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Spectral Asymptotics for Multi-Quasi-Elliptic Operators in $\mathbb{R}^n$

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0. – Introduction

The estimation of the growth of the number of eigenvalues for a given operator in $L^2(\mathbb{R}^n)$ plays an important rôle in Physics and is a central theme in Spectral Analysis.

In this paper we give a precise estimate for the asymptotic behavior of the eigenvalues counting function $N(\lambda)$ for global multi-quasi-elliptic operators in $\mathbb{R}^n$.

Global multi-quasi-elliptic pseudo-differential operators in $\mathbb{R}^n$ are a generalization of the multi-quasi-elliptic differential operators with constant coefficients defined by Friberg [7], Mihaĭlov [11] and Volevič-Gindikin [16] and have been studied by several authors among which Cattabriga [6], Pini [13] and Zanghirati [17]. They have been introduced and studied by Boggiatto [2], [3] and are an important example of the global hypoelliptic operators in $\mathbb{R}^n$ considered by Berezin and Shubin and many other authors in connection with mathematical questions in Quantum Mechanics. See [1] for a brief survey of the theory.

Multi-quasi-elliptic operators are defined in Section 1. They form a class containing quasi-elliptic operators and closed with respect to composition. Their definition is based on a weight function $w_{\mathcal{P}}$ associated with a convex polyhedron $\mathcal{P} \subset (\mathbb{R}_0^+)^N$ satisfying suitable hypotheses (see Section 1). An operator which is multi-quasi-elliptic with respect to $\mathcal{P}$ is called $\mathcal{P}$-elliptic.

Our main results are Theorems 2.1 and 3.4. In Theorem 3.4 we give an asymptotic computation of the Weyl term

$$V(\lambda) = (2\pi)^{-n} \int_{|a(z)| \leq \lambda} dz$$

associated with a $\mathcal{P}$-elliptic symbol $a(z)$ with polynomial principal symbol.

Under the assumption that the characteristic polyhedron $\mathcal{P}$ is non-degenerate, i.e. the intersection of the boundary of $\mathcal{P}$ with the diagonal of $\mathbb{R}^N$ is an internal point to a face F_ω of $\mathcal{P}$ of equation $\omega \cdot z = 1$, we obtain the following asymptotic

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expansion:

$$V(\lambda) = \left[V_0 + \mathcal{O}(\tilde{V}(\lambda)\right]\lambda^{|\omega|}, \quad \text{as} \quad \lambda \to \infty \tag{1}$$

where

$$V_0 = (2\pi)^{-n} \int\limits_{|a_0(z)|\le 1} dz,$$

a_0 is the part of the principal symbol which "lies" on the face F_ω (for the precise definition of a_0 see (8)) and the remainder $\tilde{V}$ is given by (15).

An asymptotic estimate of $V(\lambda)$ for multi-quasi-elliptic polynomial symbols is also contained in [8], however in a less explicit way, without the estimate of the remainder and using a completely different approach.

In Theorem 2.1, thanks to the estimate (1), we are able to extend the asymptotic expansion of the eigenvalues counting function $N(\lambda)$, due to Tulovskiĭ and Shubin (see [14] and [15]), to the case of multi-quasi-elliptic operators. As a matter of facts, if A is a global $\mathcal{P}$-elliptic operator in $\mathbb{R}^n$, then we have

$$N(\lambda) = [V_0 + \mathcal{O}(\lambda^{-\epsilon})]\lambda^{|\omega|}, \quad \text{as} \quad \lambda \to +\infty, \tag{2}$$

with ϵ satisfying (9), (10) and (11).

Tulovskiĭ-Shubin result is based on the assumption that the Weyl term satisfies the estimate

$$V(\lambda + \lambda^{1-\epsilon}) = V(\lambda)(1 + \mathcal{O}(\lambda^{-\epsilon})), \quad \text{as} \quad \lambda \to \infty, \tag{3}$$

for some $\epsilon > 0$ (see Theorem 3.1). In order to meet this condition, Tulovskiĭ and Shubin make the following assumption on the symbol a of the operator:

$$\left|\sum_{j=1}^{2n} z_j \partial_{z_j} a(z)\right| \ge C|a(z)|^\delta, \quad \text{for} \quad |z| \ge R, \tag{4}$$

for some $C, R > 0$ and $0 < \delta \le 1$ (see [14], Proposition 28.3). Condition (4) looks rather restrictive: in fact it is not verified even for quasi-elliptic symbols. For example the symbol in $\mathbb{R}^2$

$$a(x, \xi) = x^8 - \frac{\sqrt{97}}{5} x^4 \xi^6 + \xi^{12}$$

is quasi-elliptic because $\sqrt{97}/5 < 2$, but it does not satisfy (4). In fact

$$x\partial_x a(x, \xi) + \xi \partial_\xi a(x, \xi) = 8x^8 - 2\sqrt{97} x^4 \xi^6 + 12\xi^{12}$$

vanishes along the curve $12\xi^6 = (\sqrt{97} + 1)x^4$.

Luckily, our estimate (1) shows that for multi-quasi-elliptic operators, $V(\lambda)$ satisfies (3) apart from (4), which consequently can be eliminated.

Finally it is worth to remark that our $\mathcal{P}$-elliptic classes allow us to give a slight better estimate of the remainder in (3) with respect to the one could be obtained by Tulovskiĭ-Shubin classes (see Remark 3.3).

For example, the self-adjoint ordinary differential operator in $\mathbb{R}$

$$A = x^{2h_0} + D^{k_1}(x^{2h_1}D^{k_1}) + D^{2k_2},$$

with

$$h_0 > h_1 > k_1, \quad k_2 > k_1 > 0, \quad \text{and} \quad \frac{h_1}{h_0} + \frac{k_1}{k_2} > 1,$$

is globally $\mathcal{P}$-elliptic with respect to the non-degenerate polyhedron $\mathcal{P}$ of vertices $(0,0)$, $(2h_0,0)$, $(2h_1,2k_1)$, $(0,2k_2)$. As a consequence, we have the following asymptotic formula for the eigenvalues counting function:

$$N(\lambda) = \left[\frac{1}{\pi(h_1-k_1+k_2)}B\left(\frac{h_1-k_1}{2h_1k_2}, \frac{1}{2h_1}\right) + \mathcal{O}(\lambda^{-\epsilon})\right]\lambda^{\frac{h_1-k_1+k_2}{2h_1k_2}}, \quad \text{as } \lambda \to \infty,$$

where B is the Euler Beta function,

$$0 < \epsilon < \min\{p, q\},$$
$$p = \min\left\{\frac{h_0-h_1}{2h_0k_1}, \frac{k_2-k_1}{2k_1k_2}, 1 - \frac{(k_2-k_1)h_0}{h_1k_2}, 1 - \frac{h_1-k_1+k_2}{h_1k_2}\right\}$$

and

$$q = \frac{(k_2-k_1)(h_1-k_1)}{1-(k_2-k_1)(h_1-k_1)} \frac{h_1-k_1+k_2}{2h_1k_2} \frac{p}{1-p}$$

(see Example 2.5).

As a second example consider the Schrödinger operator in $\mathbb{R}^2$ with multi-quasi-elliptic potential:

$$A = -\Delta + \sum_{j=0}^{m} c_j x^{2h_j} y^{2kj}$$

with $m > 1$,

$$h_0, \ldots, h_m, k_0, \ldots, k_m \in \mathbb{N}$$

and

$$h_0 > h_1 > \ldots > h_m = 0,$$
$$0 = k_0 < k_1 < \ldots < k_m.$$

Assume that

$$\frac{k_j - k_{j-1}}{h_j - h_{j-1}} < \frac{k_{j+1} - k_j}{h_{j+1} - h_j}, \quad \text{for} \quad 1 \le j < m$$

and that there exists $l < m$ such that

$$\begin{aligned} h_j > k_j \quad &\text{for} \quad 1 \leq j \leq l, \\ h_j < k_j \quad &\text{for} \quad l < j \leq m. \end{aligned}$$

Under these hypotheses A is $\mathcal{P}$-elliptic with respect to the polyhedron of vertices

$$(0, 0, 0, 0), (2, 0, 0, 0), (0, 2, 0, 0), (0, 0, 2h_0, 2k_0), \dots, (0, 0, 2h_m, 2k_m),$$

and the eigenvalues counting function has the following asymptotic expansion:

$$N(\lambda) = \left[\frac{B(r, s)}{4\pi(r+s)[(r+s)+1](h_l k_{l+1} - h_{l+1} k_l) c_l^s c_{l+1}^r} + \mathcal{O}(\lambda^{-\epsilon}) \right] \lambda^{1+r+s},$$

as $\lambda \to \infty$, where

$$r = \frac{h_l - k_l}{2(h_l k_{l+1} - h_{l+1} k_l)}, \qquad s = \frac{k_{l+1} - h_{l+1}}{2(h_l k_{l+1} - h_{l+1} k_l)},$$

$$\epsilon < \min\{p, q\},$$

with

$$p = \min\left\{ \frac{1}{\mu}, 1 - \max_{j \neq l, l+1} \frac{(k_{l+1} - k_l)h_j + (h_l - h_{l+1})k_j}{h_l k_{l+1} - h_{l+1} k_l} \right\}$$

$$q = \min\left\{ \frac{2(k_{l+1} - k_l)r}{1 - 2(k_{l+1} - k_l)r}, \frac{2(h_l - h_{l+1})s}{1 - 2(h_l - h_{l+1})s} \right\} \frac{(r+s)^2}{1+r+s} \frac{p}{1-p}$$

and

$$\mu = \max\left\{ \frac{2h_0 k_1}{h_0 - h_1}, \frac{2h_{m-1} k_m}{k_m - k_{m-1}} \right\}$$

(see Example 2.6).

These two examples are not quasi-elliptic and therefore are not included in those considered by Helffer-Robert [9], [10] and Mohamed [12].

As already announced in [1], in a subsequent paper we shall consider also the case in which the characteristic polyhedron is degenerate and give better error estimates in the spirit of those obtained by Helffer-Robert [9], [10] for quasi-elliptic operators.

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1. – Globally multi-quasi-elliptic operators

We begin by recalling some known facts about convex polyhedra in $\mathbb{R}^N$ (see [7], [4], and [5]). A *convex polyhedron* $\mathcal{P}$ is the convex hull of a finite set of points in $\mathbb{R}^N$. With each polyhedron $\mathcal{P}$ we can associate a set $V(\mathcal{P})$ of convex-linearly independent generators, called the *vertices of* $\mathcal{P}$. Let us consider a polyhedron $\mathcal{P}$ such that

1) $\mathcal{P} \subset (\mathbb{R}_0^+)^N$ (1),
2) $\mathcal{P}$ has dimension N,
3) $V(\mathcal{P}) \subset \mathbb{N}^N$,
4) $z \in \mathcal{P}, 0 \leq y \leq z \implies y \in \mathcal{P}$,

where $y \leq z$ means that $y_j \leq z_j$ for $j = 1, \ldots, N$. For such a $\mathcal{P}$ there exists a non empty finite set $H(\mathcal{P}) \subset (\mathbb{R}_0^+)^N$ such that:

$$\mathcal{P} = \bigcap_{\omega \in H(\mathcal{P})} \{z \in (\mathbb{R}_0^+)^N | \omega \cdot z \leq 1\}$$

with $\omega \cdot z = \sum_{j=1}^N \omega_j z_j$.

Let

$$F_\omega(\mathcal{P}) = \{z \in \mathcal{P} | \omega \cdot z = 1\}, \quad F(\mathcal{P}) = \bigcup_{\omega \in H(\mathcal{P})} F_\omega(\mathcal{P}).$$

We say that $F_\omega(\mathcal{P})$ is the *face* of $\mathcal{P}$ on the hyperplane ω.

A polyhedron $\mathcal{P}$ is called *complete* if for every $y \in (\mathbb{R}_0^+)^N$ and $z \in \mathcal{P}$ such that $y \leq z$ and $y \neq z$ we have $y \in \mathcal{P} \setminus F(\mathcal{P})$. This means that the polyhedron has no faces parallel to the coordinate hyperplanes, i.e. $H(\mathcal{P}) \subset (\mathbb{R}^+)^N$

DEFINITION 1.1. *Let us denote by P_N the family of complete polyhedra satisfying hypotheses* 1) *to* 4).

With a polyhedron $\mathcal{P} \in P_N$ we associate the weight function

$$w_{\mathcal{P}}(z) = \left(\sum_{\gamma \in V(\mathcal{P})} z^{2\gamma} \right)^{\frac{1}{2}},$$

and define the formal order

$$\mu = \max_{\omega \in H} \max_{1 \leq j \leq N} \omega_j^{-1},$$

and the maximum and minimum order

$$\nu = \max_{\gamma \in V(\mathcal{P})} |\gamma|, \quad \nu_0 = \min_{\gamma \in V(\mathcal{P}) \setminus 0} |\gamma|. \quad (2)$$

We say that $\mathcal{P}$ is the characteristic polyhedron *associated with the weight $w_{\mathcal{P}}$.*

(1) $\mathbb{R}^+ = \{z \in \mathbb{R} | z > 0\}$, $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$, $\mathbb{N} = \{n \in \mathbb{Z} | n \geq 0\}$.

(2) We mean $|\gamma| = \gamma_1 + \cdots + \gamma_N$, when γ is a multi-index in $\mathbb{N}^N$ and $|z| = (z_1^2 + \cdots + z_N^2)^{1/2}$, when z is a point in $\mathbb{R}^N$.

DEFINITION 1.2. *For any* $m \in \mathbb{R}$, $\rho \in]0, \frac{1}{\nu_0}]$ *and* $\mathcal{P} \in P_N$ *we denote by* $\Lambda^m_{\mathcal{P},\rho}(\mathbb{R}^N)$ *the class of symbols* $a(z) \in \mathcal{C}^\infty(\mathbb{R}^N)$ *such that for each* $\gamma \in \mathbb{N}^N$ *there exists* $C_\gamma > 0$ *for which we have:*

$$|\partial^\gamma a(z)| \leq C_\gamma (w_{\mathcal{P}}(z))^{m-\rho|\gamma|}, \quad \textit{for all} \quad z.$$

DEFINITION 1.3. *A symbol* $a \in \Lambda^m_{\mathcal{P},\rho}(\mathbb{R}^N)$ *is called* $\mathcal{P}$*-elliptic of order* (m, ρ) *in* $\mathbb{R}^N$ *if*

$$w^m_{\mathcal{P}}(z) = \mathcal{O}(a(z)), \quad \textit{as} \quad |z| \to \infty.$$

Let us denote by $E\Lambda^m_{\mathcal{P},\rho}(\mathbb{R}^N)$ the set of $\mathcal{P}$-elliptic symbols of order (m, ρ) in $\mathbb{R}^N$. The union of all the classes $E\Lambda^m_{\mathcal{P},\rho}(\mathbb{R}^N)$ forms the set of *multi-quasi-elliptic symbols* in $(\mathbb{R}^N)$.

REMARK 1.4 If $F(\mathcal{P})$ is made of a single face F_ω, then a $\mathcal{P}$-elliptic symbol is *quasi-elliptic*; in particular, if F_ω is orthogonal to the diagonal, the symbol is *elliptic*.

One easily proves the following

PROPOSITION 1.5. *We have*

$$|z|^{\nu_0} = \mathcal{O}(w_{\mathcal{P}}(z)) \quad \textit{and} \quad w_{\mathcal{P}}(z) = \mathcal{O}(|z|^\nu),$$

as $|z| \to \infty$. □

In the following proposition we clarify the relationship between our classes of multi-quasi-elliptic symbols and the Tulovskiĭ-Shubin classes $\Gamma^h_\sigma(\mathbb{R}^N)$ and $H\Gamma^{h,h_o}_\sigma(\mathbb{R}^N)$ (see [14], § 23, 25).

PROPOSITION 1.6. *For* $m \in \mathbb{R}$, $\rho \in \left]0, \frac{1}{\nu_0}\right]$, $h \in \mathbb{R}$ *and* $\sigma \in]0, 1]$ *we have*

$$w_{\mathcal{P}} \in E\Lambda^1_{\mathcal{P},\frac{1}{\mu}}(R^N),$$

$$\Lambda^m_{\mathcal{P},\rho}(\mathbb{R}^N) \subset \Gamma^l_{\rho\nu_0}(\mathbb{R}^N), \quad E\Lambda^m_{\mathcal{P},\rho}(\mathbb{R}^N) \subset H\Gamma^{l,l_0}_{\rho\nu_0}(\mathbb{R}^N),$$

with

$$l = \max\{m\nu, m\nu_0\}, \quad l_0 = \min\{m\nu, m\nu_0\},$$

and

$$\Gamma^h_\sigma(\mathbb{R}^N) \subset \Lambda^k_{\mathcal{P},\frac{\sigma}{\nu}}(\mathbb{R}^N)$$

with

$$k = \max\left\{\frac{h}{\nu}, \frac{h}{\nu_0}\right\}.$$

PROOF. We prove the first inclusion, the other ones are a trivial consequence of Proposition 1.5.

Let $0 \leq \beta \leq \gamma \in V(\mathcal{P})$, then $(\gamma - \beta) \cdot \omega \leq 1 - \frac{1}{\mu}|\beta|$, for all $\omega \in H(\mathcal{P})$. This implies that there exists a constant $C_{\gamma-\beta} > 0$ such that

$$z^{\gamma-\beta} \leq C_{\gamma-\beta}(w_{\mathcal{P}}(z))^{1-\frac{1}{\mu}|\beta|}, \quad \text{for all } z.$$

By induction it follows that $w_{\mathcal{P}} \in \Lambda^1_{\mathcal{P},\frac{1}{\mu}}(\mathbb{R}^N)$. □

In particular, for each $\tau \in \mathbb{R}$ and $a \in \Lambda^m_{\mathcal{P},\rho}(\mathbb{R}^N)$, according to Shubin [14], § 23, we let $N = 2n$, $z = (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and define a *global pseudo-differential operator A in $\mathbb{R}^N$* of τ-symbol $a(x, \xi)$ by the formula:

$$Au(x) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a((1-\tau)x + \tau y, \xi) u(y) dy\, d\xi. \tag{5}$$

Here we use the term global to signify that (5) defines a closed linear operator in $L^2(\mathbb{R}^n)$ with domain $\mathcal{S}(\mathbb{R}^n)$. We write $A = \mathrm{Op}_\tau(a)$; for $\tau = 0$ we have the usual pseudo-differential operator of symbol $a(x, \xi)$, called by Shubin *left-symbol*; for $\tau = \frac{1}{2}$ we have the so-called *Weyl symbol.*

We say that an operator is *globally $\mathcal{P}$-elliptic of order (m, ρ) in $\mathbb{R}^n$* if it has τ-symbol belonging to $E\Lambda^m_{\mathcal{P},\rho}(\mathbb{R}^{2n})$. Global $\mathcal{P}$-elliptic operators form the set of *global multi-quasi-elliptic operators* in $\mathbb{R}^n$.

Thanks to the following proposition the above definitions are independent from τ:

Proposition 1.7. *If $a, b \in \Gamma^{mv}_{\rho 0}(\mathbb{R}^{2n})$ are such that $\mathrm{Op}_\sigma(a) = \mathrm{Op}_\tau(b)$, then we have*

$$a \in \Lambda^m_{\mathcal{P},\rho}(\mathbb{R}^{2n}) \iff b \in \Lambda^m_{\mathcal{P},\rho}(\mathbb{R}^{2n}), \quad a \in E\Lambda^m_{\mathcal{P},\rho}(\mathbb{R}^{2n}) \iff b \in E\Lambda^m_{\mathcal{P},\rho}(\mathbb{R}^{2n})$$

and

$$a - b \in \Lambda^{m-2\rho}_{\mathcal{P},\rho}(\mathbb{R}^{2n}).$$

Proof. Thank to Theorem 23.3 of [14], we have the following asymptotic expansion:

$$b(x, \xi) \sim \sum_\alpha \frac{(\sigma - \tau)^{|\alpha|}}{\alpha!} \partial^\alpha_\xi D^\alpha_x a(x, \xi),$$

which, together with Proposition 1.5 and 1.6 implies the result. □

2. – Asymptotic behavior of the eigenvalues

Let us consider a formally self-adjoint globally $\mathcal{P}$-elliptic operator A of order (m, ρ) in $\mathbb{R}^n$. By Proposition 1.6 we know that each τ-symbol of A belongs to $H\Gamma^{l,l_0}_{\rho v_0}(\mathbb{R}^{2n})$. According to Theorem 26.3 of [14], we have that the spectrum of A consists of an unbounded sequence of real semi-simple eigenvalues of finite multiplicity.

In order to study the asymptotic behavior of the spectrum, as usual, we introduce the *eigenvalues counting function*:

$$\begin{cases} N : \mathbb{R}^+ \to \mathbb{R}, \\ N(\lambda) = \sum_{|\lambda_j| \le \lambda} 1, \end{cases}$$

where $\{\lambda_j\}$ is the sequence of the eigenvalues of A *repeated according to their multiplicity.*

Given a polyhedron $\mathcal{P} \in P_{2n}$ and an hyperplane $\omega \in H(\mathcal{P})$, for each $t \in [0,1]$ consider the convex hull $T_\omega(t)$ of the set

$$\left\{ \frac{t}{\omega_j}\delta_{(j)} + \frac{(l-t)}{|\omega|}\delta | 1 \leq j \leq 2n \right\},$$

where

$$\begin{aligned} &\delta = (1, \dots, 1) \in \mathbb{R}^{2n}, \\ (6) \qquad &\delta_{(j)} = (0, \dots, 0, \underset{j\text{-entry}}{1}, 0, \dots, 0) \in \mathbb{R}^{2n}, \quad \text{for } j = 1, \dots, 2n. \end{aligned}$$

We say that $\mathcal{P} \in P_{2n}$ is *non-degenerate* if the intersection of $F(\mathcal{P})$ with the diagonal is an internal point to a face F_ω of $\mathcal{P}$. This means that there exists a *unique* $\omega \in H(\mathcal{P})$ such that

$$(7) \qquad s = \max\{t \in [0, 1] | T_\omega(t) \subset F_\omega\} > 0.$$

Our main result is summarized in the following theorem we prove in the next section.

THEOREM 2.1. *Given a non-degenerate polyhedron $\mathcal{P} \in P_{2n}$, let $A = \mathrm{Op}_\tau(a)$ with $a \in E\Lambda^1_{\mathcal{P},\rho}(\mathbb{R}^{2n})$ be a formally self-adjoint pseudo-differential operator.*

Assume that A has a polynomial principal symbol, i.e. there exists a polynomial

$$a_1(z) = \sum_{\gamma \in \mathcal{G}} c_\gamma z^\gamma$$

with $\mathcal{G} \subset F(\mathcal{P})$, such that

$$a - a_1 \in \Lambda^{1-\rho}_{\mathcal{P},\rho}(\mathbb{R}^{2n}).$$

Let $\omega \in H(\mathcal{P})$ be the unique hyperplane for which (7) is satisfied and

$$V_0 = (2\pi)^{-n} \int_{|a_0(z)| \leq 1} dz,$$

with

$$(8) \qquad a_0(z) = \sum_{\gamma \in \mathcal{G} \cap F_\omega} c_\gamma z^\gamma.$$

Then we have

$$N(\lambda) = [V_0 + \mathcal{O}(\lambda^{-\epsilon})]\lambda^{|\omega|}, \quad as \to +\infty,$$

where

(9)
$$0 < \epsilon < \min\left\{1 - \tilde{s}, \frac{(1-\tilde{s})s}{(1-s)\tilde{s}}|\omega|\right\},$$

(10)
$$\tilde{s} = \max\{s', 1-\rho\}$$

and

(11)
$$s' = \begin{cases} \max\{\omega \cdot \gamma | \gamma \in \mathcal{G} \setminus F_\omega\}, & \text{if } \mathcal{G} \setminus F_\omega \neq \emptyset, \\ 0 & \text{if } \mathcal{G} \setminus F_\omega = \emptyset. \end{cases}$$

REMARK 2.2.
1) Thanks to Proposition 1.6, a_1 is independent of τ.
2) The case $\mathcal{G} \setminus F_\omega = \emptyset$ corresponds to the results of Helffer-Robert [9], [10] and Mohamed [12] concerning quasi-elliptic operators, for which they have a remainder sharper than ours.

It is not too restrictive to assume in Theorem 2.1 that a_1 is a polynomial thanks to the following

PROPOSITION 2.3. *If $A = \mathrm{Op}_\tau(a)$ with $a \in \Gamma^l_\sigma(\mathbb{R}^{2n})$ is a differential operator, then a is a polynomial.*

PROOF. The hypothesis implies that $a(x, \xi)$ is a polynomial in ξ:

$$a(x, \xi) = \sum_{|\alpha| \leq p} a_\alpha(x)\xi^\alpha$$

with $p \leq l$. On the other side $a \in \Gamma^l_\sigma(\mathbb{R}^{2n})$ implies that

$$\partial^\beta_x a(x, \xi) = \mathcal{O}(|\xi|^{l-|\beta|\sigma}), \quad \text{as } |\xi| \to \infty.$$

Therefore $\partial^\beta_x a_\alpha = 0$, for $\beta > \frac{l}{\sigma}$, so a is a polynomial. □

Moreover it easy to generalize Theorem 2.1 to operators with principal symbol given by a power of a polynomial:

COROLLARY 2.4. *Given a non-degenerate polyhedron $\mathcal{P} \in P_{2n}$, let $A = \mathrm{Op}_\tau(a)$ with $a \in E\Lambda^m_{\mathcal{P},\rho}(\mathbb{R}^{2n})$ and $m > 0$, be a formally self-adjoint pseudo-differential operator.*

Assume that A has a principal symbol which is the m-power of a polynomial, i.e. there exists a polynomial

$$a_1(z) = \sum_{\gamma \in \mathcal{G}} c_\gamma z^\gamma$$

with $\mathcal{G} \subset F(\mathcal{P})$ and such that

$$a - a_1^m \in \Lambda^{m-\rho}_{\mathcal{P},\rho}(\mathbb{R}^{2n}). \quad (3)$$

Let $\omega \in H(\mathcal{P})$ be the unique hyperplane for which (7) *is satisfied and*

$$V_0 = (2\pi)^{-n} \int_{|a_0(z)|\leq 1} dz,$$

with

$$a_0(z) = \sum_{\gamma \in \mathcal{G} \cap F_\omega} c_\gamma z^\gamma$$

Then we have

$$N(\lambda) = [V_0 + \mathcal{O}(\lambda^{-\epsilon/m})]\lambda^{|\omega|/m}, \qquad as\ \lambda \to +\infty,$$

where ϵ satisfies inequality (9). □

We end this section with two examples.

EXAMPLE 2.5. Let consider the ordinary self-adjoint differential operator in $\mathbb{R}$

$$A = \sum_{j=0}^{m} c_j D^{k_j}(x^{2h_j} D^{k_j}),$$

with

$$h_0, \dots, h_m, k_0, \dots, k_m \in \mathbb{N}$$

and

$$h_0 > h_1 > \dots > h_m = 0,$$
$$0 = k_0 < k_1 < \dots < k_m.$$

In particular we have $m \geq 1$.

Corresponding to A we consider the polyhedron $\mathcal{P}$ of vertices $(0,0)$, $(2h_0, 0), \dots, (2h_j, 2k_j), \dots, (0, 2k_m)$. We assume that $\mathcal{P}$ belongs to P_2, that, in this case, means

$$\frac{k_j - k_{j-1}}{h_j - h_{j-1}} < \frac{k_{j+1} - k_j}{h_{j+1} - h_j}, \qquad \text{for } 1 \leq j < m, \text{ if } m > 1.$$

Moreover we assume that $\mathcal{P}$ is non-degenerate, that is, if $m > 1$, that there exists $l < m$ such that

$$h_j > k_j \quad \text{for } 1 \leq j \leq l,$$
$$h_j < k_j \quad \text{for } l < j \leq m.$$

(3) Because $a_1 \in E\Lambda^1_{\mathcal{P},\rho}(\mathbb{R}^{2n})$ we may assume that $a_1(z)$ is positive for all z so that $(a_1(z))^m$ is well defined.

The Weyl symbol of A is given by

$$a(x,\xi) = \sum_{i=0}^{m} c_i \sum_{j=0}^{\min\{h_i,k_i\}} d_{ij} x^{2(h_i-j)} \xi^{2(k_i-j)}$$

where

$$d_{ij} = (-1)^j (2j)! \binom{2h_i}{2j} \left[\sum_{j'} \binom{k_j}{2j-j'} \binom{2k_j - 2j + j'}{j'} \left(\frac{-1}{2}\right)^{j'} \right].$$

We have $a \in E\Lambda^1_{\mathcal{P},1/\mu}(\mathbb{R}^2)$, where μ is the formal order of $\mathcal{P}$:

$$\mu = \max\left\{ \frac{2h_0 k_1}{h_0 - h_1}, \frac{2h_{m-1}k_m}{k_m - k_{m-1}} \right\}.$$

If we apply Theorem 2.1 to this operator we obtain that the eigenvalues counting function has the following asymptotic expansion:

$$N(\lambda) = \left[\frac{B(r,s)}{2\pi(h_l k_{l+1} - h_{l+1}k_l)(r+s)c_l^s c_{l+1}^r} + \mathcal{O}(\lambda^{-\epsilon}) \right] \lambda^{r+s}, \qquad \text{as } \lambda \to \infty,$$

where B is the Euler Beta function and

$$r = \frac{h_l - k_l}{2(h_l k_{l+1} - h_{l+1}k_l)}, \qquad s = \frac{k_{l+1} - h_{l+1}}{2(h_l k_{l+1} - h_{l+1}k_l)},$$

$$\epsilon < \min\{(p,q)\},$$

with

$$p = \min\left\{ \frac{1}{\mu}, 1 - \max_{j \neq l, l+1} \frac{(k_{l+1} - k_l)h_j + (h_l - h_{l+1})k_j}{h_l k_{l+1} - h_{l+1}k_l}, 1 - 2(r+s) \right\},$$

$$q = \min\left\{ \frac{2(k_{l+1} - k_l)r}{1 - 2(k_{l+1} - k_l)r}, \frac{2(h_l - h_{l+1})s}{1 - 2(h_l - h_{l+1})s} \right\} \frac{(r+s)p}{1-p}$$

(in the quasi-elliptic case, i.e. $m = 1$, we have $p = 1/\mu$).

EXAMPLE 2.6. As a second example we consider the Schrödinger operator A with multi-quasi-elliptic potential W in $\mathbb{R}^n$. Let $\mathcal{Q}$ be a non-degenerate polyhedron belonging to the class P_n, then:

$$A = -\Delta + W(x),$$

where the potential W is a real polynomial in $E\Lambda^1_{\mathcal{Q},\frac{1}{\mu}}(\mathbb{R}^n)$ and μ is the formal order of $\mathcal{Q}$. Because $\mathcal{Q}$ is non-degenerate, there exists a face F_ω for which (7) holds. Let

$$W(x) = \sum_{\alpha\in\mathcal{A}} c_\alpha x^\alpha,$$

with $\mathcal{A} \subset \mathcal{Q} \cap \mathbb{N}^n$ and

$$W_0(x) = \sum_{\alpha\in\mathcal{A}\cap F_\omega} c_\alpha x^\alpha.$$

Corresponding to A we consider the non-degenerate polyhedron $\mathcal{P} \in P_{2n}$ of vertices (see (6)):

$$\left\{2\delta_{(1)}, \dots 2\delta_{(n)}\right\} \cup \left\{(0, \alpha) | \alpha \in V(\mathcal{Q})\right\}.$$

Then A is globally $\mathcal{P}$-elliptic and by Theorem 2.1 we have

$$N(\lambda) = [V_0 + \mathcal{O}(\lambda^{-\epsilon})]\lambda^{\frac{n}{2}+|\omega|}, \qquad \text{as } \to +\infty$$

where

$$V_0 = \frac{\sigma_n}{n(2\pi)^n} \int\limits_{W_0(x)\leq 1} \left[1 - (W_0(x))\right]^{\frac{n}{2}} dx,$$

σ_n denotes the area of the unit sphere in $\mathbb{R}^n$,

$$0 < \epsilon < \min\left\{1 - \tilde{s}, \frac{(1-\tilde{s})s}{(1-s)\tilde{s}} \frac{|\omega|^2}{|\omega| + n/2}\right\},$$

$$\tilde{s} = \max\left\{s', 1 - \frac{1}{\mu}\right\}$$

and

$$s' = \begin{cases} \{\max\{\omega\cdot\alpha | \alpha \in \mathcal{A} \setminus F_\omega\}, & \text{if } \mathcal{A} \setminus F_\omega \neq \emptyset, \\ 0 & \text{if } \mathcal{A} \setminus F_\omega = \emptyset. \end{cases}$$

3. – Estimate of the Weyl term and proof of Theorem 2.1

We need the following result adapted from [14]:

THEOREM 3.1. *Given a formally self-adjoint globally hypoelliptic pseudo-differential operator A with Weyl symbol $a \in H\Gamma^{l,l_0}_\sigma(\mathbb{R}^{2n})$, $l_0 > 0$, assume that the Weyl term*

$$V(\lambda) = (2\pi)^{-n} \int\limits_{|a(z)|\leq\lambda} dz$$

satisfies the estimate

$$V(\lambda+\lambda^{1-\epsilon}) = V(\lambda)(1+\mathcal{O}(\lambda^{-\epsilon})), \quad \textit{as } \lambda \to \infty, \tag{12}$$

for some

$$\epsilon \in \left]0, \frac{\sigma}{l}\right[. \tag{13}$$

Then we have

$$N(\lambda) = V(\lambda)(1+\mathcal{O}(\lambda^{-\epsilon})), \quad \textit{as } \lambda \to \infty. \quad \textit{(Weyl formula)}. \tag{14}$$

PROOF. This is Theorem 30.1 in [14] with the hypotheses (30.4) replaced by (12): it easy to check that the proof given in [14] still holds in this case. □

COROLLARY 3.2. *If* $a \in E\Lambda^1_{\mathcal{P},\rho}(\mathbb{R}^{2n})$, *then we can replace* (13) *with*

$$\epsilon \in]0, \rho[.$$

PROOF. Thanks to Proposition 1.6 the proof in [14] still holds for our $\mathcal{P}$-elliptic classes. □

REMARK 3.3. Because $E\Lambda^1_{\mathcal{P},\rho}(\mathbb{R}^{2n}) \subset H\Gamma^{\nu,\nu_0}_{\rho\nu_0}(\mathbb{R}^{2n})$, Theorem 3.1 implies that (14) holds if we assume that there exists $\epsilon \in \left]0, \frac{\rho\nu_0}{\nu}\right[$ such that (12) is satisfied, while in the corollary we have to assume only that $\epsilon \in]0, \rho[$.

Now we estimate the Weyl term $V(\lambda)$:

THEOREM 3.4. *Under the hypothesis of Theorem 2.1 we have that*

$$V(\lambda) = [V_0 + \mathcal{O}(\tilde{V}(\lambda))]\lambda^{|\omega|}, \quad \textit{as } \lambda \to \infty,$$

where

$$\tilde{V}(\lambda) = \begin{cases} \lambda^{-(1-\tilde{s})}, & \textit{if} \quad s > \dfrac{\tilde{s}}{|\omega|+\tilde{s}}, \\ \lambda^{-(1-\tilde{s})}(\log\lambda)^{2n-1}, & \textit{if} \quad s = \dfrac{\tilde{s}}{|\omega|+\tilde{s}}, \\ \lambda^{-\frac{(1-\tilde{s})s}{(1-s)\tilde{s}}|\omega|}(\log\lambda)^{2n-2}, & \textit{if} \quad s < \dfrac{\tilde{s}}{|\omega|+\tilde{s}}. \end{cases} \tag{15}$$

Before proving this theorem we complete the

PROOF OF THEOREM 2.1. Thanks (15) we have that $V(\lambda)$ satisfies

$$V(\lambda) = (V_0 + \mathcal{O}(\lambda^{-\epsilon}))\lambda^{|\omega|}, \quad \text{as } \lambda \to \infty,$$

for any ϵ satisfying (9). In particular $V(\lambda)$ satisfies (12) for $\epsilon < 1-\tilde{s} \le \rho$. By Proposition 1.7 we may assume that a is the Weyl symbol of A. Therefore, by Corollary 3.2 and Theorem 3.4 we obtain:

$$N(\lambda) = V(\lambda)(1+\mathcal{O}(\lambda^{-\epsilon}) = (V_0 + \mathcal{O}(\lambda^{-\epsilon}))\lambda^{|\omega|}, \quad \text{as } \lambda \to \infty,$$

that is Theorem 2.1. □

In the sequel, for simplicity, we adopt the following notation. Given two functions $f(x)$ and $g(x)$, we write

$$f(x) \prec g(x), \qquad \text{for all } x \in X,$$

to mean that there exists a constant $C > 0$ such that

$$f(x) \leq Cg(x), \qquad \text{for all } x \in X.$$

PROOF OF THEOREM 3.4. By its definition a_0 satisfies the following quasi-homogeneity property:

$$a_0(\lambda^{\omega_1} z_1, \dots, \lambda^{\omega_{2n}} z_{2n}) = \lambda a_0(z), \qquad \text{for } \lambda > 0 \text{ and all } z.$$

Because $a \in E\Lambda^1_{\mathcal{P},\rho}(\mathbb{R}^{2n})$ we have

$$w_{\mathcal{P}}(z) = \mathcal{O}(a(z)), \qquad \text{as } |z| \to \infty.$$

Because $\mathcal{P}$ is not degenerate we have

$$\frac{s}{\omega_j}\delta_{(j)} + \frac{1-s}{|\omega|}\delta \in F_\omega \qquad \text{for } j = 1, \dots, 2n,$$

where δ and $\delta_{(j)}$ are defined in (6) and s is given by (7). It follows that

$$|z_1 z_2 \dots z_{2n}|^{\frac{1-s}{|\omega|}} \left(|z_1|^{\frac{s}{\omega_1}} + \dots + |z_{2n}|^{\frac{s}{\omega_{2n}}} \right) \prec w_{\mathcal{P}}(z), \qquad \text{for all } z,$$

hence

$$|z_1 z_2 \dots z_{2n}|^{\frac{1-s}{|\omega|}} \left(|z_1|^{\frac{s}{\omega_1}} + \dots + |z_{2n}|^{\frac{s}{\omega_{2n}}} \right) = \mathcal{O}(a(z)), \qquad \text{as } |z| \to \infty. \tag{16}$$

Let

$$\tilde{a}(z) = a(z) - a_0(z), \tag{17}$$

then

$$|\tilde{a}(z)| \leq |a_1(z) - a_0(z)| + |a(z) - a_1(z)| .$$

By hypothesis

$$a - a_1 \in \Lambda^{1-\rho}_{\mathcal{P},\rho}(\mathbb{R}^{2n}),$$

so

$$|a(z) - a_1(z)| \prec (w_{\mathcal{P}}(z))^{1-\rho}, \qquad \text{for all } z.$$

But

$$w_{\mathcal{P}}(z) \prec 1 + |z_1|^{\frac{1}{\omega_1}} + \dots + |z_{2n}|^{\frac{1}{\omega_{2n}}}, \qquad \text{for all } z,$$

therefore

$$(18)\qquad |a(z) - a_1(z)| \prec 1 + |z_1|^{\frac{1-\rho}{\omega_1}} + \ldots + |z_{2n}|^{\frac{1-\rho}{\omega_{2n}}}, \qquad \text{for all } z\,.$$

Let now estimate $a_1 - a_0$. If $\mathcal{G} \setminus F_\omega \neq \emptyset$, then from the definition (11) of s' we have that

$$|z^\gamma| \prec 1 + |z_1|^{\frac{s'}{\omega_1}} + \ldots + |z_{2n}|^{\frac{s'}{\omega_{2n}}}, \qquad \text{for all } z \text{ and } \gamma \in \mathcal{G} \setminus F_\omega,$$

which implies

$$(19)\qquad |a_1(z) - a_0(z)| \prec 1 + |z_1|^{\frac{s'}{\omega_1}} + \ldots + |z_{2n}|^{\frac{s'}{\omega_{2n}}}, \qquad \text{for all } z\,.$$

If $\mathcal{G} \setminus F_\omega = \emptyset$, i.e. in the quasi-elliptic case, we have $a_1 = a_0$ and (19) is trivially satisfied. Therefore from (18) and (19) we can conclude that

$$(20)\qquad |\tilde{a}(z)| \prec 1 + |z_1|^{\frac{\tilde{s}}{\omega_1}} + \ldots + |z_{2n}|^{\frac{\tilde{s}}{\omega_{2n}}}, \qquad \text{for all } z$$

with $\tilde{s}$ given by (10).

Now we estimate $V(\lambda)$ as $\lambda \to \infty$. We can limit ourselves to consider only $\int_{|a| \leq \lambda, z \geq 0} dz$. The integrals extended to the other quadrants can be transformed to the first quadrant and handled in the same way.

Let us perform the following change of variables:

$$(21)\qquad z_j = (\lambda u_j)^{\omega_j}, \qquad \text{for } j = 1, \ldots, 2n.$$

The Jacobian of (21) is given by

$$\frac{\partial z}{\partial u} = \lambda^{|\omega|} \prod_{j=1}^{2n} \omega_j u_j^{\omega_j - 1}.$$

Let

$$(22)\qquad \begin{aligned} b_0(u) &= a_0(u_1^{\omega_1}, \ldots, u_{2n}^{\omega_{2n}}), \\ \tilde{b}_\lambda(u) &= \lambda^{-1} \tilde{a}((\lambda u_1)^{\omega_1}, \ldots, (\lambda u_{2n})^{\omega_{2n}}) \end{aligned}$$

($\tilde{a}$ is defined in (17)), then

$$\int_{\substack{|a(z)| \leq \lambda \\ z \geq 0}} dz = \lambda^{|\omega|} \int_{\substack{|b_0(u) + \tilde{b}_\lambda(u)| \leq 1 \\ u \geq 0}} \prod_{j=1}^{2n} \omega_j u_j^{\omega_j - 1} du,$$

$$\int_{\substack{|a_0(z)| \leq \lambda \\ z \geq 0}} dz = \lambda^{|\omega|} \int_{\substack{|b_0(u)| \leq 1 \\ u \geq 0}} \prod_{j=1}^{2n} \omega_j u_j^{\omega_j - 1} du.$$

In order to complete the proof it suffices to show that

$$\int_{\substack{|b_0+\tilde{b}_\lambda|\le 1\\ u\ge 0}} \prod_{j=1}^{2n} u_j^{\omega_j-1}du - \int_{\substack{|b_0|\le 1\\ u\ge 0}} \prod_{j=1}^{2n} u_j^{\omega_j-1}du = \mathcal{O}(\tilde{V}(\lambda)), \qquad \text{as } \to +\infty,$$

with $\tilde{V}$ given by (15). But this is a consequence the following estimates:

$$\mathcal{R}_\sigma(\lambda) = \mathcal{O}(\tilde{V}(\lambda)), \qquad \text{as } \lambda \to +\infty, \qquad \text{for all } \sigma \in \Sigma,$$

where Σ is the set of all permutations of $(1, 2, \dots, 2n)$ and

$$\mathcal{R}_\sigma(\lambda) = \int_{\substack{|b_0+\tilde{b}_\lambda|\le 1\\ u\in U_\sigma}} \prod_{j=1}^{2n} u_j^{\omega_j-1}du - \int_{\substack{|b_0|\le 1\\ u\in U_\sigma}} \prod_{j=1}^{2n} u_j^{\omega_j-1}du,$$

with

$$U_\sigma = \{u \in \mathbb{R}^{2n} | u_{\sigma(1)} \ge u_{\sigma(2)} \ge \dots \ge u_{\sigma(2n)} \ge 0\}.$$

We limit ourselves to estimate

$$\mathcal{R}(\lambda) = \mathcal{R}_{(1,2,\dots,2n)}(\lambda) = \int_{\substack{|b_0+\tilde{b}_\lambda|\le 1\\ u\in U}} \prod_{j=1}^{2n} u_j^{\omega_j-1}du - \int_{\substack{|b_0|\le 1\\ u\in U}} \prod_{j=1}^{2n} u_j^{\omega_j-1}du, \tag{23}$$

with

$$U = U_{(1,\dots,2n)} = \{u \in \mathbb{R}^{2n} | u_1 \ge u_2 \ge \dots \ge u_{2n} \ge 0\}.$$

The estimate of the other remainders $\mathcal{R}_\sigma$ can be obtained in the same way.

From(16) and (22) we obtain that there exists $R > 0$ such that

$$(u^\omega)^{\frac{1-s}{|\omega|}} (u_1^s + \dots + u_{2n}^s) \prec |b_0(u) + \tilde{b}_\lambda(u)|, \tag{24}$$

for

$$\lambda > 0, \quad u_1 + \dots + u_{2n} \ge \frac{R}{\lambda} \quad \text{and} \quad u \ge 0.$$

Letting $\lambda \to +\infty$ in (24), by (20) we obtain that

$$(u^\omega)^{\frac{1-s}{|\omega|}} (u_1^s + \dots + u_{2n}^s) \prec |b_0(u)|, \qquad \text{for } u \ge 0. \tag{25}$$

Thank to the fact that b_0 is positive homogeneous of degree 1 and satisfies (25), one easily shows that

$$u = \frac{(1+t)\eta(\theta)}{|b_0(\eta(\theta))|}, \tag{26}$$

with

$$\theta = (\theta_1, \dots, \theta_{2n-1}),$$

$$\eta(\theta) = (\eta_1(\theta), \dots, \eta_{2n}(\theta)),$$

(27)
$$\begin{cases} \eta_1(\theta) = \cos\theta_1, \\ \eta_k(\theta) = \left(\prod_{j=1}^{k-1} \sin\theta_j\right) \cos\theta_k, \quad (2 \le k \le 2n-1), \\ \eta_{2n}(\theta) = \prod_{j=1}^{2n-1} \sin\theta_j, \end{cases}$$

is a change of co-ordinates between

$$\left\{(t, \theta) \in \mathbb{R} \times \mathbb{R}^{2n-1} | -1 < t, 0 \le \theta \le \frac{\pi}{2}\right\} \quad \text{and} \quad \{u \in \mathbb{R}^{2n} | 0 \le u\},$$

which is C^1 in the complement of a set of measure 0. Let us show that the Jacobian of (26) is given by

$$\begin{aligned} \frac{\partial u}{\partial(t, \theta)} &= (1+t)^{2n-1}(b_0(\eta(\theta)))^{-2n} \prod_{j=1}^{2n-1} (\sin\theta_j)^{2n-1-j} \\ &= (1+t)^{2n-1}(b_0(\eta(\theta)))^{-2n} \prod_{j=1}^{2n-1} \frac{\eta_j(\theta)}{\cos\theta_j}. \end{aligned}$$

Let

$$(b_0(\eta(\theta)))^{-1} = g(\theta),$$

then, by representing matrices in column-form, we have

$$\begin{aligned} \frac{\partial u}{\partial(t, \theta)} &= \det\left[g\eta, (1+t)\left(\frac{\partial g}{\partial\theta_1}\eta + g\frac{\partial\eta}{\partial\theta_1}\right), \dots, (1+t)\left(\frac{\partial g}{\partial\theta_{2n-1}}\eta + g\frac{\partial\theta}{\partial\theta_{2n-1}}\right)\right] \\ &= (1+t)^{2n-1} g \det\left[\eta, \frac{\partial g}{\partial\theta_1}\eta + g\frac{\partial\eta}{\partial\theta_1}, \dots, \frac{\partial g}{\partial\theta_{2n-1}}\eta + g\frac{\partial\theta}{\partial\theta_{2n-1}}\right] \\ &= (1+t)^{2n-1} g^{2n} \det\left[\eta, \frac{\partial\eta}{\partial\theta_1}, \dots, \frac{\partial\eta}{\partial\theta_{2n-1}}\right] \\ &= (1+t)^{2n-1} g^{2n} \prod_{j=1}^{2n-1} (\sin\theta_j)^{2n-1-j}. \end{aligned}$$

The last equality is the well-known Jacobian of spherical co-ordinates.

Let

$$(28)\qquad \begin{aligned} r_0 &= b_0\left(\frac{\eta(\theta)}{|b_0(\eta(\theta))|}\right) = \frac{b_0(\eta(\theta))}{|b_0(\eta(\theta))|},\\ \tilde r_\lambda(t,\theta) &= \tilde b_\lambda\left(\frac{(1+t)\eta(\theta)}{|b_0(\eta(\theta))|}\right), \end{aligned}$$

then, from (23), (26) and (27), we obtain

$$(29)\quad R(\lambda) = \int\limits_{\substack{|(1+t)r_0+\tilde r_\lambda|\le 1\\ \theta\in\Theta, t\ge -1}} H(\theta)(1+t)^{|\omega|-1}d\theta\,dt - \int\limits_{\theta\in\Theta,-1\le t\le 0} H(\theta)(1+t)^{|\omega|-1}d\theta\,dt,$$

with

$$(30)\qquad \Theta = \Big\{\theta\in\mathbb{R}^{2n-1}\,|0\le\theta_j\le\arctan(\sec\theta_{j+1}),\quad \text{for } 1\le j<2n-1, 0\le\theta_{2n-1}\le\frac{\pi}{4}\Big\}$$

and

$$(31)\qquad \begin{aligned} H(\theta) &= (b_0(\eta))^{-|\omega|}\prod_{j=1}^{2n}\eta_j^{\omega_j-1}\prod_{j=1}^{2n-1}\frac{\eta_j}{\cos\theta_j}\\ &= (b_0(\eta))^{-|\omega|}\left(\prod_{j=1}^{2n-1}\frac{\eta_j^{\omega_j}}{\cos\theta_j}\right)\eta_{2n}^{\omega_{2n}-1}\\ &= (b_0(\eta))^{-|\omega|}\eta_1^{\omega_1}\prod_{j=2}^{2n}\frac{\eta_j^{\omega_j}}{\sin\theta_{j-1}\cos\theta_{j-1}}. \end{aligned}$$

From $a\in E\Lambda^1_{\mathcal{P},\rho}(\mathbb{R}^{2n})$ and (22) we have

$$|b_0(u)+\tilde b_\lambda(u)| \prec \lambda^{-1}w_{\mathcal{P}}((\lambda u_1)^{\omega_1},\dots,(\lambda u_{2n})^{\omega_{2n}}),\qquad \text{for } \lambda>\ \text{and } u\ge 0.$$

By letting $\lambda\to\infty$, we obtain

$$|b_0(u)| \prec \left(\sum_{\gamma\in F_\omega} u_1^{2\gamma_1\omega_1}\cdot\ldots\cdot u_{2n}^{2\gamma_{2n}\omega_{2n}}\right)^{\frac12},\qquad \text{for } u\ge 0.$$

But

$$\left(\sum_{\gamma\in F_\omega} u_1^{2\gamma_1\omega_1}\cdot\ldots\cdot u_{2n}^{2\gamma_{2n}\omega_{2n}}\right)^{\frac12} \le \lambda^{-1}w_{\mathcal{P}}\big((\lambda u_1)^{\omega_1},\dots,(\lambda u_{2n})^{\omega_{2n}}\big),$$

for $\lambda>0$ and $u\ge 0$, therefore we have

$$|b_0(u)| \prec \lambda^{-1}w_{\mathcal{P}}\big((\lambda u_1)^{\omega_1},\dots,(\lambda u_{2n})^{\omega_{2n}}\big),$$

for $\lambda > 0$ and $u \geq 0$. On the other side, from $a \in E\Lambda^1_{\mathcal{P},\rho}(\mathbb{R}^{2n})$ we have that there exists $R > 0$ such that

$$\lambda^{-1} w_{\mathcal{P}}\big((\lambda u_1)^{\omega_1}, \dots, (\lambda u_{2n})^{\omega_{2n}}\big) \prec |b_0(u) + \tilde{b}_\lambda(u)|,$$

for $\lambda > 0$ and $|\lambda u|^{|\omega|} \geq R$. In conclusion we obtain

(32) $$|b_0(u)| \prec |b_0(u) + \tilde{b}_\lambda(u)|, \qquad \text{for } |u| \geq R^{\frac{1}{|\omega|}} \text{ and } \lambda \geq 1.$$

From (26), (28) and (32) we have that

$$1 + t \prec |(1+t) r_0(\theta) + \tilde{r}_\lambda(t,\theta)|, \qquad \text{for } \frac{(1+t)|\eta(\theta)|}{|b_0(\eta(\theta))|} \geq R^{\frac{1}{|\omega|}} \text{ and } \lambda \geq 1.$$

It follows that for $\lambda \geq 1$ we have either

$$1 + t \leq R^{\frac{1}{|\omega|}} \max_{\theta \in \Theta} \frac{|b_0(\eta(\theta))|}{|\eta(\theta)|} < \infty,$$

or

$$1 + t \prec |(1+t) r_0(\theta) + \tilde{r}_\lambda(t,\theta)|.$$

Therefore there exists a constant $T > 1$ such that

$$1 + t \leq T$$

whenever

$$|r_0(\theta)(1+t) + \tilde{r}_\lambda(t,\theta)| \leq 1 \qquad \text{and } \lambda \geq 1.$$

From (20) and (22) we have

$$|\tilde{b}_\lambda(u)| \prec \lambda^{-1}[1 + \lambda^{\tilde{s}}(u_1^{\tilde{s}} + \dots + u_{2n}^{\tilde{s}})], \qquad \text{for } \lambda > 0 \text{ and } u \geq 0.$$

But from (25) we have that there exists $C > 0$ such that

$$(u^\omega)^{\frac{1-s}{|\omega|}}(u_1^s + \dots + u_{2n}^s) \leq C|b_0(u)|, \qquad \text{for } u \geq 0,$$

so from (26) and (28) we obtain

$$\begin{aligned} |\tilde{r}_\lambda(t,\theta)| &\prec \lambda^{-1}\left[1 + \lambda^{\tilde{s}} \frac{(1+t)^{\tilde{s}}}{|b_0(\eta)|^{\tilde{s}}}(\eta_1^{\tilde{s}} + \dots + \eta_{2n}^{\tilde{s}})\right] \\ &\leq \lambda^{-1}\left[1 + \lambda^{\tilde{s}} \frac{(1+t)^{\tilde{s}}(\eta_1^{\tilde{s}} + \dots + \eta_{2n}^{\tilde{s}})}{\left(C(\eta^\omega)^{\frac{1-s}{|\omega|}}(\eta_1^s + \dots + \eta_{2n}^s)\right)^{\tilde{s}}}\right] \\ &\leq \lambda^{-1}\left[1 + \lambda^{\tilde{s}} \frac{T^{\tilde{s}}}{C^{\tilde{s}}}(\eta^\omega)^{-\frac{(1-s)\tilde{s}}{|\omega|}} \frac{(\eta_1^{\tilde{s}} + \dots + \eta_{2n}^{\tilde{s}})}{(\eta_1^s + \dots + \eta_{2n}^s)^{\tilde{s}}}\right], \end{aligned}$$

for $\lambda \geq 1, -1 \leq t \leq T$ and $0 < \theta < \frac{\pi}{2}$.

But from (27) we have that $\frac{(\eta_1^{\tilde{s}}+\ldots+\eta_{2n}^{\tilde{s}})}{(\eta_1^s+\ldots+\eta_{2n}^s)^{\tilde{s}}}$ is bounded for $\theta \in \Theta$ (see (30)), because η_1 never vanishes for $\theta \in \Theta$, and that

$$\eta^\omega = (\cos\theta_1)^{\omega_1} \cdot \ldots \cdot (\cos\theta_{2n-1})^{\omega_{2n-1}} \cdot (\sin\theta_1)^{\omega_2+\ldots+\omega_{2n}} \cdot \ldots \cdot (\sin\theta_{2n-1})^{\omega_{2n}}.$$

Hence there exits $L > 0$ such that

$$|\tilde{r}_\lambda(t,\theta)| \leq L\lambda^{\tilde{s}-1}(\theta_1^{\omega_2+\ldots+\omega_{2n}} \cdot \ldots \cdot \theta_{2n-1}^{\omega_{2n}})^{-\frac{(1-s)\tilde{s}}{|\omega|}}, \tag{33}$$

for $\lambda \geq 1$, $-1 \leq t \leq T$, and $\theta \in \Theta$.

Eventually let us estimate the integrand $H(\theta)$. From (25) and (31) we have

$$\begin{aligned} |H(\theta)| &\prec (\eta^\omega)^{s-1}(\eta_1^s + \ldots + \eta_{2n}^s)^{-|\omega|}\eta_1^{\omega_1} \prod_{j=2}^{2n} \frac{\eta_j^{\omega_j}}{\sin\theta_{j-1}\cos\theta_{j-1}} \\ &= (\eta_1^s + \ldots + \eta_{2n}^s)^{-|\omega|}\eta_1^{s\omega_1} \prod_{j=2}^{2n} \frac{n_j^{s\omega_j}}{\sin\theta_{j-1}\cos\theta_{j-1}}. \end{aligned}$$

But, by (27), $(\eta_1^s + \ldots + \eta_{2n}^s)$ never vanishes for $\theta \in \Theta$ and

$$\frac{\eta_j^{\omega_j}}{\sin\theta_{j-1}\cos\theta_{j-1}} \prec (\sin\theta_1 \ldots \sin\theta_{j-2})^{\omega_j}(\sin\theta_{j-1})^{\omega_j-1}, \qquad \text{for } \theta \in \Theta.$$

Therefore:

$$H(\theta) \prec \theta_1^{s(\omega_2+\ldots+\omega_{2n})-1} \cdot \ldots \cdot \theta_{2n-1}^{s\omega_{2n}-1}, \qquad \text{for } \theta \in \Theta. \tag{34}$$

Now we can estimate $\mathcal{R}(\lambda)$. Let

$$\bar{\omega} = (\omega_2 + \ldots + \omega_{2n}, \omega_3 + \ldots + \omega_{2n}, \ldots, \omega_{2n}) \in \mathbb{R}^{2n-1},$$

$$\delta = (1, 1, \ldots, 1) \in \mathbb{R}^{2n-1}.$$

Then from (29), (33) and (34), we obtain that

$$
\begin{aligned}
|\mathcal{R}(\lambda)| &\le \int_{\substack{|(1+t)r_0+\tilde{r}_\lambda|\ge 1\\ \theta\in\Theta,-1\le t\le 0}} H(\theta)(1+t)^{|\omega|-1}d\theta\,dt + \int_{\substack{|(1+t)r_0+\tilde{r}_\lambda|\le 1\\ \theta\in\Theta,t\ge 0}} H(\theta)(1+t)^{|\omega|-1}d\theta\,dt \\
&\prec \int_{\substack{|r_0(1+t)+\tilde{r}_\lambda|\le 1\\ 0\le\theta\frac{\pi}{2},0\le t\le T}} \theta^{s\bar{\omega}-\delta}(1+t)^{|\omega|-1}dt\,d\theta + \int_{\substack{|r_0(1+t)+\tilde{r}_\lambda|\ge 1\\ 0\le\theta\le\frac{\pi}{2},-1\le t\le 0}} \theta^{s\bar{\omega}-\delta}(1+t)^{|\omega|-1}dt\,d\theta \\
&\le \int_{\substack{t\le L\lambda^{-(1-\tilde{s})}\theta^{-\frac{(1-s)\tilde{s}}{|\omega|}\bar{\omega}}\\ 0\le t\le T,0\le\theta\le\pi/2}} \theta^{s\bar{\omega}-\delta}(1+t)^{|\omega|-1}dt\,d\theta \\
&\quad + \int_{\substack{-L\lambda^{-(1-\tilde{s})}\theta^{-\frac{(1-s)\tilde{s}}{|\omega|}\bar{\omega}}\le t\\ -1\le t\le 0,0\le\theta\le\frac{\pi}{2}}} \theta^{s\bar{\omega}-\delta}(1+t)^{|\omega|-1}dt\,d\theta \\
&= \frac{1}{|\omega|}\int_{0\le\theta\le\frac{\pi}{2}} \theta^{s\bar{\omega}-\delta}\left[\left(1+\min\left\{T, L\lambda^{-(1-\tilde{s})}\theta^{-\frac{(1-s)\tilde{s}}{|\omega|}\bar{\omega}}\right\}\right)^{|\omega|}-1\right]d\theta \\
&\quad + \frac{1}{|\omega|}\int_{0\le\theta\le\frac{\pi}{2}} \theta^{s\bar{\omega}-\delta}\left[1-\left(1-\min\left\{1, L\lambda^{-(1-\tilde{s})}\theta^{-\frac{(1-s)\tilde{s}}{|\omega|}\bar{\omega}}\right\}\right)^{|\omega|}\right]d\theta,
\end{aligned}
$$

for $\lambda \ge 1$. But it is easy to see that

$$
\left(1+\min\left\{T, L\lambda^{(1-\tilde{s})}\theta^{-\frac{(1-s)\tilde{s}}{|\omega|}\bar{\omega}}\right\}\right)^{|\omega|}-1 \prec \min\left\{T, L\lambda^{-(1-\tilde{s})}\theta^{-\frac{(1-s)\tilde{s}}{|\omega|}\bar{\omega}}\right\}
$$

and

$$
\begin{aligned}
1-\left(1-\min\left\{1, L\lambda^{-(1-\tilde{s})}\theta^{-\frac{(1-s)\tilde{s}}{|\omega|}\bar{\omega}}\right\}\right)^{|\omega|} &\le \min\left\{1, L\lambda^{-(1-\tilde{s})}\theta^{-\frac{(1-s)\tilde{s}}{|\omega|}\bar{\omega}}\right\} \\
&\le \min\left\{T, L\lambda^{-(1-\tilde{s})}\theta^{-\frac{(1-s)\tilde{s}}{|\omega|}\bar{\omega}}\right\},
\end{aligned}
$$

for $\lambda \ge 1$ and $\theta \in \Theta$.

Therefore we have

$$
\begin{aligned}
|\mathcal{R}(\lambda)| &\prec \int_{0\le\theta\le\frac{\pi}{2}} \theta^{s\bar{\omega}-\delta}\min\left\{T, L\lambda^{-(1-\tilde{s})}\theta^{-\frac{(1-s)\tilde{s}}{|\omega|}\bar{\omega}}\right\}d\theta \\
&= T\mathcal{I}_1(\lambda) + L\lambda^{-(1-\tilde{s})}\mathcal{I}_2(\lambda) \qquad \text{for } \lambda \ge 1,
\end{aligned} \tag{35}
$$

with

$$\mathcal{I}_1(\lambda) = \int\limits_{\substack{\theta^{-\frac{(1-s)\tilde{s}}{|\omega|}\bar{\omega}} \geq \frac{T}{L}\lambda^{1-\tilde{s}} \\ 0\leq\theta\leq\frac{\pi}{2}}} \theta^{s\bar{\omega}-\delta} d\theta$$

and

$$\mathcal{I}_2(\lambda) = \int\limits_{\substack{\theta^{-\frac{(1-s)\tilde{s}}{|\omega|}\bar{\omega}} \leq \frac{T}{L}\lambda^{1-\tilde{s}} \\ 0\leq\theta\leq\frac{\pi}{2}}} \theta^{\left(s-\frac{(1-s)\tilde{s}}{|\omega|}\right)\bar{\omega}-\delta} d\theta.$$

Let us estimate the first integral. If $n = 1$ a simple integration gives

$$\mathcal{I}_1(\lambda) = \mathcal{O}\left(\lambda^{-\frac{(1-\tilde{s})s}{(1-s)\tilde{s}}|\omega|}\right), \quad \text{as } \lambda \to +\infty$$

which is (36).

If $n > 1$ we proceed by induction on n. Set

$$\theta' = (\theta_1, \dots, \theta_{2n-2}),$$
$$\delta' = (1, 1, \dots, 1) \in \mathbb{R}^{2n-2},$$
$$\bar{\omega}' = (\omega_2 + \omega_3 + \dots + \omega_{2n}, \dots, \omega_{2n-1} + \omega_{2n}).$$

If $(1-s)\tilde{s} = 0$ we have $\mathcal{I}_1(\lambda) = 0$ for $\frac{T}{L}\lambda^{1-\tilde{s}} > 1$.
If $(1-s)\tilde{s} \neq 0$, that is $(1-s)\tilde{s} > 0$, we have

$$\begin{aligned}
\mathcal{I}_1(\lambda) &= \int\limits_{\substack{\theta^{\bar{\omega}} \leq K_0 \lambda^{-\frac{1-\tilde{s}}{(1-s)\tilde{s}}|\omega|} \\ 0\leq\theta\leq\frac{\pi}{2}}} \theta^{s\bar{\omega}-\delta} d\theta \\
&= \frac{1}{s\omega_{2n}} \int\limits_{0\leq\theta'\leq\frac{\pi}{2}} \theta'^{s\bar{\omega}'-\delta'} \min\left\{\left(\frac{\pi}{2}\right)^{s\omega_{2n}}, K_0^s \lambda^{-\frac{(1-\tilde{s})s}{(1-s)\tilde{s}}|\omega|} \theta'^{-s\bar{\omega}'}\right\} d\theta' \\
&= \frac{1}{s\omega_{2n}} \left(\frac{\pi}{2}\right)^{s\omega_{2n}} \int\limits_{\substack{\theta'^{\bar{\omega}'} \leq K_1 \lambda^{-\frac{1-\tilde{s}}{(1-s)\tilde{s}}|\omega|} \\ 0\leq\theta'\leq\frac{\pi}{2}}} \theta'^{s\bar{\omega}'-\delta'} d\theta' \\
&\quad + \frac{1}{s\omega_{2n}} K_0^s \lambda^{-\frac{(1-\tilde{s})s}{(1-s)\tilde{s}}|\omega|} \int\limits_{\substack{\theta'^{\bar{\omega}'} \geq K_1 \lambda^{-\frac{1-\tilde{s}}{(1-s)\tilde{s}}|\omega|} \\ 0\leq\theta'\leq\frac{\pi}{2}}} \theta'^{-\delta'} d\theta'
\end{aligned}$$

where

$$K_0 = \left(\frac{T}{L}\right)^{-\frac{|\omega|}{(1-s)\tilde{s}}},$$

$$K_1 = \left(\frac{\pi}{2}\right)^{-\omega_{2n}} K_0.$$

But

$$\begin{cases} \theta'^{\bar{\omega}'} \geq K_1 \lambda^{-\frac{1-\tilde{s}}{(1-s)\tilde{s}}|\omega|}, \\ 0 \leq \theta' \leq \frac{\pi}{2} \end{cases}$$

implies that

$$C_0 \lambda^{-c_0} \leq \theta' \leq \frac{\pi}{2},$$

for suitable $C_0 > 0$ and $c_0 > 0$. Therefore we have

$$\int\limits_{\substack{\theta'^{\bar{\omega}'} \geq K_1 \lambda^{-\frac{1-\tilde{s}}{(1-s)\tilde{s}}|\omega|} \\ 0 \leq \theta' \leq \frac{\pi}{2}}} \theta'^{-\delta'} d\theta' \leq \int\limits_{C_0 \lambda^{-c_0} \leq \theta' \leq \frac{\pi}{2}} \theta'^{-\delta'} d\theta' = \mathcal{O}((\log \lambda)^{2n-2}),$$

as $\lambda \to \infty$. Thus, by induction we obtain

$$\mathcal{I}_1(\lambda) = \mathcal{O}\left(\lambda^{-\frac{(1-\tilde{s})s}{(1-s)\tilde{s}}|\omega|} (\log \lambda)^{2n-2}\right), \quad \text{as } \lambda \to +\infty. \tag{36}$$

Now we estimate the second integral $\mathcal{I}_2(\lambda)$. If

$$s - \frac{(1-s)\tilde{s}}{|\omega|} > 0,$$

then

$$\mathcal{I}_2(\lambda) \leq \int\limits_{0 \leq \theta \leq \frac{\pi}{2}} \theta^{\left(s - \frac{(1-s)\tilde{s}}{|\omega|}\right)\bar{\omega} - \delta} d\theta < +\infty. \tag{37}$$

If

$$s - \frac{(1-s)\tilde{s}}{|\omega|} = 0,$$

we have that there exist $C_1 > 0$ and $c_1 > 0$ such that

$$\mathcal{I}_2(\lambda) \leq \int\limits_{C_1 \lambda^{-c_1} \leq \theta \leq \frac{\pi}{2}} \theta^{-\delta} d\theta = \mathcal{O}\big((\log \lambda)^{2n-1}\big), \quad \text{as } \lambda \to +\infty. \tag{38}$$

Finally, consider the case

$$s-\frac{(1-s)\tilde{s}}{|\omega|}<0.$$

If $n=1$, a simple integration yields

$$\mathcal{I}_2(\lambda)=\mathcal{O}\left(\lambda^{-\frac{1-\tilde{s}}{(1-s)\tilde{s}}|\omega|\left(s-\frac{(1-s)\tilde{s}}{|\omega|}\right)}\right),\qquad \text{as } \lambda\to+\infty,$$

which is (39)

If $n>1$, we have

$$\begin{aligned}
\mathcal{I}_2(\lambda)&=\int\limits_{\substack{\theta^{\bar{\omega}}\geq K_0\lambda^{-\frac{1-\tilde{s}}{(1-s)\tilde{s}}|\omega|}\\ 0\leq\theta\leq\frac{\pi}{2}}}\theta^{\left(s-\frac{(1-s)\tilde{s}}{|\omega|}\right)\bar{\omega}-\delta}\,d\theta\\
&=\left(s-\frac{(1-s)\tilde{s}}{|\omega|}\right)^{-1}\frac{1}{\omega_{2n}}\int\limits_{\substack{K_1\lambda^{-\frac{1-\tilde{s}}{(1-s)\tilde{s}}|\omega|}\theta'^{-\bar{\omega}'}\leq 1\\ 0\leq\theta'\leq\frac{\pi}{2}}}\theta'^{\left(s-\frac{(1-s)\tilde{s}}{|\omega|}\right)\bar{\omega}'-\delta'}.\\
&\quad\cdot\left[\left(\frac{\pi}{2}\right)^{\left(s-\frac{(1-s)\tilde{s}}{|\omega|}\right)\omega_{2n}}-\left(K_0\lambda^{-\frac{(1-\tilde{s})}{(1-s)\tilde{s}}|\omega|}\theta'^{-\bar{\omega}'}\right)^{\left(s-\frac{(1-s)\tilde{s}}{|\omega|}\right)}\right]d\theta'\\
&=\left(s-\frac{(1-s)\tilde{s}}{|\omega|}\right)^{-1}\frac{1}{\omega_{2n}}\left(\frac{\pi}{2}\right)^{\left(s-\frac{(1-s)\tilde{s}}{|\omega|}\right)\omega_{2n}}\int\limits_{\substack{\theta'^{\bar{\omega}'}\geq K_1\lambda^{-\frac{1-\tilde{s}}{(1-s)\tilde{s}}|\omega|}\\ 0\leq\theta'\leq\frac{\pi}{2}}}\theta'^{\left(s-\frac{(1-s)\tilde{s}}{|\omega|}\right)\bar{\omega}'-\delta'}\,d\theta'\\
&\quad-\left(s-\frac{(1-s)\tilde{s}}{|\omega|}\right)^{-1}\frac{1}{\omega_{2n}}K_0^{-\left(s-\frac{(1-s)\tilde{s}}{|\omega|}\right)}\lambda^{-\frac{(1-\tilde{s})}{(1-s)\tilde{s}}|\omega|\left(s-\frac{(1-s)\tilde{s}}{|\omega|}\right)}.\\
&\quad\cdot\int\limits_{\substack{\theta'^{\bar{\omega}'}\geq K_1\lambda^{-\frac{1-\tilde{s}}{(1-s)\tilde{s}}|\omega|}\\ 0\leq\theta'\leq\frac{\pi}{2}}}\theta'^{-\delta'}\,d\theta'.
\end{aligned}$$

But

$$\int\limits_{\substack{\theta'^{\bar{\omega}'}\geq K_1\lambda^{-\frac{1-\tilde{s}}{(1-s)\tilde{s}}|\omega|}\\ 0\leq\theta'\leq\frac{\pi}{2}}}\theta'^{-\delta'}\,d\theta'\leq\int\limits_{C_2\lambda^{-c_2}\leq\theta'\leq\frac{\pi}{2}}\theta'^{-\delta'}\,d\theta'=\mathcal{O}\big((\log\lambda)^{2n-2}\big),\qquad \text{as } \lambda\to+\infty,$$

for suitable $C_2 > 0$ and $c_2 > 0$. Thus, by induction we obtain

$$\mathcal{I}_2(\lambda) = \mathcal{O}\left(\lambda^{-\frac{1-\tilde{s}}{(1-s)\tilde{s}}|\omega|\left(s-\frac{(1-s)\tilde{s}}{|\omega|}\right)}(\log\lambda)^{2n-2}\right), \qquad \text{as } \lambda \to +\infty. \tag{39}$$

In conclusion, from (35), (36), (37), (38) and (39) we obtain

$$\mathcal{R}(\lambda) = \mathcal{O}\left(\lambda^{-\frac{(1-\tilde{s})s}{(1-s)\tilde{s}}|\omega|}(\log\lambda)^{2n-2}\right)$$

$$+\begin{cases} \mathcal{O}\big(\lambda^{-(1-\tilde{s})}\big), & \text{if } s - \dfrac{(1-s)\tilde{s}}{|\omega|} > 0, \\ \mathcal{O}\big(\lambda^{-(1-\tilde{s})}(\log\lambda)^{2n-1}\big), & \text{if } s - \dfrac{(1-s)\tilde{s}}{|\omega|} > 0, \\ \mathcal{O}\big(\lambda^{-\frac{(1-\tilde{s})s}{(1-s)\tilde{s}}|\omega|}(\log\lambda)^{2n-2}\big), & \text{if } s - \dfrac{(1-s)\tilde{s}}{|\omega|} < 0, \end{cases}$$

as $\lambda \to +\infty$, which implies

$$\mathcal{R}(\lambda) = \tilde{V}(\lambda), \qquad \text{as } \lambda \to +\infty,$$

with $\tilde{V}$ given by (15). □

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