \tag{4.18}
$$

Since $\operatorname{med}(v-c, B_r(x)) = \operatorname{med}(v, B_r(x)) - c$ for every $c \in \mathbb{R}$ and $\nabla(\mathcal{P}_{x,r}v) = \nabla(M_{x,r}v) = \operatorname{med}(\nabla v, B_r(x))$, then we have $\mathcal{P}_{x,r}(v - \mathcal{P}_{x,r}v) = 0$, say

$$\operatorname{med}\big(v - \mathcal{P}_{x,r}v, B_r(x)\big) = 0, \qquad \operatorname{med}\big(\nabla(v - \mathcal{P}_{x,r}v), B_r(x)\big) = \mathbf{0}.$$

We notice that there are v such that $\operatorname{med}(v, B_r(x)) \neq \operatorname{med}(\mathcal{P}_{x,r}v, B_r(x))$, take e.g. $v(x) = (x_1^2 - x_1)H(-x_1) - \frac{x_1}{2}H(x_1)$, where H is the Heaviside function.

In the following we denote by s' the conjugate exponent of s in the Hölder inequality.

THEOREM 4.3 (Compactness and lower semicontinuity). *Let $p \geq n \geq 2$, $B_r(x) \subset \mathbb{R}^n$ be an open ball, $(v_h) \subset GSBV^2(B_r(x))$. Set $L_h := \mathcal{H}^{n-1}(S_{v_h} \cup S_{\nabla v_h})$. Assume*

$$\sup_h \int_{B_r(x)} |\nabla^2 v_h|^p dy < +\infty, \tag{4.19}$$

$$\lim_h L_h = 0. \tag{4.20}$$

Then there exist $z \in W^{2,p}(B_r(x))$, a sequence $(z_h) \subset GSBV^2(B_r(x))$ and a positive constant c (depending on the left hand side of (4.19)) such that, up to a finite number of indices,

$$\big|\{z_h \neq (v_h - \mathcal{P}_{x,r}v_h)\}\big| \leq c{L_h}^{n'}, \tag{4.21}$$

$$P\big(\{z_h \neq (v_h - \mathcal{P}_{x,r}v_h)\}, B_r(x)\big) \leq cL_h. \tag{4.22}$$

Moreover there is a subsequence (z_{h_k}) such that for every $\vartheta \geq 1$

$$\lim_k z_{h_k} = z \text{ strongly in } L^{\vartheta}(B_r(x)), \tag{4.23}$$

$$\lim_k \nabla z_{h_k} = Dz \text{ strongly in } L^{\vartheta}(B_r(x), \mathbb{R}^n), \tag{4.24}$$

$$\begin{aligned}\int_{B_r(x)} |D^2 z|^p dy &\leq \liminf_k \int_{B_r(x)} |\nabla^2 z_{h_k}|^p dy \\ &\leq \liminf_k \int_{B_r(x)} |\nabla^2 v_{h_k}|^p dy,\end{aligned} \tag{4.25}$$

$$\lim_k \big(v_{h_k} - \mathcal{P}_{x,r}v_{h_k}\big) = z \text{ a.e. on } B_r(x), \tag{4.26}$$

$$\lim_k \nabla(v_{h_k} - \mathcal{P}_{x,r}v_{h_k}) = Dz \text{ a.e. on } B_r(x). \tag{4.27}$$

PROOF. We can assume $x = 0$, $\mathcal{P}_{0,r} v_h = 0$ and we extract subsequences without relabeling. We use properties of functions that hold true only up to a finite number of indices, hence by (4.20) we can assume that

$$a_h := \left(2\gamma_n L_h\right)^{n'} \leq \frac{1}{2}|B_r|.$$

Arguing separately on each component of ∇v_h we deduce that $\forall h \in \mathbb{N}$, $\forall k \in \{1, \dots, n\}$, there exist $\eta_h^k \in (0, 1)$ and c depending on the l.h.s. of (4.19) such that

$$\left|\left\{T\left(\nabla_k v_h, a_h, \eta_h^k\right) \neq \nabla_k v_h\right\}\right| \leq c {L_h}^{n'}, \tag{4.28}$$

$$P\left(\left\{T\left(\nabla_k v_h, a_h, \eta_h^k\right) \neq \nabla_k v_h\right\}, B_r\right) \leq c\left(L_h^{\frac{n'}{p'}} + \mathcal{H}^{n-1}(S_{\nabla_k v_h})\right). \tag{4.29}$$

Inequality (4.28) follows by (4.2), while (4.29) follows by Lemma 4.2 with $s = p$ applied to $\nabla_k v_h$ and estimating $\nabla T(\nabla_k v_h, a_h, 1)$ through (4.19).

We define small subsets $E_h \subset B_r$ where we have to modify v_h in order to force boundedness of ∇v_h, and we perform a former tuning of v_h. To this aim set

$$E_h = \bigcup_{k=1}^{n} \left\{y \in B_r; T\left(\nabla_k v_h, a_h, \eta_h^k\right) \neq \nabla_k v_h\right\}, \tag{4.30}$$

$$w_h = v_h \chi_{B_r \setminus E_h}. \tag{4.31}$$

Since $\operatorname{med}(v_h, B_r) = 0$ then $\operatorname{med}(w_h, B_r) = 0$ and by the definition $\operatorname{med}(\nabla w_h, B_r) = \operatorname{med}(\nabla v_h, B_r) = 0$. By (4.29) E_h has finite perimeter, hence by Lemma 2.3 we have $w_h \in GSBV^2(B_r)$ and $\nabla w_h \in SBV(B_r, \mathbb{R}^n) \cap L^\infty(B_r, \mathbb{R}^n)$. Then by summarizing

$$\begin{cases} \mathcal{P}_{0,r} w_h = 0, \\ |\nabla_k w_h| \leq \left|T\left(\nabla_k v_h, a_h, \eta_h^k\right)\right|, \quad |\nabla^2 w_h| \leq |\nabla^2 v_h| \text{ a.e. on } B_r, \\ \mathcal{H}^{n-1}\left(S_{w_h} \cup S_{\nabla w_h}\right) \leq c\left(L_h^{\frac{n'}{p'}} + L_h\right), \end{cases} \tag{4.32}$$

and by (4.20), (4.28) and (4.29)

$$\lim_h \left(|E_h| + P(E_h, B_r)\right) = 0. \tag{4.33}$$

Fix $\vartheta \geq p$. Since $0 < \eta_h^k < 1$, by (4.32), (4.6), (4.19) and (4.20) we get

$$\begin{aligned} \int_{B_r} |\nabla_k w_h|^\vartheta dy &\leq \int_{B_r} \left|T\left(\nabla_k v_h, a_h, \eta_h^k\right)\right|^\vartheta dy \\ &\leq c\left(\int_{B_r} |\nabla T(\nabla_k v_h, a_h, 0)|^p\, dy\right)^{\frac{t}{p}} + 2^\vartheta a_h \leq c < +\infty. \end{aligned} \tag{4.34}$$

By (4.32), (4.9) and by the assumptions (4.19), (4.20) and $p \geq n$, we get

$$\begin{aligned}\int_{B_r} |D\nabla_k w_h| &\leq \int_{B_r} |\nabla^2 w_h|\, dy \\ &\quad + \big(\tau''(\nabla_k w_h, a_h, B_r) - \tau'(\nabla_k w_h, a_h, B_r) + 2\big)\mathcal{H}^{n-1}(S_{\nabla_k w_h}) \\ &\leq 2|B_r|^{\frac{1}{p'}} \left(\int_{B_r} |\nabla^2 w_h|^p\, dy\right)^{\frac{1}{p}} + 2\mathcal{H}^{n-1}(S_{\nabla_k w_h}) \leq c < +\infty.\end{aligned} \tag{4.35}$$

Now we want to force boundedness of (w_h). By (4.32) we can assume

$$b_h := \big(2\gamma_n \mathcal{H}^{n-1}(S_{w_h} \cup S_{\nabla w_h})\big)^{n'} \leq \frac{1}{2}|B_r|$$

and there are $\eta_h \in (0, 1)$ and a constant, depending on the l.h.s. of (4.19), such that

$$|\{T(w_h, b_h, \eta_h) \neq w_h\}| \leq c\left(L_h^{\frac{n'}{p'}} + L_h\right)^{n'}, \tag{4.36}$$

$$P(\{T(w_h, b_h, \eta_h) \neq w_h\}, B_r) \leq c\left(L_h^{\frac{n'^2}{p'^2}} + L_h^{\frac{n'}{p'}} + L_h\right). \tag{4.37}$$

Inequality (4.36) follows by (4.2), while (4.37) follows by Lemma 4.2 with $s = p$, applied to w_h, taking into account (4.32) and (4.34). Then we define

$$W_h = \{y \in B_r;\, T(w_h, b_h, \eta_h) \neq w_h\}$$

and we perform the following tuning of the sequence (w_h):

$$z_h = T(w_h, b_h, \eta_h). \tag{4.38}$$

We notice that $z_h \in GSBV^2(B_r)$ and

$$\begin{cases} \operatorname{med}(z_h, B_r) = 0, \ \ T(z_h, b_h, \eta_h) = z_h, \ \ \{z_h \neq v_h\} = E_h \cup W_h, \\ |\nabla_k z_h| \leq |\nabla_k w_h|, \quad |\nabla^2 z_h| \leq |\nabla^2 w_h| \text{ a.e. on } B_r, \\ \mathcal{H}^{n-1}(S_{z_h} \cup S_{\nabla z_h}) \leq c\left(L_h^{\frac{n'^2}{p'^2}} + L_h^{\frac{n'}{p'}} + L_h\right). \end{cases} \tag{4.39}$$

Hence (4.21) and (4.22) follow by (4.28), (4.36), and (4.29), (4.37) respectively, taking into account the assumption $p \geq n$. By (4.7), (4.9) and (4.34) we have

$$\begin{aligned}&\int_{B_r} |Dz_h| \\ &\leq \int_{B_r} |\nabla w_h|\, dy + \big(\tau''(w_h, b_h, B_r) - \tau'(w_h, b_h, B_r) + 2\big)\mathcal{H}^{n-1}(S_{w_h}) \\ &\leq 2|B_r|^{\frac{1}{p'}} \left(\int_{B_r} |\nabla w_h|^p dy\right)^{\frac{1}{p}} + 2\mathcal{H}^{n-1}(S_{w_h}) \leq c < +\infty.\end{aligned} \tag{4.40}$$

Moreover, by (4.39) and (4.34),

$$\int_{B_r} |\nabla_k z_h|^{\vartheta} dy \leq \int_{B_r} |\nabla_k w_h|^{\vartheta} dy \leq c < +\infty, \tag{4.41}$$

and by (4.32), (4.39) we get

$$\begin{aligned}
\int_{B_r} |D\nabla_k z_h| \leq & \int_{B_r} |\nabla^2 z_h|\, dy \\
& + \left(\tau''(\nabla_k v_h, a_h, B_r) - \tau'(\nabla_k v_h, a_h, B_r) + 2\right)\mathcal{H}^{n-1}(S_{\nabla_k z_h}) \\
\leq & \; c \int_{B_r} |\nabla^2 v_h|\, dy \\
& + c\left(\tau''(\nabla_k v_h, a_h, B_r) - \tau'(\nabla_k v_h, a_h, B_r) + 2\right)\left(L_h^{\frac{n'2}{p'2}} + L_h^{\frac{n'}{p'}} + L_h\right),
\end{aligned}$$

hence by (4.11) and the assumption $p \geq n$ we have

$$\int_{B_r} |D\nabla_k z_h| \leq c < +\infty. \tag{4.42}$$

Since $0 < \eta_h < 1$, by (4.6) and (4.41)

$$\begin{aligned}
\int_{B_r} |z_h|^{\vartheta} dy \leq & \frac{1}{2}\left(\frac{4\gamma_n \vartheta (n-1)}{n}\right)^{\vartheta} \left(\int_{B_r} |\nabla T(z_h, b_h, 0)|^p\, dy\right)^{\frac{t}{p}} |B_r|^{1+\frac{\vartheta}{n}-\frac{\vartheta}{p}} \\
& + 2^{\vartheta}\left(2\gamma_n \mathcal{H}^{n-1}(S_{z_h})\right)^{n'} \leq c < +\infty.
\end{aligned} \tag{4.43}$$

By (4.41), (4.42) and the compactness theorem in $BV(B_r)$ applied to (∇z_h) there exists $f \in BV(B_r, \mathbb{R}^n)$ such that

$$\nabla z_h \to f \qquad \text{strongly in } L^s(B_r, \mathbb{R}^n) \quad \forall s \in [1, 1^*).$$

By (4.39), (4.19), (4.20) and Theorem 2.1 in [A1], $f \in SBV(B_r, \mathbb{R}^n)$ and also

$$\begin{aligned}
&\int_{B_r} |\nabla f|^p dy \leq \liminf_h \int_{B_r} |\nabla^2 z_h|^p dy, \\
&\mathcal{H}^{n-1}(S_f) \leq \lim_h \mathcal{H}^{n-1}(S_{\nabla z_h}) = 0,
\end{aligned} \tag{4.44}$$

hence $f \in W^{1,p}(B_r, \mathbb{R}^n)$. By [B], Theorem 18 in Appendix, and by (4.34) we get

$$\nabla z_h \to f \qquad \text{strongly in } L^s(B_r, \mathbb{R}^n) \quad \forall s \in [1, \vartheta). \tag{4.45}$$

By (4.40), (4.43) and the compactness theorem in $BV(B_r)$ applied to (z_h), there exists $z \in BV(B_r)$ such that

$$z_h \to z \quad \text{strongly in } L^s(B_r), \ \forall s \in [1, 1^*).$$

By (4.39), (4.41) and Theorem 2.1 in [A1] we have $z \in SBV(B_r)$ and also

$$\tag{4.46} \begin{gathered} \nabla z_h \to \nabla z \qquad \text{weakly in } L^{\vartheta}(B_r, \mathbb{R}^n), \\ \mathcal{H}^{n-1}(S_z) \leq \lim_h \mathcal{H}^{n-1}(S_{z_h}) = 0, \end{gathered}$$

hence $z \in W^{1,p}(B_r)$ and also by (4.43)

$$\tag{4.47} z_h \to z \qquad \text{strongly in } L^s(B_r) \quad \forall s \in [1, \vartheta).$$

By (4.45) and (4.46) $f = \nabla z = Dz$ so that $z \in W^{2,p}(B_r)$. By the arbitrariness of ϑ, (4.23), (4.24), (4.25) hold true and (4.26), (4.27) follow by (4.21). □

5. – Blow-up equation

The aim of this section is to prove for any minimizer v of (2.1) a faster decay estimate of the functional $\mathcal{F}_g$ around points $x \in \Omega$ such that

$$\tag{5.1} \lim_{\rho \to 0} \rho^{1-n} \mathcal{F}_g \left(v, \mu, \alpha, \beta, B_\rho(x)\right) = 0.$$

THEOREM 5.1 (Blow up equation). *Let $p \geq n \geq 2$ and $B_r(x) \subset \Omega$ be an open ball, $(v_h) \subset X(\Omega)$, $(\varphi_h) \subset L^q(\Omega)$, let (α_h), (β_h) and (μ_h) three sequences of positive numbers with $\beta_h \leq \alpha_h$ and let $z \in W^{2,p}(B_r(x))$. Assume that*

(i) *v_h is a local minimizer for $\mathcal{F}_{\varphi_h}(\cdot, \mu_h, \alpha_h, \beta_h, \Omega)$,*

(ii) *$\lim_h \mathcal{H}^{n-1}\left((S_{v_h} \cup S_{\nabla v_h}) \cap B_r(x)\right) = 0$,*

(iii) *$\lim_h \mathcal{F}_{\varphi_h}(v_h, \mu_h, \alpha_h, \beta_h, \overline{B}_\rho(x)) = \omega(\rho) < +\infty$ for almost all $\rho < r$,*

(iv) *$\lim_h \left(v_h - \mathcal{P}_{x,r} v_h\right) = z$ a.e. on $B_r(x)$, where $\mathcal{P}_{x,r}$ is defined in* (4.18).

(v) *$\lim_h \mu_h = 0, \quad \lim_h \mu_h \|\varphi_h\|^q_{L^q(B_r(x))} = 0$ and $\quad \lim_h \mu_h^{1/q} \mathcal{P}_{x,r} v_h = \xi$ uniformly in $B_r(x)$, where ξ is a polynomial of degree one.*

Then, for every $\rho < r$, z minimizes the functional

$$\int_{B_\rho(x)} |D^2 v|^p \, dy$$

over all $v \in W^{2,p}(B_r(x))$ such that $v = z$ in $B_r(x) \setminus B_\rho(x)$ and

$$\omega(\rho) = \int_{B_\rho(x)} \left(|D^2 z|^p + |\xi|^q\right) dy \qquad \textit{for almost all } \rho < r.$$

In particular z is a weak solution of the fourth order elliptic equation

$$\sum_{i,j=1}^{n} D^2_{ij}\left(D^2_{ij} z \, |D^2 z|^{p-2}\right) = 0 \qquad \textit{on } B_r(x).$$

PROOF. Assume $x = 0$ and set $\mathcal{P}v_h = \mathcal{P}_{0,r}v_h$ for brevity's sake. We will extract subsequences (without saying explicitly and without relabeling) each time is needed and we will omit the arguments μ_h, α_h, β_h in $\mathcal{F}_{\varphi_h}$. First we estimate

$$\mu_h^{1/q}|\mathcal{P}v_h| \le \mu_h^{1/q}\left(|\mathcal{P}v_h - v_h| + |v_h - \varphi_h| + |\varphi_h|\right),$$

and we have by (iv), (v)

$$\lim_h \mu_h^{1/q}|v_h - \mathcal{P}v_h| = 0 \quad \text{a.e. on } B_r$$

and even by (v),

$$\lim_h \mu_h^{1/q}|\varphi_h| = 0 \quad \text{a.e. on } B_r.$$

Hence

$$\liminf_h \mu_h|\mathcal{P}v_h|^q \le \liminf_h \mu_h|v_h - \varphi_h|^q \quad \text{a.e. on } B_r$$

and then, by Fatou's lemma,

$$\int_{B_\rho} |\xi|^q dy = \lim_h \mu_h \int_{B_\rho} |\mathcal{P}v_h|^q dy \le \liminf_h \mu_h \int_{B_\rho} |v_h - \varphi_h|^q dy. \tag{5.2}$$

By (ii) and (iii), the assumptions of Theorem 4.3 are satisfied, so that we can choose z_h and z as like as in Theorem 4.3 and z turns out to be the same of assumption (iv). By (4.25), (5.2) and (iii) we obtain

$$\begin{aligned}\int_{B_\rho} \left(|D^2 z|^p + |\xi|^q\right) dy &\le \liminf_h \int_{B_\rho} \left(|\nabla^2 v_h|^p + \mu_h|v_h - \varphi_h|^q\right) dy \\ &\le \liminf_h \mathcal{F}_{\varphi_h}(v_h, B_\rho) = \omega(\rho).\end{aligned}$$

To achieve the proof we want to show that for almost every $\rho < r$, for every $v \in W^{2,p}(B_r)$ with $v = z$ in $B_r \setminus B_\rho$

$$\int_{B_\rho} \left(|D^2 v|^p + |\xi|^q\right) dy \ge \omega(\rho).$$

By monotonicity ω is continuous for a.e. $\rho \in (0, r)$. Assume by contradiction that there exist $s < r$, $v \in W^{2,p}(B_r)$ and $\varepsilon > 0$ such that ω is continuous at s, $v = z$ in $B_r \setminus B_s$ and

$$\int_{B_s} \left(|D^2 v|^p + |\xi|^q\right) dy \leq \omega(s) - \varepsilon. \tag{5.3}$$

As like as in Theorem 4.3, we set $L_h = \mathcal{H}^{n-1}(S_{v_h} \cup S_{\nabla v_h})$ and $A_h = E_h \cup W_h = \{z_h \neq (v_h - \mathcal{P} v_h)\}$. To get a contradiction we paste together $z_h + \mathcal{P} v_h$ and v_h along the boundary of a suitably choosen ball and then we join the new function with v (which reduces the energy in B_s). To this aim, we remark that for a.e. $\rho \in (0, r)$ we have

$$\mathcal{H}^{n-1}(S_{v_h} \cap \partial B_\rho) = \mathcal{H}^{n-1}(S_{\nabla v_h} \cap \partial B_\rho) = 0,$$

and, by (4.22),

$$P(A_h, B_r) \leq c\, L_h.$$

Integrating in polar coordinates and by the isoperimetric inequality we get

$$\alpha_h \int_0^r \mathcal{H}^{n-1}\left(A_h \cap \partial B_\rho\right) d\rho \;=\; \alpha_h |A_h| \;\leq\; \alpha_h \left(\gamma_n\, P(A_h, B_r)\right)^{n'} \;\leq\; c \alpha_h\, L_h^{n'}.$$

Since by (iii) the sequence $\left(\alpha_h L_h\right)$ is bounded, then by (ii)

$$\lim_h \alpha_h L_h{}^{n'} = 0.$$

Hence, for a.e. $\rho \in (0, r)$, we have

$$\lim_h \alpha_h \mathcal{H}^{n-1}(A_h \cap \partial B_\rho) = 0. \tag{5.4}$$

Now, by the continuity of ω at s and by (5.4), we choose $t \in (s, r)$ such that

$$\begin{cases} \omega(t) - \omega(s) < \dfrac{\varepsilon}{3}, \\ \lim_h \alpha_h \mathcal{H}^{n-1}(A_h \cap \partial B_t) = 0, \\ \displaystyle\int_{B_t \setminus B_s} \left(|D^2 v|^p + |\xi|^q\right) dy \;<\; \dfrac{\varepsilon}{3}, \end{cases} \tag{5.5}$$

and we glue together v_h and $z_h + \mathcal{P} v_h$ by defining

$$u_h = (z_h + \mathcal{P} v_h)\, \chi_{B_t} + v_h\, \chi_{B_r \setminus B_t}. \tag{5.6}$$

By Lemma 3.6, $u_h \in GSBV^2(B_r)$ and

$$\begin{aligned} \mathcal{F}_{\varphi_h}(u_h, \bar{B}_r) \leq{} & \mathcal{F}_{\varphi_h}(z_h + \mathcal{P} v_h, B_t) \\ & + \mathcal{F}_{\varphi_h}(v_h, B_r \setminus \overline{B}_t) + (\alpha_h + \beta_h)\mathcal{H}^{n-1}(A_h \cap \partial B_t). \end{aligned} \tag{5.7}$$

We join u_h and $v+\mathcal{P}v_h$. Let ψ_h be the cut-off functions between B_s and B_t provided by Lemma 3.7 and set

$$U_h = \psi_h\,(v+\mathcal{P}v_h) + (1-\psi_h)\,u_h.$$

Then U_h belongs to $GSBV^2(B_r)$, $U_h = v_h$ in $B_r \setminus B_t$ and we have, by the minimality of v_h in $\overline{B}_t$, by $v = z$ on $B_r \setminus B_s$ and by (5.7),

$$\begin{aligned}
\mathcal{F}_{\varphi_h}(v_h, \overline{B}_t) &\le \mathcal{F}_{\varphi_h}(U_h, \overline{B}_t) \le (1+\delta)\big(\mathcal{F}_{\varphi_h}(v+\mathcal{P}v_h, \overline{B}_t) + \mathcal{F}_{\varphi_h}(u_h, \overline{B}_t \setminus B_s)\big) \\
&\quad + \frac{c}{(t-s)^p}\left(\int_{B_t\setminus B_s} |D(v+\mathcal{P}v_h) - \nabla u_h|^p dy\right. \\
&\quad \left. + \frac{c}{\delta s^p (t-s)^p}\int_{B_t\setminus B_s} |v+\mathcal{P}v_h - u_h|^p dy\right) \\
&\le (1+\delta)\big(\mathcal{F}_{\varphi_h}(v+\mathcal{P}v_h, \overline{B}_t) + \mathcal{F}_{\varphi_h}(z_h+\mathcal{P}v_h, \overline{B}_t\setminus B_s) + 2\alpha_h \mathcal{H}^{n-1}(A_h \cap \partial B_t)\big) \\
&\quad + \frac{c}{(t-s)^p}\left(\int_{B_t\setminus B_s} |Dz - \nabla z_h|^p dy + \frac{c}{\delta s^p (t-s)^p}\int_{B_t\setminus B_s} |z - z_h|^p dy\right).
\end{aligned}$$

By taking into account (4.22)-(4.27), (4.31), (4.38), $v \in W^{2,p}(B_r)$ and assumptions (i), (iii) and (v), we take the limit as $h \to +\infty$ and we have

$$\omega(t) \le (1+\delta)\left(\int_{B_t} (|D^2 v|^p + |\xi|^q)\,dy + \omega(t) - \omega(s)\right).$$

Then by the arbitrariness of δ and by (5.3) and (5.5) we get

$$\omega(t) \le \int_{B_s} (|D^2 v|^p + |\xi|^q)\,dy + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \le \omega(s) - \frac{\varepsilon}{3},$$

contradicting the monotonicity of ω. □

We recall the following well-known result for the biharmonic operator. For the proof we refer to [G], Ch. III, sect. 2.

THEOREM 5.2 (Estimate on the solutions of the biharmonic equation). *Let $B_r \subset \mathbb{R}^n$ and let $u \in W^{2,2}(B_r)$ be a solution of*

$$\Delta^2 u = 0 \quad \text{on } B_r.$$

Then the following inequality holds:

$$\int_{B_\rho} |D^2 u|^2 dy \le c_n \left(\frac{\rho}{r}\right)^n \int_{B_r} |D^2 u|^2 dy \qquad \forall\, \rho < r,$$

where c_n is an absolute constant depending only on the dimension.

REMARK 5.3. For $q \geq 1$ there exists a constant c depending only on n and q such that for every affine function ξ and every $0 < \rho < r$ we have

$$\int_{B_\rho} |\xi|^q dy \leq c \left(\frac{\rho}{r}\right)^n \int_{B_r} |\xi|^q dy \ ,$$

hence there exists a constant $c_{n,q}$ such that for every biharmonic function u and for every affine function ξ we have

$$(5.8) \qquad \int_{B_\rho} \left(|D^2u|^2 + |\xi|^q\right) dy \leq c_{n,q} \left(\frac{\rho}{r}\right)^n \int_{B_r} \left(|D^2u|^2 + |\xi|^q\right) dy \quad \forall \rho < r.$$

THEOREM 5.4 (Decay). *Assume* (1.2) *with $n = p = 2$. For every $k > 2$, $\eta, \sigma \in (0,1)$ with $\eta^\sigma < \frac{1}{c_{2,q}}$, there is $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ and $\overline{B}_\rho(x) \subset \Omega$, if $u \in GSBV^2(\Omega)$ is a local minimizer of $\mathcal{F}_g(\cdot, \mu, \alpha, \beta, \Omega)$ with*

$$\rho \leq \varepsilon^k, \qquad \int_{B_\rho(x)} |g|^{2q} dy \leq \varepsilon^k$$

and

$$\alpha\mathcal{H}^1(S_u \cap B_\rho(x)) + \beta\mathcal{H}^1\left((S_{\nabla u} \setminus S_u) \cap B_\rho(x)\right) \leq \varepsilon\rho,$$

then

$$\mathcal{F}_g\left(u, \mu, \alpha, \beta, B_{\eta\rho}(x)\right) \leq \eta^{2-\sigma}\mathcal{F}_g(u, \mu, \alpha, \beta, B_\rho(x)).$$

PROOF. Assume that the theorem is false. In such case, there exist $k > 2$, $\eta, \sigma \in (0,1)$ with $\eta^\sigma < \frac{1}{c_{2,q}}$, a sequence of $\varepsilon_h > 0$ with $\lim_h \varepsilon_h = 0$ and $B_{\rho_h}(x_h) \subset \Omega$, $g_h \in L^{2q}_{\mathrm{loc}}(\Omega)$, u_h local minimizers of $\mathcal{F}_{g_h}(\cdot, \mu, \alpha, \beta, \Omega)$ such that

$$\rho_h \leq {\varepsilon_h}^k, \qquad \int_{B_{\rho_h}(x_h)} |g_h|^{2q} dy \leq {\varepsilon_h}^k,$$

$$\alpha\mathcal{H}^1\left(S_{u_h} \cap B_{\rho_h}(x_h)\right) + \beta\mathcal{H}^1\left((S_{\nabla u_h} \setminus S_{u_h}) \cap B_{\rho_h}(x_h)\right) \leq \varepsilon_h\rho_h$$

and

$$\mathcal{F}_{g_h}\left(u_h, \mu, \alpha, \beta, B_{\eta\rho_h}(x_h)\right) > \eta^{2-\sigma}\mathcal{F}_{g_h}\left(u_h, \mu, \alpha, \beta, B_{\rho_h}(x_h)\right).$$

By translating into the origin and rescaling we set

$$v_h(x) = \left(\lambda_h {\rho_h}^3\right)^{-\frac{1}{2}} u_h(x_h + \rho_h x) \ ,$$

$$\varphi_h(x) = \left(\lambda_h {\rho_h}^3\right)^{-\frac{1}{2}} g_h(x_h + \rho_h x)$$

where

$$\lambda_h = \left(\rho_h^{-1}\mathcal{F}_{g_h}(u_h, \mu, \alpha, \beta, B_{\rho_h}(x_h))\right) \vee \varepsilon_h.$$

The functions v_h are local minimizers in B_1 for $\mathcal{F}_{\varphi_h}(\cdot, \mu_h, \alpha_h, \beta_h, B_1)$ where

$$\alpha_h = \frac{\alpha}{\lambda_h}, \quad \beta_h = \frac{\beta}{\lambda_h}, \quad \mu_h = \mu \lambda_h{}^{\frac{q}{2}-1} \rho_h{}^{1+\frac{3}{2}q}.$$

By Lemma 3.4 we get

$$\mathcal{F}_{g_h}\big(u_h, \mu, \alpha, \beta, B_{\rho_h}(x_h)\big) = \lambda_h \rho_h \mathcal{F}_{\varphi_h}(v_h, \mu_h, \alpha_h, \beta_h, B_1)$$

hence

$$\mathcal{F}_{\varphi_h}(v_h, \mu_h, \alpha_h, \beta_h, B_1) \leq 1, \tag{5.9}$$

$$\alpha_h \mathcal{H}^1(S_{v_h} \cap B_1) + \beta_h \mathcal{H}^1\big((S_{\nabla v_h} \setminus S_{v_h})\big) \cap B_1) \leq \varepsilon_h, \tag{5.10}$$

$$\mathcal{F}_{\varphi_h}(v_h, \mu_h, \alpha_h, \beta_h, B_\eta) > \eta^{2-\sigma} \mathcal{F}_{\varphi_h}(v_h, \mu_h, \alpha_h, \beta_h, B_1). \tag{5.11}$$

By (5.9) and by Theorem 4.3, up to a subsequence that we do not relabel, there exists $z \in W^{2,2}(B_1)$ such that

$$\lim_h (v_h - \mathcal{P}_{0,1} v_h) = z \quad \text{a.e. on } B_1.$$

Now we verify that the functions v_h and z satisfy all the remaining assumptions of Theorem 5.1. Assumption (i) follows by Lemma 3.4 and (ii) follows by (5.10). We prove (iii): let (t_j) be a sequence of radii dense in $(0, 1)$. By (5.9), by extracting for any t_j a subsequence of v_h and by diagonalizing, there exists for any t_j

$$\lim_h \mathcal{F}_{\varphi_h}(v_h, \mu_h, \alpha_h, \beta_h, B_{t_j}) = \omega(t_j).$$

Since $t_j \to \omega(t_j)$ is monotone nondecreasing, hence continuous up to a countable set, then the above limit exists for almost every radius $\rho \in (0, 1)$. To verify (v) of Theorem 5.1, we prove first that

$$\lim_h \mu_h = 0.$$

By Lemma 3.1 (λ_h) is bounded, so our claim is obvious if $q \geq 2$, and in the case $q < 2$ it follows by the assumption $\rho_h \leq \varepsilon_h{}^k$. Moreover, by changing variables and using Hölder inequality, we get

$$\begin{aligned}
\mu_h \int_{B_1} |\varphi_h|^q dy &= \frac{\mu_h}{\rho_h^2} \frac{1}{\left(\lambda_h \rho_h^3\right)^{\frac{q}{2}}} \int_{B_{\rho_h}(x_h)} |g_h|^q dy = \mu \rho_h{}^{-1} \lambda_h{}^{-1} \int_{B_{\rho_h}(x_h)} |g_h|^q dy \\
&\leq \mu \rho_h{}^{-1} \varepsilon_h{}^{-1} \left(\int_{B_{\rho_h}(x_h)} |g_h|^{2q} dy \right)^{\frac{1}{2}} \sqrt{\pi} \rho_h \leq \mu \sqrt{\pi} \varepsilon_h{}^{\frac{k}{2}-1}
\end{aligned}$$

hence

$$\lim_h \mu_h \int_{B_1} |\varphi_h|^q dy = 0. \tag{5.12}$$

Finally we prove that $\mu_h |\mathcal{P}_{0,1} v_h|^q$ is bounded in B_1. Since $\mathcal{P}_{0,1} v_h$ are affine functions, it will be enough proving that $\mu_h \int_{B_1} |\mathcal{P}_{0,1} v_h|^q dy$ is bounded, and the convergence follows possibly extracting a subsequence. In fact, referring to the functions z_h introduced in Theorem 4.3 and setting $A_h = \{v_h \neq z_h + \mathcal{P}_{0,1} v_h\}$, for every h big enough we have

$$\begin{aligned}
\mu_h \int_{B_1} |\mathcal{P}_{0,1} v_h|^q dy &\leq c\mu_h \int_{B_1 \setminus A_h} |\mathcal{P}_{0,1} v_h|^q dy \\
&\leq c\mu_h \int_{B_1 \setminus A_h} \left(|z_h + \mathcal{P}_{0,1} v_h|^q + |z_h|^q\right) dy \\
&\leq c\mu_h \int_{B_1} |v_h|^q dy + c \\
&\leq c\mu_h \int_{B_1} |\varphi_h|^q dy + c\mu_h \int_{B_1} |v_h - \varphi_h|^q dy + c \leq c \ ,
\end{aligned}$$

due to (5.9) and (5.12). By Theorem 5.1 we have

$$\begin{aligned}
\int_{B_\eta} (|D^2 z|^2 + |\xi|^q)\, dy &= \lim_h \mathcal{F}_{\varphi_h}(v_h, \mu_h, \alpha_h, \beta_h, B_\eta) \\
&\geq \eta^{2-\sigma} \lim_h \mathcal{F}_{\varphi_h}(v_h, \mu_h, \alpha_h, \beta_h, B_1) \\
&= \eta^{2-\sigma} \int_{B_1} (|D^2 z|^2 + |\xi|^q)\, dy.
\end{aligned}$$

On the other hand, by (5.8),

$$\int_{B_\eta} (|D^2 z|^2 + |\xi|^q)\, dy \leq c_{2,q} \eta^2 \int_{B_1} (|D^2 z|^2 + |\xi|^q)\, dy,$$

hence $\eta^{2-\sigma} \leq c_{2,q} \eta^2$, which contradicts the assumption on η. □

6. – Proof of the main results

In this concluding section we prove Theorems 1.1 and 1.2.

Let $v \in GSBV^2(\Omega)$ be a minimizer of $\mathcal{F}$ given by Theorem 2.9. We remark first that if B is an open ball such that $B \subset \Omega$ and $\mathcal{H}^1(B \cap (S_v \cup S_{\nabla v})) = 0$ then v is smooth in B, since we have $v \in W^{2,2}(B)$ so that by the standard regularity theory we obtain $\tilde{v} \in C^2(B)$ (see e.g. [B] for the case $q > 1$ and use an approximation argument for $q = 1$). Hence we have $\tilde{v} \in C^2(\Omega \setminus \overline{(S_v \cup S_{\nabla v})})$.

We now estimate $\mathcal{H}^1(\Omega \cap (\overline{S_v \cup S_{\nabla v}} \setminus (S_v \cup S_{\nabla v})))$. Set

$$\Omega_0 = \left\{ x \in \Omega : \lim_{\rho \to 0} \rho^{-1} \mathcal{F}_g\big(v, \mu, \alpha, \beta, B_\rho(x)\big) = 0 \right\}.$$

We prove that Ω_0 is open. In order to apply Theorem 5.4, we fix $k > 2$ and $\eta, \sigma \in (0, 1)$ with $\eta^\sigma < \frac{1}{c_{2,q}}$; let ε_0 be as in Theorem 5.4, let $x \in \Omega_0$ and let $r > 0$ be such that

$$r \leq {\varepsilon_0}^k, \qquad \int_{B_r(x)} |g|^{2q}\, dy \leq {\varepsilon_0}^k,$$

$$\mathcal{F}_g\big(v, \mu, \alpha, \beta, B_r(x)\big) \leq \varepsilon_0 \eta r.$$

We show that $B_{(1-\eta)r}(x) \subset \Omega_0$. If $z \in B_{(1-\eta)r}(x)$ then

$$\mathcal{F}_g\big(v, \mu, \alpha, \beta, B_{\eta r}(z)\big) \leq \mathcal{F}_g\big(v, \mu, \alpha, \beta, B_r(x)\big) \leq \varepsilon_0 \eta r,$$

hence we may apply Theorem 5.4 inductively to obtain

$$\mathcal{F}_g\big(v, \mu, \alpha, \beta, B_{\eta^{h+1} r}(z)\big) \leq \eta^{h(2-\sigma)} \varepsilon_0 \eta r = \eta^{h(1-\sigma)} \varepsilon_0 (\eta^{h+1} r)$$

for every $h \in \mathbb{N}$. For every $0 < t < \eta r$ there exists j such that $\eta^{j+1} r < t < \eta^j r$ so that

$$\begin{aligned} t^{-1} \mathcal{F}_g\big(v, \mu, \alpha, \beta, B_t(z)\big) &\leq t^{-1} \mathcal{F}_g\big(v, \mu, \alpha, \beta, B_{\eta^j r}(z)\big) \\ &\leq t^{-1} \eta^{(j-1)(1-\sigma)} \varepsilon_0 (\eta^j r) \\ &\leq \eta^{(j-1)(1-\sigma)-1} \varepsilon_0, \end{aligned}$$

then passing to the limit as $t \to 0$ (so that $j \to +\infty$) we get $z \in \Omega_0$. Since $S_v \cup S_{\nabla v}$ is countably $(\mathcal{H}^1, 1)$ rectifiable, by Theorem 3.2.19 in [F] we have $\mathcal{H}^1((S_v \cup S_{\nabla v}) \cap \Omega_0) = 0$ so that $\tilde{v} \in C^2(\Omega_0)$ and $(S_v \cup S_{\nabla v}) \cap \Omega_0 = \emptyset$. Since Ω_0 is open then $\Omega_0 \cap \overline{S_v \cup S_{\nabla v}} = \emptyset$. By Lemma 3.8 we have

$$\mathcal{H}^1\big(\Omega \cap (\overline{S_v \cup S_{\nabla v}} \setminus (S_v \cup S_{\nabla v}))\big) = 0 \ . \tag{6.1}$$

Setting

$$K_0 = \overline{S}_v \setminus (S_{\nabla v} \setminus S_v) \ , \quad K_1 = \overline{S}_{\nabla v} \setminus S_v \ ,$$

we have that $K_0 \cup K_1$ is closed, $\mathcal{H}^1(K_0 \cap \Omega) = \mathcal{H}^1(S_v)$ and $\mathcal{H}^1(K_1 \cap \Omega) = \mathcal{H}^1(S_{\nabla v} \setminus S_v)$, hence $K_0 \cap \Omega, K_1 \cap \Omega$ are $(\mathcal{H}^1, 1)$ rectifiable, moreover

$$F(K_0, K_1, \tilde{v}) = \mathcal{F}(v) = \min \left\{ \mathcal{F}(u) : \ u \in GSBV^2(\Omega) \right\} \ ,$$

and, by Remark 3.3, we conclude that $(K_0, K_1, \tilde{v})$ is a minimizer for F in the class of the admissible triplets. □

Theorem 1.2 follows immediately by Theorem 1.1 choosing $K = K_0 \cup K_1$ and taking into account the assumption $\alpha = \beta$.

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