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A Blow-up Mechanism for a Chemotaxis Model

MIGUEL A. HERRERO – JUAN J. L. VELÁZQUEZ

Abstract

We consider the following nonlinear system of parabolic equations:

$$\begin{cases} u_t = \Delta u - \chi \nabla(u \nabla v) & \text{for} \quad x \in B_R \quad t > 0 \,, \\ \Gamma v_t = \Delta v + u - av & \text{for} \quad x \in B_R, \quad t > 0 \,. \end{cases} \tag{1}$$

Here Γ, χ and a are positive constants, and B_R is a ball of radius $R > 0$ in $\mathbb{R}^2$. At the boundary of B_R, we impose homogeneous Neumann conditions, namely:

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{for} \ | \, x \, |= R, \quad t > 0 \,. \tag{2}$$

Problem (1), (2) is a classical model to describe chemotaxis, i.e, the motion of organisms induced by high concentrations of a chemical that they secrete. In this paper we prove that there exist radial solutions of (1), (2) that develop a Dirac-delta type singularity in finite time, a feature known in the literature as chemotactic collapse. The asymptotics of such solutions near the formation of the singularity is described in detail, and particular attention is paid to the structure of the inner layer around the unfolding singularity.

1. – Introduction

This article deals with a system of partial differential equations modelling chemotaxis. This last term is commonly used to describe the motion of organisms which have a tendency to aggregate by moving towards higher concentrations of chemical substances that they themselves produce. A classical model in chemotaxis is the so-called Keller-Segel system, which reads as follows:

$$u_t = \Delta u - \chi \nabla(u \nabla v) \quad \text{for } x \in \Omega \,, \quad t > 0 \,, \tag{1.1}$$

$$\Gamma v_t = \Delta v + u - av \quad \text{for } x \in \Omega \,, \quad t > 0 \,, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{for } x \in \partial\Omega \,, \ t > 0 \,, \tag{1.3}$$

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together with initial conditions:

(1.4a) $$u(x,0) = u_0(x) \quad \text{for } x \in \Omega.$$

(1.4b) $$v(x,0) = v_0(x) \quad \text{for } x \in \Omega.$$

Here Ω denotes a bounded and smooth open set in $\mathbb{R}^N (N \geq 2)$; a, χ and Γ are some positive constants, and $u(x,t)$ (respectively $v(x,t)$) is a suitable rescaled variable corresponding to the concentration of the biological species under consideration (respectively that of the chemical released). See for instance [KS], [JL], [N], [HV1] and [FL] for details about this and other chemotaxis models.

Concerning the mathematical analysis of problem (1.1)-(1.4), a first set of results was obtained by Jäger and Luckhaus in [JL], where the particular case corresponding to setting $\Gamma = 0$ in (1.2) was discussed in two space dimensions. Under such assumptions, it was proved in [JL] that (1.1)-(1.4) has global (in time) radial solutions when the initial values have small enough mass. It was also shown there that there exist radial solutions that blow-up at the origin in a finite time T. By this we mean that $\lim_{t \uparrow T} u(0,T) = \infty$ for some $T < \infty$. Notice, however, that $\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx$ for all $t > 0$ for which solutions are defined, provided that this last quantity is finite (cf. (1.1)-(1.3)).

The analysis started in [JL] was then pursued by Nagai in reference [N]. Assuming only mild requirements on the initial values, it was proved in [N] that blow-up never occurs in the case of one space dimension ($N = 1$). On the other hand, there is finite-time blow-up when $N \geq 3$ and $u(r,t)$ is a radial solution satisfying some conditions at $t = 0$ (cf. Theorem 3.1 in [N]). The case $N = 2$ emerges then as a bordeline one, since the results in [N] show that there is no blow-up if $N = 2, \Omega$ is a ball, $u_0(r)$ is radially symmetric and:

$$\frac{1}{|\,\Omega\,|}\int_\Omega u_0(x)dx \quad < \quad \frac{8}{\chi},$$

where $|\,\Omega\,|$ denotes the volume of the set Ω. However, blow-up for radially symmetric solutions is shown to occur if $N = 2, \Omega$ is a ball and $u_0(r)$ is a suitable radially symmetric function such that:

(1.5) $$\frac{1}{|\,\Omega\,|}\int_\Omega u_0(x)dx \; > \; \frac{8}{\chi}.$$

After the discussion performed in [JL] and [N], a question that naturally arised was that of describing the manner of blow-up when this phenomenon actually occurs. Recently, it was proved in [HV1] that when $N = 2$ and Ω is a ball, there exist radial solutions of (1.1)-(1.4) with $\Gamma = 0$ such that $u(r,t)$ blows up in a finite time T, and it does so by concentrating into a multiple of the Dirac mass centered at the origin (cf. Theorem 1 in [HV1]). This type of blow-up is usually termed as chemotatic collapse, and is a mathematical representation of aggregation into a single spora.

When $\Gamma > 0$ in (1.2), much less seems to be known about the evolution in time of solutions of (1.1)-(1.4). At the technical level, serious complications with respect to the case $\Gamma = 0$ arise. For instance, while in this last situation an auxiliary mass function can be introduced that reduces the whole system to a single nonlinear parabolic equation, no such procedure seems to be available for the case of (1.1) and (1.2). It is possible, however, to gain further insight into system (1.1)-(1.4) by means of matched asymptotic expansions methods. By using such techniques, we recently showed in [HV2] that radial solutions of (1.1)-(1.4) exist in a ball $\Omega \subset \mathbb{R}^2$, that exhibit chemotactic collapse in a finite time T. The goal of this paper is to provide a rigorous proof of the results obtained in [HV2]. More precisely, we shall prove the following:

THEOREM 1.1. *Let $R > 0$, and let $\Omega_R = \{x \in \mathbb{R}^2 :| x |< R\}$. Then there exist radial solutions of* (1.1) − (1.3) *defined in an interval* $(0, T)$ *with* $T > 0$*, and such that:*

$$u(r,t) \to \frac{8\pi}{\chi}\delta(0) + \psi(r) \quad \text{as } t \uparrow T, \tag{1.6}$$

in the sense of measures, where $\delta(0)$ is Dirac measure centered at $r = 0$, and:

$$\psi(r) = \frac{C}{r^2} e^{-2|\log r|^{1/2}}(1 + o(1)) \tag{1.7}$$

as $r \to 0$, where C is a positive constant depending on χ and Γ. At $t = T$, the profile near $r = 0$ is given by:

$$u(r,T) = \frac{8\pi}{\chi}\delta(0) + \psi(r) \quad ; \psi(r) \text{ as in } (1.7). \tag{1.8}$$

Moreover, if we set $S(t) = (T - t)\left(\sup_\Omega u(r,t)\right) \equiv (T - t)u(0,t)$, one has that $\lim_{t\uparrow T} S(t) = \infty$. *More precisely, there holds:*

$$S(t) = C_1(T-t)^{-1} \exp\left(\sqrt{2|\log(T-t)|}\right) \tag{1.9}$$

as $t \uparrow T$, for some $C_1 > 0$.

It is possible to describe in further detail how chemotactic collapse develops. To this end, we consider the stationary system:

$$\Delta u - \chi \nabla(u \nabla v) = 0 \quad \text{in } \Omega, \tag{1.10a}$$

$$\Delta v + u = 0 \quad \text{in } \Omega. \tag{1.10b}$$

One readily sees that for any constant K, the functions:

$$\overline{u}(r) = \frac{8}{\chi(1+r^2)^2} \quad , \quad \overline{v}_k(r) = -\frac{2\log(1+r^2)}{\chi} + K \tag{1.11}$$

are radial solutions of (1.10) in Ω_R. We then have:

THEOREM 1.2. *Under the assumptions of Theorem* 1.1, *there holds:*

$$(1.12)\quad u(r,t) = \frac{8}{\chi}\cdot\frac{1}{R(t)^2}\overline{u}\left(\frac{x}{R(t)}\right)(1+\mathrm{o}(1)) + \mathrm{O}\left(\frac{1}{r^2}\mathrm{e}^{-\sqrt{2}|\log(T-t)|^{1/2}}\widehat{\chi}_{R(t)}\right),$$

as $t \uparrow T$, *uniformly on sets* $|\, x \,| \leq R(t)$, *where* $\widehat{\chi}_{R(t)} = 1$ *when* $|\, x \,| > R(t)$ *and* $\widehat{\chi}_{R(t)} = 0$ *otherwise, and:*

$$(1.13)\quad R(t) = K(T-t)^{1/2}\mathrm{e}^{-\frac{\sqrt{2}}{2}|\log(T-t)|^{1/2}}\cdot(1+\mathrm{o}(1))$$

as $t \uparrow T$, *where* K *is a positive constant depending on* χ *and* Γ.

We conclude this Introduction by describing the plan of the article. Section 2 below will be devoted to introducing some notation, as well as to recalling briefly the way in which our results can be arrived at in a heuristic way. Section 3 contains a topological argument which is a crucial ingredient in our approach. We shall state a crucial technical result there (Proposition 3.1), and show how to derive Theorems 1.1 and 1.2 from it. Its implementation requires of a number of estimates which are provided in the subsequent sections. A first set of such results is given in Section 4, where the oscillation of the rescaled inner layer is estimated, and the first Fourier coefficients in the corresponding expansion for solutions of (2.23) are given. Section 5 is devoted to describing how our solutions stabilize towards their asymptotic profile in the inner layer. The analysis of the asymptotics outside such interior region is then made in Section 6. Finally, Section 7 contains a number of technical results that were required in the previous steps, and whose proofs were initially postponed to keep the flow of the arguments as smooth as possible.

2. – Preliminaries. A description of the mechanism of formation of singularities

In this section we shall introduce some relevant notation, and will briefly describe the manner in which chemotactic collapse develops. To this end, we shall borrow from the approach in [HV2] and refer to that paper for details. To start with, we define a local mass function $M(r,t)$ given by:

$$(2.1)\quad M(r,t) = \int_{|x|\leq r}(u(x,t)-1)dx\,.$$

Differentiation of (2.1) yields:

$$(2.2)\quad u(r,t) - 1 = \frac{1}{2\pi r}\cdot\frac{\partial M}{\partial r},$$

so that M satisfies:

$$M_t = M_{rr} - \frac{M_r}{r} - \chi v_r M_r - 2\pi \chi r v_r \quad ; \; 0 < r < R \; , \; t > 0 \, . \tag{2.3}$$

We now introduce self-similar variables corresponding to the natural scales of the problem:

$$M(r,t) = \Phi(y,\tau) \qquad v(r,t) = V(y,\tau) \, , \tag{2.4a}$$

where

$$y = r(T-t)^{-1/2} \qquad \tau = -\log(T-t) \, . \tag{2.4b}$$

Functions Φ and V satisfy the system of equations:

$$\Phi_\tau = \Phi_{yy} - \frac{\Phi_y}{y} - \frac{y\Phi_y}{2} - \chi V_y \Phi_y - 2\pi \chi e^{-\tau} y V_y \, , \tag{2.5a}$$

$$\Gamma V_\tau = V_{yy} + \frac{V_y}{y} - \Gamma_y \frac{V_y}{2} + \frac{\Phi_y}{2\pi y} \, . \tag{2.5b}$$

We now make the following guess. Suppose that, as $\tau \to \infty$, $\Phi(y,\tau)$ behaves in (2.5) as in the case $\Gamma = 0$ analysed in [HV1]. The reason for such assumption is that we expect all the time derivatives to be small for $\tau \gg 1$ (cf. (1.12) and the remarks made at the Introduction). In [HV1] we obtained existence of blowing up solutions of (1.1)-(1.3) with $\Gamma = 0$ such that:

$$\Phi(y,\tau) \sim \frac{8\pi}{\chi} \frac{(y/\varepsilon(\tau))^2}{\left((y/\varepsilon(\tau))^2 + 1\right)} \qquad \text{as} \quad \tau \to \infty \, , \tag{2.6a}$$

where:

$$\varepsilon(\tau) \sim K e^{-\frac{\sqrt{2}}{2}\tau^{1/2}} \qquad \text{as } \tau \to \infty \, , \tag{2.6b}$$
$$\text{for some} K > 0 \, .$$

We will look for solutions of our system with $\Gamma > 0$ having a similar behaviour for Φ. Notice that in such case:

$$\frac{\Phi_y}{2\pi y} \sim \frac{8}{\chi}\left(\frac{1}{\varepsilon(\tau)^2}\right)\left(\left(\frac{y}{\varepsilon(\tau)}\right)^2 + 1\right)^{-1} \to \frac{8\pi}{\chi}\delta(y) \tag{2.7}$$
as $\tau \to \infty$.

Then in equation (2.5b) a source of the order of a Dirac mass appears. We next consider the linear operator:

$$AV \equiv A_\Gamma V = V_{yy} + \left(\frac{1}{y} - \frac{\Gamma y}{2}\right) V_y , \tag{2.8a}$$

which is self-adjoint in the set:

$$L^2_w(\mathbb{R}^2) = \Big\{ f \in L^2_{\text{loc}}(\mathbb{R}^2) : f \text{ is radially symmetric and } \int_0^\infty r f(r)^2 e^{-\frac{\Gamma r^2}{4}} dr < \infty \Big\} , \tag{2.8b}$$

with domain:

$$H^2_w\left(\mathbb{R}^2\right) = \Big\{ f \in H^2_{\text{loc}}\left(\mathbb{R}^2\right) : f \text{ is radially symmetric and } f, f', f'' \text{ are in } L^2_w\left(\mathbb{R}^2\right) \Big\} . \tag{2.8c}$$

Notice that L^2_w is a Hilbert space when endowed with the norm:

$$\| f \|^2 = \langle f, f \rangle = \int_0^\infty r f(r)^2 e^{-\Gamma r^2/4} dr.$$

The spectrum of A consists of the eigenvalues:

$$\lambda_k = -\Gamma k \ ; \ k = 0, 1, 2 \cdots \tag{2.9}$$

and the correponding eigenfunctions can be written in the form:

$$\varphi_k(r) = \left(\frac{\Gamma}{4\pi k!}\right)^{1/2} L_k\left(\frac{\Gamma r^2}{4}\right) , \tag{2.10}$$

where $L_k(r)$ is the k^{th} Laguerre polynomial. For simplicity, to proceed further we split $V(y, \tau)$ in the form:

$$V(y,\tau) = b_0(\tau)\varphi_0 + G(y,\tau) \quad , \quad \text{where } \langle G, \varphi_0 \rangle = 0 . \tag{2.11}$$

Then:

$$\Gamma b_0 = \langle \varphi_0, \frac{\Phi_y}{2\pi y} \rangle , \tag{2.12}$$

$$\Gamma G_\tau = G_{yy} + \frac{G_y}{y} - \Gamma y \frac{G_y}{2} + \frac{\Phi_y}{2\pi y} - \varphi_0 \langle \varphi_0, \frac{\Phi_y}{2\pi y} \rangle . \tag{2.13}$$

Taking into account (2.7), we readily see that $\langle \varphi_0, \frac{\Phi_y}{2\pi y} \rangle \sim \frac{8\pi}{\chi}\varphi_0$, whence:

$$b_0(\tau) \sim \frac{8\pi}{\chi\Gamma}\varphi_0\tau \quad \text{as} \quad \tau \to \infty .$$

On the other hand, since $V_y(y,\tau) = G_y(y,\tau)$ by (2.11), we may write (2.5a) in the form:

$$\Phi_\tau = \Phi_{yy} - \frac{\Phi_y}{y} - \frac{y\Phi_y}{2} - \chi G_y \Phi_y - 2\pi\chi y e^{-\tau} G_y . \tag{2.14}$$

We shall often make use of (2.13), (2.14) (which is decoupled from $b_0(\tau)$) instead of (2.5). Since $\langle G, \varphi_0 \rangle = 0$, we expect from (2.13) that $G(y,\tau)$ will approach exponentially fast towards a stationary solution of that equation. At distances $y \sim \varepsilon(\tau)$ (cf. (2.6b)) we then rewrite (2.13) in terms of the new variable $\xi = \frac{y}{\varepsilon(\tau)}$ to obtain:

$$\Gamma\varepsilon^2 G_\tau - \varepsilon\dot{\varepsilon}\xi G_\xi = G_{\xi\xi} + \frac{G_\xi}{\xi} + \frac{G_\xi}{\xi} - \varepsilon^2\Gamma\frac{\xi G_\xi}{2\pi\xi} - \frac{\Phi_\xi}{2\pi\xi} - \varepsilon^2\varphi_0\langle \varphi_0, \frac{\Phi_y}{2\pi y} \rangle .$$

We expect those terms containing time derivatives to be negligible in the equation above when $\tau \gg 1$. It is then natural to make the following ansatz: As $\tau \to \infty$, solutions $G(y,\tau)$ of (2.13) are such that $G(y,\tau) \sim \overline{G}(y,\tau)$ for $y \sim \varepsilon(\tau)$, where:

$$\overline{G}_{yy} + \frac{\overline{G}_y}{y} - \frac{\Gamma_y \overline{G}_y}{2} + \frac{\Phi_y}{2\pi y} - \varphi_0\langle \varphi_0, \frac{\Phi_y}{2\pi y} \rangle = 0 , \tag{2.15a}$$

with:

$$\langle \overline{G}, \varphi_0 \rangle = 0 . \tag{2.15b}$$

On the other hand, it is reasonable to expect then that, as $\tau \to \infty$, (2.14) will be asymptotically equivalent to the equation:

$$\Phi_\tau = \Phi_{yy} - \frac{\Phi_y}{y} - \frac{y\Phi_y}{2} - \chi\Phi_y\overline{G}_y . \tag{2.16}$$

Equations (2.15) can be integrated explicitly to obtain:

$$\overline{G}_y(y,\tau) = y^{-1}e^{\frac{\Gamma y^2}{4}}\left(\frac{1}{2\pi}\int_y^\infty e^{-\frac{\Gamma r^2}{4}}\Phi_y(r,\tau)dr - \frac{2\varphi_0^2}{\Gamma}\langle 1, \frac{\Phi_y}{2\pi y} \rangle e^{-\frac{\Gamma y^2}{4}} \right) .$$

A quick computation reveals then that:

$$\langle 1, \frac{\Phi_y}{2\pi y} \rangle = \frac{\Gamma}{2}\int_0^\infty r\Phi(r,\tau)e^{-\frac{\Gamma r^2}{4}}dr ,$$

and:

$$\int_y^\infty \Phi_y(y,\tau)e^{-\frac{\Gamma r^2}{4}}dr = -e^{-\frac{\Gamma y^2}{4}}\Phi(y,\tau) + \frac{\Gamma}{2}\int_y^\infty r\Phi(r,\tau)e^{-\frac{\Gamma r^2}{4}}dr\,,$$

hence:

$$\begin{aligned}\overline{G}_y(y,\tau) &= -\frac{\Phi(y,\tau)}{2\pi y} + \frac{\Gamma e^{\frac{\Gamma y^2}{4}}}{4\pi y}\left(\int_y^\infty r\Phi(r,\tau)e^{-\frac{\Gamma r^2}{4}}dr\right.\\ &\qquad \left. - e^{-\frac{\Gamma y^2}{4}}\left(\int_0^\infty r\Phi(r,\tau)e^{-\frac{\Gamma r^2}{4}}dr\right)\right)\\ &\equiv -\frac{\Phi(y,\tau)}{2\pi y} + J(y,\tau)\,.\end{aligned} \tag{2.17}$$

Substituting (2.17) into (2.16), we arrive at:

$$\Phi_\tau = \Phi_{yy} - \frac{\Phi_y}{y} - \frac{y\Phi_y}{2} + \frac{\chi\Phi\Phi_y}{2\pi y} - \chi\Phi_y J\,. \tag{2.18}$$

Following [HV2], we now introduce a function $W(y,\tau)$ given by:

$$W(y,\tau) = \int_0^y r\Phi(r,\tau)dr\,. \tag{2.19}$$

so that $\Phi = \frac{W_y}{y}$, and (2.18) is then transformed into the following equation for W:

$$\begin{aligned}W_\tau &= W_{yy} + \frac{W_y}{y} - \frac{yW_y}{2} + W + \left(\frac{\chi}{4\pi}\Phi^2 - 4\Phi\right)\\ &\quad - \chi\int_0^y \Gamma\Phi_y(r,\tau)J(r,\tau)dr\,.\end{aligned} \tag{2.20}$$

As observed in [HV2], we expect that our sought-for solutions will behave asymptotically as indicated in (2.6) in an inner layer near the point $x = 0$, where the singularity will appear. In terms of W, this behaviour reads:

$$W(y,\tau) \sim \frac{4\pi}{\chi}y^2\,. \tag{2.21}$$

To obtain further insight about the asymptotics near the unfolding collapse, we linearise around the limit profile in (2.21) by setting:

$$W(y,\tau) = \frac{4\pi}{\chi}y^2 + \psi(y,\tau)\,, \tag{2.22}$$

so that $\psi(y,\tau)$ satisfies:

$$\begin{aligned}\psi_\tau = \psi_{yy} + \left(\frac{1}{y} - \frac{y}{2}\right) + \psi + \left(\frac{\chi}{4\pi}\Phi^2 - 4\Phi + \frac{16\pi}{\chi}\right) \\ - \chi \int_0^y r\Phi_y(r,\tau)J(r,\tau)dr\,.\end{aligned} \tag{2.23}$$

We now write:

$$g(y,\tau) = \left(\frac{\chi\Phi^2}{4\pi} - 4\Phi + \frac{16\pi}{\chi}\right). \tag{2.24}$$

Taking into account (2.6) and (2.7), we observe as in [HV2] that $g(y,\tau)$ can be approximated as follows:

$$g(y,\tau) \sim \gamma\varepsilon(\tau)^2\delta(y) \quad \text{as } \tau \to \infty \quad , \qquad \text{where } \gamma = \frac{16\pi^2}{\chi}. \tag{2.25}$$

We have yet to obtain a suitable approximation for the last term in the right in (2.23). To this end, we make use of (2.6) and (2.17) to observe that:

$$J(y,\tau) \sim \frac{2\Gamma e^{\frac{\Gamma y^2}{4}}}{\chi y}\varepsilon(\tau)^2\left(-\int_y^\infty \frac{re^{-\frac{\Gamma r^2}{4}}}{r^2+\varepsilon(\tau)^2}dr + e^{-\frac{\Gamma y^2}{4}}\int_0^\infty \frac{re^{-\frac{\Gamma r^2}{4}}}{r^2+\varepsilon(\tau)^2}dr\right),$$

whence:

$$I(y,\tau) \equiv \chi\int_0^y r\Phi_y(r,\tau)J(r,\tau)dr \tag{2.26}$$

$$\begin{aligned}&\sim 2\Gamma\varepsilon^2\int_0^y \left(\Phi - \frac{8\pi}{\chi}\right)_y \left(\int_0^\infty \frac{re^{-\frac{\Gamma r^2}{4}}}{r^2+\varepsilon^2}dr - e^{\frac{\Gamma\eta^2}{4}}\int_\eta^\infty \frac{re^{-\frac{\Gamma r^2}{4}}}{r^2+\varepsilon^2}dr\right)d\eta \\ &\sim 16\pi\Gamma\varepsilon^2\int_0^y \left((\eta/\varepsilon)^2+1\right)^{-1}\frac{\partial}{\partial\eta}\left(-e^{\frac{\Gamma\eta^2}{4}}\int_\eta^\infty \frac{re^{-\frac{\Gamma r^2}{4}}}{r^2+\varepsilon^2}dr\right)d\eta \\ &= 16\pi\Gamma\varepsilon^4\int_0^y \frac{1}{\eta^2+\varepsilon^2}\left(\frac{-\Gamma\eta e^{\frac{\Gamma\eta^2}{4}}}{2}\int_\eta^\infty \frac{re^{-\frac{\Gamma r^2}{4}}}{r^2+\varepsilon^2}dr + \frac{\eta}{\eta^2+\varepsilon^2}\right)d\eta \\ &\equiv I_1(y,\tau) + I_2(y,\tau)\,.\end{aligned}$$

The first term in the sum can be bounded as follows:

$$I_1(y,\tau) \le C\varepsilon^4 \int_0^y \frac{\eta e^{\frac{\Gamma\eta^2}{4}}}{\eta^2+\varepsilon^2}\left(\int_\eta^\infty \frac{re^{-\frac{\Gamma r^2}{4}}}{r^2+\varepsilon^2}dr\right)d\eta$$

(2.27)

$$= C\varepsilon^4 \int_0^y \frac{\eta e^{\frac{\Gamma\eta^2}{4}}}{\eta^2+\varepsilon^2}\left(\int_\eta^1 (\cdot) + \int_1^\infty (\cdot)\right)d\eta$$

$$\le C\varepsilon^4 \left(\int_0^y \frac{\eta}{\eta^2+\varepsilon^2}\left|\log\left(\eta^2+\varepsilon^2\right)\right| d\eta + 1\right) \le C\varepsilon^4 \mid \log\varepsilon \mid .$$

As to $I_2(y,\tau)$, we readily see that:

$$I_2(y,\tau) \sim 16\pi\Gamma\varepsilon^4 \int_0^y \frac{\eta d\eta}{(\eta^2+\varepsilon^2)^2} = \frac{\varepsilon^2 8\pi\Gamma y^2}{y^2+\varepsilon^2} = 8\pi\Gamma\varepsilon^2 - \frac{8\eta\Gamma\varepsilon^2}{y^2+\varepsilon^2}. \tag{2.28}$$

From (2.26)-(2.28) we eventually arrive at:

$$I(y,\tau) \sim 8\pi\Gamma\varepsilon(\tau)^2 \quad \text{as } \tau \to \infty. \tag{2.29}$$

Taking into account (2.25) and (2.29), (2.23) reads now:

$$\psi_\tau = \psi_{yy} + \left(\frac{1}{y} - \frac{y}{2}\right)\psi_y + \psi + \gamma\varepsilon(\tau)^2\delta(y) - 8\pi\Gamma\varepsilon^2(\tau), \quad \text{as } \tau \to \infty. \tag{2.30}$$

We now look for solutions of (2.30) in the form:

$$\begin{aligned} &\psi(y,\tau) = a_0(\tau)\varphi_0 + a_1(\tau)\varphi_1(y) + Q(y,\tau), \\ &\text{where } \langle Q(.,\tau), \varphi_k\rangle = 0 \text{ for } k = 0, 1. \end{aligned} \tag{2.31}$$

One then readily sees that:

$$\dot a_0 = a_0 + \gamma\varepsilon(\tau)^2\varphi_0 - 8\pi\Gamma\varepsilon(\tau)^2\varphi_0\langle 1, 1\rangle, \tag{2.32}$$

$$\dot a_1 = \gamma\varepsilon(\tau)^2\varphi_1(0), \tag{2.33}$$

$$Q_\tau = \Delta Q + \frac{y\nabla Q}{2} + Q + \gamma\varepsilon(\tau)^2\left(\delta(y) - \sum_{k=0}^{1}\varphi_k\langle\varphi_k, \delta(y)\rangle\right). \tag{2.34}$$

Since $\mid \dot\varepsilon(\tau) \mid \ll \varepsilon(\tau)$ as $\tau \to \infty$ (cf. (2.6b)), we now expect $Q(y,\tau) \sim \gamma\varepsilon(\tau)^2 F(y)$ for large τ, in which case F solves:

$$\Delta F - \frac{y\nabla F}{2} + F + \gamma\left(\delta(y) - \sum_{k=0}^{1}\varphi_k\langle\varphi_k, \delta(y)\rangle\right) = 0, \tag{2.35a}$$

$$\langle F, \varphi_k \rangle = 0 \quad \text{for } k = 0, 1\,. \tag{2.35b}$$

Equations (2.35) can be integrated to give:

$$F(y) = -\frac{1}{2\pi} \log y + B + 0\left(y^2 \mid \log y \mid\right) \quad \text{as} \quad \mid y \mid \to 0\,, \tag{2.36}$$

for some explicit constant $B > 0$. Recalling that we expect $\psi(y, \tau) \to 0$ as $\tau \to \infty$, we may integrate the equations for the two first Fourier coefficients (cf. (2.32), (2.33)) to obtain that:

$$a_0(\tau) = \gamma\varphi_0(2\Gamma\chi - 1) \int_\tau^\infty e^{\tau - s}\varepsilon(s)^2 ds\,, \tag{2.37a}$$

$$a_1(\tau) = -\gamma\varphi_1(0) \int_\tau^\infty \varepsilon(s)^2 ds\,. \tag{2.37b}$$

From (2.22), (2.30), (2.36) and (2.37), the following outer expansion has been obtained for $W(y, \tau)$ in regions $\varepsilon \ll\mid y \mid\leq 1$, $\tau \gg 1$:

$$\begin{aligned} W(y, \tau) \sim \frac{4\pi}{\chi} y^2 + \gamma\varphi_0^2(2\Gamma\chi - 1) \int_\tau^\infty e^{\tau - s}\varepsilon(s)^2 ds - \gamma\varphi_1^2(0) \int_\tau^\infty \varepsilon(s)^2 ds \\ + \gamma\varepsilon(\tau)^2 \left(-\frac{\log y}{2\pi} + B + 0\left(y^2 \mid \log y \mid\right)\right). \end{aligned} \tag{2.38}$$

On the other hand, by (2.6a) and (2.19), we obtain the following inner expansion for $W(y, \tau)$ in regions $y \leq \varepsilon(\tau)$ and $\tau \gg 1$:

$$W(y, \tau) \sim \frac{4\pi}{\chi} y^2 - \frac{4\pi}{\chi} \varepsilon^2 \log\left(1 + (y/\varepsilon)^2\right). \tag{2.39}$$

Matching (2.38) and (2.39) we obtain an integral equation for $\varepsilon(\tau)$, namely:

$$\begin{aligned} \frac{4\pi}{\chi} \varepsilon^2 \log \varepsilon = \gamma\varphi_0^2(2\Gamma\chi - 1) \int_\tau^\infty e^{\tau - s}\varepsilon(s)^2 ds - \gamma\varphi_1^2(0) \int_2^\infty \varepsilon(s)^2 ds \\ + B\varepsilon(\tau)^2\,. \end{aligned} \tag{2.40}$$

This equation can be solved asymptotically for $\tau \gg 1$ (cf. for instance [HV1], [HV2]), thus yielding (2.6b) and thereby concluding the formal derivation of the size of the inner layer where the singularity develops. Once the value of $\varepsilon(\tau)$ is known, we may retrace the steps in [HV2] to obtain all the remaining estimates in Theorems 1.1 and 1.2.

3. – A basic topological argument

In this section we shall describe the main argument behind the proofs of Theorems 1.1 and 1.2. To avoid winding up with details, we shall postpone most of the technicalities involved to Sections 4-6 below. As a preliminary step, we proceed to define a suitable class of functions. Let us set:

$$\bar{\varepsilon}(\tau) = K e^{-\frac{\sqrt{2}}{2}\tau} \quad , \quad \text{where } K = K(\chi, \Gamma) > 0 \text{ is as in (2.6)}. \tag{3.1}$$

For given numbers τ_0, τ_1 and μ with $\tau_0 \le \tau_1 \le \infty$, $0 < \mu < 1$, we now define the set $\mathcal{A}$ $(\tau_0, \tau_1; \mu)$ consisting of functions $\Phi(y, \tau)$ which are regular enough (say C^1) and satisfy the following estimates for $0 < y \equiv \xi\bar{\varepsilon}(\tau) < Re^{\tau/2}$ and $\tau_0 \le \tau \le \tau_1$,

$$\begin{aligned} &\left| \Phi\left(\xi\bar{\varepsilon}(\tau), \tau\right) - \frac{8\pi}{\chi}\frac{\xi^2}{\xi^2+1} \right| \\ &\quad < M\mu\left(\frac{\xi^2}{\xi^2+1} + \bar{\varepsilon}^2(\tau) \mid \log\bar{\varepsilon}(\tau) \mid^3 \widetilde{\chi}_{\bar{\varepsilon}(\tau)}(1+y)\right), \end{aligned} \tag{3.2}$$

for some large enough constant $M > 0$, where $\widetilde{\chi}_{\bar{\varepsilon}(\tau)}$ is defined as in the statement of Theorem 1.1,

$$\left|\frac{\partial \Phi}{\partial y}(\xi\bar{\varepsilon}(\tau), \tau)\right| \le M\mu\left(\frac{\xi}{\bar{\varepsilon}(\tau)(\xi^2+1)^2} + \bar{\varepsilon}^2(\tau) \mid \log\bar{\varepsilon}(\tau) \mid^3 \widetilde{\chi}_{\bar{\varepsilon}(\tau)}\left(1+y^2\right)\right), \tag{3.3}$$

$$\left|\frac{\partial \Phi}{\partial \tau}(\xi\bar{\varepsilon}(\tau), \tau)\right| < M\mu\left(\frac{1}{\sqrt{\tau}(\xi^2+1)} + \bar{\varepsilon}(\tau)^2 \mid \log\bar{\varepsilon}(\tau) \mid^3 \left(1+y^2\right)\right). \tag{3.4}$$

$$0 \le \Phi \le M\mu\,. \tag{3.5}$$

As in [HV1], for any $\Phi \in \mathcal{A}(\tau_0, \tau_1; \mu)$ we now define $\varepsilon(\tau)$ as follows:

$$\gamma\varepsilon(\tau)^2 = \int_{B(\theta)}\left(\frac{\chi\Phi^2}{4\pi} - 4\Phi + \frac{16\pi}{\chi}\right)e^{-\frac{\Gamma y^2}{4}}dy\,, \tag{3.6}$$

where $\gamma = \frac{16\pi^2}{\chi}$, and for a given number $\theta > 0$ we define

$$B(\theta) = \left\{y \in \mathbb{R}^2 : 0 \le y \le (\bar{\varepsilon}(\tau))^{2\theta}\right\}.$$

Notice that in general $\bar{\varepsilon}(\tau) \neq \varepsilon(\tau)$. We shall impose however the following condition for a function to belong to the clas $\mathcal{A}$ $(\tau_0, \tau_1; \mu)$:

$$\frac{\mu}{10}\bar{\varepsilon}(\tau) < \varepsilon(\tau) < 10\mu\bar{\varepsilon}(\tau) \quad \text{for } \tau_0 \le \tau \le \tau_1\,. \tag{3.7}$$

We shall say that $\Phi(y,\tau) \in \overline{\mathcal{A}(\tau_0,\tau_1;\mu)}$ if Φ satisfies (3.2)-(3.5) and (3.7) when strict inequalities there are replaced by the symbol $\leq$. To produce our desired solutions, we shall select a function $\Phi(y,\tau_0) \in \mathcal{A}\ (\tau_0,\tau_{0;}\mu)$ for some $\tau_0 \gg 1$ and some $\mu \in (0,1)$, and will use it as an initial value to solve (2.5a) with suitable boundary conditions at $r = 0$, $Re^{\tau/2}$ and $\tau > \tau_0$. To this end, it will be convenient to work with the auxiliary functions $W(y,\tau)$ and $\psi(y,\tau)$, given respectively in (2.19) and (2.22). We shall then consider initial values in the form:

$$(3.8)\quad \psi\,(y,\tau_0) = \alpha_0\widetilde{\varphi}_0(y,\tau_0;\alpha_0,\alpha_1) + \alpha_1\widetilde{\varphi}_1\,(y,\tau_0;\alpha_0,\alpha_1) + \frac{16\pi}{\chi^2}\overline{\varepsilon}(\tau_0)^2 F(y)\,,$$

where $F(y)$ is given in (2.36), and $\alpha_0, \alpha_1, \widetilde{\varphi}_0, \widetilde{\varphi}_1$ will be selected presently. To (3.8) we should add a condition on $G(y,\tau)$ given in (2.11), say,

$$(3.9)\qquad G(y,\tau_0) = \overline{G}(y,\tau_0)\,.$$

Functions $\widetilde{\varphi}_j$ with $j = 0,1$ will coincide with eigenfunctions $\varphi_j(y)$ given in (2.10) everywhere except at a narrow layer near $y = 0$. We may then take $\widetilde{\varphi}_j(y) = \varphi_j(y)$ for $y \geq \overline{\varepsilon}(\tau_0)^\sigma$ with $\sigma > 0$ and $j = 0,1$. In particular, $\widetilde{\varphi}_j = \varphi_j$ in the overlapping region corresponding to the matching described in Section 2, and one thus obtain:

$$\alpha_0\varphi_0(0) + \alpha_1\varphi_1(0) + \frac{16\pi^2 B}{\chi}\overline{\varepsilon}(\tau_0)^2 \sim \frac{4\pi}{\chi}\overline{\varepsilon}(\tau_0)^2 \log\left((\overline{\varepsilon}_0)^2\right)\,,$$

whence:

$$(3.10)\qquad |\,\alpha_0\,| + |\,\alpha_1\,| = o\left(\overline{\varepsilon}(\tau_0)^2 \log\overline{\varepsilon}(\tau_0)\right)\,,$$

which provides an estimate on the values of parameters $\alpha_j (i = 0,1)$ in (3.8). Near $y = 0$, $\widetilde{\varphi}_0$ and $\widetilde{\varphi}_1$ need to be redefined to avoid the logarithmic singularity that would appear in (3.8) if $\widetilde{\varphi}_j = \varphi_j$ for $j = 0,1$. This can be done by selecting these functions so that:

$$(3.11)\qquad \alpha_0\widetilde{\varphi}_0(0) + \alpha_1\widetilde{\varphi}_1(y) \sim -\frac{8\pi}{\chi}\overline{\varepsilon}(\tau_0)^2 \log y \quad \text{as } y \to 0\,.$$

Notice that relations (3.8)-(3.11) are compatible. As a matter of fact, they allow for many possible choices of the parameters involved. For $j = 0,1$, and $\tau \geq \tau_0$ we now define:

$$(3.12)\qquad \ell_j\,(\alpha_0,\alpha_1;\tau) = \langle \psi\,(y,\tau;\alpha_0,\alpha_1)\,,\varphi_j\rangle + \int_\tau^\infty e^{(1-j)(\tau-s)}\overline{\varepsilon}(s)^2 ds\,,$$

where $\psi\,(y,\tau;\alpha_0,\alpha_1)$ denotes the solution of (2.23) with initial value $\psi\,(y,\tau_0;\alpha_0,\alpha_1)$ as in (3.8)-(3.11). As in [HV1], where the case $\Gamma = 0$ was discussed, a key point in the proof of Theorem 1.1 and 1.2 is the following:

PROPOSITION 3.1. *Assume that constant M in* (3.2) – (3.5) *is selected large enough, and that $\theta > 0$ in* (3.6) *is sufficiently small. Let $\psi(y,\tau) \equiv \psi(y,\tau;\alpha_0,\alpha_1)$ be the solution of* (2.23) *defined for $\tau > \tau_0$ and corresponding to an initial value $\psi(y,\tau_0) \equiv \psi(y,\tau_0;\alpha_0,\alpha_1)$ satisfying the above requirements. Suppose also that:*

$$\psi(y,\tau) \in \overline{\mathcal{A}(\tau_0,\tau_1;1)}, \tag{3.13}$$

for $\tau \in [\tau_0,\tau_1]$ with $\tau_1 > \tau_0 \gg 1$. Then, if:

$$\ell_j(\alpha_0,\alpha_1;\tau_1) = 0 \ \textit{for}\ j = 0,1, \tag{3.14}$$

(cf. (3.12)), *it then turns out that:*

$$\psi(y,\tau) \in \mathcal{A}(\tau_0,\tau_1;1/2). \tag{3.15}$$

Let us suppose for the moment that Proposition 3.1 holds. Then we can take advantage of a topological argument quite similar to that already used in [HV1]. To this end, we argue as follows. Let $\ell_j(\alpha_0,\alpha_1;\tau)$ be the function defined in (3.12). By our choice of initial values at $\tau = \tau_0$, we then have that, for $j = 0,1$:

$$\ell_j(\alpha_0,\alpha_1;\tau_0) = \alpha_j + \delta(\alpha,\tau_0)(|\alpha_0| + |\alpha_1|) + O\left(\overline{\varepsilon}(\tau_0)^2\right).$$

where $\alpha = (\alpha_0,\alpha_1)$, $\delta(\alpha,\tau_0) \to 0$ as $\tau_0 \to \infty$, uniformly for $|\alpha| = |\alpha_0| + |\alpha_1|$ bounded, and the last term in the equality above may be assumed to be independent on α. On the other hand, by modifying if necessary our choice of initial value, we may always assume that $\ell = (\ell_0,\ell_1)$ is differentiable with respect to α_0,α_1: we shall assume henceforth that such a selection has been made. It then turns out that for $j = 0,1$, equation $\ell_j(\alpha_0,\alpha_1;\tau_0) = 0$ has a unique solution α_j such that:

$$\alpha_j = O\left(\overline{\varepsilon}(\tau_0)^2\right).$$

Let τ_1,τ_0 be such that $\tau_1 \geq \tau_0$, and define $\mathcal{U}(\tau_0,\tau_1) \subset \mathbb{R}^2$ as the open set consisting of all points $(\alpha_0,\alpha_1) \subset \mathbb{R}^2$ such that the corresponding solution of (2.23), (3.8) satisfies that $\psi(y,\tau) \in \mathcal{A}(\tau_0,\tau_1;1)$. As a matter of fact, we now have that:

$$d(\ell,\ \mathcal{U}(\tau_0,\tau_0);\ 0) = 1.$$

where for $\tau \geq \tau_0$, $d(\ell,\ \mathcal{U}(\tau_0,\tau);0)$ denotes the topological degree of the mapping ℓ in the set $\mathcal{U}(\tau_0,\tau)$ at the value zero.

Assume now that $\mathcal{U}(\tau_0,\tau) \neq \phi$ for any $\tau \in [\tau_0,\tau_1)$ with $\tau_0 > 0$, and denote by $\partial\mathcal{U}(\tau_0,\tau)$ the boundary of the open set $\mathcal{U}(\tau_0,\tau)$. We notice that, if $\ell \neq 0$ on $\cup(\partial\mathcal{U}(\tau_0,\tau))$ for $\tau_0 \leq \tau \leq \tau_1$, then $d(\ell,\mathcal{U}(\tau_0,\tau);0) = d(\ell,\mathcal{U}(\tau_0,\tau_0);0)$ for any such τ. Since $\mathcal{U}(\tau_0, \tau) \neq \phi$ for $\tau_o \leq \tau < \tau_1$, it then follows from standard continuous dependence results that:

$$\mathcal{U}(\tau_0, \tau_1) \neq \phi,$$

and:

$$d(\ell, \mathcal{U}(\tau_0, \tau_1); 0) = 1\,, \tag{3.16}$$

Actually, (3.16) holds for any $\tau_1 > \tau_0$, as far as:

$$\mathcal{U}(\tau_0, \tau_1) \neq \phi\,. \tag{3.17}$$

Indeed, suppose that there exists a first time $\tau > \tau_0$ when (3.16) fails but (3.17) holds true. In view of our previous remark, there must be a point:

$$\beta = (\beta_0, \beta_1) \in \partial\mathcal{U}(\tau_0, \tau)\,,$$

where $\ell(\beta) = 0$, and clearly $\psi(y, \tau; \beta_0, \beta_1) \in \overline{\mathcal{A}(\tau_0, \tau; 1)}$. We then use Proposition 3.1 to deduce that $\beta \in \mathcal{U}(\tau_0, \tau)$, a contradiction.

We further observe that:

$$\mathcal{U}(\tau_0, \tau) \neq \phi \quad \text{for any} \quad \tau > \tau_0, \quad \text{provided that } \tau_0 \gg 1\,. \tag{3.18}$$

To check (3.18), we define $\tau^\star = \sup\{\tau : \mathcal{U}(\tau_0, \tau) \neq \phi\}$. We already know that $\tau^\star > \tau_0$. Assume now that $\tau^\star < \infty$. By (3.16) and (3.17), we may select a sequence of times $\{\tau_n\}$ increasing to $\{\tau^\star\}$, and a sequence $\{\alpha_n\} = \{(\alpha_{0n}, \alpha_{1n})\}$ such that $\ell\,(\alpha_{0n}, \alpha_{1n}; \tau_n) = 0$ and $\alpha_n \in \mathcal{U}(\tau_0, \tau_n)$. Since $\mathcal{U}(\tau_0, \tau_{n+1}) \subset \mathcal{U}(\tau_0, \tau_n)$, one has that $\{\alpha_n\}$ is bounded. Therefore, a subsequence (still denoted by $\{\alpha_n\}$) exists, which converges to some point $\alpha^\star = \left(\alpha_0^\star, \alpha_1^\star\right)$. It then turns out that $\ell\left(\alpha_0^\star, \alpha_1^\star; \tau^\star\right) = 0$, and by Proposition 3.1, the corresponding function $\psi\left(y, \tau; \alpha_0^\star, \alpha_1^\star\right)$ remains at the interior of $\mathcal{A}\,(\tau_0, \tau^\star; 1)$: this is the point where restriction $\tau_0 \gg 1$ needs to be imposed in (3.18). By continuous dependence results, ψ would also remain at the interior of $\mathcal{A}\,(\tau_0, \tau^\star + \delta; 1)$ for some $\delta > 0$, thus contradicting the definition of $\tau^\star$.

We are now able to explain the argument leading to the existence of solutions referred to in Theorems 1.1 and 1.2. Take a sequence $\{\tau_n\}$ such that $\tau_1 > \tau_0$ and $\lim_{n\to\infty} \tau_n = \infty$. For any such $n, \mathcal{U}(\tau_0, \tau_n) \neq \phi$, and we may select $\alpha_n = (\alpha_{0n}, \alpha_{1n})$ such that $\ell\,(\alpha_{0n}, \alpha_{1n}; \tau_n) = 0$. Let $\psi_n(y, \tau) \equiv \psi_n\,(y, \tau; \alpha_{0n}, \alpha_{1n})$ be the solution of (2.23) with initial value $\psi_n(y, \tau_0) = \psi_0\,(y; \alpha_{0n}, \alpha_{1n})$ satisfying (3.8)-(3.11). By Proposition 3.1, we have that $\psi_n(y, \tau) \in \mathcal{A}\left(\tau_0, \tau_n; \frac{1}{2}\right)$. Since the sequence $\{\alpha_n\}$ is bounded, there exists a subsequence (still denoted by $\{\alpha_n\}$) and a value $\overline{\alpha} = (\overline{\alpha}_0, \overline{\alpha}_1)$ such that $\lim_{n\to\infty} \alpha_n = \overline{\alpha} \in \mathcal{U}(\tau_0, \tau_0)$. It then turns out that function $\psi\,(y, \tau; \overline{\alpha}_0, \overline{\alpha}_1)$, solution of (2.23) with initial value $\psi\,(y, \tau_0; \overline{\alpha}_0, \overline{\alpha}_1)$, provides a sought-for solution, and the proof is concluded. □

4. – The proof of Proposition 3.1

Thoroughout this section we shall derive a number of estimates that will be required in the proof of our basic Proposition. We begin as follows:

4.1. – Estimating the oscillation of $\varepsilon(\tau)$

We first observe that (3.6) provides a bound of $\dot{\varepsilon}(\tau)$. Indeed, differentiating both sides of (3.6) with respect to τ yields:

$$\begin{aligned}2\gamma\varepsilon(\tau)\dot{\varepsilon}(\tau) = &\int_{B(\theta)}\left(\frac{\chi\Phi}{2\pi}-4\right)\frac{\partial\Phi}{\partial\tau}(y,\tau)dy\\ &+ 2\pi\theta\overline{\varepsilon}(\tau)^{\theta}\left(\overline{\varepsilon}(\tau)\right)^{\theta-1}\frac{d}{d\tau}\left(\left(\frac{\chi\Phi^2}{4\pi}-4\Phi+\frac{16\pi}{\chi}\right)\right)_{y=\overline{\varepsilon}(\tau)^{2\theta}}.\end{aligned}$$

In view of (3.2) and (3.4), we easily obtain that:

$$\begin{aligned}\mid\varepsilon(\tau)\dot{\varepsilon}(\tau)\mid &\leq C_M\left(\int_{B(\theta)}\left(\tau^{1/2}\left(\left(\frac{r}{\overline{\varepsilon}(\tau)}\right)^2+1\right)\right)^{-1}dy+\tau^{-1/2}\left(\overline{\varepsilon}(\tau)\right)^{4-2\theta}\right)\\ &\leq C_M\,\tau^{-1/2}\overline{\varepsilon}(\tau)^2,\end{aligned}$$

for some constant C_M that depends on M and θ is a positive and small enough number. Recalling (3.7), we thus obtain:

$$\mid\dot{\varepsilon}(\tau)\mid\leq C_M\,\tau^{-1/2}\overline{\varepsilon}(\tau)\,. \tag{4.1}$$

4.2. – Approximating $G(y,\tau)$ by $\overline{G}(y,\tau)$

In the course of proving our results, precise estimates will be required on the difference $\mid G_y - \overline{G}_y\mid$, where $G(y,\tau)$ is defined in (2.11), and $\overline{G}(y,\tau)$ satisfies (2.15). We claim that the following result holds:

LEMMA 4.1. *Assume that $G(y,\tau_0)=\overline{G}(y,\tau_0)$, where G and $\overline{G}$ are as recalled above. Then for $\tau>\tau_0\gg 1$, there holds:*

$$\begin{aligned}&\left|G_y(y,\tau)-\overline{G}_y(y,\tau)\right|\\ &\leq CM\overline{\varepsilon}(\tau)^2\left(\left(\tau^{1/2}(y+\overline{\varepsilon}(\tau)\right)^{-1}(1+\tilde{\chi}_2|\log(y\overline{\varepsilon}(\tau))|)+|\log\overline{\varepsilon}(\tau)|^3(1+y^3)\right)\end{aligned} \tag{4.2}$$

where $\tilde{\chi}_2=1$ if $\xi=\frac{y}{\overline{\varepsilon}(\tau)}\geq 2$ and is zero otherwise.

The proof of Lemma 4.1 will be postponed to Section 7 (cf. Subsection 7.1 there).

4.3. – Analysis of the first Fourier coefficients

In view of (2.17), we may write (2.14) in the form:

$$\Phi_\tau = \Phi_{yy} - \left(\frac{1}{y} + \frac{y}{2}\right)\Phi_y + \frac{\chi}{2\pi}\frac{\Phi\Phi_y}{y} - \chi J\Phi_y - \chi\Phi_y\left(G_y - \overline{G}_y\right) - 2\pi\chi y e^{-\tau} G_y.$$

where $J \equiv J(y,\tau)$ is given in (2.17). Let now $W(y,\tau)$ be the auxiliary function defined in (2.19). A quick check reveals that the above equation for Φ transforms into the following one for W:

$$W_\tau = W_{yy} + \left(\frac{1}{y} - \frac{y}{2}\right) W_y + \left(\frac{\chi\Phi^2}{4\pi} - 4\Phi\right) + \ell(y,\tau) + m(y,\tau), \tag{4.3a}$$

where

$$\ell(y,\tau) = -\chi \int_0^y r\Phi_y(r,\tau) J(r,\tau)dr, \tag{4.3b}$$

$$m(y,\tau) = 2\pi\chi\, e^{-\tau}\int_0^y r^2 G_y(r,\tau)dr - \chi\int_0^y r\Phi_y(r,\tau)\left(G_y(r,\tau) - \overline{G}_y(r,\tau)\right)dr. \tag{4.3c}$$

The corresponding equation for $\psi(y,\tau)$ given in (2.22) reads now:

$$\psi_\tau = \psi_{yy} + \left(\frac{1}{y} - \frac{y}{2}\right)\psi_y + \psi + g(y,\tau) + \ell(y,\tau) + m(y,\tau), \tag{4.4}$$

where $g(y,\tau)$ is given in (2.24). We now write:

$$\psi(y,\tau) = a_0(\tau)\varphi_0 + a_1(\tau)\varphi_1(y) + R(y,\tau), \tag{4.5}$$

with $\langle R, \varphi_k\rangle = 0$ for $k = 0, 1$. One then sees that:

$$\dot a_0 = a_0 + \langle \varphi_0, p(\cdot,\tau)\rangle, \tag{4.6}$$

$$\dot a_1 = \langle \varphi_1, p(\cdot,\tau)\rangle, \tag{4.7}$$

$$R_\tau = R_{yy} + \left(\frac{1}{y} - \frac{y}{2}\right) R_y + R + p(y,\tau) - \sum_{k=0}^{1}\varphi_k\langle\varphi_k, p\rangle, \tag{4.8}$$

where we have set $p(y,\tau) = g(y,\tau) + \ell(y,\tau) + m(y,\tau)$. If (3.14) holds, we then deduce from (4.6) and (4.7) that:

$$a_k(\tau) = -\int_\tau^{\tau_1} e^{(1-k)(\tau-s)}\langle\varphi_k, p(\cdot,s)\rangle ds - \int_{\tau_1}^{\infty} e^{(1-k)(\tau-s)}\overline{\varepsilon}(s)^2 ds. \tag{4.9}$$

for $k = 0, 1$ and $\tau_0 \le \tau \le \tau_1$. Arguing as in Lemma 4.1 in [HV1], we obtain that:

$$\left|\langle \varphi_k, g(\cdot, s)\rangle - \gamma\varepsilon(s)^2\varphi_k(0)\right| \le C_M\overline{\varepsilon}(s)^{4-2\theta} \tag{4.10}$$

for $\tau_0 \le s \le \tau_1$ and $k = 0, 1$, where C_M is a positive constant depending on M.

We next proceed to estimate the terms $\langle\varphi_k, \ell(\cdot,\tau)\rangle$ and $\langle\varphi_k, m(\cdot,\tau)\rangle$. Assuming without loss of generality that $y > 1$, we first observe that:

$$\begin{aligned}\ell(y,\tau) &= -\chi\int_0^1 r\Phi_y(r,\tau)J(r,\tau)dr - \chi\int_1^y r\Phi_y(r,\tau)J(r,\tau)dr \\ &\equiv A(\tau) + \overline{\ell}(y,\tau)\end{aligned} \tag{4.11}$$

Notice that:

$$\begin{aligned}A(\tau) = -\chi\Gamma\int_0^1 \frac{\partial}{\partial r}\left(\Phi(r,\tau) - \frac{8\pi}{\chi}\right)\left(e^{\frac{\Gamma r^2}{4}}\int_1^\infty \lambda\left(\Phi(\lambda,\tau) - \frac{8\pi}{\chi}\right)e^{-\frac{\Gamma\lambda^2}{4}}d\lambda\right. \\ \left. - \int_0^\infty \lambda\left(\Phi(\lambda,\tau) - \frac{8\pi}{\chi}\right)e^{-\frac{\Gamma\lambda^2}{4}}d\lambda\right)dr\end{aligned}$$

whence:

$$\begin{aligned}\mid A(\tau)\mid \le C &\left(\left|\Phi(1,\tau) - \frac{8\pi}{\chi}\right|\int_1^\infty \left|\Phi(\lambda,\tau) - \frac{8\pi}{\chi}\right|\lambda e^{-\frac{\Gamma\lambda^2}{4}}d\lambda\right. \\ &\left. + \left|\Phi(1,\lambda) - \frac{8\pi}{\chi}\right|\int_0^\infty \left|\Phi(\lambda,r) - \frac{8\pi}{\chi}\right|\lambda e^{-\frac{\Gamma\lambda^2}{4}}d\lambda\right) \\ &+ \left|\int_0^1\left(\Phi(r,\tau) - \frac{8\pi}{\chi}\right)\frac{\partial}{\partial r}\left(e^{\frac{\Gamma r^2}{4}}\int_r^\infty\left(\Phi(\lambda,\tau) - \frac{8\pi}{\chi}\right)\lambda e^{\frac{-\Gamma\lambda^2}{4}}d\lambda\right)dr\right| \\ \le\; & C_M\overline{\varepsilon}(\tau)^{4-2\theta} + C\int_0^1\left|\Phi(r,\tau) - \frac{8\pi}{\chi}\right| \\ &\times r\left(\int_r^\infty\left|\Phi(\Lambda,\tau) - \frac{8\pi}{\chi}\right|\lambda e^{-\frac{\Gamma\lambda^2}{4}}d\lambda\right)dr \\ &+ C\int_0^1\left|\Phi(r,\tau) - \frac{8\pi}{\chi}\right|^2 r\,dr\,,\end{aligned}$$

hence:

$$\mid A(\tau)\mid \le C_M\left(\overline{\varepsilon}(\tau)\right)^2\,. \tag{4.12}$$

We may estimate $\overline{\ell}(y,\tau)$ as follows:

$$
\begin{aligned}
|\overline{\ell}(y,\tau)| \leq & C\left|yJ(y,\tau)\left(\Phi(y,\tau)-\frac{8\pi}{\chi}\right)\right| + C\left|J(1,\tau)\left(\Phi(1,\tau)-\frac{8\pi}{\chi}\right)\right| \\
& + C\int_1^y \left|\frac{\partial}{\partial r}(rJ(r,\tau))\right| \left|\Phi(r,\tau)-\frac{8\pi}{\chi}\right| dr \\
& \leq C\,(\overline{\varepsilon}(\tau))^{4-2\theta}\,(1+y^4) + C\int_1^y \left|\Phi(r,\tau)-\frac{8\pi}{\chi}\right| \\
& \times \quad \frac{\partial}{\partial r}\left(\mathrm{e}^{\frac{\Gamma r^2}{4}}\int_r^\infty \left(\Phi(\lambda,\tau)-\frac{8\pi}{\chi}\right)\lambda \mathrm{e}^{-\frac{\Gamma\lambda^2}{4}}d\lambda\right) dr \\
& \leq C\overline{\varepsilon}(\tau)^{4-2\theta}(1+y^4) + C_M\int_1^y r\left((r/\varepsilon)^2+1\right)^{-1} dr \\
& \leq C_M\left(\overline{\varepsilon}(\tau)^{4-2\theta}(1+y^4)\right) + \overline{\varepsilon}(\tau)^2\left(1+(y/\varepsilon(\tau))^2)^{-1}\right)
\end{aligned} \tag{4.13}
$$

whence:

$$
\left|\langle\varphi_k,\overline{\ell}(\cdot,\tau)\rangle\right| \leq C_M\,(\overline{\varepsilon}(\tau))^{4-2\theta} \quad \text{for } k=0,1 \text{ and } \tau\gg 1\,. \tag{4.14}
$$

On the other hand, it follows from (4.12) that:

$$
\langle\varphi_1,A(\tau)\rangle = 0, \quad |\,\langle\varphi_0,A(\tau)\rangle\,| \leq C_M\overline{\varepsilon}(\tau)^2 \text{ for } \tau\gg 1\,. \tag{4.15}
$$

We now take advantage of (4.2) and (2.17) to obtain:

$$
\left|\left\langle\varphi_k,\mathrm{e}^{-\tau}\int_0^y r^2G_y(r,\tau)dr\right\rangle\right| \leq C_M\overline{\varepsilon}(\tau)^4 \quad \text{for } \tau\gg 1\,. \tag{4.16}
$$

Let us set:

$$
\begin{aligned}
\int_0^y \Phi_y(r,\tau)\left(G_y(r,\tau)-\overline{G}_y(r,\tau)\right) r\,dr = & \int_1^y \Phi_y(r,\tau)\left(G_y(r,\tau)-\overline{G}_y(r,\tau)\right) r\,dr \\
+ \int_0^1 \Phi_y(r,\tau)\left(G_y(r,\tau)-\overline{G}_y(r,\tau)\right) r\,dr \equiv & \overline{m}(y,\tau) + B(\tau).
\end{aligned}
$$

Arguing as in the previous case, we readily obtain that:

$$
|\langle\varphi_k,\overline{m}(r,\tau)\rangle| \leq C_M\,(\overline{\varepsilon}(\tau))^{4-2\theta} \quad \text{for } k=0,1, \text{ and } \tau\gg 1\,. \tag{4.17}
$$

Moreover, in view of (4.2), there holds:

$$
|\,B(\tau)\,| \leq C_M\,(\overline{\varepsilon}(\tau))^2\,\tau^{-1/2} \quad \text{for } \tau\gg 1\,. \tag{4.18}
$$

Putting all these estimates together, it then turns out from (4.9) that:

$$
\left|a_k(\tau) + \int_\tau^\infty \mathrm{e}^{(1-k)(\tau-s)}\overline{\varepsilon}(s)^2 ds\right| \leq C_M\,(\overline{\varepsilon}(\tau))^{4-2\theta} \quad \text{for } k=0,1\,, \tag{4.19}
$$

where as by (4.9) and (3.14):

$$\mid \alpha_0 \mid + \mid \alpha_1 \mid \leq C\,\tau_0^{1/2}\,(\overline{\varepsilon}(\tau_0))^2\,. \tag{4.20}$$

4.4. – Analysis of $Q(y,\tau)$

We shall pay attention now to the solution $Q(y,\tau)$ of equation (2.34) for $\tau > \tau_0$ with initial datum:

$$Q(y,\tau) = \psi(y,\tau_0) - \sum_{k=0}^{1} \varphi_k \langle \varphi_{k,} \psi(,\tau_0)\rangle\,. \tag{4.21}$$

We then have:

LEMMA 4.2. *Assume that* $\Phi \in \mathcal{A}(\tau_0, \tau_1; 1)$ *for* $\tau_0 \gg 1$. *Then for* $\mid y \mid \geq (\overline{\varepsilon}(\tau))^\theta$ *there holds:*

$$\begin{aligned}
&\left| Q(y,\tau) - \gamma\varepsilon(\tau_0) S(\tau-\tau_0) F(y)\right| \\
&\leq \frac{\gamma}{\pi}\int_{\Sigma}^{\infty} \varepsilon(s)^2 \eta^{-1} e^{-\eta} d\eta \\
&+ \frac{\gamma}{\pi}\int_{\Sigma}^{\infty} \varepsilon(s)^2 (4\eta^2)^{-1} y^2 e^{-\eta} d\eta + \gamma \int_{\tau_0}^{\beta} e^{\tau_1}\varepsilon(s)^2 K(y,\tau-s) ds \\
&+ \gamma \int_{\beta}^{\tau} e^{\tau-s}\varepsilon(s)^2 \Big(A_1 + A_2\left(1 - e^{-(\tau-s)}\right)\Big) ds \\
&+ C_M \overline{\varepsilon}(\tau)^{2-\theta}\left(1+y^4\right),
\end{aligned} \tag{4.22}$$

where $\beta(\tau) = \max\{\tau-1, \tau_0\}$, $s = \tau - \log\left(1 + \frac{y^2}{4\eta^2}\right)$, $\sum = \frac{y^2}{4(e^{\beta}-1)}$, $K(y,\tau)$ *is a smooth function that satisfies* $\mid K(y,\tau) \mid \leq C e^{-2\tau}(1+y^2)$ *for* $\tau \geq 1$, A_1 *and* A_2 *are suitable real constants,* $F(y)$ *is the function defined in* (2.35), *and* $S(\tau)$ *is the semigroup correponding to the linear operator* A *in* (2.30).

The proof of this result is similar to that of Lemma 4.2 in [HV1], and will be omitted here. We shall also require in what follows an estimate for $\frac{\partial Q}{\partial \tau}$. This is provided by the following:

LEMMA 4.3. *Assume that* $\Phi \in \mathcal{A}(\tau_0, \tau_1; 1)$ *for* $\tau_0 \gg 1$. *Then:*

$$\left|\frac{\partial Q}{\partial \tau}(y,\tau)\right| \leq C\varepsilon(\tau)^2\left(1 + M\tau^{-1/2}(1 + \mid \log y \mid)\right). \tag{4.23}$$

where $C > 0$ *is independent on* M.

We shall prove Lemma 4.3 in Section 7 (see Subsection 7.2 there).

4.5. – Approximating the remainder function by $Q(y,\tau)$

As a next step, we shall aproximate function $R(y,\tau)$ given in (4.8) by $Q(y,\tau)$. To this end, we split R in two terms as follows:

$$R(y,\tau) = R_1(y,\tau) + R_2(y,\tau)\,, \tag{4.24}$$

where R_1 solves:

$$R_\tau = R_{yy} + \left(\frac{1}{y} - \frac{y}{2}\right) R_y + R + g(y,\tau) - \sum_{k=0}^{1} \varphi_k \langle \varphi_k, g\rangle\,, \tag{4.25a}$$

for $\tau > \tau_0$, and:

$$R_1(y,\tau_0) = Q(y,\tau_0)\,. \tag{4.25b}$$

On the other hand, R_2 satisfies:

$$R_\tau = R_{yy} + \left(\frac{1}{y} - \frac{y}{2}\right) R_y + R + \ell(y,\tau) + m(y,\tau) - \sum_{k=0}^{1} \varphi_k \langle \varphi_k, \ell + m\rangle\,, \tag{4.26a}$$

for $\tau > \tau_0$, and:

$$R_2(y,\tau_0) = 0\,. \tag{4.26b}$$

Function R_1 can now be analysed exactly as in [HV1], Subsection 4.2. In particular, one has that:

$$|R_1(y,\tau) - Q(y,\tau)| \le C_M(\bar{\varepsilon}(\tau))^{2+2\theta}(1+y^4) \quad \text{for } y \ge \bar{\varepsilon}(\tau)^\theta\,. \tag{4.27}$$

As to R_2, we first observe that by (4.13) we have that:

$$|\,\bar{\ell}(y,\tau)\,| \le C_M \left((\bar{\varepsilon}(\tau))^{4-2\theta}(1+y^4) + \bar{\varepsilon}(\tau)^2 \left(\left(\frac{y}{\varepsilon(\tau)}\right)^2 + 1 \right)^{-1} \right) \tag{4.28a}$$

and in a similar way we obtain that:

$$|\,\overline{m}(y,\tau)\,| \le \left(C_M \bar{\varepsilon}(\tau))^{4-2\theta}(1+y^4) + \bar{\varepsilon}(\tau)^2 \left(\left(\frac{y}{\varepsilon(\tau)}\right)^2 + 1 \right)^{-1} \right)\,. \tag{4.28b}$$

On the other hand, since:

$$(A(\tau) + B(\tau)) = \sum_{k=0}^{1} \varphi_k \langle \varphi_k, A(\tau) + B(\tau)\rangle\,,$$

a routine computation reveals that:

$$|\,R_2(y,\tau)\,| \le C_M\,(\bar{\varepsilon}(\tau))^{4-\theta}\left(1+y^4\right) \quad \text{for } y \ge \bar{\varepsilon}(\tau)^\theta\,, \tag{4.29}$$

so that by (4.26)-(4.28) we finally obtain:

$$|\,R(y,\tau) - Q(y,\tau)\,| \le C_M\,(\bar{\varepsilon}(\tau))^{2+2\theta}\left(1+y^4\right) \quad \text{for } y \ge \bar{\varepsilon}(\tau)^\theta\,. \tag{4.30}$$

5. – Stabilization in the inner region

We now turn our attention to the region where $|\, y \,| \ll 1$ and τ is large enough. Our first goal consists in obtaining some rough bounds on $\dot{a}_k(\tau)$ for $k = 0, 1$. Differentiating both sides of (4.9) with respect to τ yields:

$$|\, \dot{a}_0(\tau) \,| \leq C_M \bar{\varepsilon}(\tau)^2, \tag{5.1}$$

$$|\, \dot{a}_1(\tau) \,| \leq C \bar{\varepsilon}(\tau)^2, \tag{5.2}$$

where constant C in (5.2) does not depend on M. Notice that dependence of C_M on M in (5.1) arises from the term $A(\tau)$ in the representation formula (4.11). We shall estimate this term later, but to begin with we want to obtain suitable bounds for $\ell(y, \tau)$ and $m(y, \tau)$ (defined in (4.3)) in the region $|\, \xi \,| \leq 1$ where $\xi = \frac{y}{\varepsilon(\tau)}$. Let us set:

$$\begin{aligned} \ell(y, \tau) &= -\chi y J(y, \tau) \Phi(y, \tau) \\ &\quad + \chi \int_0^y \frac{\partial}{\partial r} (r J(r, \tau)) \, \Phi(r, \tau) dr \equiv \ell_1(y, \tau) + \ell_2(y, \tau)\,. \end{aligned} \tag{5.3}$$

In view of (2.17), we see that:

$$\begin{aligned} |\, y J(y, \tau) \,| &\leq C \left| e^{\frac{\Gamma y^2}{4}} \int_y^\infty r \Phi(r, \tau) e^{-\frac{\Gamma r^2}{4}} dr - \int_0^\infty r \Phi(r, \tau) e^{-\frac{\Gamma r^2}{4}} dr \right| \\ &\leq C \left(\left| e^{\frac{\Gamma y^2}{4}} - 1 \right| \left| \int_0^\infty r \Phi(r, \tau) e^{-\frac{\Gamma r^2}{4}} dr \right| + \left| \int_0^y r \Phi(r, \tau) e^{-\frac{\Gamma r^2}{4}} dr \right| \right) \\ &\leq C y^2\,, \end{aligned}$$

whence:

$$|\, \ell_1(y, \tau) \,| \leq C_M \varepsilon(\tau)^2 \xi^4 \quad \text{for } 0 \leq \xi \leq 1. \tag{5.4}$$

On the other hand, if $0 < y < 1$,

$$\begin{aligned} |\, \ell_1(y, \tau) \,| &\leq C \int_0^y |\, \Phi(r, \tau) \,| \left| \frac{\partial}{\partial r} \left(e^{\frac{\Gamma r^2}{4}} \int_r^\infty s \Phi(s, \tau) e^{-\frac{\Gamma s^2}{4}} ds \right) \right| dr \\ &\leq C \left(\int_0^y r \,|\, \Phi(r, \tau) \,|\, dr + \int_0^y r \,|\, \Phi(r, \tau) \,|^2 \, dr \right) \\ &\leq C_M \varepsilon(\tau)^2 \xi^4 \quad , \text{ for } 0 \leq \xi \leq 1\,. \end{aligned} \tag{5.5}$$

Putting together (5.4) and (5.5), we obtain:

$$|\, \ell(y, \tau) \,| \leq C_M \varepsilon(\tau)^2 \xi^4 \quad \text{for} \quad 0 \leq \xi \leq 1\,. \tag{5.6}$$

Combining this estimate with the bounds for $A(\tau)$ and $\overline{\ell}(y,\tau)$ obtained in (4.12) and (4.13), we deduce that:

$$|\, \ell(y,\tau)\, | \le C_M \frac{\varepsilon(\tau)^2\xi^4}{(\xi^2+1)^2} \quad \text{for} \quad 0 \le y \le 1\,. \tag{5.7}$$

We may obtain improved estimates for $m(y,\tau)$ in a similar way. In particular, using Lemma 4.1 and (2.17), we readily see that:

$$\left| e^{-\tau} \int_0^y r^2 G_y(r,\tau)dr \right| \le C\overline{\varepsilon}(\tau)^2\xi^3 \quad \text{for} \quad 0 \le \xi \le 1\,,$$

whereas from (3.3) and Lemma 4.1 we derive:

$$\left| \int_0^y r\Phi_y(r,\tau)\left(G_y(r,\tau) - \overline{G}_y(r,\tau)\right) dr \right| \le C_M \varepsilon(\tau)^2\xi^3,$$

for $0 \le \xi \le 1$. Putting together these two inequalities and (5.7), we eventually obtain:

$$|\, \ell(y,\tau) + m(y,\tau)\, | \le C_M \frac{\varepsilon(\tau)^2\xi^3}{(1+\xi^3)} \quad \text{for} \quad 0 \le y \le 1\,. \tag{5.8}$$

5.1. – Convergence towards a stationary profile

We next set out to obtain suitable sub-and supersolutions. As in [HV1], we define for any given $R > 0$:

$$w_R(\eta) = \frac{4\pi}{\chi}\left(\eta^2 - R^2\log(R^2+\eta^2) + R^2\log R^2\right) \tag{5.9}$$

which satisfies the equation:

$$w'' + \frac{w'}{\eta} + \left(\frac{\chi}{4\pi}\left(\frac{w'}{\eta}\right)^2 - 4\frac{w'}{\eta}\right) = 0\,. \tag{5.10}$$

Consider now the differential equation:

$$w'' + \frac{w'}{\eta} + \left(\frac{\chi}{4\pi}\left(\frac{w'}{\eta}\right)^2 - 4\frac{w'}{\eta}\right) - \lambda_0\left(\frac{\eta w'}{2} - w\right) = 0\,, \tag{5.11}$$

where λ_0 is a small parameter, $0 < \lambda_0 \ll 1$. Standard ODE theory shows then that there exist solutions $\widetilde{w}_R(\eta)$ such that:

$$\widetilde{w}_R(\eta) = O\left(\eta^4\right) \quad \text{as } \eta \to 0\,,$$

$$\widetilde{w}_R(\eta) = w_R(\eta) + O\left(\lambda_0 R^2 \mid \log R \mid\right)\,,$$

$$\text{as } \lambda_0 \to 0 \text{ and } R \to 0, \text{ uniformly on } \eta \in [0,1]\,.$$

Then there holds:

LEMMA. *For any* λ_0 *and* R *small enough with* $\lambda_0 \gg \overline{\varepsilon}(\tau)$, *there exist functions:*

$$Z^{\pm}(\eta) \equiv Z^{\pm}_{\lambda_0,R}(\eta) \quad \textit{and } L^{\pm}(\eta),$$

satisfying:

$$0 < w_R(\eta) - w_{BR}(\eta) \leq \; Z^{\pm}_{\lambda 0,R}(\eta) \leq \; w_{ER}(\eta) - w_R(\eta)\,, \tag{5.12}$$

for some constants $B > 1$ *and* $E \in (0, 1)$ *that depend only on* χ, *and such that the corresponding functions:*

$$\overline{w}_+(\eta,\tau) = \widetilde{w}_R(\eta) + e^{-\tau/100} Z^+(\eta) + L^+(\eta)\,, \tag{5.13a}$$

$$\overline{w}_-(\eta,\tau) = \widetilde{w}_R(\eta) - e^{-\tau/100} Z^-(\eta) + L^-(\eta)\,, \tag{5.13b}$$

are respectively super and subsolutions for the equations:

$$w_\tau = w_{\eta\eta} + \frac{w_\eta}{\eta} + \left(\frac{\chi}{4\pi}\left(\frac{w_\eta}{\eta}\right)^2 - 4\frac{w_\eta}{\eta}\right) - \lambda_0\left(\frac{\eta w_\eta}{2} - w\right) \pm C_M \frac{\varepsilon(\tau)^2\xi^2}{(1+\xi^2)} \tag{5.14}$$

on the region $0 \leq \eta \leq 1$, $\tau \geq 0$, *where:*

$$|\, L^{\pm}(\eta) \,| \leq C_M \frac{\varepsilon(\tau)^2\xi^4}{(1+\xi^2)^2}\,.$$

The proof of Lemma 4.3 is entirely similar to that of Lemma 4.4 in [HV1]. The only differences with the proof in that article arise from the presence of the term $\left(\pm C_M \frac{\varepsilon(\tau)^2\xi^2}{(1+\xi^2)}\right)$, which is easily seen to introduce very small perturbations in the region under consideration. We therefore shall omit further details.

As a next step, we shall use the sub- and supersolutions provided by Lemma 4.5 to prove that $\Phi(y, \tau)$ is close to a stationary solution when $y \leq \overline{\varepsilon}(\tau)^\theta$ with $\theta \in (0, 1)$. To this end, we observe that, by (4.23), (4.30), (5.1) and (5.2),

$$\begin{aligned} &\left| W\left(\delta(\overline{\tau}), \overline{\tau}\right) - W\left(\delta(\overline{\tau}), \tau\right)\right| \leq C_M \overline{\varepsilon}(\tau)^2\, \tau^{-1} \\ &\text{for } \;|\, \tau - \overline{\tau} \,| \leq \frac{1}{\overline{\tau}}, \;\text{ where } \delta(\tau) = (\overline{\varepsilon}(\tau))^\theta \;\text{ and } \overline{\tau} \geq \tau_0\,. \end{aligned} \tag{5.15}$$

As in [HV1], Subsection 4.5, we define:

$$w(\eta, s) = (\delta(\overline{\tau}))^{-2}\, W\left(\delta(\overline{\tau})\eta, \overline{\tau} + \delta(\overline{\tau})^2 s\right). \tag{5.16}$$

In view of (4.3), $w(\eta, \tau)$ satisfies:

$$\begin{aligned} w_s = w_{\eta\eta} + \frac{w_\eta}{\eta} - \delta(\overline{\tau})^2\left(\frac{w_\eta}{2} - w\right) + \left(\frac{\chi}{4\pi}\left(\frac{w_\eta}{\eta}\right)^2 - 4\frac{w_\eta}{\eta}\right) \\ + \ell\left(\delta(\overline{\tau})\eta, \overline{\tau} + \delta(\overline{\tau})^2 s\right) + m\left(\delta(\overline{\tau})\eta, \overline{\tau} + \delta(\overline{\tau})^2 s\right)\,. \end{aligned} \tag{5.17}$$

We shall use (5.8) and Lemma 5.1 to obtain sub- and supersolutions for (5.17). To this end, we define $R(\tau, \overline{\tau})$ as follows:

$$\widetilde{w}_{R(\tau,\overline{\tau})}(1) = W\left(\delta(\overline{\tau}), \tau\right) . \tag{5.18}$$

By Lemma 5.1 and (5.15), we now see that:

$$\begin{gathered}\left|w(\eta, \tau) - \widetilde{w}_{R(\tau,\overline{\tau})}(\eta)\right| \leq C_M \left|\widetilde{w}_{R(\tau,\overline{\tau})}(\eta) - \widetilde{w}_{BR(\tau,\overline{\tau})}(\eta)\right| \quad \text{on } 0 \leq \eta \leq 1, \\ \text{for } \frac{1}{2\overline{\tau}} \leq \mid \tau - \overline{\tau} \mid \leq \frac{1}{\overline{\tau}} \text{ and some } B > 1 .\end{gathered} \tag{5.19}$$

If $\overline{\tau} \sim \tau_0$, a similar estimate can be obtained as in Subsection 4.6 in [HV1]. Moreover, we can obtain bounds for the derivatives of Φ as follows. Define:

$$v(\lambda, s) = \frac{1}{\omega^2} W\left(\lambda\omega, \omega^2 s + \overline{\tau}\right) \quad ; \quad \overline{\varepsilon}\,(\overline{\tau}) \leq \omega \leq 1 .$$

By our previous results, W is close to a stationary solution $w_{\varepsilon(\tau)}(\eta)$ with an error of order:

$$\tau^{-3/2}\left(\mid \varepsilon^2 \log \varepsilon \mid + \varepsilon^2 \mid \log\left(\xi^2 + R^2\right) \mid\right) .$$

On its turn, $v(\lambda, s)$ satisfies a parabolic equation for $\frac{1}{2} \leq \mid \lambda \mid \leq 1$. Then by standard parabolic theory, $\frac{\partial v}{\partial \lambda} \sim \frac{\partial w}{\partial \lambda} \sim \frac{\varepsilon^2}{\omega^2} + O\left(\frac{\varepsilon^2}{\omega^2 \tau}\right)$ at the interior of such region. Hence:

$$\Phi = y^{-1} W\left(\lambda\omega, \overline{\tau} + \omega^2 s\right) = \frac{8\pi}{\chi} + \frac{v_\lambda}{\lambda} \sim \frac{8\pi}{\chi} + W_\lambda + O\left(\frac{\varepsilon^2}{\omega^2 \tau}\right),$$

on $\frac{1}{2} < \mid \lambda \mid < 1$, so that Φ also behaves as the corresponding stationary solution. Summarising, we have that:

$$\left|\Phi(y, \tau) - \frac{8\pi}{\chi}\left(\frac{y}{\varepsilon}\right)^2 \left(\left(\frac{y}{\varepsilon}\right)^2 + 1\right)^{-1}\right| \leq \frac{C_M}{\tau} \cdot \frac{\xi^2}{(\xi^2 + 1)^2} \text{ for } y \leq \overline{\varepsilon}^{\theta} . \tag{5.20}$$

5.2. – Asymptotic behaviour of $A(\tau)$

We next use (5.20) to improve our previous estimates on $A(\tau)$. To this end, we compute:

$$
\begin{aligned}
A(\tau) &= -\chi \int_0^1 r\Phi_y(r,\tau)J(r,\tau)dr = -\chi J(1,\tau)\left(\Phi - \frac{8\pi}{\chi}\right)(1,\tau) \\
&\quad + \chi \int_0^1 \left(\Phi(r,\tau) - \frac{8\pi}{\chi}\right)\frac{\partial}{\partial r}\left(rJ(r,\tau)\right)dr = \mathrm{O}\left(\overline{\varepsilon}(\tau)^{4-2\theta}\right) \\
&\quad + \frac{\Gamma\chi}{4\pi}\int_0^1 \left(\Phi(r,\tau) - \frac{8\pi}{\chi}\right)\frac{\partial}{\partial r}\left(e^{\frac{\Gamma r^2}{4}}\int_r^\infty \lambda\Phi(\lambda,\tau)e^{-\frac{\Gamma\lambda^2}{4}}d\lambda\right) \\
&= \mathrm{O}\left(\overline{\varepsilon}(\tau)^{4-2\theta}\right) - 2\Gamma\int_0^1\left(\left(\frac{r}{\varepsilon}\right)^2+1\right)^{-1} \\
&\quad \times \left(\frac{\Gamma r}{2}e^{\frac{\Gamma r^2}{4}}\int_r^\infty \lambda\Phi(\lambda,\tau)e^{-\frac{\Gamma\lambda^2}{4}}d\lambda - r\Phi(r,\tau)\right)dr \\
&\quad + \mathrm{O}\left(\frac{1}{\tau}\right)\int_0^1\left(\frac{r}{\varepsilon}\right)^2\left(\left(\frac{r}{\varepsilon}\right)^2+1\right)^{-1} \\
&\quad \times \left(\frac{\Gamma r}{2}\mathrm{e}^{\frac{\Gamma r^2}{4}}\int_r^\infty \lambda\Phi(\lambda,\tau)\mathrm{e}^{-\frac{\Gamma\lambda^2}{4}}d\lambda - r\Phi(r,\tau)\right)dr\,.
\end{aligned}
$$

We may then argue as in Subsection 4.3 to obtain that:

$$
\begin{aligned}
A(\tau) &= 2\Gamma\int_0^1\left(\left(\frac{r}{\varepsilon}\right)^2+1\right)^{-1} r\left(\Phi(r,\tau)-\frac{8\pi}{\chi}\right)dr + \mathrm{O}\left(\mathrm{C_M}\overline{\varepsilon}^2\tau^{-1/2}\right) \\
&= 16\pi\Gamma\int_0^1\left(\left(\frac{r}{\varepsilon}\right)^2+1\right)^{-2} r\,dr + \mathrm{O}\left(\mathrm{C_M}\overline{\varepsilon}^2\tau^{-1/2}\right),
\end{aligned}
$$

whence:

(5.21) $$| A(\tau) - 8\pi\Gamma\varepsilon(\tau)^2 | \leq C_M\varepsilon(\tau)^2\tau^{-1/2}.$$

5.3. – Regularizing the rescaled free boundary

To proceed further, we observe that (5.21) allows to improve (5.1) as follows:

(5.22) $$| \dot{a}_0(\tau) | \leq C\overline{\varepsilon}(\tau)^2,$$

where C is now independent of M. Given $\overline{\tau} \geq \tau_0$, we now define $n(\tau,\overline{\tau})$ by means of the relation:

(5.23) $$(\delta(\overline{\tau}))^2 n(\tau,\overline{\tau}) = \frac{4\pi}{\chi} + a_0(\tau)\varphi_0 + a_1(\tau)\varphi_1\left(\delta(\overline{\tau})\right) + Q\left(\delta(\overline{\tau}),\tau\right).$$

Taking into account (4.23), (5.2) and (5.22), we readily obtain that:

$$\left|\frac{\partial n}{\partial \tau}(\tau,\overline{\tau})\right| \leq C\left(\frac{\overline{\varepsilon}(\tau)}{\delta(\overline{\tau})}\right)^2 \quad \text{with } C \text{ independent of } M. \tag{5.24}$$

We now define $r(\tau,\overline{\tau})$ by means of the formula:

$$\widetilde{w}_{r(\tau)}(1) \equiv \widetilde{w}_{r(\tau,\overline{\tau})}(1) = n(\tau,\overline{\tau}). \tag{5.25}$$

Recalling the definition of $\widetilde{W_R}$ in (5.11) and taking advantage of (5.25), we deduce that:

$$\left|\frac{\partial r}{\partial \tau}(\tau,\overline{\tau})\right| \leq C(r \mid \log r \mid)^{-1}\left|\frac{\partial n}{\partial \tau}(r,\overline{\tau})\right| \leq C\left(\frac{\overline{\varepsilon}(\tau)}{\delta(\overline{\tau})}\right)\tau^{-1/2}. \tag{5.26}$$

We now define $\mu(\eta,\tau)$ as follows:

$$\mu(\eta,\tau) = w(\eta,\tau) - \widetilde{w}_{r(\tau)}(\eta). \tag{5.27}$$

In view of (5.17), it then turns out that:

$$\begin{aligned}\delta(\overline{\tau})^2\left((\widetilde{w}_r)_\tau + \mu_\tau\right) &= \mu_{\eta\eta} + \frac{\mu_\eta}{\eta} - \delta(\overline{\tau})^2\left(\frac{\eta\mu_\eta}{2} - \mu\right)\\ &\quad + \left(\frac{\chi}{4\pi}\left(\left(\frac{(\widetilde{w}_r)_\eta + \mu_\eta}{2}\right)^2 - \frac{((\widetilde{w}_r)_\eta)}{\eta^2}\right)^2 - 4\frac{\mu_\eta}{\eta}\right)\\ &\quad + \ell\left(\delta(\overline{\tau})\eta,\tau\right) + m\left(\delta(\overline{\tau})\eta,\tau\right).\end{aligned} \tag{5.28}$$

We next claim that:

$$|(\widetilde{w}_r)_\tau| \leq C\left(\frac{\overline{\varepsilon}(\tau)}{\delta(\overline{\tau})}\right)^2. \tag{5.29}$$

We may now use Lemma 5.1 (or, more precisely, a slight modification of it, to take into account that $\widetilde{w}_{R(\tau)}$ changes with time), to prove that $\mu(\eta,\tau)$ decays exponentially fast in intervals, say, of the type $\mid \tau - \overline{\tau} \mid \leq 1$. The term $\left(\delta(\overline{\tau})^2(\widetilde{w}_r)_\tau\right)$ in (5.28) yields a correction of order $\varepsilon(\tau)^2$, and the boundary data give terms of order $\overline{\varepsilon}^{2-\theta}\left(\delta(\overline{\tau})\right)^{-2}$. Assuming enough regularity on the correponding initial values, we then obtain that:

$$\mid \mu(\eta,\tau) \mid \leq C_M\left(\overline{\varepsilon}(\tau)\right)^{2-\theta}\frac{\xi^4}{(\xi^2+1)^2}. \tag{5.30}$$

Suitable bounds for the derivatives of μ can be obtained by rescaling. Actually, if we define:

$$v(\sigma,\tau) = \beta^{-2}\cdot\mu(\beta\sigma,\tau) \quad \text{for } \beta \leq 1.$$

Then v satisfies:

$$\begin{aligned}\beta^2\left(\delta(\overline{\tau})\right)^2 v_\tau &= v_{\sigma\sigma} + \frac{v_\sigma}{\sigma} - \beta^2\left(\delta(\overline{\tau})\right)^2\left(\frac{\sigma v_\sigma}{2} - v\right)\\ &+ \left(\frac{\chi}{4\pi}\left(\left(\frac{(\widetilde{w}_{r/\beta})_\sigma + v_\sigma}{\sigma}\right)^2 - \left(\frac{(\widetilde{w}_{r/\beta})_\sigma}{\sigma}\right)^2\right) - 4\frac{v_\sigma}{\sigma}\right)\\ &+ \mathrm{O}\left(\frac{\overline{\varepsilon}^2\xi^2}{\xi^2+1}\right).\end{aligned}$$

Taking into account (5.30), we deduce by regularizing effects for parabolic equations that:

$$|\ v_\sigma\ | \leq C_M\frac{(\overline{\varepsilon})^{2-\sigma}}{\beta^2},$$

on the set $\frac{1}{2} \leq |\ \sigma\ | \leq 1$. By our choice of function v above, one has that $\mu_\eta = \beta v_\sigma(\frac{\eta}{\beta}, \tau)$, so that:

$$\sup_{\frac{\beta}{2}\leq|\eta|\leq\beta}\ |\ \mu_\eta(\eta,\tau)\ | \leq C_M\frac{(\overline{\varepsilon})^{2-\theta}}{\beta}. \tag{5.31}$$

Set now $\varphi(\eta,\tau) = \frac{w_\eta}{\eta}$. By (5.27) and (5.31), one then has that:

$$\left|\varphi(\eta,\tau) - \frac{8\pi}{\chi}\cdot\frac{\eta^2}{\eta^2+r(\tau)^2}\right| \leq C_M\frac{(\overline{\varepsilon}(\tau))^{2-\theta}}{\eta^2}.$$

Back to the original variables, this reads:

$$\left|\Phi(y,\tau) - \frac{8\pi}{\chi}y^2\left(y^2+(\delta(\overline{\tau})r(r,\overline{\tau}))^2\right)^{-1}\right| \leq C_M(\overline{\varepsilon})^{2-\theta}y^2\left(y^2+\overline{\varepsilon}^2\right)^{-2}.$$

Let us write now:

$$\widetilde{\varepsilon}(\tau,\overline{\tau}) = \delta(\overline{\tau})r(\tau,\overline{\tau}).$$

In view of (5.26), we now have that:

$$\left|\frac{\partial\widetilde{\varepsilon}}{\partial\tau}(\tau,\overline{\tau})\right| \leq C\overline{\varepsilon}(\tau)\tau^{-1/2}. \tag{5.32}$$

By our previous definition, function $\widetilde{\varepsilon}(\tau,\overline{\tau})$ is such that:

$$\left|\Phi(y,\tau) - \frac{8\pi}{\chi}y^2\left(y^2+\widetilde{\varepsilon}(\tau,\overline{\tau})^2\right)^{-1}\right| \leq C_M\overline{\varepsilon}(\tau)^{2+\theta}y^2\left(y^2+\overline{\varepsilon}(\tau)^2\right)^{-2}. \tag{5.33}$$

Notice that we may define $\widetilde{\varepsilon}(\tau,\overline{\tau})$ for any $\overline{\tau} \geq \tau_0$ and any τ such that $\mid \tau - \overline{\tau} \mid \leq \frac{1}{\overline{\tau}}$. Suppose that we have $\overline{\tau}_1, \overline{\tau}_2$, both larger or equal than τ_0 and such that $\mid \overline{\tau}_1 - \overline{\tau}_2 \mid \leq \frac{1}{\overline{\tau}_1}$. We then set:

$$\widetilde{\varepsilon}_1(\tau) = \widetilde{\varepsilon}(\tau, \overline{\tau}_1) \quad , \quad \widetilde{\varepsilon}_2(\tau) = \widetilde{\varepsilon}(\tau, \tau_2) .$$

Taking advantage of (5.33), we then obtain:

$$\left| \frac{y^2}{y^2 + (\widetilde{\varepsilon}_1)^2} - \frac{y^2}{y^2 + (\widetilde{\varepsilon}_2)^2} \right| \leq C_M \cdot \overline{\varepsilon}^{2+\theta} \cdot \frac{y^2}{(y^2 + \overline{\varepsilon}^2)^2} ,$$

whence:

$$\frac{y^2}{(y^2 + \overline{\varepsilon}^2)^2} \left| \widetilde{\varepsilon}_1{}^2 - \widetilde{\varepsilon}_2{}^2 \right| \leq C \overline{\varepsilon}^{2+\theta} \frac{y^2}{(y^2 + \overline{\varepsilon}^2)^2} ,$$

which yields at once the estimate:

$$\mid \widetilde{\varepsilon}_1(\tau) - \widetilde{\varepsilon}_2(\tau) \mid \leq C \left(\overline{\varepsilon}(\tau)\right)^{1+\theta} .$$

Since (5.33) continues to hold for any linear interpolation of $\widetilde{\varepsilon}_1(\tau)$ and $\widetilde{\varepsilon}_2(\tau)$ in their common region of validity, we easily deduce that it is possible to deform $\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_2$ to obtain a globally defined function $\widetilde{\varepsilon}(\tau)$ such that:

$$\left| \Phi(y,\tau) - \frac{8\pi}{\chi} \frac{y^2}{y^2 + (\widetilde{\varepsilon}(\tau))^2} \right| \leq C_M (\overline{\varepsilon}(\tau))^{2+\theta} \frac{y^2}{(y^2 + (\overline{\varepsilon}(\tau))^2)^2} , \tag{5.34}$$

and moreover:

$$\mid \dot{\widetilde{\varepsilon}}(\tau) \mid \leq C \mid \widetilde{\varepsilon}(\tau) \mid \tau^{-1/2} , \tag{5.35}$$

where C is now independent of M. Just as before, (5.34) gives:

$$\mid \widetilde{\varepsilon}(\tau) - \varepsilon(\tau) \mid \leq C(\overline{\varepsilon}(\tau))^{1-\sigma} \quad \text{for some } \sigma > 0 . \tag{5.36}$$

Indeed, if (5.36) fails to hold we would derive a contradiction with the definition of $\varepsilon(\tau)$ in (3.6).

5.4. – Deriving a crucial integral equation

At this juncture, we may repeat the steps in [HV1], Subsection 4.5 to obtain that:

$$\left| W(\delta(\tau),\tau) - (\delta(\tau))^2 w_{r(\tau)}(1) \right| \leq C \frac{\overline{\varepsilon}(\tau)^2}{\tau} .$$

This equation is entirely analogous to (4.31) in [HV1]. As remarked in Subsection 4.5 in that paper, it can be used to obtain that:

$$\lim_{\tau\to\infty} \left| \frac{\varepsilon(\tau)}{\overline{\varepsilon}(\tau)} \right| = 1 ,$$

where $\overline{\varepsilon}(\tau)$ is given in (3.1). This provides the crucial asymptotic estimate on the size of the inner layer for $\tau \gg 1$.

6. – Analysis of the solutions outside the inner layer

In this section we shall describe the behaviour of our solutions in regions where $y \gg \bar{\varepsilon}(\tau)$ and $\tau \gg 1$. To this end, two separate cases will be considered. In the first place, we shall examine:

6.1. – The case where $y \gg (\bar{\varepsilon}(\tau))^{\theta}$ for some $\theta > 0$ and $y = O(1)$

Let us assume first that $(\bar{\varepsilon}(\tau))^{\theta} \ll y \leq D$ for some $D > 0$ large but fixed. Keeping to the notation introduced in Subsection 4.5, we set:

$$Z = R_1 - Q,$$

where R_1 and Q are respectively given in (4.25) and (2.31). In particular $Z(y,\tau)$ satisfies:

$$Z_\tau = Z_{yy} + \left(\frac{1}{y} - \frac{y}{2}\right) Z_y + Z + \left(g(\cdot,\tau) - \gamma\varepsilon(\tau)^2\delta(y)\right)$$
$$- \sum_{k=0}^{1} \varphi_k \left(\langle\varphi_k, g\rangle - \gamma\varepsilon(\tau)^2\langle\varphi_k, \delta(y)\rangle\right) .$$

On the other hand, by (4.27) there holds:

$$| Z(y,\tau) | \leq C_M\,(\bar{\varepsilon}(\tau))^{2+2\theta} \quad \text{for } \bar{\varepsilon}(\tau)^{\theta} \leq y \leq 1 .$$

Using a rescaling argument similar to that in Subsection 5.4, coupled with standard regularizing effects for parabolic equations, we obtain:

$$\frac{1}{y}\left|\frac{\partial Z}{\partial y}\right| \leq C_M(\bar{\varepsilon})^{2+2\theta} \cdot y^{-2} \qquad \text{for } \bar{\varepsilon}^{\theta} \leq y \leq 1 .$$

Let $R_2(y,\tau)$ be the function defined in (4.24)-(4.26). By (4.29):

$$| R_2(y,\tau) | \leq C_M\,(\bar{\varepsilon}(\tau))^{4-\theta} \qquad \text{for } \bar{\varepsilon}^{\theta} \leq y \leq 1 .$$

As in the case of $R_1(y,\tau)$, one then has that:

$$\frac{1}{y}\left|\frac{\partial Z}{\partial y}\right| \leq C_M(\bar{\varepsilon})^{4-\theta} y^{-2} \qquad \text{for } \bar{\varepsilon}^{\theta} \leq y \leq 1 .$$

Recalling the definition of $\Phi(y,\tau)$ in (2.19), we then obtain:

$$\left|\Phi(y,\tau) - a_1(\tau)\frac{\varphi_1'(y)}{y} - \frac{Q_y(y,\tau)}{y}\right| \leq C_M\,(\bar{\varepsilon}(\tau))^{2+2\theta}\, y^{-2}, \tag{6.1}$$
for $\bar{\varepsilon}(\tau)^{\theta} \leq y \leq 1$.

Let us set now:

$$Q(y,\tau) = \widetilde{Q}(y,\tau) + P(y,\tau),$$

where $\widetilde{Q}(y,\tau)$ is defined in (2.34) with $\varepsilon(\tau)$ replaced by $\widetilde{\varepsilon}(\tau)$ there (this last function has been defined and analysed in Subsection 5.3). At $\tau = \tau_0$, we impose accordingly:

$$\widetilde{Q}(y,\tau_0) = \gamma\widetilde{\varepsilon}(\tau_0)^2 F(y).$$

Recalling (5.36), one then has that:

$$| P(y,\tau) | \leq C\,(\overline{\varepsilon}(\tau))^{2+\sigma/2} \qquad \text{for } \overline{\varepsilon}^{\theta} \leq y \leq D.$$

Using the differential equations satisfied by Q and $\widetilde{Q}$, rescaling techniques and regularizing results yield:

$$\left|\frac{1}{y}\left(Q_y(y,\tau) - \widetilde{Q}_y(y,\tau)\right)\right| \leq C\,(\overline{\varepsilon}(\tau))^{2+\sigma/2}\cdot y^{-2}, \qquad \text{for } \overline{\varepsilon}(\tau)^{\theta} \leq y \leq 1.$$

To estimate $\widetilde{Q}(y,\tau)$, we write:

$$\widetilde{Q}(y,\tau) = \gamma\,(\widetilde{\varepsilon}(\tau))^2\,F(y) + S(y,\tau),$$

so that S satisfies:

$$S_\tau = S_{yy} + \left(\frac{1}{y} - \frac{y}{2}\right) S_y + S - 2\gamma\widetilde{\varepsilon}\,\dot{\widetilde{\varepsilon}} F(y), \quad \text{for } y > 0, \tau > \tau_0\,. \tag{6.2a}$$

$$S(y,\tau_0) = 0,\; S_y(0,\tau) = 0 \quad \text{for } \tau \leq \tau_0\,, \tag{6.2b}$$

where $\langle S, \varphi_k\rangle = 0$ for $k = 0, 1$. We may then bound $S(y,\tau)$ by means of classical estimates on caloric kernels and standard regularizing effects. To this end, we observe that the source term in (6.2a) satisfies the bound:

$$| \widetilde{\varepsilon}\dot{\widetilde{\varepsilon}} | \leq C\overline{\varepsilon}^2\tau^{1/2},$$

whereas $F(y)$ has a logarithmic singularity at $y = 0$. For any $\alpha \in (0,1)$, S_y can be estimated in the C^α-norm to obtain that:

$$| S_y(y,\tau) | \leq C\overline{\varepsilon}(\tau)^2\tau^{-1/2}\cdot y^{\alpha},$$

while on the other hand:

$$\left|\frac{1}{y}\left(\widetilde{Q}_y(y,\tau) - \gamma\widetilde{\varepsilon}(\tau)^2\overline{\Gamma}(y)\right)\right| \leq C\overline{\varepsilon}(\tau)^2\tau^{-1/2}y^{\alpha-1},$$

Plugging these estimates in (6.1), we arrive at:

$$\left|\Phi(y,\tau) - a_1(\tau)\frac{\varphi_1'(y)}{y} - \gamma\tilde{\varepsilon}(\tau)^2\frac{F_y(y)}{y}\right| \leq C\left(\frac{\overline{\varepsilon}(\tau)^2}{\sqrt{\tau}}y^{\alpha-1} + \frac{\overline{\varepsilon}(\tau)^{2+\beta}}{y^2}\right), \tag{6.3}$$

for some $\beta > 0$, $\overline{\varepsilon}(\tau)^\theta \leq y \leq D$ and $\tau \gg 1$.

6.2. – Estimating derivatives of $\Phi(y,\tau)$

As a next step, we proceed to derive bounds on $\frac{\partial\Phi}{\partial y}$, $\frac{\partial\Phi}{\partial\tau}$, in the region considered in our previous paragraph. To this end, some approximations to the actual solutions will be in order. We first define $\Phi_1(s)$ as the (unique) solution of the problem:

$$\Phi'' - \frac{\Phi'}{s} + \frac{4s}{s^2+1}\Phi' + \frac{8s}{(s^2+1)^2}\Phi = \frac{8\pi}{\chi}\left(\frac{s^2}{(s^2+1)^2} + \frac{2\Gamma\log(s^2+1)}{(s^2+1)^2}\right), \tag{6.4a}$$

such that:

$$\Phi(s) = \mathrm{O}(\mathrm{s}^4) \quad \text{as } \mathrm{s} \to 0. \tag{6.4b}$$

It is then easily seen that:

$$\Phi_1(s) \sim \frac{r\pi}{\chi}\log s \quad \text{as } s \to \infty.$$

Set now $\xi = \dfrac{y}{\tilde{\varepsilon}(\tau)}$, and consider the function $\tilde{\Phi}(\xi,\tau)$ given by:

$$\tilde{\Phi}(\xi,\tau) = \frac{8\pi}{\chi}\cdot\frac{\xi^2}{\xi^2+1} + \tilde{\varepsilon}(\tau)^2\Phi_1(\xi). \tag{6.5}$$

A quick check reveals that:

$$\left|\frac{\partial\tilde{\Phi}}{\partial\tau}\right| \leq C\left(\frac{\xi^2}{\sqrt{\tau}(\xi^2+1)^2} + \tilde{\varepsilon}(\tau)^2(1+|\log y|)\right), \tag{6.6}$$

with C independent of M. On the other hand, let $\tilde{G}(\xi,\tau)$ be the function defined in (2.15) with Φ replaced by $\tilde{\Phi}$ there. We now claim that:

$$\left|\frac{\partial\tilde{G}}{\partial\tau}\right| \leq C\left(\tau^{-1/2}\left(\frac{\tilde{\chi}_1}{\xi^2} + (1+|\log\xi|)\overline{\chi}_1\right)\right) + \tilde{\varepsilon}^2|\log\tilde{\varepsilon}|(1+y^2)(1+|\log y|^2), \tag{6.7}$$

where C is independent of M, and $\tilde{\chi}_1$ (respectively $\overline{\chi}_1$) is equal to one if $\xi > 1$ and zero otherwise (respectively $\overline{\chi}_1 = 1 - \tilde{\chi}_1$). The proof of (6.7) is similar to that of Lemma 4.1 in Section 7, and will therefore be omitted. We next observe that, by the definition of $\widetilde{G}$, there holds:

$$\widetilde{G}_{\xi\xi} + \frac{\widetilde{G}_\xi}{\xi} - \frac{\Gamma}{2}\widetilde{\varepsilon}^2 \cdot \xi \widetilde{G}_\xi + \frac{\widetilde{\Phi}_\xi}{2\pi\xi} - \widetilde{\varepsilon}^2 \varphi_0 \langle \varphi_0, \frac{\widetilde{\Phi}_y}{2\pi y} \rangle = 0$$

Integrating this equation gives:

$$\widetilde{G}_\xi = -\frac{\exp\left(\frac{\Gamma}{4}\widetilde{\varepsilon}(\tau)^2\xi^2\right)}{2\pi\xi} \int_0^\xi \exp\left(-\frac{\Gamma}{4}\widetilde{\varepsilon}(\tau)^2 s^2\right) \widetilde{\Phi}_y(s,\tau) ds$$
$$- \frac{2\varphi_0}{\Gamma\xi} \langle \varphi_0, \frac{\widetilde{\Phi}_1}{2\pi y} \rangle \left(1 - \exp\left(\frac{\Gamma \widetilde{\varepsilon}(\tau)^2 \xi^2}{4}\right)\right).$$

On integrating by parts, this yields:

$$\widetilde{G}_\xi = -\frac{\widetilde{\Phi}(\xi, r)}{2\pi\xi} - \frac{\Gamma\widetilde{\varepsilon}(\tau)^2}{4\pi\xi} \exp\left(\frac{\Gamma}{4}\widetilde{\varepsilon}(\tau)^2\xi^2\right) \int_0^\xi \exp\left(-\frac{\Gamma\widetilde{\varepsilon}(\tau)^2 s^2}{4}\right) \widetilde{\Phi}(s,\tau) s ds$$
$$+ \frac{\Gamma\widetilde{\varepsilon}(\tau)^2}{4\pi\xi} \left(\int_0^\infty \widetilde{\Phi}(s,\tau) \exp\left(-\frac{\Gamma\widetilde{\varepsilon}(\tau)^2 s^2}{4}\right) s ds\right) \left(\exp\left(\frac{\Gamma\widetilde{\varepsilon}(\tau)^2\xi^2}{4}\right) - 1\right).$$

It will be convenient to rewrite the previous equation in the following form:

$$\begin{aligned} \widetilde{G}_\xi &= -\frac{\widetilde{\Phi}(\xi,\tau)}{2\pi\xi} - \frac{\Gamma\widetilde{\varepsilon}(\tau)^2}{4\pi\xi} \exp\left(\frac{\Gamma\widetilde{\varepsilon}(\tau)^2\xi^2}{4}\right) \\ &\quad \times \int_0^\xi \exp\left(-\frac{\Gamma\widetilde{\varepsilon}(\tau)^2 s^2}{4}\right) \left(\widetilde{\Phi} - \frac{8\pi}{\chi}\right) s ds \\ &\quad + \frac{\Gamma\widetilde{\varepsilon}(\tau)^2}{4\pi\xi} \left(\int_0^\infty \left(\widetilde{\Phi}(s,\tau) - \frac{8\pi}{\chi}\right) \exp\left(-\frac{\Gamma\widetilde{\varepsilon}(\tau)^2 s^2}{4}\right) s ds\right) \\ &\quad \times \left(\exp\left(\frac{\Gamma\widetilde{\varepsilon}(\tau)^2\xi^2}{4}\right) - 1\right). \end{aligned} \tag{6.8}$$

We now compute:

$$\begin{aligned} I_1 &\equiv \int_0^\infty \left(\widetilde{\Phi}(s,\tau) - \frac{8\pi}{\chi}\right) \exp\left(-\frac{\Gamma\widetilde{\varepsilon}(\tau)^2 s^2}{4}\right) s ds \\ &= -\frac{8\pi}{\chi} \int_0^\infty \frac{s}{s^2+1} \exp\left(-\frac{\Gamma\widetilde{\varepsilon}(\tau)^2 s^2}{4}\right) ds \\ &\quad + \widetilde{\varepsilon}(\tau)^2 \int_0^\infty \widetilde{\Phi}_1(s) \exp\left(-\frac{\Gamma\widetilde{\varepsilon}(\tau)^2 s^2}{4}\right) s ds = -\frac{8\pi}{\chi} \int_0^\infty \frac{y\exp(-\frac{\Gamma y^2}{4})}{y^2 + \widetilde{\varepsilon}(\tau)^2} dy \\ &\quad + \int_0^\infty \widetilde{\Phi}\left(y/\widetilde{\varepsilon}\right) y \exp\left(-\frac{\Gamma y^2}{4}\right) dy. \end{aligned}$$

whence:

$$| I_1 | \leq C \mid \log \widetilde{\varepsilon}(\tau) \mid .$$

On the other hand:

$$\begin{aligned} I_2 &= \int_0^{\xi} \exp\left(-\frac{\Gamma\widetilde{\varepsilon}(\tau)^2 s^2}{4}\right)\left(\widetilde{\Phi}(s,\tau) - \frac{8\pi}{\chi}\right) s ds \\ &= -\frac{8\pi}{\chi}\int_0^{\xi} \frac{s \exp\left(-\frac{\Gamma\widetilde{\varepsilon}(\tau)^2 s^2}{4}\right)}{s^2+1} ds + \widetilde{\varepsilon}(\tau)^2 \int_0^{\xi} \Phi_1(s) \exp\left(-\frac{\Gamma\widetilde{\varepsilon}(\tau)^2 s^2}{4}\right) ds . \end{aligned}$$

Let us assume now that $\widetilde{\varepsilon}(\tau)^2\xi^2 \leq 1$. Then:

$$I_2 = -\frac{4\pi}{\chi}\log\left(\xi^2+1\right) + O\left(\widetilde{\varepsilon}(\tau)^2\xi^2\left(\log\left(1+\xi^2\right)\right)\right).$$

In this case, we easily obtain from (6.8) the following estimate:

$$\widetilde{G}_{\xi} = -\frac{\widetilde{\Phi}(\xi,\tau)}{2\pi\xi} + \frac{\Gamma\widetilde{\varepsilon}(\tau)^2}{\chi}\cdot\frac{\log(\xi^2+1)}{\xi} + \frac{1}{\xi} O\left(\widetilde{\varepsilon}(\tau)^4\xi^2\log(1+\xi^2)\right), \tag{6.9}$$

provided that $\widetilde{\varepsilon}(\tau)^2\xi^2 \leq 1$.

To proceed further, we rewrite (6.8) in the form:

$$\begin{aligned} \widetilde{G}_{\xi} = &-\frac{\widetilde{\Phi}(\xi,\tau)}{2\pi\xi} + \frac{\Gamma\widetilde{\varepsilon}(\tau)^2}{4\pi\xi}\exp\left(\frac{\Gamma(\tau)^2\xi^2}{4}\right) \\ &\times \int_{\xi}^{\infty}\left(\widetilde{\Phi}(s,\tau) - \frac{8\pi}{\chi}\right)\exp\left(-\frac{\Gamma\widetilde{\varepsilon}(\tau)^2 s^2}{4}\right) s ds \\ &-\frac{\Gamma\widetilde{\varepsilon}(\tau)^2}{4\pi\xi}\int_0^{\infty}\left(\widetilde{\Phi}(s,\tau) - \frac{8\pi}{\chi}\right)\exp\left(-\frac{\Gamma\widetilde{\varepsilon}(\tau)^2 s^2}{4}\right) s ds . \end{aligned}$$

Recalling the definition of $\widetilde{\Phi}$ as well as our previous estimate on I_1 we arrive at:

$$\left|\widetilde{G}_{\xi} + \frac{\widetilde{\Phi}(\xi,\tau)}{2\pi\xi}\right| \leq \frac{C}{\xi}\widetilde{\varepsilon}(\tau)^2 \mid \log\widetilde{\varepsilon}(\tau) \mid (1+ \mid \log y \mid). \tag{6.10}$$

For $\theta > 0$, we now use (6.9) for $\xi \leq \widetilde{\varepsilon}^{\,\theta-2}$ and (6.10) for $\xi > \overline{\varepsilon}^{\theta-2}$ to eventually obtain that:

$$\left|\widetilde{G}_{\xi} + \frac{\widetilde{\Phi}(\xi,\tau)}{2\pi\xi} - \frac{\Gamma\widetilde{\varepsilon}(\tau)^2\log(\xi^2+1)}{\chi\xi}\right| \leq C(\overline{\varepsilon}(\tau))^{2+\frac{\theta}{2}} . \tag{6.11}$$

for $y \leq 1$.

Set now $\Phi_0(\xi) = \frac{8\pi}{\chi} \cdot \frac{\xi^2}{\xi^2+1}$. By our choice of $\widetilde{\Phi}$ in (6.5), we may now take advantage of (6.11) to obtain:

$$
\begin{aligned}
J &\equiv \widetilde{\Phi}_{yy} - \frac{\widetilde{\Phi}_y}{y} - \frac{y\widetilde{\Phi}_y}{2} - \chi \widetilde{G}_y \widetilde{\Phi}_y \\
&= \widetilde{\Phi}_{yy} - \frac{\widetilde{\Phi}_y}{y} - \frac{y\widetilde{\Phi}_y}{2} + \frac{\chi}{2\pi y}\widetilde{\Phi}\widetilde{\Phi}_y - \frac{\Gamma}{\xi}\widetilde{\varepsilon}(\tau)^2 \log(\xi^2+1)\widetilde{\Phi}_y + O\left(\overline{\varepsilon}^{2+\theta/2}\right)\widetilde{\Phi}_y \\
&= (\Phi_0)_{yy} - \frac{(\Phi_0)_y}{y} - \frac{J(\Phi_0)_y}{2} + \widetilde{\varepsilon}(\tau)^2\left((\Phi_1)_{yy} - \frac{(\Phi_1)_y}{y} - \frac{y(\Phi_1)_y}{2}\right) \\
&\quad + \frac{\chi}{2\pi y}(\Phi_0)(\Phi_0)_y + \frac{\chi\widetilde{\varepsilon}(\tau)^2}{2\pi y}(\Phi_0)(\Phi_1)_y + \frac{\chi\widetilde{\varepsilon}(\tau)^2}{2\pi y}\Phi_1(\Phi_0)_y + \frac{\chi\widetilde{\varepsilon}(\tau)^4}{2\pi y}\Phi_1(\Phi_1)_y \\
&\quad - \frac{\Gamma}{\xi}\widetilde{\varepsilon}(\tau)^2\log(\xi^2+1)(\Phi_0)_y - \frac{\Gamma}{\xi}\widetilde{\varepsilon}(\tau)^4\log(\xi^2+1)(\Phi_1)_y + O\left(\overline{\varepsilon}^{2+\theta/2}\right)\widetilde{\Phi}_y \\
&= \widetilde{\varepsilon}(\tau)^2\Bigg((\Phi_1)_{yy} - \frac{(\Phi_1)_y}{y} + \frac{\chi}{2\pi y}\Phi_0(\Phi_1)_y + \frac{\chi}{2\pi y}(\Phi_0)_y\Phi_1 \\
&\quad - \frac{\Gamma\log(\xi^2+1)}{\xi}(\Phi_0)_y\Bigg) - \frac{8\pi}{\chi}\frac{\xi^2}{(\xi^2+1)^2} + \frac{\chi\widetilde{\varepsilon}(\tau)^4}{2\pi y}\Phi_1(\Phi_1)_y \\
&\quad - \frac{\Gamma}{\xi}\widetilde{\varepsilon}(\tau)^4\log(\xi^2+1)(\Phi_1)_y - \frac{\widetilde{\varepsilon}(\tau)^2}{2}y(\Phi_1)_y + O\left(\overline{\varepsilon}^{2+\theta/2}\right)\widetilde{\Phi}_y \\
&= \frac{\chi\widetilde{\varepsilon}(\tau)^4}{2\pi y}\Phi_1(\Phi_1)_y - \frac{\widetilde{\varepsilon}(\tau)^2}{2}\xi(\Phi_1)_\xi - \frac{\Gamma}{\xi}\widetilde{\varepsilon}(\tau)^4\log(\xi^2+1)(\Phi_1)_y + O\left(\overline{\varepsilon}^{2+\theta/2}\right)\widetilde{\Phi}_y.
\end{aligned}
$$

Summarizing, we have that:

$$
\begin{aligned}
J = {} & \frac{\chi\widetilde{\varepsilon}(\tau)^4\Phi_1(\Phi_1)_y}{2\pi y} - \frac{\widetilde{\varepsilon}(\tau)^2\xi(\Phi_1)_\xi}{2} - \frac{\Gamma\widetilde{\varepsilon}(\tau)^4\log(\xi^2+1)(\Phi_1)_y}{\xi} \\
& + O\left(\overline{\varepsilon}^{2+\theta/2}\right)\widetilde{\Phi}_y\,.
\end{aligned} \tag{6.12}
$$

Let us set now:

$$z(y,\tau) = \Phi(y,\tau) - \widetilde{\Phi}(\xi,\tau)\,, \tag{6.13}$$

$$f(y,\tau) = G(y,\tau) - \widetilde{G}(\xi,\tau)\,. \tag{6.14}$$

By (2.14) and (6.12), we obtain:

$$
\begin{aligned}
z_\tau = {} & z_{yy} - \frac{z_y}{y} - \frac{yz_y}{2} - \chi\left(G_y\Phi_y - \widetilde{G}_y\widetilde{\Phi}_y\right) - 2\pi\chi e^{-\tau}yG_y \\
& + J - \widetilde{\Phi}_\tau = z_{yy} - \frac{z_y}{y} - \frac{yz_y}{2} - \chi\left(f_y\Phi_y + \widetilde{G}_y z_y\right) \\
& - 2\pi\chi e^{-\tau}yG_y + J - \widetilde{\Phi}_\tau\,,
\end{aligned} \tag{6.15}
$$

as well as:

$$f_\tau = f_{yy} + \frac{f_y}{y} - \frac{\Gamma_y f_y}{2} + \frac{z_y}{2\pi y} - \varphi_0 \langle \varphi_0, \frac{\varphi_y}{2\pi y} \rangle - \widetilde{G}_\tau . \tag{6.16}$$

Notice that from (5.34), (6.1) and (6.5), we have that:

$$| z(y,\tau) | \le C_M \left(\frac{(\overline{\varepsilon}(\tau))^{2+\beta} y^2}{\left(y^2 + \overline{\varepsilon}(\tau)^2\right)^2} + (\overline{\varepsilon}(\tau))^2 \mid \log \overline{\varepsilon}(\tau) \mid \widetilde{\chi}_1 \right) \tag{6.17}$$

for $0 < y \le 1$ and some $\beta > 0$, where $\widetilde{\chi}_1(\xi) = 1$ if $\xi > 1$ and $\widetilde{\chi}_1(\xi) = 0$ otherwise.

Concerning $f(y,\tau)$, we claim that the following estimate holds:

$$| f(y,\tau) | \le C\overline{\varepsilon}(\tau)^2 \left(\mid \log \overline{\varepsilon}(\tau) \mid^3 + \mid \log \overline{\varepsilon}(\tau) \mid^2 \mid \log \xi \mid \right) . \tag{6.18}$$

To keep the flow of the main arguments here, we shall postpone the proof of (6.18) until the next section (cf. Subsection 7.3 there) and continue. We now define the rescaled variables:

$$\overline{z}(\lambda,\tau) = z(R\lambda,\tau) ,$$
$$\overline{f}(\lambda,\tau) = f(R\lambda,\tau) ,$$

where $4\overline{\varepsilon} \le R \le 1$. A quick computation reveals that $\overline{z}$ and $\overline{f}$ satisfy:

$$\begin{aligned} R^2 \overline{z}_\tau = \overline{z}_{\lambda\lambda} - \frac{\overline{z}_\lambda}{\lambda} - \frac{R^2}{2}\lambda \overline{z}_\lambda - \chi \left(f_\lambda \Phi_\lambda + \widetilde{G}_\lambda \overline{z}_\lambda \right) \\ - 2\pi \chi R^2 e^{-\tau} \lambda G_\lambda + R^2 J - R^2 \widetilde{\Phi}_\tau . \end{aligned} \tag{6.19}$$

$$R^2 \overline{f}_\tau = \overline{f}_{\lambda\lambda} + \frac{\overline{f}_\lambda}{\lambda} - \frac{\Gamma R^2}{2}\lambda \overline{f}_\lambda + \frac{\overline{z}_\lambda}{2\pi\lambda} - R^2 \varphi_0 \langle \varphi_0, \frac{z_\lambda}{2\pi y} \rangle - R^2 \widetilde{G}_\tau , \tag{6.20}$$

where $\widetilde{G}_\lambda, \Phi_\lambda, \widetilde{\Phi}_\tau$ and $\widetilde{G}_\tau$ are written in terms of the variables (λ,τ). In view of our previous estimates, the following bounds hold when $\frac{1}{4} \le \lambda \le 4$:

$$| \Phi_\tau | \le C \left(\frac{\widetilde{\varepsilon}^2}{\sqrt{\tau} R^2} + \widetilde{\varepsilon}^2 \mid \log \widetilde{\varepsilon} \mid \right) ,$$

$$| \widetilde{G}_\tau | \le C \left(\frac{\widetilde{\varepsilon}^2}{\sqrt{\tau} R^2} + \widetilde{\varepsilon}^2 \mid \log \widetilde{\varepsilon} \mid^3 \right) ,$$

$$| \overline{z} | \le C \left((\overline{\varepsilon})^{2+\beta/2} R^{-2} + (\overline{\varepsilon})^2 \mid \log \overline{\varepsilon} \mid^2 \right) ,$$

$$| \overline{f} | \le C \overline{\varepsilon}^2 \mid \log \overline{\varepsilon} \mid^3 .$$

whereas by (6.5),

$$\left|\frac{\partial^k \widetilde{\Phi}}{\partial \lambda^k}\right| \leq C\frac{\overline{\varepsilon}^2}{R^2} \qquad \text{for } R \geq \overline{\varepsilon} \text{ and } k = 1, 2, \ldots.$$

$$\left|\frac{\partial}{\partial \tau}\left(\frac{\partial^k \widetilde{\Phi}}{\partial \lambda^k}\right)\right| \leq C\frac{\overline{\varepsilon}^2}{\sqrt{\tau}R^2} \qquad \text{for } R \geq \overline{\varepsilon} \text{ and } k = 1, 2, \ldots.$$

Taking into account (6.11), we also obtain that:

$$\left|\frac{\partial \widetilde{G}}{\partial \lambda}\right| \leq C \qquad \text{for } \frac{1}{4} \leq \lambda \leq 4,$$

$$\left|\frac{\partial}{\partial \tau}\left(\frac{\partial \widetilde{G}}{\partial \lambda}\right)\right| \leq \frac{C}{\sqrt{\tau}} \qquad \text{for } \frac{1}{4} \leq \lambda \leq 4.$$

Finally, from (6.12) we deduce the estimate:

$$| J | \leq C\left(\frac{\overline{\varepsilon}^4 \mid \log \overline{\varepsilon} \mid}{R^2} + \overline{\varepsilon}^2\right),$$

as well as similar estimates for the derivatives $\left|\frac{\partial^k J}{\partial \lambda^k}\right|$ when $k \geq 1$. On rescaling time by, say, setting $\widetilde{\tau} = \frac{\tau}{R^2}$, (6.19), (6.20) can be considered as a uniformly parabolic system. Putting all the above estimates together, standard parabolic theory then gives:

$$\text{(6.21)} \qquad | \overline{z}_\lambda | + | \overline{f}_\lambda | \leq C\left(\frac{\overline{\varepsilon}^{2+\beta/2}}{R^2} + \overline{\varepsilon}^2 \mid \log \overline{\varepsilon} \mid^3\right), \quad \text{for } \lambda \in \left(\frac{1}{2}, 2\right).$$

Estimate (6.21) will be instrumental in deriving suitable bounds for $\frac{\partial z}{\partial y}, \frac{\partial z}{\partial \tau}$. To this end, we now consider the system (6.19), (6.20) in regions $\frac{1}{4} \leq \lambda \leq 4$, and denote by $\overline{z}_1, \overline{f}_1$ the solutions of the correponding Cauchy problem for (6.19), (6.20) when restricted to the set $\frac{1}{2} \leq \lambda \leq 2$. Using Eidelman's estimates for Green's functions of linear parabolic systems (cf. [E]) we obtain that:

$$\text{(6.22)} \qquad \left|\frac{\partial \overline{z}_1}{\partial \tau}\right| + \left|\frac{\partial \overline{f}_1}{\partial \tau}\right| \leq C\left(\frac{\overline{\varepsilon}^2}{\sqrt{\tau}R^2} + \overline{\varepsilon}^2 \mid \log \overline{\varepsilon} \mid^3\right) \text{ for } \lambda \in \left(\frac{1}{2}, 2\right),$$

whereas on the other hand,

$$| \overline{z}_1 | + | \overline{f}_1 | \leq C\overline{\varepsilon}^2 \mid \log \overline{\varepsilon} \mid^3 R^2.$$

If we now set $\overline{z}_2 = \overline{z} - \overline{z}_1$, $\overline{f}_2 = \overline{f} - \overline{f}_1$, we readily see that the pair $\left(\overline{z}_2, \overline{f}_2\right)$ satisfies a parabolic system of the type:

$$R^2 \left(\overline{z}_2\right)_\tau = \left(\overline{z}_2\right)_{\lambda\lambda} + a_1(\lambda,\tau)\left(\overline{z}_2\right)_\lambda + b_1(\lambda,\tau)\left(\overline{f}_2\right)_\lambda ,$$
$$R^2 \left(\overline{f}_2\right)_\tau = \left(\overline{f}_2\right)_{\lambda\lambda} + a_2(\lambda,\tau)\left(\overline{z}_2\right)_\lambda + b_2(\lambda,\tau)\left(\overline{f}_2\right)_\lambda ,$$

where, for $i = 1, 2$, functions $a_i(\lambda, \tau)$ and $b_i(\lambda, \tau)$ (as well as their derivatives with respect to λ) are uniformly bounded for $\frac{1}{2} \le \lambda \le 2$. Moreover, one also has that:

$$|\,\overline{z}_2\,| + |\,\overline{f}_2\,| \le C\left(\frac{(\overline{\varepsilon})^{2+\beta/2}}{R^2} + \overline{\varepsilon}^2\,|\log\overline{\varepsilon}\,|^3\right).$$

We next restrict our attention to an interval $\lambda \in \left(1 - \frac{\delta}{2}, 1 + \frac{\delta}{2}\right)$, where $\delta > 0$ is small enough, and rescale space and time by setting $\eta = \frac{\lambda - 1}{\delta}$, $s = \frac{\tau}{R^2}$. It then turns out that $\left(\overline{z}_2, \overline{f}_2\right)$ satisfy now:

$$\delta^2 \left(\overline{z}_2\right)_s = \left(\overline{z}_2\right)_{\eta\eta} + \delta\left(a_1\left(\delta_\eta, s\right)\left(\overline{z}_2\right)_\eta + b_1\left(\delta_\eta, s\right)\left(\overline{f}_2\right)_\eta\right), \tag{6.23}$$

$$\delta^2 \left(\overline{f}_2\right)_s = \left(\overline{f}_2\right)_{\eta\eta} + \delta\left(a_2\left(\delta_\eta, s\right)\left(\overline{z}_2\right)_\eta + b_2\left(\delta_\eta, s\right)\left(\overline{f}_2\right)_\eta\right), \tag{6.24}$$

on the interval $-\frac{1}{2} < \eta < \frac{1}{2}$. The Green function for such system has properties much alike those of the heat kernel. In particular (cf. for instance [E]) it turns out that if we denote by $H(\eta, s)$ the corresponding (vector) Green function, one has that:

$$|H_s| \le C\mathrm{e}^{-\nu/s} \text{ for some } \nu > 0, \text{ whenever } -\frac{1}{4} < \eta < \frac{1}{4} \text{ and } 0 < s \le 1. \tag{6.25a}$$

Furthermore, a similar property holds for the kernel corresponding to representing $\left(\overline{z}_2, \overline{f}_2\right)$ in terms of their boundary data. Denoting such kernel by $B(\eta, \tau)$, we also have that:

$$|B_s| \le C\,e^{-\nu/s}, \text{ whenever } -\frac{1}{4} < \eta < \frac{1}{4} \text{ and } 0 < s \le 1, \tag{6.25b}$$

where ν can be taken to be the same constant appearing in (6.25a). If $\delta > 0$ is small enough (but fixed), we have exponential decay on intervals of size O(1), and by iterating the corresponding kernels we eventually arrive at:

$$|H_s| + |B_s| \le C\,\mathrm{e}^{-\gamma/s}\cdot\mathrm{e}^{\beta s} \text{ for some } \gamma > 0 \text{ and } \beta > 0,$$
$$\text{whenever } -\frac{1}{4} < \eta < \frac{1}{4} \text{ and } 0 < s\,.$$

We then may use representation formulae from parabolic potentials to obtain:

$$\left|\frac{\partial \overline{z}_2}{\partial \tau}\right| + \left|\frac{\partial \overline{f}_2}{\partial \tau}\right| \le C \left(\frac{\overline{\varepsilon}^{2+\beta/2}}{R^2} + \overline{\varepsilon}^2 \mid \log \overline{\varepsilon} \mid^3\right) \frac{1}{R^2} \int_0^1 \mathrm{e}^{-\gamma R^2/\tau} \mathrm{e}^{-\beta\tau/R^2} d\tau$$
$$\le C \left(\frac{\overline{\varepsilon}^{2+\beta/2}}{R^2} + \overline{\varepsilon}^2 \mid \log \overline{\varepsilon} \mid^3\right).$$

Together with (6.22), this implies that:

$$\text{(6.26)} \qquad \left|\frac{\partial \overline{z}}{\partial \tau}\right| + \left|\frac{\partial \overline{f}}{\partial \tau}\right| \le C \left(\frac{\overline{\varepsilon}^2}{\sqrt{\tau} R^2} + \overline{\varepsilon}^2 \mid \log \overline{\varepsilon} \mid^3\right).$$

Recalling the definition of $\overline{z}$ and $\overline{f}$, it follows from (6.26) and (6.21) that:

$$\text{(6.27a)} \qquad \left|\frac{\partial z}{\partial \tau}(y,\tau)\right| \le C \left(\frac{\overline{\varepsilon}(\tau)^{2+\beta/2}}{y^3} + \frac{\overline{\varepsilon}(\tau)^2 \mid \log \overline{\varepsilon}(\tau) \mid^3}{y}\right),$$

$$\text{(6.27b)} \qquad \left|\frac{\partial f}{\partial \tau}(y,\tau)\right| \le C \left(\frac{\overline{\varepsilon}(\tau)^2}{\sqrt{\tau} y^2} + \overline{\varepsilon}(\tau)^2 \mid \log \overline{\varepsilon}(\tau) \mid^3\right),$$

for $\overline{\varepsilon}(\tau) \le y \le 1$, where C is independent of M. Notice that (6.17) gives quadratic bounds for $y \le \overline{\varepsilon}$. We may therefore slightly modify our previous argument to obtain similar estimates as $\xi \to 0$ (with some extra decay rate). Since on the other hand $\widetilde{\Phi}$ satisfies similar bounds with constants which do not depend on M, we readily see that estimates (3.2)-(3.4) are recovered if we take, say, $\mu \le \frac{1}{4}$ and M is initially selected large enough.

6.3. – The external region

We shall conclude this section by discussing the case where $y \ge 1$. As a first step, we notice that (2.15) implies that:

$$\overline{G}_y \sim -\frac{2}{\Gamma y} \varphi_0 \langle \varphi_0, \frac{\Psi_y}{2\pi y} \rangle \quad \text{as } \tau \to \infty .$$

Using (5.34) (for $y \le 1$) as well as (3.2) (for $y > 1$), one easily obtains that:

$$\text{(6.28)} \qquad |\overline{G}_y| \le \frac{C}{y},$$

where C is now independent of M, provided that $\tau_0 \gg 1$. On the other hand, one deduces from (2.15) and (6.28) that:

$$\text{(6.29)} \qquad |\overline{G}_{yy}| \le \frac{C}{y^2}.$$

Since $G(y,\tau)$ given in (2.13) is taken to be quite close to $\overline{G}$ at $\tau=\tau_0$, we may assume that (6.28) and (6.29) hold for $G(y,\tau)$ at $\tau=\tau_0$. We then have that

$$y \mid G_y(y,\tau)\mid + y^2 \mid G_{yy}(y,\tau)\mid \le C\,, \tag{6.30}$$

for $1 \le y \le R\,e^{\tau/2}$ and $\tau_0 \le \tau \le \tau_0+\delta$ with $\delta>0$ small enough. We may now adapt the arguments in [HV1], Subsection 4.6, in order to derive suitable bounds for $y \ge 1$. As a first step, we notice that:

$$W(y,\tau) = \frac{4\pi}{\chi}y^2 + a_1(\tau)\varphi_0 + a_1(\tau)\varphi_0(y) + \mathrm{O}\left(\overline{\varepsilon}^2\right)\,,$$
$$\text{for } 1 \le y \le D\,(D \gg 1 \text{ but fixed }), \text{ and } \tau \gg 1\,.$$

By standard parabolic theory, the derivatives of W satisfy the estimate obtained by differentiating (formally) above. As in [HV1], we may now use (6.30) to produce suitable sub- and supersolutions for $\Phi(y,\tau)$ given in (2.5). In this way, we arrive at:

$$\begin{aligned} &\left|\Phi(y,\tau) - \frac{8\pi}{\chi}\right| \le C\overline{\varepsilon}(\tau_0)^2 \mid \log\overline{\varepsilon}(\tau_0)\mid \quad \text{for } 1 \le y \le \frac{R}{2}\mathrm{e}^{\tau/2}\,,\\ &\Phi(y,\tau) \text{ smooth and bounded for } \frac{R}{2}\mathrm{e}^{\tau/2} \le y \le R\,\mathrm{e}^{\tau/2},\\ &\text{where } \tau \ge \tau_0 \text{ is as in (6.30)}\,. \end{aligned} \tag{6.31}$$

We next improve (6.31) by means of a comparison argument involving sub- and supersolutions of the type:

$$\widehat{\Phi}(y,\tau) = \frac{8\pi}{\chi} + g(\xi) + \mathrm{e}^{-(\tau-\overline{\tau})}h(\xi).$$

where $\xi = y\,\mathrm{e}^{-(\frac{\tau-\overline{\tau}}{2})-B}$, $\overline{\tau} \ge \tau_0$ and $B>0$ is a sufficiently large constant. Since the term $-\chi V_y\Phi_y = -\chi G_y \Phi_y$ is negligible as far as $y \mid G_y \mid \le C$, we may now select functions $y(\xi)$ and $h(\xi)$ as in [HV1] to eventually obtain by the maximum principle that:

$$\left|\Phi(y,\tau) - \frac{8\pi}{\chi}\right| \le C\overline{\varepsilon}(\tau)^2 \mid \log\overline{\varepsilon}(\tau)\mid y^{1/2} \quad \text{for } y \ge 1\,. \tag{6.32}$$

The local mass function $M(r,t)$ given in (2.1) is then such that:

$$\begin{aligned} &\left|M(r,t) - \frac{8\pi}{\chi}\right| \le C\min\left\{\overline{\varepsilon}(\tau)^2 \mid \log\overline{\varepsilon}(\tau)\mid y^{1/2}, \delta_0\right\}\,,\\ &\text{for } r \ge (T-t)^{1/2} \text{ and } \delta_0 > 0 \text{ small enough}\,. \end{aligned} \tag{6.33}$$

A rescaling argument as that in the previous section then reveals that:

$$|M_r(r,t)| \le \frac{C}{r}\min\left\{\overline{\varepsilon}(\tau)^2 \mid \log\overline{\varepsilon}(\tau) \mid y^{1/2}, \delta_0\right\} \quad \text{for } r \ge (T-t)^{1/2}.$$

Recalling (2.4), we have thus obtained that:

$$\left|\Phi_y(y,\tau)\right| \le C\min\left\{\overline{\varepsilon}(\tau)^2 \mid \log\overline{\varepsilon}(\tau) \mid y^{-1/2}, \delta_0 y^{-1}\right\} \quad \text{for } y \ge 1. \tag{6.34}$$

Moreover, by rewriting (6.30) in terms of v and using Eidelman's estimates in [E], we readily see that:

$$|M_t(r,t)| \le \frac{C}{r^2}\min\left\{\overline{\varepsilon}(\tau)^2 \mid \log\overline{\varepsilon}(\tau) \mid y^{1/2}, \delta_0\right\} \quad \text{for } y \ge 1.$$

whence:

$$|\Phi_\tau(y,\tau)| \le C\min\left\{\overline{\varepsilon}(\tau)^2 \mid \log\overline{\varepsilon}(\tau) \mid y^{-3/2}, \delta_0\, y^{-2}\right\} \quad \text{for } y \ge 1. \tag{6.35}$$

To proceed further, we argue as in [HV1], Subsection 4.7. Namely, we introduce a new variable $U(y,\tau)$ given by:

$$u(x,t) = (T-t)^{-1}U\left(\frac{x}{\sqrt{T-t}}, -\log(T-t)\right).$$

Since u satisfies:

$$u_t = \Delta u - \chi\nabla u\nabla v - u\Delta v, \tag{6.36}$$

(cf. (1.1)), we may now produce sub - and supersolutions in the form:

$$\overline{U}(y,\tau) = \mathrm{e}^{-(\tau-\overline{\tau})}a(\xi) + \mathrm{e}^{-2(\tau-\overline{\tau})}b(\xi),$$

where $\xi = y, \mathrm{e}^{-(\frac{\tau-\overline{\tau}}{2})-B}$, $B \gg 1$ and $\overline{\tau} \ge \tau_0$. Functions $a(\xi)$ and $b(\xi)$ are selected so as to have $\overline{U}_\tau - \mathcal{L}\overline{U} \le 0$ (respectively ≤ 0), $\overline{\mathcal{L}}$ being the corresponding differential operator obtained from (6.36) after changing u into U there. Once again, (6.30) is used to control the nonlinear terms there. As to functions $a(\xi)$ and $b(\xi)$, we take them in the form:

$$\begin{aligned} a(\xi) &= K\varepsilon(\tau)^2(1\pm\delta) && \text{for } \xi \le 1, \\ a(\xi) &\le C\xi^{-2} && \text{for all } \xi, \end{aligned}$$

where K, C are suitable constants. Concerning $b(\xi)$, it suffices to have that:

$$b(\xi) \ll a(\xi) \quad \text{for } \xi \le 1, \quad b(\xi) \le C\xi^{-2} \quad \text{for any } \xi.$$

One then eventually obtains that:

$$U(x,t) \le C\left(\varepsilon\left(2\log\frac{1}{x}\right)\right)^2 x^{-2} \text{ for } x \ge (T-t)^{1/2}. \tag{6.37}$$

where C can be selected independent of M if $\tau_0 \gg 1$ and constant B in the definition of ξ is also large enough. We may now use (6.37) to estimate v_r (or, in a equivalent way, G_y). We do this by using a standard representation formula for solutions of (1.2) via caloric kernels, and noticing that the constant factor -1, as well as boundary terms, yield negligible contributions there. For times $t \in (T-\delta, T)$, one then obtains an estimate of the type:

$$\begin{aligned} |\nabla v| \le C &\left|\int_{T-\delta}^{t} ds \int_{B_R(0)} \frac{|x-\xi|^2}{(t-s)^2} \exp\left(-\frac{A|x-\xi|^2}{(t-s)}\right) u(\xi,s)d\xi\right| \\ &+ (\text{contribution from initial value}) + (\text{smaller terms}), \end{aligned} \tag{6.38}$$

where A is a positive constant. We shall split the integral above as follows:

$$\int_{T-\delta}^{t} ds \int_{|x|\le(T-t)^{1/2}} (\ \ ) + \int_{T-\delta}^{t} \int_{(T-t)^{1/2}\le|x|\le R} (\ \ ) \equiv I_1 + I_2 .$$

Assume that $|x| \ge 2(T-t)^{1/2}$. In view of (6.34), we have that $|x-\xi| \sim |x|$ in such region, and after some simple manipulations we obtain:

$$|I_1| \le \frac{C}{|x|} \text{ with } C \text{ independent of } M .$$

We now proceed to estimate I_2. To this end, we split that term again into two, denoted by $I_{2,1}$ and $I_{2,2}$, where integration is performed respectively in the regions $|x-\xi| \le \frac{|x|}{2}$ and $|x-\xi| \ge \frac{|x|}{2}$. The term, $I_{2,2}$ can be estimated exactly as I_1. As to $I_{2,1}$, one has that $u(\xi,s) \sim u(x,s)$ there, so that using (6.37) we obtain that:

$$\begin{aligned} |I_{2,1}| &\le \frac{C}{|x|^2}\int_0^1 s^{-2}ds \int_{|x-\xi|\le\frac{|x|}{2}} |x-\xi|\, \mathrm{e}^{-A\frac{|x-\xi|^2}{s}} d\xi \\ &\le \frac{C}{|x|^2}\int_{|\lambda|\le\frac{|x|}{2}} \mathrm{e}^{-A\lambda} d\lambda \le \frac{C}{|x|} . \end{aligned}$$

Concerning contribution from the initial values, we may suppose that $G(y,\tau_0) \sim \overline{G}(y,\tau_0)$, and this in turn provides a bound of order $\mathrm{O}\left(\frac{1}{|x|}\right)$ in (6.38) (with a constant independent on M). We have thus obtained:

$$|\nabla v| \le \frac{C}{|x|} \text{ with } C \text{ independent of } M . \tag{6.39}$$

At this juncture, we may differentiate in (1.2) to obtain:

$$\Gamma(\nabla v)_t = \Delta(\nabla v) + \nabla u\,. \tag{6.40}$$

together with (6.36), one then has a parabolic system for which bounds (6.37) and (6.38) hold. On rescaling as in the previous cases, classical regularity theory yields now that:

$$\left|\frac{\partial^2 v}{\partial x_1^2}\right| + \left|\frac{\partial^2 v}{\partial x_2^2}\right| \leq \frac{C}{\mid x \mid^2}\,,$$

with C independent of M. In variables (G, y) this amounts to:

$$y \mid G_y \mid + y^2 \mid G_{yy} \mid \leq C\,. \tag{6.41}$$

Since the argument just described is independent of the constant C appearing in (6.30), it follows by a standard continuation argument that (6.40) continues to hold as far as the solution exists.

END OF THE PROOF OF PROPOSITION 3.1.

Putting together our previous results, the proof of the Proposition is now concluded. Indeed, (3.2) follows from (5.34), (6.3) and (6.31). We have recovered (3.3) and (3.4) in (6.34), (6.35). From (5.34) and Subsection 5.4, we obtain (3.7). Finally, (3.5) follows from (6.31). □

7. – Some technical results.

In this final section, we provide the proofs of Lemmata 4.1 and 4.3, as well as that of estimate (6.18).

7.1. – The proof of Lemma 4.1

By (2.17), we see that:

$$\overline{G}(y,\tau) = A(\tau) - \int_1^y \frac{\Phi(s,\tau)}{2\pi s} ds + \int_1^y J(s,\tau) ds, \tag{7.1}$$

where $A(\tau)$ is such that $\langle \overline{G}, \varphi_0 \rangle = 0$. This gives:

$$\langle 1, 1\rangle A(\tau) = \Big\langle 1, \int_1^y \frac{\Phi(s,\tau)}{2\pi s} ds\Big\rangle - \Big\langle 1, \int_1^y J(s,\tau) ds\Big\rangle. \tag{7.2}$$

To proceed further, a bound on $\overline{G}_\tau$ will be required. Notice that, by (3.4):

$$\left|\int_1^y \frac{\Phi_\tau(s,\tau)}{2\pi s}ds\right| \le C\,M\left(\tau^{-1/2}\left(\xi^{-2}\widetilde{\chi}_1 + \mid\log\xi\mid(1-\widetilde{\chi}_1)\right)\right) + \overline{\varepsilon}^2 \mid\log\overline{\varepsilon}\mid^3\left(1+y^2\right)\mid\log y\mid, \tag{7.3}$$

$$\left|\int_0^\infty \frac{\partial\Phi}{\partial\tau}(s,\tau)e^{-\frac{\Gamma s^2}{4}}ds\right| \le CM\left(\tau^{-1/2}\overline{\varepsilon}^2\mid\log\overline{\varepsilon}^2\mid\log\overline{\varepsilon}\mid^3\right), \tag{7.4}$$

$$\left|\int_y^\infty \frac{\partial\Phi}{\partial\tau}(s,\tau)e^{-\frac{\Gamma s^2}{4}}ds\right| \le CM\,e^{-\frac{\Gamma s^2}{4}}\left(\tau^{-1/2}\overline{\varepsilon}^2\mid\log\xi\mid+\overline{\varepsilon}^2\mid\log\overline{\varepsilon}\mid^3\right). \tag{7.5}$$

From (7.4) and (7.5), it follows that:

$$\left|\frac{\partial J}{\partial\tau}\right| \le \frac{CM\overline{\varepsilon}^2}{y^2}\left(\tau^{-1/2}\mid\log\xi\mid+\mid\log\overline{\varepsilon}\mid^3\right). \tag{7.6}$$

whence:

$$\left|\int_1^y \frac{\partial J}{\partial\tau}(s,\tau)ds\right| \le CM\overline{\varepsilon}^2\mid\log\overline{\varepsilon}\mid^3\mid\log y\mid^2. \tag{7.7}$$

Using (7.2), (7.3) and (7.7), it turns out that:

$$\mid\dot{A}(\tau)\mid \le CM\overline{\varepsilon}^2\mid\log\overline{\varepsilon}\mid^3, \tag{7.8}$$

and (7.1), (7.3) and (7.7) readily give that:

$$\left|\frac{\partial\overline{G}}{\partial\tau}\right| \le CM\left(\tau^{-1/2}\left(\xi^{-2}\widetilde{\chi}_1+\mid\log\xi\mid(1-\widetilde{\chi}_1)\right)\right) + \overline{\varepsilon}^2\mid\log\overline{\varepsilon}\mid^3\left(1+y^2\right)\left(1+\mid\log y\mid^2\right). \tag{7.9}$$

Let us define now

$$\widetilde{g}(y,\tau) = G(y,\tau) - \overline{G}(y,\tau).$$

In view of the definitions of G and $\widetilde{G}$, one has that:

$$\widetilde{g}_\tau = \widetilde{g}_{yy} + \frac{1}{y}\widetilde{g}_y - \frac{\Gamma y}{2}\cdot\overline{g}_y - \overline{G}_\tau, \tag{7.10}$$

where $\langle \widetilde{g}, \varphi_0 \rangle =< \overline{G}_\tau, \ \varphi_0 >= 0$ and $\widetilde{g}(y, \tau_0) = 0$. Equation (7.10) can be viewed as a nonhomogeneous, linear equation with forcing term $(-\overline{G}_\tau)$. Denoting by S_Γ the semigroup generated by the linear differential operator in the right of (7.10), we may now write:

$$\widetilde{g}(y,\tau) = -\int_{\tau_0}^{\tau} S_\Gamma(\tau - s)\overline{G}_\tau(\cdot, s)ds = -\int_{\tau_0}^{\tau-1} S_\Gamma(\tau - s)\overline{G}_\tau(\cdot, s)ds - \int_{\tau-1}^{\tau} S_\Gamma(\tau - s)\overline{G}_\tau(\cdot, s) \equiv J_1 + J_2 \,.$$

From (7.9), we readily check that:

$$\left|\frac{\partial J_1}{\partial y}\right| \leq CM\overline{\varepsilon}^2 \mid \log \overline{\varepsilon} \mid^3 \left(1 + y^3\right) . \tag{7.11}$$

Since:

$$S_\Gamma(\tau)f(y) = \frac{\Gamma}{\left(4\pi\left(1 - \mathrm{e}^{-\Gamma\tau}\right)\right)} \int_{\mathbb{R}^2} \exp\left(-\frac{\Gamma\left(y\,\mathrm{e}^{-\frac{\Gamma\tau}{2}} - r\right)^2}{4(1 - \mathrm{e}^{-\Gamma\tau})}\right) f(r)dr \,,$$

a straightforward computation yields:

$$\left|\frac{\partial J_2}{\partial y}\right| \leq C \int_{\tau-1}^{\tau} \frac{\mathrm{e}^{-\frac{\Gamma(\tau-s)}{2}}}{\left(1 - \mathrm{e}^{-\Gamma(\tau-s)}\right)^2} ds \int_{\mathbb{R}^2} \mid y\,\mathrm{e}^{-\frac{\Gamma(\tau-s)}{2}} - r \mid \times \exp\left(-\frac{\Gamma\left(y\,\mathrm{e}^{-\frac{\Gamma(\tau-s)}{2}} - r\right)^2}{4\left(1 - \mathrm{e}^{-\Gamma(\tau-s)}\right)}\right) \mid \overline{G}_\tau(r, s) \mid dr \,.$$

In view of (7.9), one then has that:

$$\begin{aligned}\left|\frac{\partial J_2}{\partial y}\right| \leq K_1 + K_2 \equiv \frac{CM}{\tau^{1/2}} \int_{\tau-1}^{\tau} \frac{ds}{\left(1 - \mathrm{e}^{-\Gamma(\tau-s)}\right)^2} \int_{\mathbb{R}^2} \left|y\mathrm{e}^{-\frac{\Gamma(\tau-s)}{2}} - r\right| \\ \times \exp\left(-\frac{\Gamma\left(y\,\mathrm{e}^{-\frac{\Gamma(\tau-s)}{2}} - r\right)^2}{4\left(1 - e^{-\Gamma(\tau-s)}\right)}\right) \left(\frac{\widetilde{\chi}_1}{\xi^2} + \mid \log \xi \mid (1 - \widetilde{\chi}_1)\right) dr \\ + CM\varepsilon^{-2} |\log \overline{\varepsilon}|^3 \int_{\tau-1}^{\tau} \frac{ds}{\left(1 - \mathrm{e}^{-\Gamma(\tau-s)}\right)^2} \int_{\mathbb{R}^2} \left|y\,\mathrm{e}^{-\frac{\Gamma(\tau-s)}{2}} - r\right| \\ \times \exp\left(-\frac{\Gamma\left(y\,\mathrm{e}^{-\frac{\Gamma(\tau-s)}{2}} - r\right)^2}{4\left(1 - \mathrm{e}^{-\Gamma(\tau-s)}\right)}\right) \left(1 + r^2\right) \left(1 + \mid \log r \mid^2\right) dr.\end{aligned}$$

Regularizing results for parabolic equations readily give that:

$$| K_2 | \leq CM\overline{\varepsilon} \mid \log \overline{\varepsilon} \mid^3 \left(1 + y^3\right) .$$

On the other hand, changing variables in the form $r \to r\,\mathrm{e}^{-\frac{\Gamma s}{2}}$, we may estimate K_1 as follows:

$$\begin{aligned} | K_1 | \leq & \frac{CM}{\tau^{1/2}} \int_0^1 \frac{ds}{s^2} \int_{\mathbb{R}^2} | r - y | \,\mathrm{e}^{-\frac{A(r-y)^2}{s}} \xi^{-2} \widetilde{\chi}_1 dr \\ & + \frac{CM}{\tau^{1/2}} \int_0^1 \frac{ds}{s^2} \int_{\mathbb{R}^2} | r - y | \,\mathrm{e}^{-\frac{A(r-y)^2}{s}} \mid \log \xi \mid (1 - \widetilde{\chi}_1)\, dr \equiv K_{1,1} + K_{1,2}. \end{aligned}$$

We now compute:

$$\begin{aligned} | K_{1,1} | & \leq \frac{CM\overline{\varepsilon}^2}{\tau^{1/2}} \int_{|r| \geq \overline{\varepsilon}} \frac{| r - y |}{r^2} \left(\int_0^1 \mathrm{e}^{-\frac{A(r-y)^2}{s}} s^{-2} ds \right) dr \\ & = \frac{CM\overline{\varepsilon}^2}{\tau^{1/2}} \int_{|r| \geq \varepsilon} \frac{1}{r^2 \mid r - y \mid} \left(\int_0^{\frac{1}{|r-y|^2}} \frac{\mathrm{e}^{-A/v}}{v^2} dv \right) dr \\ & \leq \frac{CM\overline{\varepsilon}^2}{\tau^{1/2}} \int_{|r| \geq \varepsilon} \frac{dr}{r^\tau \mid r - y \mid} = \frac{CM\overline{\varepsilon}^2}{\tau^{1/2} y} \int_{|v| \geq \frac{\varepsilon}{y}} \frac{dv}{\mid v \mid^2 \mid v - 1 \mid} \\ & = \frac{CM\overline{\varepsilon}^2}{\tau^{1/2}(y + \overline{\varepsilon})} \left(\xi^2 (1 - \widetilde{\chi}_2) + \mid \log \xi \mid \widetilde{\chi}_2 \right) . \end{aligned}$$

As to $K_{1,2}$, we notice that:

$$\begin{aligned} | K_{1,2} | & \leq \frac{CM}{\tau^{1/2}} \int_{|r| \leq \overline{\varepsilon}} | r - y |^{-1} \log \left(\frac{| r |}{\overline{\varepsilon}} \right) dr \int_0^{\frac{1}{|r-y|^2}} \mathrm{e}^{-A/v} \mid v \mid^{-2} dv \\ & \leq \frac{CM\overline{\varepsilon}}{\tau^{1/2}} \int_{|t| \leq 1} \frac{| \log t |}{| t - y/\varepsilon |} dt \leq \frac{CM\overline{\varepsilon}}{\tau^{1/2}(| \xi | + 1)} \leq \frac{CM\overline{\varepsilon}^2}{\tau^{1/2}(y + \overline{\varepsilon})} . \end{aligned}$$

Summing up, we have obtained:

$$\left| \frac{\partial J_2}{\partial y} \right| \leq CM\overline{\varepsilon}^2 \left(\frac{1}{\tau^{1/2}(y + \overline{\varepsilon})} (1 + \widetilde{\chi}_2 \mid \log \xi \mid) + \mid \log \overline{\varepsilon} \mid^3 \left(1 + y^3\right) \right) . \tag{7.12}$$

Taking into account (7.11), we then arrive at:

$$\left| \frac{\partial \widetilde{g}}{\partial y} \right| \leq CM\overline{\varepsilon}^2 \left(\frac{1}{\tau^{1/2}(y + \overline{\varepsilon})} (1 + \widetilde{\chi}_2 \mid \log \xi \mid) + \mid \log \overline{\varepsilon} \mid^3 \left(1 + y^3\right) \right) ,$$

which concludes the proof. □

7.2. – The proof of Lemma 4.2

Let us define:

$$\widetilde{\delta}(y) = \delta(y) - \sum_{k=0}^{1} \varphi_k \langle \varphi_k, \delta \rangle .$$

Denoting by $S(\tau)$ the semigroup associated to the homogeneous part of (2.34), we now have that:

$$\begin{aligned}
Q(\cdot,\tau) &= S(\tau-\tau_0)\left(\psi(\cdot,\tau_0) - \sum_{k=0}^{1} \varphi_k \langle \varphi_k, \psi \rangle\right) \\
&\quad + \gamma \int_{\tau_0}^{\tau} S(\tau-s)\varepsilon(s)^2 \widetilde{\delta} ds \\
&= S(\tau-\tau_0)\left(\sum_{k=0}^{1} \alpha_k \widetilde{\varphi}_k - \sum_{k=0}^{1}\sum_{\ell=0}^{1} \alpha_k \langle \varphi_k, \widetilde{\varphi}_\ell \rangle \varphi_k\right) \\
&\quad + \gamma \varepsilon(\tau_0)^2 S(\tau-\tau_0) F(y) + \gamma \varepsilon(\tau_0)^2 \int_{\tau_0}^{\tau} S(\tau-s) \widetilde{\delta} ds \\
&\quad + \gamma \int_{\tau_0}^{\tau} S(\tau-s)\left(\varepsilon(s)^2 - \varepsilon(\tau_0)^2\right) \widetilde{\delta} ds \\
&= \widetilde{Q}_1 + \gamma \varepsilon(\tau_0)^2 F + \widetilde{Q}_2 ,
\end{aligned}$$

whence $\frac{\partial Q}{\partial \tau} = \frac{\partial \widetilde{Q}_1}{\partial \tau} + \frac{\partial \widetilde{Q}_2}{\partial \tau}$.

Keeping to the notations in Lemma 4.2, we write $\widetilde{Q}_2$ in the form:

$$\begin{aligned}
\widetilde{Q}_2 &= \frac{\gamma}{\pi} \int_{\Sigma}^{\infty} \eta^{-1} e^{-\eta} \varepsilon(s)^2 d\eta + \frac{\gamma}{\pi} \int_{\Sigma}^{\infty} (4\eta^2)^{-1} y^2 e^{-\eta} \varepsilon(s)^2 d\eta \\
&\quad + \gamma \int_{\tau_0}^{\beta} e^{\tau-s} \varepsilon(s)^2 K(y, \tau-s) ds \\
&\quad + \gamma \int_{\beta}^{\tau} e^{\tau-s} \varepsilon(s)^2 \left(A_1 + A_2\left(1 - e^{-(\tau-s)}\right)\right) \\
&\equiv Q_{2,1} + Q_{2,2} + Q_{2,3} + Q_{2,4} .
\end{aligned}$$

Recalling (3.7), one readily checks that:

$$\left|\frac{\partial Q_{2,3}}{\partial \tau}\right| + \left|\frac{\partial Q_{2,4}}{\partial \tau}\right| \le C \overline{\varepsilon}(\tau)^2 ,$$

where C does not depend on M. On the other hand, we see that:

$$\begin{aligned}
\left|\frac{\partial Q_{2,1}}{\partial \tau}\right| &\le C\,(\varepsilon(\tau-\beta))^2 \frac{e^{-\Sigma}}{\Sigma} \mid \dot{\Sigma} \mid + C \int_{\Sigma}^{\infty} \varepsilon(s) \mid \dot{\varepsilon}(s) \mid \eta^{-1} e^{-\eta} d\eta \\
&\le C\,(\overline{\varepsilon}(\tau))^2 \left(1 + M\tau^{-1/2}(1 + \mid \log y \mid)\right) .
\end{aligned}$$

A similar estimate is easily seen to hold for $\frac{\partial Q_{2,2}}{\partial \tau}$. Summarizing, we have that:

$$\left| \frac{\partial \widetilde{Q}_2}{\partial \tau} \right| \leq C\,(\overline{\varepsilon}(\tau))^2 \left(1 + M\tau^{-1/2}(1 + \log y\,|)\right) . \tag{7.13}$$

In order to estimate $\frac{\partial Q_1}{\partial \tau}$, we write:

$$\begin{aligned} \widetilde{Q}_1(\cdot, \tau_0) &= \sum_{j=0}^{1} \alpha_j \left(\widetilde{\varphi}_j - \varphi_j\right) + \sum_{k=0}^{1} \varphi_k \left(\sum_{j=0}^{1} \alpha_j \left(\delta_{j,k} - \langle \varphi_k, \varphi_j \rangle\right) \right) \\ &\equiv \widetilde{Q}_{1,1}(\cdot, \tau_0) + \widetilde{Q}_{1,2}(\cdot, \tau_0) , \end{aligned}$$

where $\delta_{j,k} = 1$ if $j = k$, and $\delta_{j,k} = 0$ otherwise. Arguing as in [HV1] we readily estimate the second term to obtain:

$$\left| \widetilde{Q}_{1,2}(\cdot, \tau_0) \right| \leq C_M\,(\overline{\varepsilon}(\tau_0)^{2+\theta} \left(1 + y^2\right) ,$$

for some $\theta > 0$. Moreover, differentiating with respect to y yields:

$$\left| \frac{\partial^2}{\partial y^2} \widetilde{Q}_{1,2}(\cdot, \tau_0) \right| \leq C_M\,(\overline{\varepsilon}(\tau_0))^{2+\theta} .$$

One may then invoke standard parabolic theory to derive that:

$$\left| \frac{\partial}{\partial \tau} \widetilde{Q}_{1,2}(\cdot, \tau_0) \right| \leq C_M\,(\overline{\varepsilon}(\tau_0))^{2+\theta} .$$

It remains yet to estimate $\left| \frac{\partial \widetilde{Q}_{1,1}}{\partial \tau}(\cdot, \tau) \right|$. To this end, we first recall that, by our choice of initial values:

$$\sum_{j=0}^{1} \alpha_j \left(\widetilde{\varphi}_j - \widetilde{\varphi}_j\right) = \frac{4\pi}{\chi} \varepsilon(\tau)^2 \left(\log\left(y^2\right) - \log\left(y^2 + \varepsilon^2\right)\right) + \mathrm{O}\left(\varepsilon^2 y^2\right)$$

On the other hand, $\frac{\partial Q_{1,1}}{\partial \tau}$ satisfies the same differential equation as $Q_{1,1}$, with a different initial value at $\tau = \tau_0$. As a matter of fact, one has that:

$$\frac{\partial Q_{1,1}}{\partial \tau}(\cdot, \tau_0) \sim \frac{\partial^2 Q_{1,1}}{\partial y^2}(\cdot, \tau_0) + \frac{1}{y} \frac{\partial Q_{1,1}}{\partial y}(\cdot, \tau_0) .$$

In then turns out that the factor $\frac{4\pi}{\chi} \varepsilon(\tau)^2 \log(y^2)$ generates a Dirac delta at $y = 0$ and $\tau = \tau_0$, with the same mass as that arising from the term $\frac{4\pi}{\chi} \varepsilon(\tau)^2 \log(y^2 + \varepsilon(\tau)^2)$. We may then argue as in [HV1], Subsection 5.2, to derive the bound:

$$\left| \frac{\partial Q_{1,1}}{\partial \tau}(\cdot, \tau) \right| \leq C\overline{\varepsilon}(\tau)^{2+\theta} \quad \text{for } |\, y \,| \geq \overline{\varepsilon}(\tau)^{\theta} .$$

and from this and (7.13), the result follows. □

7.3. – The proof of (6.18)

Since $f(y,\tau_0)=\overline{G}(y,\tau_0)-\widetilde{G}(y,\tau_0)$, one may use the fact that $\varepsilon(\tau)=\widetilde{\varepsilon}(\tau)(1+O(\overline{\varepsilon}^{\sigma}))$ for some $\sigma>0$, to remark that:

$$|f(y,\tau_0)| \le C\overline{\varepsilon}(\tau_0)^{2+\sigma}\left(1+\widetilde{\chi}_{\widetilde{\varepsilon}}\,|\log y\,|\right).$$

We may then use variation of constants formula in (6.16) to write:

$$\begin{aligned} f(y,\tau) &= S_\Gamma(\tau-\tau_0)f(\cdot,\tau_0)+\int_{\tau_0}^{\tau} S_\Gamma(\tau-s)\left(\frac{\varphi_y}{2\pi y}-\widetilde{\varphi}_0\langle\widetilde{\varphi}_0,\frac{\varphi_y}{2\pi y}\rangle\right)ds \\ &\quad -\int_{\tau_0}^{\tau} S_\Gamma(\tau-s)\widetilde{G}_\tau(\cdot,s)ds \equiv L_1+L_2+L_3\,. \end{aligned}$$

We readily check that:

$$|\,L_1\,| \le C\overline{\varepsilon}^{2+\sigma/2}e^{-\frac{\Gamma\tau}{2}} \quad \text{for } 0\le y\le 1\,.$$

We next split L_3 as follows:

$$L_3=-\int_{\tau_0}^{\tau-1} S_\Gamma(\tau-s)\widetilde{G}_\tau(\cdot,s)ds-\int_{\tau-1}^{\tau} S_\Gamma(\tau-s)\widetilde{G}_\tau(\cdot,s)ds\equiv L_{3,1}+L_{3,2}\,.$$

In view of (6.7), we readily obtain:

$$|\,L_{3,1} \le C\widetilde{\varepsilon}(\tau)^2\,|\log\widetilde{\varepsilon}(\tau)\,|^2 \quad \text{for } 0\le y\le 1\,.$$

On the other hand, using (6.7) and properties of the caloric kernel, we see that for $y\le 1$,

$$\begin{aligned} |\,L_{3,2}\,| &\le C\int_0^1 ds\int_{\mathbb{R}^2} s^{-1}e^{-\frac{A(y-\lambda)^2}{s}}\left(|\,\lambda\,|^{-2}\,\overline{\varepsilon}^2\widetilde{\chi}_1+(1+|\log\xi\,|)(1-\widetilde{\chi}_1)\right)d\lambda \\ &\quad + C\overline{\varepsilon}^2\,|\log\overline{\varepsilon}\,|^2\equiv S+C\overline{\varepsilon}^2\,|\log\overline{\varepsilon}\,|^2\,. \end{aligned}$$

Hence:

$$\begin{aligned} S &\le C\int_0^1 ds\int_{\mathbb{R}^2} s^{-1}e^{-\frac{|\lambda|^2}{s}}\left(|\,\lambda\,|^{-2}\,\overline{\varepsilon}^2\widetilde{\chi}_1+(1+|\log\xi\,|)(1-\widetilde{\chi}_1)\right)d\lambda \\ &\le C\int_{\mathbb{R}^2}\left(|\,\lambda\,|^{-2}\,\overline{\varepsilon}^2\widetilde{\chi}_1+(1+|\log\xi\,|)(1-\widetilde{\chi}_1)\right)d\lambda \\ &\le C\overline{\varepsilon}^2\,|\log\overline{\varepsilon}\,|^2\,. \end{aligned}$$

Summing up, we have obtained:

$$|\,L_3\,|\le C\overline{\varepsilon}^2\,|\log\overline{\varepsilon}\,|^2\,.$$

To deal with L_2, we perform a similar splitting, and set $L_2 = L_{2,1} + L_{2,2}$, where integration in $L_{2,1}$ (respectively in $L_{2,2}$) is done from τ_0 to $\tau - 1$ (respectively from $\tau - 1$ to τ). In $L_{2,1}$ we use the exponential decay of the corresponing semigroup to obtain that:

$$| L_{2,1} | \leq C\bar{\varepsilon}^2 \, | \log \bar{\varepsilon} \, |^3 .$$

As to $L_{2,2}$, we notice that, arguing as for $L_{2,1}$, one obtains:

$$\left| \int_{\tau-1}^{\tau} S_\Gamma(\tau - s) \left(\widetilde{\varphi}_0 \langle \widetilde{\varphi}_0, \frac{\varphi_y}{2\pi y} \rangle \right) ds \right| \leq C\bar{\varepsilon}^2 \, | \log \bar{\varepsilon} \, |^3 .$$

Concerning the remaining term in $L_{2,1}$, we compute:

$$\begin{aligned}
&\left| \int_{\tau-1}^{\tau} S_\Gamma(\tau - s) \frac{\varphi_y(\cdot, s)}{2\pi y} ds \right| \\
&\leq C \left| \int_{\tau-1}^{\tau} \frac{ds}{(1 - \mathrm{e}^{-\Gamma(\tau-s)})} \int_{\mathbb{R}^2} \exp\left(- \frac{\Gamma(y\mathrm{e}^{-\frac{\Gamma(y-s)}{2}} - \lambda)^2}{4(1 - \bar{\mathrm{e}}^{\Gamma(\tau-s)})} \right) \frac{\varphi_y(\lambda, s)}{|\lambda|} dy \right| \\
&\leq C \int_{\tau-1}^{\tau} \frac{ds}{(1 - \mathrm{e}^{-\Gamma(\tau-s)})} \int_{\mathbb{R}^2} \exp\left(- \frac{\Gamma(y\mathrm{e}^{-\frac{\Gamma(y-s)}{2}} - \lambda)^2}{4(1 - \bar{\mathrm{e}}^{\Gamma(\tau-s)})} \right) \frac{|\varphi(\lambda, s)|}{|\lambda|^2} d\lambda \\
&\leq C \int_{\tau-1}^{\tau} \int_{\mathbb{R}^2} \frac{|y\,\mathrm{e}^{-\frac{\Gamma(\tau-s)}{2}} - \lambda|}{(1 - \mathrm{e}^{-\Gamma(\tau-s)})^2} \exp\left(- \frac{\Gamma(y\,\mathrm{e}^{-\frac{\Gamma(\tau-s)}{2}} - \lambda)^2}{4(1 - \bar{\mathrm{e}}^{\Gamma(\tau-s)})} \right) \frac{|\varphi(\lambda, s)|}{|\lambda|} d\lambda \\
&\leq C_M \int_0^1 \frac{ds}{s} \int_{\mathbb{R}^2} \frac{\mathrm{e}^{-\frac{A(y-\lambda)^2}{s}}}{|\lambda|^2} \left(\frac{\bar{\varepsilon}^{2+\beta}|\lambda|^2}{|\lambda|^2 + \bar{\varepsilon}^2} + \bar{\varepsilon}^2 |\log \bar{\varepsilon}| \widetilde{\chi}_1 \right) d\lambda \\
&\quad + C_M \int_0^1 \frac{ds}{s^2} \int_{\mathbb{R}^2} \frac{|y - \lambda|}{|\lambda|} \mathrm{e}^{-\frac{A(y-\lambda)^2}{s}} \left(\frac{\bar{\varepsilon}^2|\lambda|^2}{|\lambda|^2 + \bar{\varepsilon}^2} + \bar{\varepsilon}^2 |\log \bar{\varepsilon}| \widetilde{\chi}_1 \right) d\lambda \\
&\leq C_M \int_{|\lambda| \leq 1} |\lambda|^{-2} \left(\frac{\bar{\varepsilon}^{2+\beta}|\lambda|^2}{|\lambda|^2 + \bar{\varepsilon}^2} + \bar{\varepsilon}^2 |\log \bar{\varepsilon}| \widetilde{\chi}_{\bar{\varepsilon}} \right) \left(\int_0^{\frac{1}{|\lambda - y|}} s^{-1} \mathrm{e}^{-A/s} ds \right) d\lambda \\
&\quad + C_M \int_{|\lambda| \leq 1} (|\lambda| \, |y - \lambda|)^{-1} \left(\frac{\bar{\varepsilon}^{2+\beta}|\lambda|^2}{|\lambda|^2 + \bar{\varepsilon}^2} + \bar{\varepsilon}^2 |\log \bar{\varepsilon}| \widetilde{\chi}_{\bar{\varepsilon}} \right) \\
&\quad \times \left(\int_0^{\frac{1}{|\lambda - y|}} s^{-1} \mathrm{e}^{-A/s} ds \right) d\lambda + C_M \bar{\varepsilon}^2 |\log \bar{\varepsilon}| \\
&\leq C_M \bar{\varepsilon}^{2+\beta} \int_{|\lambda| \leq 1} (|\lambda|^2 + \bar{\varepsilon}^2)^{-1} (1 + \left| \log |\lambda - y| \right|) d\lambda
\end{aligned}$$

$$
\begin{aligned}
&+ C_M\overline{\varepsilon}^2|\log\overline{\varepsilon}| \int_{\overline{\varepsilon}\le|\lambda|\le 1} |\lambda|^{-2}(1+\big|\log|\lambda-y|\,\big|)d\lambda \\
&+ C_M\overline{\varepsilon}^{2+\beta} \int_{|\lambda|\le 1} \left(|\lambda-y|\left(|\lambda|^2+\overline{\varepsilon}^2\right)\right)|\lambda|(1+|\log|\lambda-y|\,|)d\lambda \\
&+ C_M\overline{\varepsilon}^2|\log\overline{\varepsilon}| \int_{\overline{\varepsilon}\le|\lambda|\le 1} (|y-\lambda|\,|\lambda|)^{-1}(1+|\log|\lambda-y|\,|)d\lambda + C_M\overline{\varepsilon}^2|\log\overline{\varepsilon}| \\
\le\ & C_M\overline{\varepsilon}^2|\log\overline{\varepsilon}|\,(|\log\overline{\varepsilon}|+|\log(y/\overline{\varepsilon})\,|) \\
&+ C_M\overline{\varepsilon}^2|\log\overline{\varepsilon}| \int_{|\lambda|\le\frac{5|y|}{\overline{\varepsilon}}} (|u-y/\varepsilon\,\|\,u|)^{-1}\,(1+|\log\overline{\varepsilon}|+|\log(u-y/\varepsilon)\,|)\,du \\
\le\ & C_M\overline{\varepsilon}^2|\log\overline{\varepsilon}|\,(|\log\overline{\varepsilon}|+|\log(y/\overline{\varepsilon})\,|)\ .
\end{aligned}
$$

Putting all these estimates together, we eventually obtain:

$$
|\ f(y,\tau)\ |\le C\overline{\varepsilon}\left(|\ \log\overline{\varepsilon}\ |^3+|\ \log\overline{\varepsilon}\ |^2|\ \log\xi\ |\right).
$$

where C may now be selected independently on M. This concludes the proof. □

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