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MASAMI FUJIMORI

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Rational Points of a Curve Which has a Nontrivial Automorphism

MASAMI FUJIMORI

0. – Introduction

Let k be a finite extension field of the rational number field, C a nonsingular complete curve over k of genus greater than one, and $\bar{k}$ an algebraic closure of k. If there exists a nontrivial automorphism of C, it induces an automorphism of the Jacobian variety J of C leaving stable the image of a canonical map f of C into J. This will be seen to place a restriction on the distribution of $C(\bar{k})$ in $J(\bar{k})$ under f.

Manin [6, Propositions 15 and 19] was aware that if the rank of the Néron-Severi group (group of line bundles modulo algebraic equivalence, which is finitely generated) of the Jacobian variety J is larger than one, then the image of $C(\bar{k})$ in $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ is contained in a region near some quadric hypersurface defined by a height function on J. When a projective variety V over a number field k is nonsingular, boundedness of a height function $h_V(\mathcal{L}, \cdot)$ on V attached to $\mathcal{L} \in \operatorname{Pic} V$ is equivalent to the condition that the invertible sheaf $\mathcal{L}$ is a torsion sheaf (see [9, Section 3.11]). By the additive nature with respect to the tensor operation on $\operatorname{Pic} V$, when V is regular, a $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Pic} V$ can be considered as a $\mathbb{Q}$-subspace of the space of real valued functions on $V(\bar{k})$ modulo bounded functions under the functor $h_V : \mathcal{L} \mapsto h_V(\mathcal{L}, \cdot)$. Applying the functoriality to a canonical morphism $f: C \to J$, we see the Néron-Tate height functions associated with nonzero elements of the kernel of the $\mathbb{Q}$-linear map $f^*: \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Pic} J \to \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Pic} C$ are polynomial functions of degree two on $J(\bar{k})$, and are bounded on $C(\bar{k})$.

THEOREM 0.1 (Manin). *Let r be the rank of the Néron-Severi group* $\mathrm{NS}(J)$ *of the Jacobian variety J of the curve C. There exist $r-1$ quadric hypersurfaces in $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ defined as the zero loci of some Néron-Tate height functions, with $\mathbb{Q}$-linearly independent defining equations such that the image of $\bar{k}$-valued points of*

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C under a canonical map $f: C \to J$ is in the intersection of the neighborhoods of the hypersurfaces.

Over the algebraic closure $\bar{k}$, the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{NS}(A \times_k \bar{k})$ for an abelian variety A over k is identified with the subalgebra of elements in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}(A \times_k \bar{k})$ fixed by an involution (called a Rosati involution). When the curve C has a nontrivial automorphism over the base field, it induces an automorphism of the Jacobian variety J and this may yield a nontrivial element of the endomorphism algebra $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}(J)$, hence $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{NS}(J)$. Of an automorphism of C the author constructed a line bundle on J whose inverse image under the canonical morphism f ((4) in the first section) becomes the structure sheaf on C. The associated Néron-Tate height is fairly explicitly described.

Let Δ be the diagonal divisor on $C \times C$. For an automorphism ψ of C over k different from the identity map, we set

$$D := (1_C, \psi)^* \Delta \in \mathrm{Div}\, C \quad \text{and} \quad d := \deg D,$$

where $(1_C, \psi): C \to C \times C$ is the morphism whose composition with the first projection is the identity map of C and with the second, ψ. The superscript * is indicating the pull-back by a morphism. We provide the infinite dimensional real vector space $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ with the inner product $\langle \cdot, \cdot \rangle$ associated with a theta divisor. The orthogonal transformation of the infinite dimensional normed real vector space $(\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k}), \langle \cdot, \cdot \rangle)$ induced naturally by ψ is denoted as Ψ (cf. the first section). The automorphisms ψ and Ψ are compatible with the canonical morphism $f: C \to J$. The genus of the curve C is denoted by g and let $\Omega_{C/k}$ be the invertible sheaf of regular differentials of C over k

THEOREM 0.2. *There exists a positive constant $M = M(C, \psi)$ such that the image of $C(\bar{k})$ under the canonical map $C(\bar{k}) \xrightarrow{f} J(\bar{k}) \to \mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ is situated in the region near a quadric hypersurface of the infinite dimensional normed real vector space $(\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k}), \langle \cdot, \cdot \rangle)$ defined as*

$$\left| \left\langle x, \left(\Psi + \frac{d-2}{2g} \right) x + \mathcal{O}_C((2g-2)D) \otimes \Omega_{C/k}^{\otimes(-d)} \right\rangle \right| < M, \qquad x \in \mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k}),$$

where $\mathcal{O}_C((2g-2)F) \otimes \Omega_{C/k}^{\otimes(-d)}$ is viewed as an element of $J(k) \simeq \mathrm{Pic}^\circ C$.

Denoting by $\| \cdot \|$ the norm attached to the inner product $\langle \cdot, \cdot \rangle$, we see the following.

COROLLARY 0.3. *For $P \in C(\bar{k})$,*

$$\frac{\langle f(P), f(\psi(P)) \rangle}{\|f(P)\| \|f(\psi(P))\|} \longrightarrow \frac{2-d}{2g} \quad \textit{as} \quad \|f(P)\| = \|f(\psi(P))\| \longrightarrow \infty.$$

This leads to a new proof of a fact which is usually an application of the Riemann-Hurwitz formula.

COROLLARY 0.4. *The number of fixed points of a nontrivial automorphism of a curve over a number field of genus* $g > 1$ *is at most* $2g + 2$.

If we are given a concrete curve, then we may be able to say something more. When C is a plane curve over k defined by

$$C\colon X^4 + Y^4 = aZ^4 \tag{1}$$

with a nonzero constant a in k, we will obtain

PROPOSITION 0.5. *There exist absolute positive constants* c_1, c_2, *and* c_3 *such that the image* $f(C(\bar{k}))$ *in* $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ *lies in the intersection of three regions near quadric hypersurfaces defined as*

$$\begin{cases} \left| \|u\|^2 - \|v\|^2 \right| < c_1 \\ \left| \|w\|^2 - \|u\|^2 \right| < c_2 \\ \left| \langle u, Tv \rangle \right| < c_3, \end{cases}$$

where $u \in U, v \in V, w \in W$; U, V, W *are subspaces of* $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ *such that* $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k}) = U \oplus V \oplus W$ *is an orthogonal decomposition, and* $T : V \xrightarrow{\sim} U$ *is a metric linear transformation with respect to the induced norms* (*cf.* *Proposition* 2.6).

As a consequence, we obtain another proof of the theorem of Dem'yanenko.

THEOREM 0.6 (Dem'yanenko [1, Example 1]). *Let* C *be the plane quartic curve over* k *defined by* (1) *and* E *an elliptic curve given by a Weierstrass equation*

$$E\colon y^2 = x^3 - ax.$$

If the rank of $E(k)$ *is at most one, then the canonical heights of rational points on the curve* C *are bounded by an absolute constant. Especially, the number of rational points is finite.*

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Notation and Terminology The ring of rational integers, the field of rational numbers, and the field of real numbers are respectively denoted by $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. A finite extension field of $\mathbb{Q}$ is called a number field. Let k be a field and $\bar{k}$ an algebraic closure of k. For a scheme over k, we denote by $\bar{X}$ the scheme $X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ over $\bar{k}$. The automorphism group of X over k is denoted by $\operatorname{Aut}_k X$. When X is a nonsingular variety over k and D is a divisor on X, we define $\mathcal{O}_X(D)$ as the invertible sheaf on X associated with D. In particular, $\mathcal{O}_X = \mathcal{O}_X(0)$ is the structure sheaf on X. Given morphisms $\psi^1 : X^1 \to Y^1$ and $\psi^2 : X^2 \to Y^2$ of schemes over a base scheme, let $\psi^1 \times \psi^2$ be the induced map $X^1 \times X^2 \to Y^1 \times Y^2$. If $X^1 = X^2 = X$, then (ψ^1, ψ^2) is another natural morphism $X \to Y^1 \times Y^2$. By a curve over k, we mean a regular one-dimensional geometrically irreducible proper scheme over k. If C is such a curve, $\operatorname{Pic}^\circ C$ is the Picard group of degree zero. The symbol $O(1)$ represents a bounded function of an appropriate variable.

1. – Viewpoint and a result

In the first section, we find a canonically defined height function on a curve over a number field of genus at least two. These height functions are invariant under isomorphisms of curves over the algebraic closure of the ground field. After that, we make clear our standpoint (Problem 1.5), observe a general result (Theorem 1.7), and make a simple application (Corollary 1.10).

Let k be a number field, C a nonsingular complete curve over k of genus $g > 1$, J the Jacobian variety of C over k, and $\bar{k}$ an algebraic closure of k. (For the definition and properties of a Jacobian variety, see, for example, [8].)

Fix a point $P_0 \in C(\bar{k})$. A divisor Θ on $\bar{J} = J \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ is defined by

$$(2) \qquad \begin{aligned} \Theta(\bar{k}) : &= \{\mathcal{O}_{\bar{C}}(Q_1 + \cdots + Q_{g-1} - (g-1)P_0) \mid Q_j \in C(\bar{k})\} \\ &\subset \bar{J}(\bar{k}) \simeq \operatorname{Pic}^\circ(\bar{C}), \end{aligned}$$

where $\bar{C} = C \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$. We denote by s, p and $q : \bar{J} \times \bar{J} \to \bar{J}$ the sum, the projections onto the first and the second factors, respectively. Define an invertible sheaf $\mathcal{N}_0$ on $\bar{J} \times \bar{J}$ by

$$(3) \qquad \mathcal{N}_0 := s^*\mathcal{O}_{\bar{J}}(\Theta) \otimes p^*\mathcal{O}_{\bar{J}}(-\Theta) \otimes q^*\mathcal{O}_{\bar{J}}(-\Theta).$$

Lemma 1.1. *The isomorphism class of the invertible sheaf $\mathcal{N}_0$ on an abelian variety $\bar{J} \times \bar{J}$ over $\bar{k}$ does not depend on the choice of P_0.*

Proof. See Lemma 1.15 (i) below. □

The Néron-Tate height function $\langle\cdot,\cdot\rangle$ attached to $\mathcal{N}_0$ is a symmetric bilinear form on $(J\times J)(\bar{k})\simeq J(\bar{k})\times J(\bar{k})$ which is non-degenerate and non-negative on $\mathbb{R}\otimes_{\mathbb{Z}} J(\bar{k})\times\mathbb{R}\otimes_{\mathbb{Z}} J(\bar{k})$. (For general facts about Néron-Tate height functions, see [4] or [9].) The induced inner product on $\mathbb{R}\otimes_{\mathbb{Z}} J(\bar{k})$ is also denoted by $\langle\cdot,\cdot\rangle$. The norm on $\mathbb{R}\otimes_{\mathbb{Z}} J(\bar{k})$ associated with $\langle\cdot,\cdot\rangle$ is denoted by $\|\cdot\|$. Since Néron-Tate height functions are uniquely determined by invertible sheaves, Lemma 1.1 shows that for a and $b\in J(\bar{k})$, the quantities $\langle a,b\rangle$ and $\|a\|$ are independent of the the base point of Θ and are canonically defined. We say the functions $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are *associated with a theta divisor* or are *attached to a theta divisor*.

REMARK 1.2. We note that height functions are defined in the first place for *varieties over the algebraic closure* $\bar{k}$ of a number field k. They are determined by invertible sheves up to bounded functions of $\bar{k}$-valued points. In particular, a Néron-Tate height function is unique up to constant functions. (Usually it is normalized so that the identity element takes the value zero. We follow this convention.) When we are speaking of a height function of rational points, it is only the restriction of a height function obtained by the base change to the algebraic closure of the ground field. The height function itself may not be defined over the base field.

Let $\Omega_{C/k}$ be the invertible sheaf of regular differentials of C over k. We have a canonical morphism $f\colon C\to J$ given by

$$C(\bar{k})\ni P\mapsto\Omega_{C/k}\otimes\mathcal{O}_{\bar{C}}(-(2g-2)P)\in\operatorname{Pic}^{\circ}(\bar{C})\simeq J(\bar{k}). \tag{4}$$

For another curve C' over k, define J', Θ', and $\mathcal{N}_0'$, similarly. An isomorphism ϕ of $\bar{C}$ onto $\bar{C}'$ over $\bar{k}$ induces an isomorphism Φ of $\bar{J}$ onto $\bar{J}'$ over $\bar{k}$:

$$J(\bar{k})\simeq\operatorname{Pic}^{\circ}(\bar{C})\ni\mathcal{L}\mapsto\Phi(\mathcal{L}):=\phi_*\mathcal{L}\in\operatorname{Pic}^{\circ}(\bar{C}')\simeq J'(\bar{k})$$

and the next diagram is commutative:

$$\begin{array}{ccc}\bar{C} & \xrightarrow{f} & \bar{J}\\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi}\\ \bar{C}' & \xrightarrow[f]{} & \bar{J}'\end{array}$$

Here, by abuse of notation, $f\colon\bar{C}\to\bar{J}$ denotes the canonical morphism over $\bar{k}$ obtained by the base change of the canonical morphism $f\colon C\to J$ over k. The norm $\|\cdot\|$ on $\mathbb{R}\otimes_{\mathbb{Z}} J(\bar{k})$ (respectively $\mathbb{R}\otimes_{\mathbb{Z}} J'(\bar{k})$) was defined with the help of the Néron-Tate height function associated with the invertible sheaf $\mathcal{N}_0$ (respectively $\mathcal{N}_0'$) on $\bar{J}\times\bar{J}$ (respectively $\bar{J}'\times\bar{J}'$). Since the divisor Θ on $\bar{J}$ is mapped under Φ to a translate of Θ', the sheaf $(\Phi\times\Phi)_*\mathcal{N}_0$ on $\bar{J}'\times\bar{J}'$ is $\mathcal{N}_0'$ with Θ' replaced by the translate of Θ'. Lemma 1.1 says this is isomorphic to the previous $\mathcal{N}_0'$, therefore Φ is norm-preserving by the functoriality of Néron-Tate height functions (cf. Lemma 1.13):

$$\langle \Phi(\cdot), \Phi(\cdot)\rangle = \langle \cdot, \cdot\rangle$$

In particular, for elements P and Q of $C(\bar{k})$

$$\langle f(\phi(P)), f(\phi(Q))\rangle = \langle f(P), f(Q)\rangle \quad \text{and} \quad \|f(\phi(P))\| = \|f(P)\|. \tag{5}$$

PROPOSITION 1.3. *There exists a canonically defined scalar product $\langle \cdot, \cdot\rangle$ on the real vector space $\mathbb{R}\otimes_{\mathbb{Z}} J(\bar{k})$ which is preserved from the isomorphisms of Jacobian varieties over $\bar{k}$ induced by isomorphisms of curves over $\bar{k}$. Letting $\|\cdot\|$ be the associated norm, we have*

$$\begin{aligned}\langle f(P), f(Q)\rangle = {} & h_{\bar{C}}(\Omega_{\bar{C}/\bar{k}}^{\otimes(2g-2)}, P) + h_{\bar{C}}(\Omega_{\bar{C}/\bar{k}}^{\otimes(2g-2)}, Q)\\ & - h_{\bar{C}\times\bar{C}}\left(\mathcal{O}_{\bar{C}\times\bar{C}}((2g-2)^2\Delta), (P, Q)\right) + O(1)\\ & \qquad \textit{for } (P, Q) \in (C\times C)(\bar{k})\end{aligned}$$

and

$$\|f(P)\|^2 = h_{\bar{C}}(\Omega_{\bar{C}/\bar{k}}^{\otimes 2(2g-2)g}, P) + O(1) \quad \textit{for } P \in C(\bar{k}),$$

where $f: C \to J$ is the canonical map defined by (4), $h_V(\mathcal{L}, \cdot)$ is a logarithmic absolute height function on a projective variety V over $\bar{k}$ attached to an invertible sheaf $\mathcal{L}$ on V, $\Omega_{\bar{C}/\bar{k}}$ is the sheaf of differentials of $\bar{C}$ over $\bar{k}$, and Δ is the diagonal divisor on $C\times C$.

PROOF. See Lemma 1.17. □

We call the height function $\|f(\cdot)\|^2$ attached to the invertible sheaf $\Omega_{C/k}^{\otimes 2(2g-2)g}$ on the curve C the *canonical height function* on C.

We are in the situation of the next proposition.

PROPOSITION 1.4. *We have a canonically defined orthogonal representation of the automorphism group* $\mathrm{Aut}_k C$ *of C over k on the infinite dimensional real vector space $\mathbb{R}\otimes_{\mathbb{Z}} J(\bar{k})$ normed with the Néron-Tate height attached to a theta divisor. It is compatible with a nonconstant canonical map $f: C \to J$. The representation leaves stable the filtration on $\mathbb{R}\otimes_{\mathbb{Z}} J(\bar{k})$ of finite dimensional subspaces $\mathbb{R}\otimes_{\mathbb{Z}} J(K)$ induced by the filtration on $\bar{k}$ of the finite extensions K of k.*

PROBLEM 1.5. Describe the above representation, calculate the effect on $f(C(\bar{k}))$, and investigate the configuration of $f(C(\bar{k}))$ in $\mathbb{R}\otimes_{\mathbb{Z}} J(\bar{k})$.

Trivial Example 1.6. If C is hyperelliptic, the hyperelliptic involution induces -1 in the orthogonal transformation group of $(\mathbb{R}\otimes_{\mathbb{Z}} J(\bar{k}), \langle \cdot, \cdot\rangle)$.

The following theorem is an answer to the problem.

Let Δ be the diagonal divisor on $C\times C$. Suppose we have an automorphism ψ of C over k different from the identity map, and set

$$D := (1_C, \psi)^*\Delta \in \operatorname{Div} C \quad \text{and} \quad d := \deg D.$$

In a sense, D is the divisor of fixed points with multiplicities of the morphism ψ. The orthogonal transformation of $(\mathbb{R}\otimes_{\mathbb{Z}} J(\bar{k}), \langle \cdot, \cdot\rangle)$ induced by ψ is denoted as Ψ.

THEOREM 1.7. *The image $f(C(\bar{k}))$ under the canonical map $f: C \to J$ in the normed real vector space $(\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k}), \langle\cdot,\cdot\rangle)$ is contained in the neighborhood of a quadric hypersurface defined as*

$$\left|\left\langle v, \left(\Psi + \frac{d-2}{2g}\right) v + \mathcal{O}_C((2g-2)D) \otimes \Omega_{C/k}^{\otimes(-d)}\right\rangle\right| \leq \text{const.}, \quad v \in \mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k}),$$

where $\Omega_{C/k}$ is the sheaf of differentials of C over k and $\mathcal{O}_C((2g-2)D) \otimes \Omega_{C/k}^{\otimes(-d)}$ is considered as an element of $J(k) \simeq \operatorname{Pic}^\circ C$.

REMARK 1.8. The function of degree two on the left is what was called a *null height* by Manin [6, p. 339] (cf. Remark 1.19).

COROLLARY 1.9. *For $P \in C(\bar{k})$, we have $\|f(\psi(P))\| = \|f(P)\|$. The angle made by $\psi(P)$ and P under f in $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ is* $\arccos((2-d)/(2g) + O(1) \cdot \|f(P)\|^{-1})$.

PROOF. The former part is already confirmed true (cf. (5)). Here is another proof. There exists a finite extension field K of k such that P can be considered as in $C(K)$. The automorphism Ψ induced by ψ of the finite dimensional real vector space $\mathbb{R}\otimes_{\mathbb{Z}} J(K)$ is of finite order, hence is a Euclidean motion. Therefore we obtain $\|f(\psi(P))\| = \|\Psi(f(P))\| = \|f(P)\|$. The latter half is easy because by the theorem, we see

$$\frac{\langle f(P), f(\psi(P))\rangle}{\|f(P)\|\|f(\psi(P))\|} = \frac{\langle f(P), \Psi(f(P))\rangle}{\|f(P)\|\|\Psi(f(P))\|} \longrightarrow \frac{2-d}{2g} \quad \text{as} \quad \|f(P)\| \longrightarrow \infty. \square$$

COROLLARY 1.10. *The number of fixed points of a nontrivial automorphism of a curve whose genus g is at least two is not larger than $2g+2$.*

REMARK 1.11. Ordinarily, this follows from the Riemann-Hurwitz formula.

PROOF. Values on $\mathbb{R}$ of the cosine function are not less than -1. So by the previous corollary, the degree d of the divisor of fixed points must not be greater than $2g+2$. □

To prove the theorem, we need some notation and several lemmas.

For a projective variety V over $\bar{k}$ and an invertible sheaf $\mathcal{L}$ on V, we denote by $h_V(\mathcal{L}, \cdot)$ a height function on V attached to $\mathcal{L}$. We have the ambiguity of bounded functions in choosing a height function. When V is abelian, we agree to choose the canonical height *all the time* and for $u \in V(\bar{k})$, the translation-by-u-map is denoted by $t_u: V \to V$.

LEMMA 1.12 (Additive property). *For a projective variety V over $\bar{k}$ and invertible sheaves $\mathcal{L}$ and $\mathcal{M}$ on V, we have*

$$h_V(\mathcal{L} \otimes \mathcal{M}, x) = h_V(\mathcal{L}, x) + h_V(\mathcal{M}, x) + O(1), \quad x \in V(\bar{k}).$$

PROOF. See, for example, [9, Theorem of Section 2.8]. □

LEMMA 1.13 (Functoriality). *For a morphism $\phi: V \to W$ of projective varieties over $\bar{k}$ and an invertible sheaf $\mathcal{L}$ on W, we have*

$$h_V(\phi^*\mathcal{L}, x) = h_W(\mathcal{L}, \phi x) + O(1), \quad x \in V(\bar{k}).$$

For a homomorphism $\chi: A \to B$ of abelian varieties over $\bar{k}$ and an invertible sheaf $\mathcal{M}$ on B, we have

$$h_A(\chi^*\mathcal{M}, x) = h_B(\mathcal{M}, \chi x), \quad x \in A(\bar{k}).$$

For $u \in B(\bar{k})$ and the translation-by-u-map $t_u: B(\bar{k}) \ni x \mapsto x + u \in B(\bar{k})$,

$$h_B(t_u^*\mathcal{M}, x) = h_B(\mathcal{M}, x + u) - h_B(\mathcal{M}, u).$$

PROOF. For the first part, see, for example, [9, Section 2.8]. For the second, see [9, Section 3.2] and for the third, [9, Lemma of Section 3.4]. □

LEMMA 1.14. *For an invertible sheaf $\mathcal{L}$ on an abelian variety A and a rational integer n,*

$$n^*\mathcal{L} \simeq \mathcal{L}^{\otimes(n+1)n/2} \otimes (-1)^*\mathcal{L}^{\otimes(n-1)n/2}.$$

PROOF. See, for example, [7, Corollary 6.6]. □

Fix a point $P_0 \in C(\bar{k})$. Define a divisor Θ on $\bar{J}$ as (2), an invertible sheaf $\mathcal{N}_0$ on $\bar{J} \times \bar{J}$ by (3), and a morphism $f_0: \bar{C} \to \bar{J}$ over $\bar{k}$ as

$$C(\bar{k}) \ni P \mapsto \mathcal{O}_{\bar{C}}(P - P_0) \in \operatorname{Pic}^\circ \bar{C} \simeq J(\bar{k}).$$

Let $\mathcal{M}_0 \in \operatorname{Pic}(\bar{C} \times \bar{J})$ be the universal divisorial correspondence between (C, P_0) and $(J, 0)$ (cf. [8, Section 1]).

LEMMA 1.15. (i) *For $u \in J(\bar{k})$,*

$$s^*\mathcal{O}_{\bar{J}}(t_u^*\Theta) \otimes p^*\mathcal{O}_{\bar{J}}(-t_u^*\Theta) \otimes q^*\mathcal{O}_{\bar{J}}(-t_u^*\Theta) \simeq \mathcal{N}_0,$$

where s, p and $q: \bar{J} \times \bar{J} \to \bar{J}$ are respectively the sum, the projections onto the first and the second factors.

(ii) *For u and $v \in J(\bar{k})$, we have $t_u^*\mathcal{O}_{\bar{J}}(t_v^*\Theta - \Theta) \simeq \mathcal{O}_{\bar{J}}(t_v^*\Theta - \Theta)$.*

(iii) *For $u \in J(\bar{k})$, we have $\mathcal{O}_{\bar{J}}(t_u^*\Theta - \Theta) \simeq (1_J, u)^*\mathcal{N}_0 \simeq (u, 1_J)^*\mathcal{N}_0$.*

(iv) $(f_0 \times 1_J)^*\mathcal{N}_0 \simeq \mathcal{M}_0^{\otimes(-1)}$.

(v) $(1_C \times f_0)^*\mathcal{M}_0 \simeq \mathcal{O}_{\bar{C}\times\bar{C}}(\Delta - P_0 \times C - C \times P_0)$.

PROOF. (i) An application of Theorem of the cube [7, Corollary 6.4]. (ii) Theorem of the square [7, Theorem 6.7].(iii) Easily follows from the definition. (iv)(v) See [8, Summary 6.11]. □

LEMMA 1.16. (i) $(-1)^*\mathcal{N}_0 \simeq \mathcal{N}_0$.

(ii) *For $u \in J(\bar{k})$, we have $(-1)^*\mathcal{O}_{\bar{J}}(t_u^*\Theta - \Theta) \simeq \mathcal{O}_{\bar{J}}(\Theta - t_u^*\Theta)$.*

PROOF. (i) By the Riemann-Roch theorem, we know

$$(-1)^*\Theta = t_w^*\Theta \quad \text{for some } w \in J(\bar{k})$$

(see, for example, [9, Section 5.6 (1)]). Therefore, from the previous lemma,

$$\begin{aligned}(-1)^*\mathcal{N}_0 &\simeq (-1)^*\left[s^*\mathcal{O}_{\bar{J}}(\Theta) \otimes p^*\mathcal{O}_{\bar{J}}(-\Theta) \otimes q^*\mathcal{O}_{\bar{J}}(-\Theta)\right] \\ &\simeq s^*\mathcal{O}_{\bar{J}}((-1)^*\Theta) \otimes p^*\mathcal{O}_{\bar{J}}(-(-1)^*\Theta) \otimes q^*\mathcal{O}_{\bar{J}}(-(-1)^*\Theta) \\ &\simeq s^*\mathcal{O}_{\bar{J}}(t_w^*\Theta) \otimes p^*\mathcal{O}_{\bar{J}}(-t_w^*\Theta) \otimes q^*\mathcal{O}_{\bar{J}}(-t_w^*\Theta) \\ &\simeq \mathcal{N}_0.\end{aligned}$$

(ii) See [7, Section 9]. □

LEMMA 1.17. *We have for* $(P, Q) \in (C \times C)(\bar{k})$

$$\begin{aligned}\langle fP, fQ\rangle = {}& h_{\bar{C}}(\Omega_{C/k}^{\otimes(2g-2)}, P) + h_{\bar{C}}(\Omega_{C/k}^{\otimes(2g-2)}, Q) \\ & - h_{\bar{C}\times\bar{C}}\left(\mathcal{O}_{C\times C}((2g-2)^2\Delta), (P, Q)\right) + O(1),\end{aligned}$$

in particular,

$$\|fP\|^2 = h_{\bar{C}}(\Omega_{C/k}^{\otimes 2(2g-2)g}, P) + O(1), \quad P \in C(\bar{k}).$$

PROOF. We are going to show

$$(f \times f)^*\mathcal{N}_0 \simeq p^*\Omega_{C/k}^{\otimes(2g-2)} \otimes q^*\Omega_{C/k}^{\otimes(2g-2)} \otimes \mathcal{O}_{C\times C}(-(2g-2)^2\Delta).$$

Here p and $q: C \times C \to C$ are respectively the projections onto the first and the second factors. Then by the functoriality of heights, we obtain the first relation. The second relation is an immediate consequence of the first because as well-known

$$(1_C, 1_C)^*\mathcal{O}_{C\times C}(-\Delta) \simeq \Omega_{C/k},$$

where $(1_C, 1_C): C \to C \times C$ is the diagonal map.

Since $fP = fP_0 - (2g-2)f_0P$ for $P \in C(\bar{k})$, setting $a := fP_0 \in J(\bar{k})$, we have $f = t_a \circ (2-2g) \circ f_0$ hence

$$(f \times f)^*\mathcal{N}_0 \simeq (f_0 \times f_0)^*(2-2g)^*(t_a \times t_a)^*\mathcal{N}_0.$$

By definition and Lemmas 1.15 (i) and 1.15 (ii),

$$\begin{aligned}(t_a \times t_a)^*\mathcal{N}_0 &\simeq (t_a \times t_a)^*\left[s^*\mathcal{O}_{\bar{J}}(\Theta) \otimes p^*\mathcal{O}_{\bar{J}}(-\Theta) \otimes q^*\mathcal{O}_{\bar{J}}(-\Theta)\right] \\ &\simeq s^*\mathcal{O}_{\bar{J}}(t_{2a}^*\Theta) \otimes p^*\mathcal{O}_{\bar{J}}(-t_a^*\Theta) \otimes q^*\mathcal{O}_{\bar{J}}(-t_a^*\Theta) \\ &\simeq \left[s^*\mathcal{O}_{\bar{J}}(t_{2a}^*\Theta) \otimes p^*\mathcal{O}_{\bar{J}}(-t_{2a}^*\Theta) \otimes q^*\mathcal{O}_{\bar{J}}(-t_{2a}^*\Theta)\right] \\ &\quad \otimes p^*\mathcal{O}_{\bar{J}}(t_{2a}^*\Theta - t_a^*\Theta) \otimes q^*\mathcal{O}_{\bar{J}}(t_{2a}^*\Theta - t_a^*\Theta) \\ &\simeq \mathcal{N}_0 \otimes p^*t_a^*\mathcal{O}_{\bar{J}}(t_a^*\Theta - \Theta) \otimes q^*t_a^*\mathcal{O}_{\bar{J}}(t_a^*\Theta - \Theta) \\ &\simeq \mathcal{N}_0 \otimes p^*\mathcal{O}_{\bar{J}}(t_a^*\Theta - \Theta) \otimes q^*\mathcal{O}_{\bar{J}}(t_a^*\Theta - \Theta),\end{aligned}$$

where by abuse of notation, the morphisms p and q are the projections of $\bar{J} \times \bar{J}$. By Lemma 1.14 and Lemma 1.16,

$$\begin{aligned}
&(2-2g)^*(t_a \times t_a)^*\mathcal{N}_0 \\
&\simeq (2-2g)^*\left[\mathcal{N}_0 \otimes p^*\mathcal{O}_{\bar{J}}(t_a^*\Theta - \Theta) \otimes q^*\mathcal{O}_{\bar{J}}(t_a^*\Theta - \Theta)\right] \\
&\simeq (2-2g)^*\mathcal{N}_0 \otimes p^*(2-2g)^*\mathcal{O}_{\bar{J}}(t_a^*\Theta - \Theta) \otimes q^*(2-2g)^*\mathcal{O}_{\bar{J}}(t_a^*\Theta - \Theta) \\
&\simeq \mathcal{N}_0^{\otimes(2-2g)^2} \otimes p^*\mathcal{O}_{\bar{J}}(t_a^*\Theta - \Theta)^{\otimes(2-2g)} \otimes q^*\mathcal{O}_{\bar{J}}(t_a^*\Theta - \Theta)^{\otimes(2-2g)} \\
&\simeq \mathcal{N}_0^{\otimes(2-2g)^2} \otimes p^*(1_J, a)^*\mathcal{N}_0^{\otimes(2-2g)} \otimes q^*(a, 1_J)^*\mathcal{N}_0^{\otimes(2-2g)}.
\end{aligned}$$

We further pull back this invertible sheaf by $f_0 \times f_0$.

$$\begin{aligned}
&(f_0 \times f_0)^*(2-2g)^*(t_a \times t_a)^*\mathcal{N}_0 \\
&\simeq (f_0 \times f_0)^*\left[\mathcal{N}_0^{\otimes(2-2g)^2} \otimes p^*(1_J, a)^*\mathcal{N}_0^{\otimes(2-2g)} \otimes q^*(a, 1_J)^*\mathcal{N}_0^{\otimes(2-2g)}\right] \\
&\simeq (f_0 \times f_0)^*\mathcal{N}_0^{\otimes(2-2g)^2} \\
&\qquad \otimes p^*(1_C, a)^*(f_0 \times 1_J)^*\mathcal{N}_0^{\otimes(2-2g)} \otimes q^*(a, 1_C)^*(1_J \times f_0)^*\mathcal{N}_0^{\otimes(2-2g)} \\
&\simeq (1_C \times f_0)^*(f_0 \times 1_J)^*\mathcal{N}_0^{\otimes(2-2g)^2} \\
&\qquad \otimes p^*(1_C, a)^*(f_0 \times 1_J)^*\mathcal{N}_0^{\otimes(2-2g)} \otimes q^*(a, 1_C)^*(1_J \times f_0)^*\mathcal{N}_0^{\otimes(2-2g)} \\
&\simeq (1_C \times f_0)^*(f_0 \times 1_J)^*\mathcal{N}_0^{\otimes(2-2g)^2} \\
&\qquad \otimes p^*(1_C, a)^*(f_0 \times 1_J)^*\mathcal{N}_0^{\otimes(2-2g)} \otimes q^*(1_C, a)^*(f_0 \times 1_J)^*\mathcal{N}_0^{\otimes(2-2g)}.
\end{aligned}$$

The last transformation is permitted due to the symmetry of $\mathcal{N}_0$. By Lemma 1.15 (iv),

$$\begin{aligned}
&(f_0 \times f_0)^*(2-2g)^*(t_a \times t_a)^*\mathcal{N}_0 \\
&\simeq (1_C \times f_0)^*\mathcal{M}_0^{\otimes-(2-2g)^2} \otimes p^*(1_C, a)^*\mathcal{M}_0^{\otimes(2g-2)} \otimes q^*(1_C, a)^*\mathcal{M}_0^{\otimes(2g-2)}.
\end{aligned}$$

We have Lemma 1.15 (v). And, since the universality of $\mathcal{M}_0$ tells us $(1_C, y)^*\mathcal{M}_0 \simeq y$ for $y \in J(\bar{k}) \simeq \mathrm{Pic}^\circ(\bar{C})$,

$$\begin{aligned}
&(f_0 \times f_0)^*(2-2g)^*(t_a \times t_a)^*\mathcal{N}_0 \\
&\simeq \mathcal{O}_{C\times C}(P_0 \times C + C \times P_0 - \Delta)^{\otimes(2-2g)^2} \otimes p^*a^{\otimes(2g-2)} \otimes q^*a^{\otimes(2g-2)} \\
&\simeq p^*\mathcal{O}_C(P_0)^{\otimes(2g-2)^2} \otimes q^*\mathcal{O}_C(P_0)^{\otimes(2g-2)^2} \otimes \mathcal{O}_{C\times C}(-\Delta)^{\otimes(2g-2)^2} \\
&\qquad \otimes p^*\left[\Omega_{C/k} \otimes \mathcal{O}_C(-(2g-2)P_0)\right]^{\otimes(2g-2)} \otimes q^*\left[\Omega_{C/k} \otimes \mathcal{O}_C(-(2g-2)P_0)\right]^{\otimes(2g-2)} \\
&\simeq \mathcal{O}_{C\times C}(-\Delta)^{\otimes(2-2g)^2} \otimes p^*\Omega_{C/k}^{\otimes(2g-2)} \otimes q^*\Omega_{C/k}^{\otimes(2g-2)}. \qquad \square
\end{aligned}$$

LEMMA 1.18. *Let $\mathcal{L} \in J(\bar{k}) \simeq \mathrm{Pic}^\circ(\bar{C})$. We have*

$$\langle fP, \mathcal{L}\rangle = h_{\bar{C}}(\mathcal{L}^{\otimes(2g-2)}, P) + O(1), \quad P \in C(\bar{k}).$$

PROOF. We have only to show

$$(f, \mathcal{L})^*\mathcal{N}_0 \simeq \mathcal{L}^{\otimes(2g-2)}.$$

Since $f = t_a \circ (2-2g) \circ f_0$, where $a = fP_0$, and $(f, \mathcal{L}) = (1_J, \mathcal{L}) \circ f$, we see

$$(f, \mathcal{L})^*\mathcal{N}_0 \simeq f_0^*(2-2g)^* t_a^*(1_J, \mathcal{L})^*\mathcal{N}_0.$$

By Lemma 1.15 (iii), Lemma 1.15 (ii), Lemma 1.14, and Lemma 1.16 (ii), we gain

$$\begin{aligned}(2-2g)^* t_a^*(1_J, \mathcal{L})^*\mathcal{N}_0 &\simeq (2-2g)^* t_a^* \mathcal{O}_{\bar{J}}(t_{\mathcal{L}}^*\Theta - \Theta) \\ &\simeq (2-2g)^* \mathcal{O}_{\bar{J}}(t_{\mathcal{L}}^*\Theta - \Theta) \\ &\simeq \mathcal{O}_{\bar{J}}(t_{\mathcal{L}}^*\Theta - \Theta)^{\otimes(2-2g)} \\ &\simeq (1_J, \mathcal{L})^*\mathcal{N}_0^{\otimes(2-2g)}.\end{aligned}$$

Pulling this back by f_0, we have from Lemma 1.15 (iv) and the property of the universal divisorial correspondence $\mathcal{M}_0$,

$$\begin{aligned}f_0^*(2-2g)^* t_a^*(1_J, \mathcal{L})^*\mathcal{N}_0 &\simeq f_0^*(1_J, \mathcal{L})^*\mathcal{N}_0^{\otimes(2-2g)} \\ &\simeq (1_C, \mathcal{L})^*(f_0 \times 1_J)^*\mathcal{N}_0^{\otimes(2-2g)} \\ &\simeq (1_C, \mathcal{L})^*\mathcal{M}_0^{\otimes(2g-2)} \\ &\simeq \mathcal{L}^{\otimes(2g-2)}.\end{aligned}$$

□

PROOF OF THE THEOREM. By Lemma 1.17 and the functoriality of heights, we see

$$\begin{aligned}\langle fP, \Psi(fP)\rangle &= \langle fP, f(\psi P)\rangle \\ &= h_{\bar{C}}(\Omega_{C/k}^{\otimes(2g-2)}, P) + h_{\bar{C}}(\Omega_{C/k}^{\otimes(2g-2)}, \psi P) \\ &\quad - h_{\bar{C}\times\bar{C}}(\mathcal{O}_{C\times C}((2g-2)^2\Delta), (P, \psi P)) + O(1) \\ &= h_{\bar{C}}(\Omega_{C/k}^{\otimes(2g-2)}, P) + h_{\bar{C}}(\psi^*\Omega_{C/k}^{\otimes(2g-2)}, P) \\ &\quad - h_{\bar{C}}((1_C, \psi)^*\mathcal{O}_{C\times C}((2g-2)^2\Delta), P) + O(1)\end{aligned}$$

as functions of the $\bar{k}$-valued points P on $C(\bar{k})$. Since $\psi^*\Omega_{C/k} \simeq \Omega_{C/k}$,

$$\langle fP, \Psi(fP)\rangle = h_{\bar{C}}(\Omega_{C/k}^{\otimes 2(2g-2)} \otimes \mathcal{O}_C(-(2g-2)^2 D), P) + O(1).$$

By the second equality of Lemma 1.17 and Lemma 1.18,

$$\begin{aligned}&2g\langle fP, \Psi(fP)\rangle + (d-2)\|fP\|^2 \\ &= h_{\bar{C}}(\Omega_{C/k}^{\otimes 4(2g-2)g} \otimes \mathcal{O}_C(-2(2g-2)^2 gD), P) \\ &\quad + h_{\bar{C}}(\Omega_{C/k}^{\otimes 2(2g-2)(d-2)g}, P) + O(1) \\ &= h_{\bar{C}}(\Omega_{C/k}^{\otimes 2(2g-2)gd} \otimes \mathcal{O}_C(-2(2g-2)^2 gD), P) + O(1) \\ &= 2g \cdot h_{\bar{C}}\left(\left[\Omega_{C/k}^{\otimes d} \otimes \mathcal{O}_C(-(2g-2)D)\right]^{\otimes(2g-2)}, P\right) + O(1) \\ &= 2g\langle fP, \Omega_{C/k}^{\otimes d} \otimes \mathcal{O}_C(-(2g-2)D)\rangle + O(1).\end{aligned}$$

□

REMARK 1.19. Using the additive property and functoriality of height functions, and setting $\mathcal{L}_\psi := \mathcal{O}_C((2g-2)D) \otimes \Omega_{C/k}^{\otimes(-d)}$, we get for $v \in J(\bar{k})$,

$$\begin{aligned}&2g\langle v, \Psi v\rangle + (d-2)\|v\|^2 + 2g\langle v, \mathcal{L}_\psi\rangle \\ &= h_{\bar{J}}((1_J, \Psi)^*\mathcal{N}_0^{\otimes 2g} \otimes (1_J, 1_J)^*\mathcal{N}_0^{\otimes(d-2)} \otimes (1_J, \mathcal{L}_\psi)^*\mathcal{N}_0^{\otimes 2g}, v).\end{aligned}$$

This is a canonical height on J and after all, we have proved

$$f^*\left[(1_J, \Psi)^*\mathcal{N}_0^{\otimes 2g} \otimes (1_J, 1_J)^*\mathcal{N}_0^{\otimes(d-2)} \otimes (1_J, \mathcal{L}_\psi)^*\mathcal{N}_0^{\otimes 2g}\right] \simeq \mathcal{O}_{\bar{C}}.$$

2. – Twisted Fermat curve of degree four

In this section, we describe in detail the regions where canonical images lie for a particular family of curves. As a consequence, we obtain another proof of a certain well-known finiteness criterion (Theorem 0.6).

Let k be a number field and a, b, c elements of k different from zero. We call the curve Q in the projective plane $\mathbb{P}_k^2$ over k defined by the homogeneous equation

$$Q: aX^4 + bY^4 + cZ^4 = 0 \tag{6}$$

a twisted Fermat curve of degree four. We define also an elliptic curve E_X over k given by a Weierstrass equation

$$E_X: S^2T = R^3 + a^2bcT^2R,$$

where R, S, and T are the homogeneous coordinates of $\mathbb{P}_k^2$. This is a quotient curve of Q by a subgroup of order four of the automorphism group of $\bar{Q} = Q \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ over $\bar{k}$, where $\bar{k}$ is an algebraic closure of k. The quotient map $\phi_X: Q \to E_X$ is given over k by

$$Q(\bar{k}) \ni (x:y:z) \mapsto (r:s:t) = (-aby^2z : a^2bx^2y : z^3) \in E_X(\bar{k}).$$

There exists a homomorphism Ψ_X over k of E_X into the Jacobian variety J of Q induced by ϕ_X such that

$$\Psi_X: E_X(\bar{k}) \simeq \operatorname{Pic}^\circ(\bar{E}_X) \ni \mathcal{L} \mapsto \phi_X^*\mathcal{L} \in \operatorname{Pic}^\circ(\bar{Q}) \simeq J(\bar{k}).$$

The image of Ψ_X is one-dimensional, for we have a natural homomorphism $\Phi_X: J \to E_X$ satisfying $\Phi_X \circ \Psi_X = 4$ (cf. [8, Proposition 6.1]).

We denote by μ the group of square roots of unity in k. Then $\mu \times \mu \times \mu$ acts on Q as follows: For $(\xi, \eta, \zeta) \in \mu \times \mu \times \mu$, the action is given by

$$Q(\bar{k}) \ni (x : y : z) \mapsto (\xi x : \eta y : \zeta z) \in Q(\bar{k}).$$

This action yields an action on E_X compatible with the quotient map ϕ_X of Q onto E_X. Let γ_X, γ_Y, and γ_Z be the respective automorphisms of Q over k corresponding to the elements $(-1, 1, 1)$, $(1, -1, 1)$, and $(1, 1, -1)$ of $\mu \times \mu \times \mu$. The next diagrams are commutative:

$$\begin{tikzcd} Q \arrow[r, "\phi_X"] \arrow[d, "\gamma_X"'] & E_X \arrow[d, equal] \\ Q \arrow[r] & E_X \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r] \arrow[d, "\gamma_Y"'] & E_X \arrow[d, "-1"] \\ Q \arrow[r] & E_X \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r] \arrow[d, "\gamma_Z"'] & E_X \arrow[d, "-1"] \\ Q \arrow[r] & E_X \end{tikzcd}$$

Denoting the induced automorphisms of J over k respectively by Γ_X, Γ_Y, and Γ_Z, we obtain the following commutative diagrams:

$$\begin{tikzcd} J \arrow[d, "\Gamma_X"'] & E_X \arrow[l, "\Psi_X"'] \arrow[d, equal] \\ J & E_X \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} J \arrow[d, "\Gamma_Y"'] & E_X \arrow[l] \arrow[d, "-1"] \\ J & E_X \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} J \arrow[d, "\Gamma_Z"'] & E_X \arrow[l] \arrow[d, "-1"] \\ J & E_X \arrow[l] \end{tikzcd}$$

We define in a cyclic manner E_Y, $\Psi_Y : E_Y \to J$, E_Z, and $\Psi_Z : E_Z \to J$. We see for i and $j \in \{X, Y, Z\}$

$$\Gamma_i \circ \Psi_j = (-1)^{\delta_{ij}-1} \Psi_j,$$

where δ_{ij} is Kronecker's delta function.

Now consider the map $\Psi := \Psi_X \circ p_1 + \Psi_Y \circ p_2 + \Psi_Z \circ p_3$ of $E_X \times E_Y \times E_Z$ into J, where p_i is the projection onto the i-th factor. The subvariety $\Psi(E_X \times E_Y \times 0)$ of J includes a curve $\Psi(E_X \times 0 \times 0) = \Psi_X(E_X)$, and the action of γ_X on $\Psi(E_X \times 0 \times 0)$ is trivial but not so on $\Psi(E_X \times E_Y \times 0) = \Psi_X(E_X) + \Psi_Y(E_Y)$. Therefore $\Psi(E_X \times E_Y \times 0)$ must be two-dimensional. By the same sort of reasoning, $\Psi(E_X \times E_Y \times E_Z)$ is three-dimensional. Since the dimension of J is also three, Ψ is an isogeny. Accordingly, we get an isomorphism of $\mathbb{R}$-vector spaces

$$\Psi : \bigoplus_{i=X,Y,Z} \mathbb{R} \otimes_{\mathbb{Z}} E_i(\bar{k}) \xrightarrow{\sim} \mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k}).$$

We identify $\mathbb{R} \otimes_{\mathbb{Z}} E_i(\bar{k})$ with the corresponding subspace of $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ by this isomorphism.

Provide $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$ attached to a theta divisor. The elements of $\mathbb{R} \otimes_{\mathbb{Z}} E_i(\bar{k})$ are simultaneous eigenvectors of $\mu \times \mu \times \mu$ and each eigenspace $\mathbb{R} \otimes_{\mathbb{Z}} E_i(\bar{k})$ corresponds to a different character of $\mu \times \mu \times \mu$. Since eigenvectors of an orthogonal transformation with different eigenvalues are orthogonal to each other, the above decomposition of $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ into $\mathbb{R} \otimes_{\mathbb{Z}} E_i(\bar{k})$ is in addition orthogonal with respect to $\langle \cdot, \cdot \rangle$.

For $v = v_X + v_Y + v_Z \in \mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$, where $v_i \in \mathbb{R} \otimes_{\mathbb{Z}} E_i(\bar{k})$, we have

$$\begin{aligned} \langle v, \Gamma_X v \rangle &= \langle v, \Gamma_X v_X + \Gamma_X v_Y + \Gamma_X v_Z \rangle \\ &= \langle v_X + v_Y + v_Z, v_X - v_Y - v_Z \rangle \\ &= \|v_X\|^2 - \|v_Y\|^2 - \|v_Z\|^2. \end{aligned}$$

Similarly, we gain

$$\langle v, \Gamma_Y v \rangle = -\|v_X\|^2 + \|v_Y\|^2 - \|v_Z\|^2$$

and

$$\langle v, \Gamma_Z v \rangle = -\|v_X\|^2 - \|v_Y\|^2 + \|v_Z\|^2.$$

PROPOSITION 2.1. *Let Q be a twisted Fermat curve of degree four, J its Jacobian variety, and $f\colon Q \to J$ the canonical morphism given as* (4). *We equip $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ with the norm $\|\cdot\|$ associated with a theta divisor. Then there exist absolute constants c_1 and c_2 and an orthogonal decomposition $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k}) = V_X \oplus V_Y \oplus V_Z$ into subspaces such that the image $f(Q(\bar{k}))$ is contained in the region of $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ defined by*

$$\begin{cases} \left| \|v_X\|^2 - \|v_Y\|^2 \right| & \leq c_1 \\ \left| \|v_Z\|^2 - \|v_X\|^2 \right| & \leq c_2, \end{cases}$$

where $v_X \in V_X$, $v_Y \in V_Y$, and $v_Z \in V_Z$.

PROOF. For $P \in Q(\bar{k})$, let $f(P) = v_X + v_Y + v_Z$, $v_i \in V_i := \mathbb{R} \otimes_{\mathbb{Z}} E_i(\bar{k})$. By Proposition 2.3 below, we see

$$\begin{aligned} \langle f(P), \Gamma_X(f(P)) \rangle = \langle f(P), f(\gamma_X(P)) \rangle &= -\frac{1}{3}\|f(P)\|^2 + O(1) \\ &= -\frac{1}{3}(\|v_X\|^2 + \|v_Y\|^2 + \|v_Z\|^2) + O(1) \end{aligned}$$

and

$$\langle f(P), \Gamma_Y(f(P)) \rangle = -\frac{1}{3}(\|v_X\|^2 + \|v_Y\|^2 + \|v_Z\|^2) + O(1)$$

with $O(1)$ terms bounded by absolute constants. Combining these with the equalities before Proposition 2.1, we obtain

$$2\|v_X\|^2 - \|v_Y\|^2 - \|v_Z\|^2 = O(1)$$

and

$$-\|v_X\|^2 + 2\|v_Y\|^2 - \|v_Z\|^2 = O(1).$$

Eliminations of appropriate terms give the result. □

LEMMA 2.2. *Let F be a plane curve of degree four over k defined as*

$$F: -X^4 - Y^4 + Z^4 = 0 \tag{7}$$

and γ an automorphism of F over k which acts as multiplications by -1 or 1 of X, Y, and Z coordinates. If γ is different from the identity map, then for $P \in F(\bar{k})$,

$$\langle f(P), f(\gamma(P))\rangle = -\frac{1}{3}\|f(P)\|\|f(\gamma(P))\| + O(1).$$

PROOF. Similar to the proof of Lemma 2.5 below (cf. [2, Proposition 6.4]). □

PROPOSITION 2.3. *Let Q be a twisted Fermat curve of degree four given by* (6) *and γ an automorphism of Q over k which acts as multiplications by -1 or 1 of X, Y, and Z coordinates. If γ is not the identity map, then for $P \in Q(\bar{k})$,*

$$\langle f(P), f(\gamma(P))\rangle = -\frac{1}{3}\|f(P)\|\|f(\gamma(P))\| + O(1)$$

with $O(1)$ bounded by absolute constants. Here $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are respectively the scalar product and the norm on $\mathbb{R}\otimes_{\mathbb{Z}} J(\bar{k})$ attached to a theta divisor.

PROOF. Choose elements ξ, η, and ζ of $\bar{k}$ such that $a = -\xi^4$, $b = -\eta^4$, and $c = \zeta^4$. There exists an isomorphism ϕ of $\bar{Q}$ onto $\bar{F}$, where F is the curve (7), given by

$$Q(\bar{k}) \ni (x : y : z) \mapsto (\xi x : \eta y : \zeta z) \in F(\bar{k}).$$

The morphism ϕ is compatible with the respective automorphisms γ of Q and F, that is, $\gamma\circ\phi = \phi\circ\gamma$. By the invariance of heights (5), for $P \in Q(\bar{k})$

$$\begin{aligned}\langle f(P), f(\gamma(P))\rangle &= \langle f(\phi(P)), f(\phi(\gamma(P)))\rangle\\ &= \langle f(\phi(P)), f(\gamma(\phi(P)))\rangle\\ &= -\frac{1}{3}\|f(\phi(P))\|^2 + O(1)\\ &= -\frac{1}{3}\|f(P)\|^2 + O(1).\end{aligned}$$

The last $O(1)$ is the composition of ϕ with $O(1)$ in the previous lemma hence is absolutely bounded. □

When the coefficients of X^4 and Y^4 of the defining equation (6) are the same, we receive some more information about the distribution of rational points of the curve in the Jacobian variety.

Assume $a = b = -1$. In this case, there is another automorphism τ of Q given by the exchange of X and Y coordinates. The automorphism τ yields isomorphisms $\tau_X: E_Y \to E_X$, $\tau_Y: E_X \to E_Y$ (not as groups), and an automorphism τ_Z of E_Z (not as a group) compatible with the quotient maps $\phi_i: Q \to E_i$. Explicitly, these morphisms are defined as

$$\tau_X: E_Y(\bar{k}) \ni (x : y : z) \mapsto (-czx : -cyz : x^2) \in E_X(\bar{k}),$$
$$\tau_Y: E_X(\bar{k}) \ni (x : y : z) \mapsto (-czx : -cyz : x^2) \in E_Y(\bar{k}),$$

and

$$\tau_Z: E_Z(\bar{k}) \ni (x : y : z) \mapsto (c^2zx : c^2yz : x^2) \in E_Z(\bar{k}).$$

They induce group isomorphisms $T: J \to J$, $T_X: E_Y \to E_X$, $T_Y: E_X \to E_Y$, and $T_Z: E_Z \to E_Z$ all compatible with $\Psi_i: E_i \to J$. Since τ_X and τ_Y are inverse to each other, so are T_X and T_Y. On the other hand, $T_Z = -1$, because τ_Z is the multiplication-by-(-1)-map plus a two-torsion point $(0 : 0 : 1)$.

PROPOSITION 2.4. *Let Q be a twisted Fermat curve of degree four whose coefficients of X^4 and Y^4 of the defining equation* (6) *are the same, τ the automorphism of Q exchanging the X and Y coordinates, and f the canonical map of Q into the Jacobian variety J of Q defined as* (4). *For $P \in Q(\bar{k})$, we have*

$$\langle f(P), f(\tau(P))\rangle = -\frac{1}{3}\|f(P)\|\|f(\tau(P))\| + O(1),$$

where $O(1)$ is bounded by absolute constants.

PROOF. Follows from the next lemma in the same way as Proposition 2.3 followed from Lemma 2.2. □

LEMMA 2.5. *Let F be the plane curve* (7), *τ the automorphism of F exchanging the X and Y coordinates, and f the canonical map of F into the Jacobian variety of F defined as* (4). *For $P \in F(\bar{k})$, we have*

$$\langle f(P), f(\tau(P))\rangle = -\frac{1}{3}\|f(P)\|\|f(\tau(P))\| + O(1).$$

PROOF. Since F is a plane curve of degree four, the canonical sheaf $\Omega_{F/k}$ is isomorphic to the inverse image of $\mathcal{O}_{\mathbb{P}^2}(1)$ (cf. [3, II 8.20.3]), where $\mathbb{P}^2$ denotes the ambient projective plane. According to Theorem 1.7, we have only to show for the diagonal Δ on $F \times F$ the divisor $(1_F, \tau)^*\Delta$ is linearly equivalent to a hyperplane section.

We compute the inverse image under $(1_F, \tau)$ of the ideal sheaf $\mathcal{O}_{F\times F}(-\Delta)$ of the diagonal subvariety of $F \times F$. Let $X_1, Y_1, Z_1; X_2, Y_2, Z_2$ be the bihomogeneous coordinates of $\mathbb{P}^2 \times \mathbb{P}^2$ and X, Y, Z the homogeneous coordinates of $\mathbb{P}^2$. We naturally regard $F \times F$ as in $\mathbb{P}^2_k \times \mathbb{P}^2_k$ and F, in $\mathbb{P}^2_k$. Fixed points of τ are not mapped to the closed subscheme $\{Z_1Z_2 = 0\}$ by the map $(1_F, \tau): F \to F \times F$, hence it suffices to see the affine open subvariety $\{Z_1Z_2 \neq 0\}$. Since

$$\begin{aligned}\Gamma(\{Z_1Z_2 \neq 0\}, \mathcal{O}_{F\times F}(-\Delta)) &\simeq (X_1/Z_1 - X_2/Z_2, Y_1/Z_1 - Y_2/Z_2)\\ &\bmod \left(-(X_1/Z_1)^4 - (Y_1/Z_1)^4 + 1, -(X_2/Z_2)^4 - (Y_2/Z_2)^4 + 1\right),\end{aligned}$$

where the right hand side is an ideal of $\Gamma(\{Z_1Z_2 \neq 0\}, \mathcal{O}_{F\times F})$, we have

$$\Gamma\left(\{Z \neq 0\}, (1_F, \tau)^*\mathcal{O}_{F\times F}(-\Delta)\right) \simeq (X/Z - Y/Z) \bmod \left(-(X/Z)^4 - (Y/Z)^4 + 1\right).$$

From this, we see $(1_F, \tau)^*\mathcal{O}_{F\times F}(-\Delta)$ is naturally isomorphic to the ideal sheaf of a hyperplane section $\{X - Y = 0\}$, which is the desired result. □

PROPOSITION 2.6. *Notation being the same as in Proposition 2.1, suppose the coefficients of X^4 and Y^4 of the defining equation* (6) *are both minus one. Then, besides the neighborhoods of hypersurfaces in Proposition 2.1, the image $f(Q(\bar{k}))$ is included in the region near another quadric hypersurface in $\mathbb{R} \otimes_{\mathbb{Z}} J(\bar{k})$ given by*

$$|\langle v_X, T_X v_Y\rangle| \leq c_3$$

with an absolute constant c_3, where $T_X\colon V_Y \to V_X$ is a metric linear isomorphism. Let E be an elliptic curve defined by a Weierstrass equation

$$y^2 = x^3 - cx. \tag{8}$$

Then we can take V_X and V_Y as $\mathbb{R}\otimes_{\mathbb{Z}} E(\bar{k})$, and T_X is induced by an automorphism of E over k.

PROOF. For $P \in Q(\bar{k})$, let $f(P) = v_X + v_Y + v_Z$, $v_i \in \mathbb{R} \otimes_{\mathbb{Z}} E_i(\bar{k})$. Then we see

$$\begin{aligned}\langle f(P), T(f(P))\rangle &= \langle v_X + v_Y + v_Z, T_Y v_X + T_X v_Y + T_Z v_Z\rangle \\ &= \langle v_X, T_X v_Y\rangle + \langle v_Y, T_Y v_X\rangle + \langle v_Z, T_Z v_Z\rangle \\ &= \langle v_X, T_X v_Y\rangle + \langle T_X v_Y, v_X\rangle - \|v_Z\|^2,\end{aligned}$$

because $T_X = T_Y^{-1}$ does not change norm and $T_Z = -1$. From Proposition 2.4, we know

$$\langle f(P), T(f(P))\rangle = -\frac{1}{3}(\|v_X\|^2 + \|v_Y\|^2 + \|v_Z\|^2) + O(1)$$

with an absolutely bounded function $O(1)$ of P on $Q(\bar{k})$. Therefore we have

$$6\langle v_X, T_X v_Y\rangle + \|v_X\|^2 + \|v_Y\|^2 - 2\|v_Z\|^2 = O(1).$$

Add appropriate times the inequalities in Proposition 2.1 to this equality. □

COROLLARY 2.7 (Dem'yanenko [1, Example 1], [9, § 5.3]). *If the rank of $E(k)$ is not larger than one, then the canonical heights of rational points on Q are bounded by an absolute constant.*

PROOF. Note first that the whole story was occurring over the base field k. So in Proposition 2.1 and Proposition 2.6, we can replace $\bar{k}$ with k.

For $P \in Q(k)$, let $f(P) = v_X + v_Y + v_Z$, $v_i \in V_i$. When $\dim V_X = \dim V_Y = 1$, we see $v_X = 0$ or $T_X v_Y = r v_X$ for some $r \in \mathbb{Q}$. In the latter situation, we have $\|v_Y\| = |r| \cdot \|v_X\|$, for T_X preserves the norm. If $|r| < 1/2$, then Proposition 2.1 says $\|v_X\|$ is absolutely bounded. If $|r| \geq 1/2$, then Proposition 2.6 still asserts $\|v_X\|$ is absolutely bounded. Anyway, by Proposition 2.1, $\|v_i\|$'s are all absolutely bounded, hence $\|f(P)\|$, too. □

REMARK 2.8. Manin [5, Example 1] got the same kind of result as Dem'yanenko's. He has constructed a null height on the product of two copies of an elliptic curve on which lies a special twisted Fermat curve.

REMARK 2.9. The quantity $\langle v_X, T_X v_Y \rangle$ for a point in $f(Q(k))$ does not necessarily vanish. Consider the example $a = b = -1$ and $c = 2$, in other words, $Q: X^4 + Y^4 = 2Z^4$ and $E_X, E_Y: y^2 = x^3 - 2x$. In this case, we know the ranks of $E_X(\mathbb{Q})$ and $E_Y(\mathbb{Q})$ are one. For $P = (1 : 1 : 1) \in Q(\mathbb{Q})$, let $f(P) = v_X + v_Y + v_Z$ as above. We can see

$$v_X = \frac{1}{4} \otimes \phi_X(P) \in \mathbb{R} \otimes_{\mathbb{Z}} E_X(\mathbb{Q}) \quad \text{and} \quad v_Y = \frac{1}{4} \otimes \phi_Y(P) \in \mathbb{R} \otimes_{\mathbb{Z}} E_Y(\mathbb{Q}).$$

Since P is a fixed point of τ,

$$\begin{aligned} T_X v_Y &= \frac{1}{4} \otimes [\tau_X(\phi_Y(P)) - \tau_X(\infty)] \\ &= \frac{1}{4} \otimes \phi_X(\tau(P)) - \frac{1}{4} \otimes (0,0) \\ &= \frac{1}{4} \otimes \phi_X(P) \\ &= v_X. \end{aligned}$$

Consequently, $\langle v_X, T_X v_Y \rangle = \|v_X\|^2$. On the other hand, $\phi_X(P) = (-1,-1) \in E_X(\mathbb{Q})$ is not a torsion point, therefore $v_X \neq 0$.

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Mathematical Institute
Tohoku University
Sendai 980-77
Japan