ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

FAUSTO ACANFORA
STEFANO MORTOLA

Non semicontinuous quadratic integral functionals with continuous coefficients

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 25, nº 1-2 (1997), p. 1-9
<http://www.numdam.org/item?id=ASNSP_1997_4_25_1-2_1_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXV (1997), pp. 1-9

Non Semicontinuous Quadratic Integral Functionals with Continuous Coefficients

FAUSTO ACANFORA - STEFANO MORTOLA

1. - The counterexample

The main aim of this note is to construct some quadratic forms of the type

$$F(u) = \int_\Omega \sum_{i,j=1}^{n} a_{i,j}(x) D_i u(x) D_j u(x) dx \tag{1}$$

with the coefficients uniformly continuous in an open bounded set Ω of $\mathbb{R}^n$, such that the matrix $(a_{i,j})$ is positive semidefinite, this means:

$$\sum_{i,j=1}^{n} a_{i,j}(x)\lambda_i\lambda_j \geq 0$$

for every $x \in \Omega$, for every $\lambda \in \mathbb{R}^n$, in such a way that $F(u)$ is not lower semicontinuous with respect of the topology of the uniform convergence.

The example we will show has been explicitily suggested by De Giorgi, through some conjectures that will be proved in this note.

We look for a quadratic form of the type

$$\int_\Omega (au_x + bu_y)^2 dxdy$$

where a and b are functions defined in an open bounded set of $\mathbb{R}^2$, uniformly continuous and we will suppose that a is strictly positive.

The example will be a consequencce of the following two results:

THEOREM. *Given the square $\Omega = (-1, 1)^2$ in $\mathbb{R}^2$ and $f \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $f = (a, b)$ where $a(x, y) > 0$ for every $(x, y) \in \bar{\Omega}$, for every $u \in C^1(\mathbb{R}^2)$ there exists $f_n \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\lim_{n\to\infty} f_n = f$ uniformly in Ω and there exists $u_n \in C^1(\mathbb{R}^2)$ with $\lim_{n\to\infty} u_n = u$ uniformly in Ω such that*

$$\lim_{n\to\infty} \int_\Omega (f_n \cdot Du_n)^2 dxdy = 0\,.$$

The meaning of the theorem is that the global relaxed functional of $F(u) = \int_\Omega (f \cdot Du)^2 dxdy$ is identically 0, where by global relaxed we mean the biggest functional non greater than F, lower semicontinuous with respect to both f and u.

LEMMA. *Let Y be a metric space, let X be a complete metric space and let $B : X \times Y \to \mathbb{R}^+ = [0, +\infty)$ be a function continuous with respect to the first variable and such that for every $(x, y) \in D \times Y$, with D dense in X, we have $B(x_n, y_n) \to 0$ for a suitable (x_n, y_n) sequence that converge to (x, y).*

Then, for every $(x, \bar{y}) \in D \times Y$, for every $\epsilon > 0$ there exists $z \in X$ such that $d(x, z) < \varepsilon$ and $B(z, y_n) \to 0$ for a suitable y_n that converges to $\bar{y}$.

PROOF OF THE LEMMA. Let us denote by d_X the metric in X and d_Y the metric in Y. We know that for every $\varepsilon > 0$ there exists $x_1 \in D$, $y_1 \in Y$ such that

$$d_X(x_1, x) < \frac{\varepsilon}{2}, \ d_Y(y_1, \bar{y}) < \frac{1}{2}, \ B(x_1, y_1) < 1/2,$$

for our hypothesis on B.

There exists a point $x_2 \in D$, $y_2 \in Y$ such that

$$d_X(x_2, x_1) < \varepsilon/2^2, \ d_Y(y_2, \bar{y}) < 1/2^2, \ B(x_2, y_2) < 1/2^2, \ B(x_2, y_1) < 1/2$$

where the last inequality turns out from the continuity of B with respect to x and thanks to $B(x_1, y_1) < \frac{1}{2}$.

Proceeding in the same way we can find two sequences $(x_n, y_n) \in D \times Y$ such that:

$$d_X(x_n, x_{n-1}) < \varepsilon/2^n, \ d_Y(y_n, \bar{y}) < 1/2^n, \ B(x_n, y_h) < 1/2^h,$$

for $h = 1, 2, \ldots, n$.

This implies that x_n converges to a point $z \in X$ such that $d_X(x, z) < \varepsilon$, due to $d_X(x_1, x) + \sum_{h=1}^{\infty} d_X(x_h, x_{h+1}) < \varepsilon$ and the completeness of X.

Moreover y_n converges to $\bar{y}$ and, for the continuity of B, we have $B(z, y_h) \leq \frac{1}{2^h}$ for every h. This gives our thesis.

As a consequence of the two previous results, we have the existence of quadratic functionals of the type $F(u) = \int_\Omega (f \cdot Du)^2 dx$ with $f = (a, b)$ continuous, such that F is not lower semicontinuous with respect to the uniform convergence in the variable u. In fact, we take $B(f, u) = F_f(u)$, with $f \in C(\bar{\Omega}, \mathbb{R}^2)$, $u \in C^1(\bar{\Omega})$ and $D = C^\infty(\bar{\Omega}, \mathbb{R}^2)$: the theorem tells us that we may apply the lemma and, starting from a pair $(f, u) \in D \times C(\bar{\Omega})$ such that $F_f(u) \neq 0$, we consider $\varepsilon > 0$ so small that $F_g(u) > 0$ for every $g \in C(\Omega, \mathbb{R}^2)$ with $\|f - g\|_\infty < \varepsilon$. The lemma allows us to construct our counterexample.

PROOF OF THE THEOREM. Let us fix $a, b \in C^\infty(\mathbb{R}^2)$, $u \in C^1(\mathbb{R}^2)$ with compact support in Ω and let us assume that $\Omega = (-1, 1)^2 \subset \mathbb{R}^2$, and that the function a be strictly positive in $\bar{\Omega}$.

Now, let us consider the sequence of functions u_h solutions of the following system:

$$\text{(2)} \qquad \begin{cases} av_x + bv_y = 0 \\ v(0,y) = \frac{1}{h^\alpha}\sin(hy) \end{cases}$$

and let $\phi_h(x,y) = -(au_x + bu_y)\frac{Du_h}{h^\alpha + |Du_h|^2}$, where $\alpha > 0$ (α will be chosen later).

Let us verify these three facts:

1) $\phi_h \to 0$ uniformly

2) $u_h \to 0$ uniformly

3) $\int_\Omega \big((f + \phi_h)\cdot D(u + u_h)\big)^2 dxdy \to 0$

The fact 2) is immediate: it is enough to observe that $|u_h(x,y)| \le \frac{1}{h^\alpha}$, for the maximum principle, consequence of the method of characteristics, as we will see later.

For the point 1), let us observe that the function $\frac{y}{h+y^2}$ has its maximum value equal to $\frac{1}{2h^{1/2}}$ and so

$$\frac{|Du_h|}{h^\alpha + |Du_h|^2} \le \frac{1}{2h^{\alpha/2}}.$$

This implies that the sequence ϕ_h tends to 0 uniformly.

Concerning the point 3), let us develop the function

$$\big((f + \phi_h)\cdot D(u + u_h)\big)^2 = \big((f\cdot Du) + (f\cdot Du_h) + (\phi_h\cdot Du) + (\phi_h\cdot Du_h)\big)^2.$$

Let us observe that the second term is always 0 by construction, the third one tends to 0 for the point 1), and so we have only to check that $(\phi_h\cdot Du_h) \to -(f\cdot Du)$ in $L^2(\Omega)$.

Let us consider the following Cauchy problem:

$$\begin{cases} y' = \dfrac{b(x,y)}{a(x,y)} \\ y(0) = y_0 \end{cases}$$

and let us denote by $T_x(y_0)$ the solution at the point x.

The function $y_0 \to T_x(y_0)$ is a diffeomorfism and let us denote by T_x^{-1} its inverse.

Let us observe that, if $y = T_x(y_0)$:

$$\frac{d}{dx}u_h(x, T_x(y_0)) = (u_h)_x + (u_h)_y\frac{b(x,y)}{a(x,y)} = 0$$

because u_h are solutions of (2).

This means that $u_h(x, T_x(y_0))$ is constant with respect to x and so its value is $u_h(0, y_0) = \frac{1}{h^\alpha}\sin(hy_0)$.

From this it turns out that

$$u_h(x,y) = u_h(0, T_x^{-1}(y)) = \frac{1}{h^\alpha}\sin(hT_x^{-1}(y)).$$

In particular we have that $u_h \to 0$ uniformly in Ω, as previously stated. Moreover we have:

$$\int_\Omega [(f \cdot Du) + (\phi_h \cdot D(u_h)]^2 dxdy = \int_\Omega [(f \cdot Du) - (f \cdot Du)\frac{|Du_h|^2}{h^\alpha + |Du_h|^2}]^2 dxdy$$
$$= \int_\Omega [(f \cdot Du)\frac{h^\alpha}{h^\alpha + |Du_h|^2}]^2 dxdy.$$

Let us study the behaviour of the sequence of functions

$$\frac{h^\alpha}{h^\alpha + |Du_h|^2} = \frac{1}{1 + \frac{1}{h^\alpha}(h^{2-2\alpha}\cos^2(hT_x^{-1}(y))|DT_x^{(-1)}(y)|^2}$$

where $DT_x^{-1}(y)$ denotes the gradient of the function $(x,y) \to T_x^{-1}(y)$, with respect to the two variables x, y.

To prove that this sequence tends to 0 in $L^2(\Omega)$, we may break Ω in two regions, $\Omega_{1,\varepsilon,h}$ and $\Omega_{2,\varepsilon,h}$: in $\Omega_{1,\varepsilon,h}$ the function $\cos^2(hT_x^{-1}(y))$ is bigger than a fixed $\varepsilon > 0$ and so it is small if the exponent of h (i.e. $2 - 3\alpha$) is positive, this happens when $\alpha < 2/3$. For the integral corresponding to $\Omega_{2,\varepsilon,h}$, where $\cos^2(hT_x^{-1}) < \varepsilon$, let us observe that its measure is small when ε is small, uniformly with respect to h, because T_x is a diffeomorfism and the sequence is equibounded (due to the fact that $|DT_x^{-1}(y)|$ is bigger than 0), this implies that the integral is small when h is large.

REMARK. Even when the functional $F(u) = \int_\Omega (f \cdot Du)^2$ is not lower semicontinuous in the uniform topology, it is possible always to prove that the set of functions where the functional assume its minimum value, this means the set of the solutions of the first order linear equation:

$$(f(x) \cdot Du) = 0, \tag{3}$$

is closed in $C^1(\bar{\Omega})$ with respect to the uniform topology.

Let us write the simple proof of this fact, due to P. Majer.

If a sequence of functions u_n solutions of (3) converges uniformly to a function $u \in C^1$ then, if we fix a point x_0 in Ω and we fix a solution $x(t)$ of the Cauchy problem $x' = f(x), x(0) = x_0$ we have that $u_n(x(t))$ is constant in the neighborood of 0 and this implies that also $u(x(t))$ is constant and so u is a solution of (3): this means that the space of solutions is closed with respect to the uniform topology. Actually this proof shows that the space of solutions is closed also with respect to pointwise convergence.

As F. Cipriani pointed out, this kind of proof allows to prove that also for the functionals

$$F(u) = \int_{\Omega} \sum_{i,j} a_{i,j} D_i u D_j u dx \tag{4}$$

the set of minimum points is closed when $a_{i,j}$ are continuous, because it is possible to consider the matrix $(b_{i,j})$ square root of $(a_{i,j})$, i.e. such that $a_{i,j} = \sum_{h=1}^{n} b_{i,h} b_{h,j}$ with $b_{i,h} = b_{h,i}$ continuous and $(b_{i,j})$ semidefinite positive. In this way we have:

$$\sum_{i,j} a_{i,j} D_i u D_j u = 0 \Longleftrightarrow \sum_{i,j} \sum_{h} b_{i,h} D_i u b_{j,h} D_j u = 0 \Longleftrightarrow \sum_{h=1}^{n} (b^h \cdot Du)^2 = 0$$

where b^h are the vector functions whose components are $b_{i,h}$.

So we have that u is a minimum point for (4) if and only if u is a solution of $(b^h \cdot Du) = 0$ for every $h = 1, 2, \dots, n$ and so the set of solutions is closed because it is the intersection of n closed sets.

2. – Cases of lower semicontinuity

In this section we examine the main cases in which one has lower semicontinuity. Our functionals $F(u) = \int_{\Omega} \sum a_{i,j} D_i u D_j u dx$ are considered for $u \in C^1(\bar{\Omega})$.

A) The coercive case

The most classical case is when $(a_{i,j})$ is positive definite, i.e. when there exists $c > 0$ such that

$$\sum_{i,j} a_{i,j}(x) \lambda_i \lambda_j \geq c|\lambda|^2$$

for every $x \in \bar{\Omega}$, for every $\lambda \in \mathbb{R}^n$.

In this case if $u_h \in C^1(\bar{\Omega})$ converges to $u \in C^1(\bar{\Omega})$ uniformly and it is such that $F(u_h)$ is bounded, it turns out that u_h is also bounded in $H^1(\Omega)$ and so it weakly converges to u in H^1. The functional F is continuous with respect to the strong topology of H^1 and so, being a convex functional, it is weakly lower semicontinuous.

Actually in this case the continuity of the coefficients $(a_{i,j})$ is not important: even in the case of measurable and bounded coefficients the semicontinuity holds.

B) The unidimensional case and the isotropic case.

Let us suppose that $(a_{i,j})$ is isotropic, i.e. of the form $a(x)I$ where I denotes the identity matrix and $a(x)$ is a continuous function in Ω, non negative.

In this case the functional is pointwise the least upper bound of the functionals

$$F_\varepsilon(u) = \int_{\Omega_\varepsilon} a(x)|Du|^2 dx$$

where $\Omega_\varepsilon = \{x \in \Omega \mid a(x) > \varepsilon\}$.

The functionals F_ε are lower semicontinuous in $L^\infty(\Omega_\varepsilon)$ because they are coercive and it is immediate to observe that they are lower semicontinuous also in $L^\infty(\Omega)$.

Let us observe that the unidimensional case $\int_\alpha^\beta a(x)u'(x)^2 dx$ is always isotropic and so the continuity of a is enough to prove the lower semicontinuity of the functional.

If a is not continuous but only in the class L^∞, we can have lack of lower semicontinuity, as one can verify even in the unidimensional case: let A be a dense open set in $(0, 1)$ with measure $\lambda < 1$ and let a the function with value 0 in A and 1 in $[0, 1] \setminus A$. In this case, taking the function $u(x) = x$, we have $F(u) = \int_{[0,1]\setminus A} u'(x)^2 dx = 1 - \lambda > 0$ but it is easy to approximate u uniformly with functions u_h such that $u_h' = 0$ in $[0, 1] \setminus A$: in this way we have $F(u_h) = 0$ even if $F(u) > 0$. (This example is considered by Carbone and Sbordone in [2]).

C) The diagonal case

A generalisation of the isotropic case is that of diagonal matrices:

$$F(u) = \sum_{i=1}^{n} \int_\Omega a_i(x) \left(\frac{\partial u}{\partial x_i}\right)^2 dx$$

where $a_i \in C(\bar{\Omega})$ and $a_i \geq 0$ for $i = 1, 2 \ldots, n$.

To prove the lower semicontinuity of F it is enough to consider the case of just a single term, and in particular:

$$F_1(u) = \int_\Omega a_1(x) \left(\frac{\partial u}{\partial x_1}\right)^2 dx$$

If $u_h \to u$ in $L^\infty(\Omega)$, for every $x = (x_1, x_2, \ldots, x_n) \in \Omega$ let us consider $\Omega(x_2, x_3, \ldots, x_n) = \{y \in \mathbb{R} \mid (y, x_2, x_3, \ldots, x_n) \in \Omega\} \subset \mathbb{R}$.

For the result of the unidimensional case we have

$$\int_{\Omega(x_2,x_3,\ldots,x_n)} a_1(y, x_2, \ldots, x_n)(D_1 u)^2 dy \leq \liminf_{h\to\infty} \int_{\Omega(x_2,x_3,\ldots,x_n)} a_1(D_1 u_h)^2 dy\,.$$

If we integrate with respect to $(x_2, \dots, x_n)$ and using Fatou lemma we find

$$\begin{aligned}\int_\Omega a_1(D_1u)^2 dx_1 \dots dx_n &= \int_{\mathbb{R}^{n-1}} dx_2 \dots dx_n \int_{\Omega(x_2,x_3,\dots,x_n)} a_1(D_1u)^2 dy \\ &\le \int_{\mathbb{R}^{n-1}} dx_2 \dots dx_n \liminf_{h\to\infty} \int_{\Omega(x_2,x_3,\dots,x_n)} a_1(D_1u)^2 dy \\ &\le \liminf_{h\to\infty} \int_\Omega a_1(D_1u)^2 dy\,.\end{aligned}$$

Another way to show the lower semicontinuity for the diagonal case is the following. We approximate F_1 with the family of functionals

$$F_\varepsilon(u) = \int_{\Omega_\varepsilon} a_1(D_1u)^2 dx$$

where $\Omega_\varepsilon = \{x \in \Omega \mid a_1(x) > \varepsilon\}$. For every $\varepsilon > 0$, F_ε are lower semicontinuous because in Ω_ε the function a_1 is the limit of an increasing sequence of C^1 positive functions: for the case of C^1 coefficients the lower semicontinuity will be proved in the following paragraph.

D) The regular case

When the coefficients $a_{i,j}$ are C^1 functions (or even local Lipshitzian) we can prove that the functionals are lower semicontinuous even with respect to the $L^1(\Omega)$ topology. In fact

$$F(u) = \sup_{\psi\in C_0^1, 0\le\psi\le 1} \int_\Omega \sum_{i,j} a_{i,j}(x) D_i u(x) D_j u(x) \psi(x) dx$$

so it is enough to prove L^1-lower semicontinuity for the functionals

$$G(u) = \int_\Omega \sum_{i,j} a_{i,j}(x) D_i u(x) D_j u(x) \psi(x) dx$$

where ψ is a C^1 function with compact support in Ω and with values in $[0, 1]$.

For these functionals we have:

$$\sqrt{G(u)} = \sup_{\{\phi\in C_0^1(\Omega) \mid \int_\Omega \sum_{i,j} a_{i,j} D_i\phi D_j\phi\psi dx \le 1\}} \int_\Omega \sum_{i,j} a_{i,j} D_i u D_j \phi \psi dx$$

Let us note that the functionals

$$u \to \int_\Omega \sum_{i,j} a_{i,j} D_i u D_j \phi \;\; \psi dx = -\int_\Omega \sum_{i,j} D_i(a_{i,j}\psi D_j\phi) u dx$$

are linear and continuous with respect to the L^1-topology and so also the functional G is L^1-lower semicontinuous.

3. – Some open questions

In this last section we formulate some natural questions whose solutions so far we ignore.

1) Is it possible to find an integral quadratic functional with the coefficients $a_{i,j}$ continuous and not identically 0 such that the lower semicontinuous envelope is the functional identically 0?

2) It is known that the lower semicontinuous envelope of an integral quadratic functional is still a functional of the same type and it is also known a formula for the coefficients of the relaxed functional, as proved in [5] and in [8]. It is not clear if, starting from continuous coefficients, the relaxed functional has still continuous coefficients.

3) The set of functions u that are critical points for our functional $F(u)$, namely the functions u such that for every regular function ϕ with compact support in Ω we have

$$\int_{\Omega} \sum_{i,j} a_{i,j}(x) D_i u(x) D_j \phi(x) dx = 0$$

is closed in $C^1(\Omega)$ with respect to the uniform topology?

4) The following problem has been suggested by De Giorgi: let $(a_{i,j})$ be a semidefinite positive matrix with regular coefficients and let $(b_{i,j})$ be another matrix with continuous coefficients such that

$$\sum_{i,j} a_{i,j}(x)\lambda_i\lambda_j \leq \sum_{i,j} b_{i,j}(x)\lambda_i\lambda_j \leq C \sum_{i,j} a_{i,j}(x)\lambda_i\lambda_j$$

for every $x \in \Omega$, for every $\lambda \in \mathbb{R}^n$, for some $C > 1$. Then the quadratic integral functional corresponding to the matrix $(b_{i,j})$ is lower semicontinuous with respect to the uniform convergence.

REFERENCES

[1] G. BUTTAZZO, "Semicontinuity, relaxation and integral representation in the calculus of variations", Longman Scientific & Technical, 1989.

[2] L. CARBONE – C. SBORDONE, *Some properties of Γ-limits of integrals functionals*, Ann. Mat. Pura Appl. **122** (1979), 1-60.

[3] G. DAL MASO, "An Introduction to Γ-convergence", Birkhäuser, 1993.

[4] G. DAL MASO, *Integral representation on $BV(\Omega)$ of Γ-limits of variational integrals*, Manuscripta Math. **30** (1980), 387-413.

[5] N. FUSCO – G. MOSCARIELLO, *L^2-lower semicontinuity of functionals of quadratic type*, Ann. Mat. Pura Appl. **129** (1981), 305-326.

[6] C. Y. Pauc, “La Méthode Métrique en Calcul des Variations”, Hermann, Paris, 1941.

[7] J. Serrin, *On the definition and properties of certain variational integrals*, Trans. Amer. Math. Soc. **101** (1961), 139-167.

Seconda Univ. degli studi di Napoli
Via Roma 29
Aversa
Caserta, Italy

Dipartimento di Matematica
Politecnico di Milano
P.za Leonardo da Vinci 32
20133 Milano
Italy
e-mail: mortola@mate.polimi.it