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On Nonstationary Stokes Problem in Exterior Domains

P. MAREMONTI - V. A. SOLONNIKOV

To Professor O. A. Ladyzhenskaya on her 75th birthday

Abstract: The paper is concerned with L_p−estimates for solutions of n−dimensional exterior Stokes problem. The main result of the paper are new $L_p - L_q$ estimates

$$|\mathbf{v}(t)|_q \leq C_1 |\mathbf{v}_0|_p t^{-\mu}, \tag{1}$$

$$|\mathbf{v}_t(t)|_q \leq C_2 |\mathbf{v}_0|_p t^{-\mu'}, \tag{2}$$

$$|\nabla \mathbf{v}(t)|_q \leq C_3 |\mathbf{v}_0|_p t^{-\hat{\mu}}, \tag{3}$$

for the solution of a homogeneous Stokes problem with the initial condition $\mathbf{v}(x, 0) = \mathbf{v}_0(x)$; $|\cdot|_p$ is the L_p−norm in an exterior domain $\Omega \subseteq \mathbb{R}^n$. We prove that estimate (1) holds with $\mu = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$ for arbitrary p, q satisfying the conditions $1 \leq p \leq q \leq \infty$, $p + q > 2$ for $n > 2$, $1 < p \leq q < \infty$ for $n = 2$. Estimate (2) holds with $\mu' = 1 + \mu$ and $n \geq 3$. Finally, inequality (3) holds with $\hat{\mu} = \frac{1}{2} + \mu$ for $q \in [p, n]$ and $\hat{\mu} = \frac{n}{2p}$ for $q \in (n, \infty)$. The constants C_i are independent of $t > 0$. We show also that in formulas (1) and (3) μ, $\hat{\mu}$ are exact, in particular, that $\hat{\mu} < \frac{1}{2} + \mu$ for $q > n > 2$. The method of the proof of (1)-(3) is quite elementary and relies on energy estimates, imbedding theorems, $L_p - L_q$ estimates for the Cauchy problem and duality arguments.
In addition, we give a new proof of $W^{2,1}_{p,r}(Q_T)$− estimates of derivatives of the solution of the Stokes problem (here $Q_T = \Omega \times (0, T)$, $p, r > 1$), obtained by Y. Giga and H. Sohr [13], [14]. Inequality (1) allows us to show that the constant in this estimate can be taken independent of T, if $n > 2$, $p < \frac{n}{2}$, and we prove that the condition $p < \frac{n}{2}$ can not be relaxed.

1. – Introduction

Throughout this paper the symbol Ω means an unbounded domain in $\mathbb{R}^n$, $n \geq 2$, exterior to a finite number of compact regions. Its boundary $\partial\Omega$ is assumed to belong to the class C^m where m is an even positive number such

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that $2m > n$. By $\mathcal{C}_0(\Omega)$ we denote the set of all soleinoidal vector fields $\varphi(x) \in C_0^\infty(\Omega)$. $J_p(\Omega)$ is the closure of $\mathcal{C}_0(\Omega)$ in $L_p(\Omega)$. The norm in $L_p(\Omega)$ is denoted by $|\cdot|_p$. It is well known (cf. [17,9] and Remark 3.2 in Section 3 also) that $L_p(\Omega) \equiv J_p(\Omega) \oplus G_p(\Omega)$, where $G_p(\Omega) = \{\nabla\psi(x) \in L_p(\Omega),\ \psi(x) \in L_{p,\mathrm{loc}}(\Omega)\}$. If $(\mathbf{f}(x), \nabla\psi(x)) \in J_p(\Omega) \times G_{p'}(\Omega)$ where $p' = \frac{p}{p-1}$, then $(\mathbf{f}, \nabla\psi) = \int_\Omega \mathbf{f}(x) \cdot \nabla\psi(x)dx = 0$. By the symbol $P\cdot$ we denote the projector from $L_p(\Omega)$ into $J_p(\Omega)$. By $W_p^m(\Omega)$ we mean the Sobolev space of functions (or vector fields) with the norm $|\mathbf{u}|_{m,p} = \left(\sum_{j=0}^m |D^j\mathbf{u}|_p^p\right)^{1/p}$ where $D^j\mathbf{u}(x)$ is the vector consisting of all derivatives of $\mathbf{u}(x)$ of the order j. $J_p^1(\Omega)$ is the closure of $\mathcal{C}_0(\Omega)$ in $W_p^1(\Omega)$. We consider also spaces of functions $\mathbf{u}(x,t)$ defined in $Q_T = \Omega\times(0,T)$ ($x \in \Omega, t \in (0,T)$) and possessing different regularity properties with respect to x and t. The symbol $L_{p,r}(Q_T)$ denotes the space $L_r((0,T); L_p(\Omega))$ with the norm $|\mathbf{f}|_{L_{p,r}(Q_T)} = \left(\int_0^T |\mathbf{f}(t)|_p^r dt\right)^{1/r}$, $p, r > 1$. By $W_{p,r}^{2,1}(Q_T)$ we mean the space with the norm $|\mathbf{u}|_{W_{p,r}^{2,1}(Q_T)} = \left(\int_0^T \left[|\mathbf{u}(t)|_{2,p}^r + |\mathbf{u}_t(t)|_p^r\right] dt\right)^{1/r}$, in other words, $W_{p,r}^{2,1}(Q_T) = L_r((0,T); W_p^2(\Omega)) \cap W_r^1((0,T); L_p(\Omega))$. In the case of $r = p$ we set $W_{p,r}^{2,1}(Q_T) = W_p^{2,1}(Q_T)$. The space of traces of functions from $W_{p,r}^{2,1}(Q_T)$ on the cross-section of Q_T by the plane $t = $ const. coincides with the Besov space $B_{p,r}^{2-\frac{2}{r}}(\Omega)$. The norm $B_{p,r}^\ell(\Omega)$ for arbitrary $\ell > 0$ may be defined by the formula (see [2])

$$|\mathbf{u}|_{B_{p,r}^\ell(\Omega)}^r = \sum_{j=0}^{\bar\ell} |D^j\mathbf{u}|_p^r + \int_\Omega |z|^{-n-r(\ell-\bar\ell)} \left(\int_{\Omega(z)} |\Delta^k(z) D^{\bar\ell}\mathbf{u}(x)|^p dx\right)^{\frac{r}{p}} dz.$$

Here $\bar\ell$ is a maximal integer which is less than ℓ, $\Delta^k(z)\mathbf{u}(x)$ is a finite difference of $\mathbf{u}(x)$ of the order $k > \ell - \bar\ell$ (so it is possible to take $k = 1$ in the case of non-integral ℓ and $k = 2$ when ℓ is integral), $\Delta^1(z)\mathbf{u}(x) = \mathbf{u}(x+z) - \mathbf{u}(x)$, $\Delta^2(z)\mathbf{u}(x) = \mathbf{u}(x+z) - 2\mathbf{u}(x+\frac{z}{2}) + \mathbf{u}(x)$, and $\Omega(z) = \{x \in \Omega : x + tz \in \Omega, t \in [0,1]\}$. In the case $r = p$ we set $B_{p,p}^\ell(\Omega) \equiv B_p^\ell(\Omega)$; finally, we introduce the norm

$$|||\mathbf{u}|||_p = \left(|\mathbf{u}|_{B_p^{2-2/p}(\Omega')}^p + \int_{\mathbb{R}^n} |\Delta^2(z)\mathbf{u}(z)w(z)|_{L_p(\mathbb{R}^n)}^p |z|^{2-n-2p} dz\right)^{\frac{1}{p}},$$

where Ω' is a bounded subdomain of Ω such that $\mathrm{dist}(\Omega - \Omega', \partial\Omega) > 0$, and $w(x)$ is a smooth function equal to 1 for $x \in \Omega - \Omega'$ and to zero near $\partial\Omega$ and in $\mathbb{R}^n - \Omega$.

We consider exterior initial-boundary value problem

$$\begin{aligned} &\mathbf{v}_t(x,t) - \Delta\mathbf{v}(x,t) = \nabla\pi(x,t) + \mathbf{f}(x,t), \quad \text{in } \Omega\times(0,T),\\ &\nabla\cdot\mathbf{v}(x,t) = 0, \quad \text{in } \Omega\times(0,T),\\ &\mathbf{v}(x,t)_{|\partial\Omega} = 0, \quad \mathbf{v}(x,t) \to 0 \text{ for } |x| \to \infty, \quad \forall t > 0,\\ &\mathbf{v}(x,0) = \mathbf{v}_0(x), \end{aligned} \tag{1.1}$$

and prove the following theorems:

THEOREM 1.1. *Assume that* $\mathbf{f}(x,t) \in L_p(Q_T)$, $\mathbf{v}_0(x) \in B_p^{2-\frac{2}{p}}(\Omega) \cap J_p(\Omega)$, $p > 1$, *and that the following compatibility conditions hold:*

$$\mathbf{v}_0(x)_{|\partial\Omega} = 0, \text{ if } \quad 2 - \frac{2}{p} > \frac{1}{p},$$
$$\mathbf{v}_0^{(0)}(x) \in B_p^{2-\frac{2}{p}}(\mathbb{R}^n), \text{ if } \quad 2 - \frac{2}{p} = \frac{1}{p} \quad \left(\text{i.e. } p = \frac{3}{2}\right);$$

here $\mathbf{v}_0^{(0)}(x) = \mathbf{v}_0(x)$ *for* $x \in \Omega$, $\mathbf{v}_0^{(0)}(x) = 0$ *for* $x \in \mathbb{R}^n - \Omega$. *Then problem* (1.1) *has a unique solution* $\mathbf{v}(x,t) \in W_p^{2,1}(Q_T)$, $\nabla\pi(x,t) \in L_p(Q_T)$, *and this solution satisfies the inequalities*

$$\begin{aligned} &\int_0^T \left(|\mathbf{v}_t(t)|_p^p + |D^2\mathbf{v}(t)|_p^p + |\nabla\pi(t)|_p^p\right) dt \\ &\qquad \le C_1 \left[\int_0^T \left(|\mathbf{f}(t)|_p^p + |\mathbf{v}(t)|_{L_p(\Omega')}^p\right) dt + |\!|\!|\mathbf{v}_0|\!|\!|_p^p + N_1\right], \end{aligned} \tag{1.2}$$

$N_1 = |\mathbf{v}_0^{(0)}|^{3/2}_{B^{2/3}_{3/2}(\tilde{\Omega})}$, $\tilde{\Omega} = (\mathbb{R}^n - \Omega) \cup \Omega'$, *if* $p = \frac{3}{2}$, *otherwise* $N_1 = 0$, *and*

$$\begin{aligned} &\int_0^T \left(|\mathbf{v}_t(t)|_p^p + |D^2\mathbf{v}(t)|_p^p + |\nabla\pi(t)|_p^p\right) dt \\ &\qquad \le C_2(T) \left(\int_0^T |\mathbf{f}(t)|_p^p dt + |\mathbf{v}_0|^p_{B_p^{2-\frac{2}{p}}} + N_1\right). \end{aligned} \tag{1.3}$$

The constant C_1 *is independent of* T. *The constant* $C_2(T)$ *is independent of* T, *if* $n > 2$, $p \in (1, \frac{n}{2})$. *If* $p \ge \frac{n}{2}$, *then* $C_2(T) = C(1 + T^{1+\varepsilon p - \frac{n}{2p}})$, $\forall \varepsilon > 0$.

We consider further the solution of problem (1.1) with $\mathbf{f}(x,t) = 0$ and establish the following estimates

THEOREM 1.2. *Let* $\mathbf{v}(x,t)$ *be a solution of problem* (1.1) *with* $\mathbf{f}(x,t) = 0$. *Then there exist such constants* C_1, C_2 *that*

$$|\mathbf{v}(t)|_q \le C_1 |\mathbf{v}(s)|_p (t-s)^{-\mu}, \; t - s > 0, \tag{1.4}$$

where $\mu = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$, $q > 1$,

$$q \in [p, \infty], \; p \ge 1, \; \text{if } n \ge 3,$$
$$q \in [p, \infty), \; p > 1, \; \text{if } n = 2;$$

and

$$|\nabla\mathbf{v}(t)|_q \le C_2 |\mathbf{v}(s)|_p (t-s)^{-\hat{\mu}}, \tag{1.5}$$

with $q > 1$,

$$(1.6)\qquad \hat{\mu} = \begin{cases} \dfrac{1}{2}+\mu, & \text{if } q\in[p,n], t-s>0, n>2, p\geq 1, \text{ or } n\geq 2, p>1; \\ \dfrac{1}{2}+\mu, & \text{if } q\in[p,\infty), t-s\in(0,1], n\geq 2; \\ \dfrac{n}{2p}, & \text{if } q\geq p\geq 1, q\geq n, t-s\geq 1, n\geq 3; \\ \dfrac{1}{p}-\delta, & \text{if } q\geq p, q\in(2,\infty), t-s\geq 1, n=2, \end{cases}$$

where $\delta > 0$ *can be chosen arbitrarily small. Moreover, estimate* (1.5) *for* $q \geq p$, $q > n$, $p \geq \frac{n}{2}$, $n \geq 3$, *is sharp, in the sense that it is not possible to replace the exponent* $\hat{\mu}$ *with* $\hat{\mu} + \varepsilon$ *for any* $\varepsilon > 0$. *In particular it is not possible to have* $\hat{\mu} = \frac{1}{2} + \mu$. *As a consequence, also estimates* (1.4) *for* $q \geq p \geq \frac{n}{2}$ *are sharp.*

Finally, there exists a constant C_3 *such that*

$$|\mathbf{v}_t(t)|_q \leq |\mathbf{v}(s)|_p (t-s)^{-\mu'},$$

with $q > 1$, *and*

$$\mu' = \begin{cases} 1+\mu, & \text{if } n>2, q\in[p,\infty], t-s>0, p\geq 1; \\ 1+\mu, & \text{if } n=2, q\in[p,\infty), 1\geq t-s>0, p>1; \\ 1+\mu, & \text{if } n=2, q=2, t-s>0, p\in(1,2]; \\ \dfrac{1}{2}+\dfrac{1}{p}-\delta, & \text{if } n=2, q\in[p,\infty)\cap(2,\infty), t-s\geq 1, p>1; \\ \dfrac{3}{2}+\dfrac{1}{p}-\dfrac{2}{q}-\delta, & \text{if } n=2, q\in[p,2], t-s\geq 1, p>1, \end{cases}$$

where $\delta > 0$ *can be chosen arbitrarily small.*

COROLLARY 1.1. *Let* $\mathbf{v}(x,t)$ *be a solution of problem* (1.1) *with* $\mathbf{f}(x,t) = 0$. *Then*

$$(1.7)\qquad \lim_{t\to\infty} |\mathbf{v}(t)|_p = 0;$$

$$(1.8)\qquad |\mathbf{v}(t)|_q = o(t^{-\mu}),\quad |\mathbf{v}_t(t)|_q = o(t^{-\mu'}),\quad |\nabla\mathbf{v}(t)|_q = o(t^{-\hat{\mu}}),$$

where μ, $\hat{\mu}$, μ' *are defined in* (1.5)-(1.6).

THEOREM 1.3. *Let* $\mathbf{v}(x,t)$ *be a solution of problem* (1.1) *with* $\mathbf{v}_0(x) = 0$, $\mathbf{f}(x,t) \in L^r((0,T); L^p(\Omega))$ *and with the following restrictions:*

$$\begin{aligned} &\text{if}\quad p > n/2, \quad \text{then}\quad r\in(1, 2p/(2p-n)) \\ &\text{if}\quad p\in(1, n/2], \quad \text{then}\quad r > 1. \end{aligned}$$

Then there exists a constant C such that

$$\left(\int_0^T |\mathbf{v}(t)|_q^s dt\right)^{\frac{1}{s}} \le C\left(\int_0^T |\mathbf{f}(t)|_p^r dt\right)^{\frac{1}{r}}, \forall T > 0, \tag{1.9}$$

where $q \ge p$, $s \ge r$, $\frac{1}{s} - \frac{1}{r} \ge \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - 1$. *The constant C is independent of T, if* $\frac{1}{s} - \frac{1}{r} = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - 1$, *otherwise* $C = C_1 T^b$, $b = \frac{1}{s} - \frac{1}{r} - \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) + 1$, C_1 *is independent of T.*

In addition,

$$\left(\int_0^T |\mathbf{v}(t)|^r_{L_q(\Omega\cap S_R)} dt\right)^{\frac{1}{r}} \le C(R)\left(\int_0^T |\mathbf{f}(\tau)|_p^r d\tau\right)^{\frac{1}{r}},$$

where $S_R = \{x \in \mathbb{R}^n : |x| < R\}$, $p \in (1, \frac{n}{2})$, $q \in (1, \frac{nq}{n-2q})$ *and* $r > 1$ *is arbitrary. The constant* $C(R)$ *is independent of T.*

Finally, as a consequence of Theorem 1.1 and Theorem 1.3, we give $L_{p,r}$ estimates for the solutions of problem (1.1) (and prove their optimality)

THEOREM 1.4. *Assume that* $\mathbf{f}(x,t) \in L_{p,r}(Q_T)$ $\mathbf{v}_0(x) \in B_{p,r}^{2-\frac{2}{r}}(\Omega) \cap J_p(\Omega)$ *and that the following compatibility conditions are satisfied:*

$$\mathbf{v}_0(x)_{|\partial\Omega} = 0, \quad \text{if} \quad 2 - \frac{2}{r} > \frac{1}{p},$$

$$\mathbf{v}_0^{(0)}(x) \in B_{p,r}^{2-\frac{2}{r}}(\mathbb{R}^n), \quad \text{if} \quad 2 - \frac{2}{r} > \frac{1}{p}.$$

Then problem (1.1) *has a unique solution* $\mathbf{v}(x,t) \in W^{2,1}_{p,r}(Q_T)$ $\nabla\pi(x,t) \in L_{p,r}(Q_T)$ *and*

$$\begin{aligned}&\int_0^T \left(|\mathbf{v}_t(t)|_p^r + |\mathbf{v}(t)|^r_{W^2_p(\Omega)} + |\nabla\pi(t)|_p^r\right) dt \\ &\quad \le C_1(T)\left(\int_0^T |\mathbf{f}(t)|_p^r dt + |\mathbf{v}_0|^r_{B_{p,r}^{2-\frac{2}{r}}} + N_2\right),\end{aligned} \tag{1.10}$$

$N_2 = |\mathbf{v}_0^{(0)}|^r_{B_{p,r}^{2-\frac{2}{r}}(\mathbb{R}^n)}$, *if* $2 - \frac{2}{r} = \frac{1}{p}$, *otherwise* $N_2 = 0$. *Moreover, there holds the inequality*

$$\int_0^T \left(|\mathbf{v}_t(t)|_p^r + |D^2\mathbf{v}(t)|_p^r + |\nabla\pi(t)|_p^r\right) dt \le C_2\left(\int_0^T |\mathbf{f}(t)|_p^r dt + |||\mathbf{v}_0|||_p^r + N_2\right); \tag{1.11}$$

with C_2 *independent of T in the case* $p \in (1, \frac{n}{2})$, $n \ge 3$; *otherwise* $C_2^{\frac{1}{r}} = 1 + CT^b$, $b = 1 + \eta - \frac{n}{2p}$, $\forall \eta > 0$, *C is independent of T.*

For $p \ge \frac{n}{2}$ *and* $\forall r \in (1, \infty)$ C_2 *can not be constant with respect to T.*

Finally, for arbitrary $r, p \in (1, \infty)$, $n \ge 3$,

$$\int_0^T \left(|\mathbf{v}_t(t)|_p^r + |P\Delta\mathbf{v}(t)|_p^r\right) dt \le C_3\left(\int_0^T |\mathbf{f}(t)|_p^r dt + |\mathbf{v}_0|^r_{B_p^{2-\frac{2}{p}}(\Omega)} + N_2\right), \tag{1.12}$$

with C_3 *independent of T.*

The solvability of problem (1.1), both in a generalized and in a classical sense, was established by O.A. Ladyzhenskaya (see her book [17]). The estimate (1.2) in bounded and exterior domains, and also $W_p^{2m,m}(Q_T)$-estimates for bounded domains, were obtained in the papers [26]-[27], moreover, as a consequence of these estimates, $L_p - L_q$ estimates of the type (1.4)-(1.5) were proved in a bounded time interval, and the analyticity of a semigroup generated by the Stokes operator was established. However, all these estimates were not uniform with respect to T. At the elimination of this defect and at the further analysis of the exterior problem for the Stokes and Navier-Stokes equation the papers [4]-[7], [12]-[15] were aimed. Estimate (1.10) with arbitrary $p, r > 1$ was proved by V. I. Yudovich [31] (for bounded domains) and by Y. Giga and H. Sohr [13] for bounded and exterior domains. The latter authors have shown also that the constant in this estimate may be taken independent of T, if $n > 2, 1 < p < \frac{n}{2}$ and established the inequality (1.12).

Our proof of the estimates (1.11)-(1.12) seems to be more elementary. Making use of the idea of V.I. Yudovich we obtain estimate (1.11) as a consequence of inequality (1.3), and we show with the help of (6.1) that the constant C can be taken independent of T, if $n > 2, p < \frac{n}{2}$, and that the latter restriction can not be relaxed (we prove that otherwise the estimate

$$|\mathbf{w}|_{W_{1,p}(\Omega')} + |D^2\mathbf{w}|_p \leq C|\mathbf{f}|_p$$

for the solution of the exterior Stokes problem would be true for $p \geq \frac{n}{2}$ which is not the case for arbitrary $\mathbf{f}(x) \in J_p(\Omega)$, see [21]). We can give an estimate of the constant $C(T)$ in (1.11): $C^{\frac{1}{r}}(T) = O(T^{1+\eta-\frac{n}{2p}}), \forall \eta > 0$. Estimate (1.12) easily follows from (1.11) by arguments of duality.

As far as the inequality (1.2) is concerned, we obtain it, as in [27], by Schauder's method, moreover, we show that the arguments of the paper [27] carried out in the three-dimensional case, apply practically without any changes to the case of arbitrary $n \geq 2$. This refers also to some auxiliary propositions such as the Helmholtz-Weyl decomposition of $\mathbf{f}(x) \in L_p(\Omega)$.

In the paper [22] we obtain (1.11) by a somewhat different procedure connected with the estimates of heat potentials in these spaces with the subsequent application of Schauder's method.

Estimates of the type (1.4)-(1.5) were obtained in [4]-[6], [12], [15], [16], [20] (for analogous question related to Navier-Stokes system see also [10], [19]-[20], [24]). More precisely, in the paper [15], it has been proved for $n \geq 3$

$$\begin{aligned} &|\mathbf{v}(t)|_q \leq C|\mathbf{v}_0|_p t^{-\sigma}, q \in [p, \infty), \sigma = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right), \forall t > 0; \\ &|\nabla \mathbf{v}(t)|_r \leq C|\mathbf{v}_0|_p t^{-\sigma-1/2}, r \in (p, n], \forall t > 0; \end{aligned} \tag{1.13}$$

and [5], [6], [16], [20] for $n = 2$

$$\begin{aligned} &|\mathbf{v}(t)|_q = o(t^{1/q-1/p}), p \in (1, 2), q \in [2, \infty); \\ &|\nabla \mathbf{v}(t)|_2 = o(t^{-1/p}), p \in (1, 2]. \end{aligned}$$

In Theorem 1.2 we extend estimate $(1.13)_1$ to the cases of $q = \infty$ and $p = 1$; moreover, estimates $(1.13)_2$ to any $r \geq p$. We give also estimate of time derivative of the solution. Finally, we extend inequality (1.13) to the two dimensional case. We prove that the estimate (1.5) is sharp, in particular that (1.5) does not hold with $\hat{\mu} = \mu + \frac{1}{2}$ for $q > n$. For bounded time interval, we deduce (1.4)-(1.5), as in [27], from estimate (1.2) (for the case of finite $p, q > 1$). Subsequent arguments are based on the estimate for the solution of the Cauchy problem for the heat equation, on some auxiliary energy estimates and on duality arguments. These arguments are classical in uniqueness theorems, it is enough to mention the famous Holmgren theorem for the systems of the Cauchy-Kowalewsky type. In the paper of Foias [8] they were used (together with some estimates $(5.2)_1$) in the proof of the uniqueness of the solution to the Cauchy problem for the Navier-Stokes equations. We observe finally that inequalities (1.4)-(1.5) have important applications to the Navier-Stokes equations (see [19], [20]).

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2. – Auxiliary propositions

We begin this section with the consideration of some half-space problems, namely

$$\begin{aligned} &\mathbf{u}_t(x,t) - \Delta \mathbf{u}(x,t) = \mathbf{f}(x,t), \text{ in } \mathbb{R}^n_+ \times (0,T)\,, \\ &\mathbf{u}(x,t)_{|x_n=0} = 0, \forall t > 0,\ \mathbf{u}(x,0) = \mathbf{u}_0(x)\,. \end{aligned} \tag{2.1}$$

$$\begin{aligned} &-\Delta \mathbf{u}(x) + \nabla q(x) = 0, \text{ in } \mathbb{R}^n_+\,, \\ &\nabla \cdot \mathbf{u}(x) = r(x) \text{ in } \mathbb{R}^n_+, \quad \mathbf{u}(x)_{|x_n=0} = 0\,. \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\mathbf{w}_t(x,t) - \Delta \mathbf{w}(x,t) + \nabla s(x,t) = \mathbf{g}(x,t)\,, \\ &\nabla \cdot \mathbf{w}(x,t) = 0, \quad \text{in } \mathbb{R}^n_+ \times (0,T)\,, \\ &\mathbf{w}(x,t)_{|x_n=0} = 0,\ \forall t \geq 0,\ \mathbf{w}(x,0) = 0\,. \end{aligned} \tag{2.3}$$

If the data decay at infinity sufficiently rapidly, then the solutions of these problems can be written explicitly in terms of fundamental solutions and of the Green matrices of corresponding boundary value problems. These explicit formulas have a form

$$\text{(2.4)}\qquad \mathbf{u}(x,t)=\int_0^t\int_{\mathbb{R}^n}\Gamma(x-y,t-\tau)\mathbf{f}^{(-)}(y,\tau)dyd\tau + \int_{\mathbb{R}^n}\Gamma(x-y,t)\mathbf{u}_0^{(-)}(y)dy\,,$$

$$\text{(2.5)}\qquad \mathbf{u}(x)=\int_{\mathbb{R}^n}\mathcal{L}(x,y)r(y)dy\,,$$

$$\text{(2.6)}\qquad \mathbf{w}(x,t)=\int_0^t\int_{\mathbb{R}^n_+}\mathcal{G}(x,y,t-\tau)P\mathbf{g}(y,\tau)dyd\tau\,,$$

(we omit formulas for $q(x)$ and $s(x,t)$). Here $\mathbf{f}^{(-)}(x,t)$ is an odd (with respect to x_n) extension of the function $\mathbf{f}(x,t)$, $x\in\mathbb{R}^n_+$, into the domain $\mathbb{R}^n_-(x_n<0)$, $\Gamma(x,t)$ is a fundamental solution of the heat equation:

$$\Gamma(x,t)=(4\pi t)^{-\frac{n}{2}}e^{-\frac{|x|}{4t}},\ (t>0),\ \Gamma(x,t)=0\ (t<0)\,;$$
$$\mathcal{L}(x,y)=\left(\frac{\partial L_1(x,y)}{\partial x_1},\dots,\frac{\partial L_n(x,y)}{\partial x_n}\right)$$

where

$$L_i(x,y)=\mathcal{E}(x-y)+\mathcal{E}(x-y^*)-2x_n\frac{\partial}{\partial x_n}\mathcal{E}(x-y^*)+(4\delta_{in}-2)\mathcal{E}(x-y^*)\,;$$

$y^*=(y_1,\dots,y_{n-1},-y_n)$, and $\mathcal{E}(z)$ is a fundamental solution of the Laplace equation:

$$\mathcal{E}(z)=\begin{cases}\dfrac{1}{(n-2)|S_1|}|x|^{2-n}, & \text{if}\quad n>2\,,\\ \dfrac{1}{2\pi}\ln\dfrac{1}{|x|}, & \text{if}\quad n=2\,,\end{cases}$$

$|S_1|=2\pi^{\frac{n}{2}}/\Gamma(n/2)$. Finally,

$$P\mathbf{g}(x,t)=\mathbf{g}(x,t)-\nabla_x\int_{\mathbb{R}^n_+}\nabla_y\left[\mathcal{E}(x-y)+\mathcal{E}(x-y^*)\right]\cdot\mathbf{g}(y,t)dy$$

(P is the projection of $\mathbf{g}(x,t)$ onto $J_p(\mathbb{R}^n_+)$) and $\mathcal{G}(x,y,t)$ is the matrix with the elements

$$G_{ij}(x,y,t)=\delta_{ij}\left[\Gamma(x-y,t)\ -\Gamma(x-y^*,t)\right] + 4(1-\delta_{jn})\frac{\partial}{\partial x_j}\int_0^{x_n}\int_{\mathbb{R}^{n-1}}\frac{\partial}{\partial x_i}\mathcal{E}(x-z)\Gamma(z-y^*,t)dz\,.$$

From the estimates of the heat potentials obtained in [18], [28] and from the Calderon-Zygmund theorem there follow the inequalities

$$\int_0^T\left(|\mathbf{u}_t(t)|^p_{p,\mathbb{R}^n_+}+|D^2\mathbf{u}(t)|^p_{p,\mathbb{R}^n_+}\right)dt$$

$$\leq C_1\left(\int_0^T|\mathbf{f}(t)|^p_p dt+\int_{\mathbb{R}^n}|z|^{2-n-2p}\int_{\mathbb{R}^n}|\Delta^2(z)\mathbf{u}_0^{(-)}(x)|^p dx\,dz\right) \tag{2.7}$$

$$\leq C_1\int_0^T|\mathbf{f}(t)|^p_p dt+C_2\left(\int_{\mathbb{R}^n_+}|z|^{2-n-2p}\int_{\mathbb{R}^n_+}|\Delta^2(z)\mathbf{u}_0(x)|^p dx\,dz+I_p[\mathbf{u}_0]\right).$$

$$|D^2\mathbf{u}|^p_{p,\mathbb{R}^n_+}+|\nabla q|^p_{p,\mathbb{R}^n_+}\leq C_4|\nabla r|^p_{p,\mathbb{R}^n_+}, \tag{2.8}$$

$$\int_0^T\left(|\mathbf{w}_t(t)|^p_{p,\mathbb{R}^n_+}+|D^2\mathbf{w}(t)|^p_{p,\mathbb{R}^n_+}+|\nabla s(t)|^p_{p,\mathbb{R}^n_+}\right)dt\leq C_3\int_0^T|\mathbf{g}(t)|^p_p dt. \tag{2.9}$$

moreover, if $r(x)=\nabla\cdot\mathbf{R}(x)$, $\mathbf{R}_n(x)_{|x_n=0}=0$, then

$$|\mathbf{u}|_p\leq C_5|\mathbf{R}|_p. \tag{2.9'}$$

The constant C_1-C_5 are independent of T. By $I_p[\mathbf{u}_0]$ we mean the integral

$$\int_{\mathbb{R}^n}|z|^{2-n-2p}\int_{\mathbb{R}^n}|\Delta^2(z)\mathbf{u}_0^{(0)}(x)|^p dx\,dz$$

where $\mathbf{u}^{(0)}(x)=\mathbf{u}(x)$ for $x_n>0$, $\mathbf{u}^{(0)}(x)=0$ for $x_n<0$. As shown in [27], $I_p[\mathbf{u}_0]$ can be omitted, if $p\neq\frac{3}{2}$, since in this case it can be evaluated by

$$\int_{\mathbb{R}^n}|z|^{2-n-2p}\int_{\mathbb{R}^n}|\Delta^2(z)\mathbf{u}_0(x)|^p dx\,dz$$

We observe that formula (2.6) for the solution of problem (2.3) was found in [27] (see also [29]) for the most important three-dimensional case, but exactly the same formula holds for arbitrary n and the proof of (2.6) presented in [27] applies without any changes. We verify below in the Appendix that (2.6) is indeed a solution of problem (2.3). The second derivatives of $L_i(x,y)$ contain the term $D^2(x_h\frac{\partial}{\partial x_h}\mathcal{E}(x-y^*))$ which is not of Calderon-Zygmund type, but this term can be evaluated by $C|x-y^*|^{-n}$ and the corresponding integral operators are bounded in $L_p(\mathbb{R}^n_+)$.

In fact, for the proof of estimate (2.8) we need to consider a slightly more general half-space problem than (2.3), namely,

$$\mathbf{w}_t(x,t)-\Delta\mathbf{w}(x,t)+\nabla s(x,t)=\mathbf{g}(x,t),$$
$$\nabla\cdot\mathbf{w}(x,t)=\rho(x,t),\quad \text{in } \mathbb{R}^n_+\times(0,T),$$
$$\mathbf{w}(x,t)_{|x_n=0}=0,\ \forall t\geq 0,\ \mathbf{w}(x,0)=\mathbf{w}_0(x).$$

We assume that

$$\rho(x,t)=\nabla\cdot\mathbf{R}(x,t),\quad \mathbf{R}(x,t)\cdot\vec{n}_{|x_n=0}=0. \tag{2.10}$$

The following proposition was actually used in [27] in the proof of estimate (2.8).

LEMMA 2.1. *Under the hypothesis* (2.10), $(\mathbf{w}(x,t), s(x,t))$ *satisfies the inequality*

$$\begin{aligned}
&\int_0^T\Big(|\mathbf{w}_t(t)|_p^p+|D^2\mathbf{w}(t)|_p^p+|\nabla s(t)|_p^p\Big)\,dt\\
&\le C\int_0^T\Big(|\mathbf{g}(t)|_p^p+|\nabla\rho(t)|_p^p+|\mathbf{R}_t(t)|_p^p\Big)\,dt\\
&+C|\int_{\mathbb{R}^n_+}|z|^{2-n-2p}\int_{\mathbb{R}^n}|\Delta^2(z)\mathbf{w}_0(x)|^p dxdz+I_p[\mathbf{w}_0]\,.
\end{aligned}\tag{2.11}$$

with the constant C independent of T.

PROOF. It is easy to verify that $\mathbf{w}(x,t)=\mathbf{w}^1(x,t)+\mathbf{w}^2(x,t)+\mathbf{w}^3(x,t)$, $s(x,t)=s^2(x,t)+s^3(x,t)$ where $\mathbf{w}^1(x,t)=\mathbf{u}(x,t)$ is a solution of (2.1) with $\mathbf{f}=\mathbf{g}$, $(\mathbf{w}^2(x,t), s^2(x,t))$ is a solution of (2.2) with $r(x,t)=\rho(x,t)-\nabla\cdot\mathbf{w}^1(x,t)$, and $(\mathbf{w}^3(x,t), s^3(x,t))$ is a solution of (2.3) with $\mathbf{g}(x,t)=-\frac{\partial}{\partial t}\mathbf{w}^2(x,t)$. Hence, estimate (2.11) follows from (2.7)-(2.9).

Analogous proposition holds for the Cauchy problem

$$\begin{aligned}
&\mathbf{w}_t(x,t)-\Delta\mathbf{w}(x,t)+\nabla s(x,t)=\mathbf{g}(x,t),\\
&\nabla\cdot\mathbf{w}(x,t)=\rho(x,t),\quad \text{in } \mathbb{R}^n\times(0,T),\\
&\mathbf{w}(x,0)=\mathbf{w}_0(x).
\end{aligned}\tag{2.12}$$

LEMMA 2.2. *If* $\rho(x,t)=\nabla\cdot\mathbf{R}(x,t)$, *then the solution of problem* (2.12) *satisfies the inequality*

$$\begin{aligned}
&\int_0^T\Big(|\mathbf{w}_t(t)|_p^p+|D^2\mathbf{w}(t)|_p^p+|\nabla s(t)|_p^p\Big)\,dt\\
&\le C\int_0^T\Big(|\mathbf{g}(t)|_p^p+|\nabla\rho(t)|_p^p+|\mathbf{R}_t(t)|_p^p\Big)\,dt\\
&+C\int_{\mathbb{R}^n}|z|^{2-n-2p}\int_{\mathbb{R}^n}|\mathbf{w}_0(x+2z)-2\mathbf{w}_0(x+z)+\mathbf{w}_0(x)|^p dx\,dz\,.
\end{aligned}\tag{2.13}$$

Consider the Neumann problem and the Stokes problem in an exterior domain Ω:

$$\begin{aligned}
&\Delta\phi(x)=0,\ \text{in }\Omega,\quad \frac{d}{d\vec{n}}\phi(x)_{|\partial\Omega}=b(x)\,,\\
&\phi(x)\to 0 \text{ for } |x|\to\infty\ (\phi(x)\to \text{const for } |x|\to\infty\ n=2)\,.
\end{aligned}\tag{2.14}$$

$$\begin{aligned}
&-\Delta\mathbf{u}(x)+\nabla p(x)=\mathbf{f}(x),\quad \nabla\cdot\mathbf{u}(x)=0,\ \text{in }\Omega,\ \mathbf{u}(x)_{|\partial\Omega}=0,\\
&\mathbf{u}(x)\to 0 \text{ for } |x|\to\infty\ (\mathbf{u}(x)\to \text{const for } |x|\to\infty\ n=2)\,.
\end{aligned}\tag{2.15}$$

LEMMA 2.3. *Let* $\partial\Omega \in C^2$ *and*

$$b(x) = \sum_{i,k=1}^{n} \left(n_i \frac{\partial}{\partial x_k} - n_k \frac{\partial}{\partial x_i} \right) a_{ik}(x)\,, \tag{2.16}$$

If $a_{ik}(x) \in B_p^{1-1/p}(\partial\Omega)$, *then the solution of* (2.14) *satisfies the inequality*

$$|\phi|_{L^p(\Omega')} \le C \sum_{i,k=1}^{n} |a_{ik}|_{B^{\lambda,p}(\partial\Omega)}\,, \tag{2.17}$$

where λ *is an arbitrary positive number beloging to* $(0, 1)$ *and* $\Omega' \subset \Omega$ *is an arbitrary bounded domain such that* $\operatorname{dist}(\Omega - \Omega', \partial\Omega) > 0$. *Moreover,*

$$|\nabla\phi|_p^p \le C \sum_{i,k=1}^{n} \int_{\partial\Omega}\int_{\partial\Omega} \frac{|a_{ik}(x) - a_{ik}(y)|^p}{|x-y|^{n-2+p}} d\sigma_x d\sigma_y \le C|a_{ik}|^p_{B_p^{1-\frac{1}{p}}} \tag{2.18}$$

This proposition is an analogue of Lemma 2.1 in [27] and it is proved in the same way. Another, more direct proof of (2.17)-(2.18) are given in the Appendix. Condition (2.16) in the three-dimensional case is equivalent to $\mathbf{b}(x) = \operatorname{rot} \mathbf{A}(x)\cdot\vec{n}$ with $\mathbf{A}(x) = (a_{23}(x) - a_{32}(x), a_{31}(x) - a_{13}(x), a_{12}(x) - a_{21}(x))$.

As pointed out in [27], the following proposition is a consequence of Lemma 2.3

LEMMA 2.4. *Arbitrary* $\mathbf{f}(x) \in L_q(\Omega)$, $q > 1$, *can be represented in a unique way in the form*

$$\mathbf{f}(x) = \mathbf{f}_1(x) + \nabla\varphi(x)\,, \tag{2.19}$$

where $\mathbf{f}_1(x)$ *satisfies the relation* $(\mathbf{f}_1, \nabla\eta) = 0$ *for arbitrary smooth* $\eta(x)$ *with a compact support, and*

$$|\mathbf{f}_1|_q + |\nabla\varphi|_q \le C|\mathbf{f}|_q\,.$$

Formula (2.19) is often refered to as the Helmhotz-Weyl decomposition (cf. also [9] the references there), and the subspaces of functions $\mathbf{f}(x)$ and $\nabla\varphi(x)$ are denoted by $J_q(\Omega)$ and $G_q(\Omega)$. The set $\mathcal{C}_0(\Omega)$ of all divergence free vector fields from $C_0^\infty(\Omega)$ is dense in $J_q(\Omega)$, therefore for arbitrary $\mathbf{v}(x) \in J_q(\Omega)$ $|\mathbf{v}|_q = \sup|(\mathbf{v}, \varphi)|$ where supremum is taken over all vector fields $\varphi(x) \in \mathcal{C}_0(\Omega)$ with $|\varphi|_{q'} = 1$, in other words, $|(\mathbf{v}, \varphi)| \le M|\varphi|_{q'}$, $\forall\,\varphi(x) \in \mathcal{C}_0(\Omega)$ implies $|\mathbf{v}|_q \le M$.

LEMMA 2.5. *Let* $\partial\Omega \in C^2$, $\mathbf{f}(x) \in L_q(\Omega)$ *and let* $\mathbf{v}(x) \in L_{q,\mathrm{loc}}(\Omega)$ *be a solution of* (2.15) *with* $D^2\mathbf{v}(x) \in L_q(\Omega)$. *Then*

$$|D^2\mathbf{v}|_q \le C(|\mathbf{f}|_q + |\mathbf{v}|_{L_q(\Omega')})\,, \tag{2.20}$$

where $\Omega' \subset \Omega$ is a bounded domain such that $\mathrm{dist}(\Omega - \Omega', \partial\Omega) >$. *The norm of* $\mathbf{v}(x)$ *in the right hand side may be omitted, if* $q < \frac{n}{2}$. *In addition,*

$$|D^2\mathbf{v}|_{L_q(\Omega'')} \leq C(|\mathbf{f}|_{L_q(\Omega')} + |\mathbf{v}|_{L_q(\Omega')}), \tag{2.20*}$$

where $\Omega' \subset \Omega'$, $\mathrm{dist}(\Omega' - \Omega', \partial\Omega) > 0$.

If $\mathbf{f}(x) \in L_q(\Omega) \cap L_r(\Omega)$, $\nabla\mathbf{v}(x) \in L_q(\Omega) \cap L_r(\Omega)$, $D^2\mathbf{v}(x) \in L_q(\Omega)$, *then* $D^2\mathbf{v}(x) \in L_r(\Omega)$.

Let $\partial\Omega \in C^m$ *and let* $\mathbf{v}(x)$ *be a solution of* (2.15) *with* $D^k\mathbf{v}(x) \in L_q(\Omega)$, $k = 1, \cdots, m$. *Then for* $j \geq 2$

$$|D^j\mathbf{v}|_q \leq C(|D^{j-2}\mathbf{f}|_q + |\mathbf{v}|_{L_q(\Omega')}), \tag{2.21}$$

PROOF. See [21]

LEMMA 2.6. *Let* $\partial\Omega \in C^2$, $n \geq 3$, $\mathbf{\Phi}(x) \in L_p(\Omega)$, $p > \frac{n}{n-1}$ *and*

$$|(\mathbf{\Phi}, \varphi)| \leq M|\varphi|_{q'}, \forall\, \varphi(x) \in \mathcal{C}_0(\Omega), \tag{2.22}$$

for some $q' = \frac{q}{q-1} > p'$ *i.e.* $q < p$. *If* $\mathbf{\Phi}(x) \in J_p(\Omega)$, *then* $\mathbf{\Phi}(x) \in J_q(\Omega)$ *and* $|\mathbf{\Phi}|_q \leq M$. *If* $\nabla \cdot \mathbf{\Phi}(x) = 0$ *in a weak sense and* $q \in (\frac{n}{n-1}, p)$, *then* $\mathbf{\Phi}(x) \in L_q(\Omega)$.

PROOF. Assume first that $\mathbf{\Phi}(x) \in J_p(\Omega)$, and consider the family of domains $\Omega_R = \{x \in \Omega : |x| < R\}$, and the functions $\mathbf{\Phi}_R(x) = \mathbf{\Phi}(x)_{|\Omega_R}$ and $\mathbf{\Phi}^1_R(x) = P_R\mathbf{\Phi}_R(x)$ where P_R is the projector onto $J_p(\Omega_R)$. We extend these functions by zero into $\mathbb{R}^n - \Omega_R$. Clearly, $\mathbf{\Phi}^1_R(x) \in J_p(\Omega) \cap J_q(\Omega)$ and the norms $|\mathbf{\Phi}^1_R|_p$ and $|\mathbf{\Phi}^1_R|_q$ are uniformly bounded. Hence, there exists a subsequence $\mathbf{\Phi}^1_{R_n}$ converging weakly to $\mathbf{\Phi}^1(x) \in J_p(\Omega) \cap J_q(\Omega)$ for which the inequality $|\mathbf{\Phi}^1|_q \leq M$ holds. From the equation

$$(\mathbf{\Phi}_{R_n} - \mathbf{\Phi}^1_{R_n}, \varphi) = 0,\ \forall\, \varphi(x) \in \mathcal{C}_0(\Omega),$$

we conclude that $(\mathbf{\Phi} - \mathbf{\Phi}^1, \varphi) = 0$, which shows that $\mathbf{\Phi}^1(x) = \mathbf{\Phi}(x)$ [25]. Hence, $\mathbf{\Phi}(x) \in J_q(\Omega)$ and $|\mathbf{\Phi}|_q \leq M$.

Now assume that $\nabla \cdot \mathbf{\Phi}(x) = 0$, i.e. $(\mathbf{\Phi}, \nabla\eta) = 0$ for arbitrary $\eta(x) \in C_0^\infty(\Omega)$. This means that $\nabla\Pi(x) = (I - P)\mathbf{\Phi}(x)$ satisfies the relation $(\nabla\Pi, \nabla\eta) = 0$, i.e. the function $\Pi(x)$ is harmonic. Since $\nabla\Pi(x) \in L_p(\Omega)$, $\nabla\Pi(x) = O(|x|^{-n+1})$ for large x, and $\nabla\Pi(x) \in L_q(\Omega)$. The vector field $\mathbf{\Phi}'(x) = P\mathbf{\Phi}(x) \in J_p(\Omega)$ satisfies the inequality (2.22), so $|\mathbf{\Phi}'|_q \leq M$. The lemma is proved.

LEMMA 2.7. *Let Ω be a bounded domain, whose boundary* $\partial\Omega \in C^1$. *Assume* $v(x) \in W^2_p(\Omega)$ *and* $v(x)_{|\Sigma} = 0$, *where* $\emptyset \neq \Sigma \subseteq \partial\Omega$, *meas*$(\Sigma) > 0$ *and* $\vec{n}$ *is not constant on* Σ. *Then, there exists a constant C such that*

$$|v|_p + |\nabla v|_p \leq C|D^2 v|_p.$$

PROOF. If $|v|_p \leq C|D^2 v|_p$ with C independent of $v(x)$, then from the well known inequality

$$|\nabla v|_p \leq C(|D^2 v|_p + |v|_p),$$

it is very simple to obtain the desidered estimate. To prove that $|v|_p \leq C|D^2 v|_p$ we argue *ab absurdo*. We assume that for any $m \in N$ there exists $u_m(x)$ such that $|u_m|_p > m|D^2 u_m|_p$. Then there holds $|v_m|_p = 1$ and $|D^2 v_m|_p < \frac{1}{m}$ for $v_m = \frac{u_m(x)}{|u_m|_p}$. Therefore there exists a subsequence $\{v_{mk}(x)\}$ which converging weakly in $W_p^2(\Omega)$ and strongly in $L_p(\Omega)$ to a function $v(x)$ such that $|v|_p = 1$ and $|D^2 v|_p = 0$. This last property implies $v(x) = a + \mathbf{b} \cdot x$, $\forall x \in \Omega$, with a, $\mathbf{b} = (b_1, \ldots, b_n)$ constants. Since $\nabla v(x) - \vec{n}(\vec{n} \cdot \nabla v(x))_{|\Sigma} = 0$, we conclude that $b_i - (\sum_{k=1}^n b_k n_k(x)) n_i(x)_{|\Sigma} = 0$, which yields $|\mathbf{b}|^2 = |\mathbf{b} \cdot \vec{n}|^2, \forall x \in \Sigma$. On the other hand $\vec{n}$ is not a constant, so the above relation is possible only for $\mathbf{b} = 0$. Therefore $v(x) = \text{const} = 0$ (since $v(x)_{|\Sigma} = 0$) which contradicts to $|v|_p = 1$. The lemma is proved.

We shall often make use of Green's identity. Let $(\mathbf{v}(x,t), \pi(x,t))$ be a solution of problem (1.1) and let $(\varphi(x,t), p(x,t))$ be a solution of the same problem with $\mathbf{f}(x,t) = 0$ and $\varphi_0(x) = 0$. Multiplying the equation $\mathbf{v}_t(x,t) - \Delta \mathbf{v}(x,t) = \nabla \pi(x,t) + \mathbf{f}(x,t)$ by $\varphi(x, t-\tau)$ and integrating with respect to $(x,\tau) \in \Omega \times (0,t)$, we obtain

$$(\mathbf{v}(t), \varphi_0) = \int_0^t (\mathbf{f}(\tau), \varphi(t-\tau)) d\tau + (\mathbf{v}_0, \varphi(t)). \tag{2.23}$$

If $\mathbf{v}(x,t)$ satisfies non-homogeneous boundary condition $\mathbf{v}(x,t)_{|\partial\Omega} = \mathbf{a}(x,t)$, there appear surface integrals in the right hand side:

$$\begin{aligned}(\mathbf{v}(t), \varphi_0) = & \int_0^t (\mathbf{f}(\tau), \varphi(t-\tau)) d\tau + (\mathbf{v}_0, \varphi(t)) \\ & - \int_0^t \int_{\partial\Omega} \mathbf{a}(x,\tau) \vec{n} \cdot \nabla \varphi(x, t-\tau) d\sigma d\tau \\ & - \int_0^t \int_{\partial\Omega} \mathbf{a}(x,t) \cdot \vec{n}\, p(x, t-\tau) d\sigma d\tau .\end{aligned} \tag{2.24}$$

3. – The proof of Theorem 1.1

We prove at first a-priori estimate (1.2) assuming that $\mathbf{f}(x,t)$ belongs to the space $J_p(\Omega)$ for all $t \in (0,T)$. This does not restrict the generality because the projector onto $J_p(\Omega)$ is bounded in $L_p(\Omega)$ and the gradient part of $\mathbf{f}(x,t)$ can be incorporated into $\nabla \pi(x,t)$. As in [27], we use Schauder's method.

Let ξ be an arbitrary point of $\partial\Omega$. Without restriction of generality we may assume that ξ coincides with the origin and that the x_n-axis is directed along the interior normal $\vec{n}(\xi)$. In the ball $K'_d = \{x' \equiv (x_1, \cdots, x_{n-1}) : |x'| < d\}$ the boundary is given by the equation $x_n = F(x')$ where $F(x') \in C^2(K'_d)$, $F(0) = 0$, $\nabla F(0) = 0$. Let us introduce new coordinates

$$z' = x',\ z_n = x_n - F(x'),$$

and functions $\mathbf{V}(z,t) = \mathbf{v}(x(z),t)\zeta(z)$, $P(z,t) = p(x(z),t)\zeta(z)$ where $\zeta(z)$ is a smooth function equal 1 for $|z| \leq d/2$ and to 0 for $|z| \geq 3d/4$, and satisfying the inequality $0 \leq \zeta(z) \leq 1$. If we extend $\mathbf{V}(z,t)$, $P(z,t)$ by zero into domain $|z| \geq 3d/4$, $z_n > 0$, then these functions can be regarded as a solution of the half space problem

$$\begin{aligned}\mathbf{V}_t(z,t) - \Delta\mathbf{V}(z,t) + \nabla P(z,t) &= -2\nabla'\zeta(z)\cdot\nabla'\mathbf{v}(x,t) - \mathbf{v}(x,t)\Delta'\zeta(z)\\ &\quad + p(z,t)\nabla'\zeta(z) - (\Delta - \Delta')\mathbf{V}(z,t)\\ &\quad + (\nabla - \nabla')P(z,t) + \mathbf{f}(z,t)\zeta(z) \equiv \mathbf{g}(z,t),\end{aligned}$$

$$\begin{aligned}\nabla\cdot\mathbf{V}(z,t) &= \mathbf{v}(x,t)\cdot\nabla'\zeta(z) + (\nabla-\nabla')\cdot\mathbf{V}(z,t) = \rho(z,t)\\ \mathbf{V}(z,t)_{|z_n=0} &= 0,\ \mathbf{V}(z,0) = \mathbf{v}_0(z)\zeta(z),\end{aligned}$$

where

$$\begin{aligned}&\nabla' = \nabla - \nabla F\frac{\partial}{\partial z_n} = \left(\frac{\partial}{\partial z_1} - \frac{\partial F}{\partial z_1}\frac{\partial}{\partial z_n}, \dots, \frac{\partial}{\partial z_{n-1}} - \frac{\partial F}{\partial z_{n-1}}\frac{\partial}{\partial z_n}, \frac{\partial}{\partial z_n}\right),\\ &\Delta' = \nabla'\cdot\nabla'.\end{aligned}$$

We are in a position to apply Theorem 1.3. The function $\rho(z,t)$ can be represented in the form $\rho(z,t) = \nabla\cdot\mathbf{R}(z,t)$ where

$$\begin{aligned}&R_i(z,t) = \delta_{in}\sum_{j=1}^{n-1}F_{z_j}(z')V_j(z,t) + \int_{\mathbb{R}^n_+}\frac{\partial}{\partial z_i}N(z,y)\mathbf{v}(y,t)\cdot\nabla'\zeta(y)dy\\ &N(z,y) = -[\mathcal{E}(z-y) + \mathcal{E}(z-y^*)].\end{aligned}$$

Clearly, $R_{n|z_n=0} = 0$, moreover, since $\int_{\mathbb{R}^n_+}\mathbf{v}(y,t)\cdot\nabla'\zeta(y)dy = \int_\Omega \mathbf{v}(x,t)\cdot\nabla\zeta(x)dx = 0$, we can write $R_i(z,t)$ in the form

$$\begin{aligned}R_i(z,t) &= \delta_{in}\sum_{j=1}^{n-1}F_{z_j}(z')V_j(z,t)\\ &\quad + \int_{\mathbb{R}^n_+}\left(\frac{\partial}{\partial z_i}N(z,y) - (1-\zeta(2z))\frac{\partial}{\partial z_i}N(z,0)\right)\mathbf{v}(y,t)\cdot\nabla'\zeta(y)dy.\end{aligned}$$

Taking into account that $\mathbf{f}(x,t) \in J_p(\Omega)$ for almost all t, we obtain

$$R_{it}(z,t) = \delta_{in}\sum_{j=1}^{n-1} F_{z_j}(z')V_{jt}(z,t) + \int_{\mathbb{R}^n_+}\left(\frac{\partial}{\partial z_i}N(z,y) - (1-\zeta(2z))\frac{\partial}{\partial z_i}N(z,0)\right)\mathbf{v}_t(y,t)\cdot\nabla'\zeta(y)dy$$

$$= \delta_{in}\sum_{j=1}^{n-1} F_{z_j}(z')V_{jt}(z,t) + \int_{\mathbb{R}^n_+}\left(\frac{\partial N(z,y)}{\partial z_i} - (1-\zeta(2z))\frac{\partial N(z,0)}{\partial z_i}\right)\big(\nabla'\cdot[\mathbf{f}(y,t)\zeta(y)] - \nabla' p(y,t)\cdot\nabla'\zeta(y) + \Delta'\mathbf{v}(y,t)\cdot\nabla'\zeta(y)\big)dy\,.$$

After integration by parts we can write the integral as the sum

$$-\frac{\partial}{\partial z_i}\left[\int_{\mathbb{R}^n_+}\nabla'_y N(z,y)\cdot(\mathbf{f}(y,t) - p(y,t)\nabla'\zeta(y))dy - \int_{\mathbb{R}^n_+}\sum_{k=1}^{n}(\nabla'_y)_k N(z,y)\nabla'\zeta(y)\cdot\nabla'_k\mathbf{v}(y,t)dy\right]$$

$$+\int_{\mathbb{R}^n_+}\left(\frac{\partial N(z,y)}{\partial z_i} - (1-\zeta(2z))\frac{\partial N(z,0)}{\partial z_i}\right)\big(p(y,t)\Delta'\zeta(y) - \sum_{k=1}^{n}\nabla'_k\nabla'\zeta(y)\cdot\nabla'_k\mathbf{v}(y,t)\big)dy + \int_{\mathbb{R}^{n-1}}\left(\frac{\partial N(z,y)}{\partial z_i} - (1-\zeta(2z))\frac{\partial N(z,0)}{\partial z_i}\right)$$

$$\cdot\left[p(y,t)\big(\nabla'_n\zeta(y) - \sum_{j=1}^{n-1}F_{y_j}(y)\nabla'_j\zeta(y)\big) - \nabla'\zeta(y)\cdot\nabla'_n\mathbf{v}(y,t) + \sum_{j=1}^{n-1}F_{y_j}(y)\nabla'\zeta(y)\cdot\nabla'_j\mathbf{v}(y,t)\right]_{|y_n=0} dy$$

Applying the theorem of Calderon-Zigmund and taking into account that $|\frac{\partial N(z,y)}{\partial z_i} - (1-\zeta(2z))\frac{\partial N(z,0)}{\partial z_i}| = O(|z|^{-n})$ for large $|z|$, we easily obtain

$$|\mathbf{R}_t(t)|_{L_p(\mathbb{R}^n_+)} \le C(|\mathbf{f}(t)|_{L_p(K)} + |p(t)|_{L_p(K)} + |\nabla\mathbf{v}(t)|_{L_p(K)} + |p(t)|_{L_p(K')} + |\nabla\mathbf{v}(t)|_{L_p(K')}) + Cd|\mathbf{v}_t(t)|_{L_p(\mathbb{R}^n_+)}\,,$$

where $K = \{x \in \Omega : |x - y| < d\}$, $K' = K \cap \partial\Omega$. Hence, in virtue of (2.11), for small d

$$\int_0^T \left(|\mathbf{V}_t(t)|^p_{L_p(\mathbb{R}^n_+)} + |D^2\mathbf{V}(t)|^p_{L_p(\mathbb{R}^n_+)} + |\nabla P(t)|^p_{L_p(\mathbb{R}^n_+)}\right)dt$$

$$\leq C\left(\int_0^T |\mathbf{f}_1(t)|^p_{L_p(\mathbb{R}^n_+)}dt + |\mathbf{v}_0\zeta|^p_{2-\frac{2}{p},p}\right)$$

$$+C\int_0^T \left(|\nabla\mathbf{v}(t)|^p_{L_p(K)} + |\mathbf{v}(t)|^p_{L_p(K)} + |\nabla\mathbf{v}(t)|^p_{L_p(K')} + |p(t)|^p_{L_p(K)} + |p(t)|^p_{L_p(K')}\right)dt.$$

This inequality holds for arbitrary $\xi \in \partial\Omega$. Similar estimates (without integrals over K') hold for $\xi \in \Omega$, $\text{dist}(\xi, \partial\Omega) > d_1/2$. Taking an appropriate finite set of the points $\{\xi_k\}$ and summing up the corresponding estimates, we arrive at the inequality

$$\int_0^T \left(|\mathbf{v}_t(t)|^p_{L_p(\Omega'')} + |D^2\mathbf{v}(t)|^p_{L_p(\Omega'')} + |\nabla p(t)|^p_{L_p(\Omega'')}\right)dt$$

$$\leq C\left\{\int_0^T |\mathbf{f}(t)|^p_{L_p(\Omega')}dt + |\mathbf{v}_0|^p_{B_p^{2-\frac{2}{p}}(\Omega')} + N_1\right. \tag{3.1}$$

$$\left. +\int_0^T \left(|\mathbf{v}(t)|^p_{W^1_p(\Omega')} + |p(t)|^p_{L_p(\Omega')} + |\nabla\mathbf{v}(t)|^p_{L_p(\partial\Omega)} + |p(t)|^p_{L_p(\partial\Omega)}\right)dt\right\},$$

where $\Omega'' \subset \Omega' \subset \Omega$, $\text{dist}(\partial\Omega, \Omega' - \Omega'') > 0$. It remains to evaluate the solution in an infinite domain $\Omega - \Omega''$. Let $w(x)$ be a smooth function equal to 1 in $\Omega - \Omega''$, to zero in the neighbourhood of $\partial\Omega$ and in $\mathbb{R}^n - \Omega$. Clearly, $\mathbf{u}(x,t) = \mathbf{v}(x,t)w(x)$ and $s(x,t) = \pi(x,t)w(x)$ can be considered as solutions of the Cauchy problem

$$\mathbf{u}_t(x,t) - \Delta\mathbf{u}(x,t) + \nabla s(x,t) = \mathbf{f}(x,t)w(x) + p(x,t)\nabla w(x)$$
$$- 2\nabla w(x)\cdot\nabla\mathbf{v}(x,t) - \mathbf{v}(x,t)\Delta w(x),$$

$$\nabla\cdot\mathbf{u}(x,t) = \mathbf{v}(x,t)\cdot\nabla w(x) = \rho(x,t), \quad \mathbf{u}(x,0) = \mathbf{v}_0(x)w(x).$$

Since $\rho = \nabla\cdot\mathbf{R}(x,t)$, $\mathbf{R}(x,t) = \int_{\mathbb{R}^n}\nabla\mathcal{E}(x-y)\mathbf{v}(y,t)\cdot\nabla w(y)dy = \int_{\mathbb{R}^n}[\nabla\mathcal{E}(x-y) - \zeta(x-x_0)\nabla\mathcal{E}(x)]\mathbf{v}(y,t)\cdot\nabla w(y)dy$, $x_0 \in \mathbb{R}^n - \Omega$, so that

$$|\mathbf{R}_t(t)|_{L_p(\mathbb{R}^n)} \leq C(|\mathbf{f}(t)w|_{L_p(\mathbb{R}^n)} + |\nabla\mathbf{v}(t)|_{W^1_p(\Omega'')} + |p(t)|_{L_p(\Omega'')},$$

we can apply Lemma 2.2, which gives

$$\int_0^T \left(|\mathbf{v}_t(t)|^p_{L_p(\Omega-\Omega'')} + |D^2\mathbf{v}(t)|^p_{L_p(\Omega-\Omega'')} + |\nabla p(t)|^p_{L_p(\Omega-\Omega'')}\right)dt$$

$$\leq C\left[\int_0^T |\mathbf{f}(t)|^p_p dt + \|\mathbf{v}_0\|^p + \int_0^T \left(|\mathbf{v}(t)|^p_{W^1_p(\Omega')} + |p(t)|^p_{L_p(\Omega')}\right)dt\right]$$

When we add this inequality and (3.1), we obtain

$$\begin{aligned}&\int_0^T \left(|\mathbf{v}_t(t)|_p^p + |D^2\mathbf{v}(t)|_p^p + |\nabla p(t)|_p^p\right) dt \\ (3.2)\qquad &\leq C\Big[\mathcal{N}(T) \\ &+ \int_0^T \left(|\mathbf{v}(t)|^p_{W^1_p(\Omega')} + |p(t)|^p_{L_p(\Omega')} + |\nabla \mathbf{v}(t)|^p_{L_p(\partial\Omega)} + |p(t)|^p_{L_p(\partial\Omega)}\right) dt\Big]\end{aligned}$$

where $\mathcal{N}(T)$ is the sum of the norms of the data in (1.2).

Let us evaluate the norms of $p(x, t)$ which is a solution of the Neumann problem

$$\Delta p(x,t) = 0, \text{ in } \Omega, \ \frac{d}{dn} p(x,t) = \Delta \mathbf{v}(x,t) \cdot \vec{n}_{|\partial\Omega} = \sum_{i,k=1}^n \left(n_i \frac{\partial}{\partial x_k} - n_k \frac{\partial}{\partial x_i}\right) \frac{\partial v_i(x)}{\partial x_k},$$

$$p(x,t) \to 0 \quad (p(x,t) \to \text{const. for } n = 2) \quad \text{for } |x| \to \infty .$$

In virtue of Lemma 2.3

$$(3.3)\qquad |p(t)|_{L_p(\Omega')} \leq C |\nabla \mathbf{v}(t)|_{B_p^\lambda(\partial\Omega)}, \ \lambda \in \left(0, 1 - \frac{1}{p}\right).$$

Now, making use of interpolation inequalities

$$\begin{aligned}|p(t)|_{L_p(\partial\Omega)} &\leq \varepsilon_1 |\nabla p(t)|_p + C(\varepsilon_1) |p(t)|_{L_p(\Omega')} \\ &\leq \varepsilon_1 |\nabla p(t)|_{L_p(\Omega)} + C |\nabla \mathbf{v}(t)|_{B_p^\lambda(\partial\Omega)}, \\ |\nabla \mathbf{v}(t)|_{B_p^\lambda(\partial\Omega)} &\leq \varepsilon_2 |D^2 \mathbf{v}(t)|_p + C(\varepsilon_2) |\mathbf{v}(t)|_{L_p(\Omega')} \\ |\nabla \mathbf{v}(t)|_{L_p(\Omega')} &\leq \varepsilon_3 |D^2 \mathbf{v}(t)|_p + C(\varepsilon_3) |\mathbf{v}(t)|_{L_p(\Omega')},\end{aligned}$$

and taking ε_i small enough, we deduce (1.2) from (3.2), (3.3). Applying Gronwall's lemma we easily deduce (1.3) with $C_2(T) = C'e^{C''T}$ from (1.2).

To prove (1.3), we should estimate the norm $\int_0^T |\mathbf{v}(t)|^q_{L_p(\Omega')} dt$. Clearly, $\mathbf{v}(x,t) = \mathbf{v}_1(x,t) + \mathbf{v}_2(x,t)$ where $\mathbf{v}_1(x,t)$ is the solution of (1.1) with $\mathbf{v}_0(x) = 0$ and $\mathbf{v}_2(x,t)$ is the solution of (1.1) with $\mathbf{f}(x,t) = 0$. $\mathbf{v}_1(x,t)$ is evaluated in Section 6 (see Lemma 6.1 and Remark 6.1)[(1)]. Further, in virtue of (1.4),

$$\begin{aligned}\int_0^T |\mathbf{v}_2(t)|^p_{L_p(\Omega')} dt &\leq |\Omega'|^{r(-\sigma+1/p)} \int_1^T |\mathbf{v}_2(t)|^p_{L_{1/\sigma}(\Omega')} dt + \int_0^1 |\mathbf{v}_2(t)|^p_{L_p(\Omega')} dt \\ &\leq C|\mathbf{v}_0|_p \Big(1 + \int_1^T t^{-p\frac{n}{2}(-\sigma+1/p)} dt\Big) \leq C|\mathbf{v}_0|_p ,\end{aligned}$$

[(1)]Of course, the results of Section 6 are independent of estimate (1.3).

for σ sufficiently small. Combining this estimate with (6.1) we obtain

$$\int_0^T |\mathbf{v}(t)|^p_{L_p(\Omega')} dt \le C(|\mathbf{v}_0|^p_p + \int_0^T |\mathbf{f}(t)|^p_p dt),$$

with the constant C independent of T in the case $p < \frac{n}{2}$ and with $C = O(T^{1+\varepsilon p - \frac{n}{2}})$ ($\forall \varepsilon > 0$) in the case $p \ge \frac{n}{2}$. Estimate (1.3) is proved.

Let us turn to the proof of the solvability of problem (1.1). We consider at first the case $\mathbf{v}_0(x) = 0$, and we assume that $\mathbf{f}(x,t)$ is a smooth vector field with a compact support, so $\mathbf{f}(x,t) \in L_p(Q_T)$ with arbitrary $p > 1$.

It is well known (see [17]) that problem (1.1) has a unique solution $\mathbf{v}(x,t) \in W^{2,1}_2(Q_T)$, $\nabla p(x,t) \in L_2(Q_T)$. Let us show that $\mathbf{v}(x,t) \in W^{2,1}_p(Q_T)$, $\nabla p(x,t) \in L_p(Q_T)$ for arbitrary fixed $p > 1$. We set $\mathbf{v}(x,t) = 0$, $p(x,t) = 0$, $\mathbf{f}(x,t) = 0$ for $t < 0$ and introduce the mollified functions

$$\mathbf{f}_\varepsilon(x,t) = \int_0^\varepsilon \omega_\varepsilon(\tau)\mathbf{f}(x,t-\tau)d\tau,$$
$$\mathbf{v}_\varepsilon(x,t) = \int_0^\varepsilon \omega_\varepsilon(\tau)\mathbf{v}(x,t-\tau)d\tau,$$
$$p_\varepsilon(x,t) = \int_0^\varepsilon \omega_\varepsilon(\tau)p(x,t-\tau)d\tau,$$

where $$\omega_\varepsilon(\tau) = \frac{1}{\varepsilon}\omega\left(\frac{\tau}{\varepsilon}\right), \ \text{supp}\{\omega\} \in (0,1), \ \int_0^1 \omega(\tau) = 1, \ \varepsilon \in (0,1).$$

It is easily seen that $(\mathbf{v}_\varepsilon(x,t), p_\varepsilon(x,t))$ is a solution to the problem

$$\begin{aligned} &\mathbf{v}_{\varepsilon t}(x,t) - \Delta\mathbf{v}_\varepsilon(x,t) + \nabla p_\varepsilon(x,t) = \mathbf{f}_\varepsilon(x,t), \quad \nabla\cdot\mathbf{v}_\varepsilon(x,t) = 0,\\ &\mathbf{v}_\varepsilon(x,0) = 0, \quad \mathbf{v}_\varepsilon(x,t)_{|\partial\Omega} = 0, \ \mathbf{v}_\varepsilon(x,t) \to 0 \text{ for } |x| \to \infty, \end{aligned} \tag{3.4}$$

and that $D_t\mathbf{v}_\varepsilon(x,t) \in C((0,T); W^2_2(\Omega))$. Hence, in virtue of S.L. Sobolev imbedding theorem, $\mathbf{v}_\varepsilon(x,t), D_t\mathbf{v}_\varepsilon(x,t) \in C((0,T); L_{p_1}(Q_T))$ where p_1 satisfies the inequality $\frac{1}{p_1} \ge \frac{1}{2} - \frac{2}{n}$ (if $\frac{1}{q} - \frac{2}{n} \le 0$, p_1 is an arbitrary number greater than 1). We consider $\mathbf{v}_\varepsilon(x,t)$ as a solution of the exterior Stokes problem

$$\begin{aligned} &-\Delta\mathbf{v}_\varepsilon(x,t) + \nabla p_\varepsilon(x,t) = \mathbf{f}_\varepsilon(x,t) - \mathbf{v}_{\varepsilon t}(x,t), \quad \nabla\cdot\mathbf{v}_\varepsilon(x,t) = 0,\\ &\mathbf{v}_\varepsilon(x,t)_{|\partial\Omega} = 0, \quad \mathbf{v}_\varepsilon(x,t) \to 0 \text{ for } |x| \to \infty, \end{aligned}$$

and conclude from Lemma 2.5 that $\mathbf{v}_\varepsilon(x,t) \in W^2_{p_1}(\Omega)$. Now, inequality (1.2) gives a uniform (with respect to ε) estimate for $\mathbf{v}_\varepsilon(x,t)$, so, taking the limit as $\varepsilon \to 0$, we show that $\mathbf{v}(x,t) \in W^{2,1}_{p_1}(Q_T)$. If $\frac{1}{p_1} \le \frac{1}{2} - \frac{2}{n}$, then we may repeat the above argument and show that $\mathbf{v}(x,t) \in W^{2,1}_{p_2}(Q_T)$ with $\frac{1}{p_2} \ge \frac{1}{p_1} - \frac{2}{n}$. After a finite number of steps we arrive at the conclusion that $\mathbf{v}(x,t) \in W^{2,1}_p(Q_T)$. To prove that $\mathbf{v}(x,t) \in W^{2,1}_p(Q_T)$, with $p < 2$, we evaluate it in an infinite domain $\Omega - \Omega'$ introduced above. We consider the Cauchy problem (2.12) with $\mathbf{v}_0(x) = 0$, whose solution is given by the formula

$$
\begin{aligned}
\mathbf{u}(x,t) = \int_0^t \int_{\mathbb{R}^n} &\mathbf{T}(x-y, t-\tau)\cdot(\mathbf{f}(y,t)w(y) + p(y,t)\nabla w(y) \\
&- 2\nabla w(y)\cdot\nabla\mathbf{v}(y,t) - \mathbf{v}(y,t)\Delta w(y))dy\,d\tau \\
&+ \int_{\mathbb{R}^n} \nabla\mathcal{E}(x-y)\mathbf{v}(y,t)\cdot\nabla w(y)dy\,,
\end{aligned}
\tag{3.5}
$$

where $\mathbf{T}(z,s)$ is the Oseen's tensor with the elements

$$
T_{ij}(z,s) = \delta_{ij}\Gamma(z,t) + \frac{\partial^2}{\partial z_i \partial z_j} \int_{\mathbb{R}^n} \mathcal{E}(z-\xi)\Gamma(\xi,s)d\xi\,.
$$

Since $\int_{\mathbb{R}^n} \mathbf{v}(x,t)\cdot\nabla w(x)dx = 0$, it is easily seen that $\mathbf{u}(x,t) = O(|x|^{-n})$ for large $|x|$, so $\mathbf{v}(x,t) \in L_p(Q_T)$ with arbitrary $p > 1$. As it has been already pointed out, this implies that $\mathbf{v}(x,t) \in W_p^{2,1}(Q_T)$. Let us consider now problem (1.1) with $\mathbf{f}(x,t) = 0$ and with $\mathbf{v}_0(x) \in \mathcal{C}_0(\Omega)$. Its solution may be found in the form $\mathbf{v}(x,t) = \mathbf{v}_0(x) + \mathbf{z}(x,t)$. where $\mathbf{z}(x,t)$ is a solution to the problem

$$
\begin{aligned}
&\mathbf{z}_t(x,t) - \Delta\mathbf{z}(x,t) + \nabla p(x,t) = -\Delta\mathbf{v}_0(x), \quad \nabla\cdot\mathbf{z}(x,t) = 0\,, \\
&\mathbf{z}(x,0) = 0, \quad \mathbf{z}(x,t)_{|\partial\Omega} = 0,\ \mathbf{z}(x,t) \to 0 \ \text{ for } \ |x| \to \infty\,.
\end{aligned}
$$

As shown above, this problem is solvable, and $\mathbf{z}(x,t) \in W_s^{2,1}(Q_T)$, $\nabla p(x,t) \in L_s(Q_T)$ with arbitrary $s > 1$, hence, $\mathbf{v}(x,t) \in W_p^{2,1}(Q_T)$.

We have proved the solvability of problem (1.1) for smooth $\mathbf{f}(x,t)$ and $\mathbf{v}_0(x)$ with a compact support, but this class is dence in $L_p(Q_T)$ and in $B_p^{2-2/p}(\Omega)$, therefore the statement of Theorem 1.1 concerning the solvability of problem (1.1) holds in a general case.

The uniqueness of the solution follows from the solvability of the adjoint problem, so Theorem 1.1 is finally established.

We observe that the above arguments prove the following proposition:

LEMMA 3.1. *The solution* $(\mathbf{v}(x,t), \pi(x,t))$ *of problem* (1.1) *with* $\mathbf{f}(x,t) \in C((0,T);\mathcal{C}_0(\Omega))$ *and with* $\mathbf{v}_0(x) \in \mathcal{C}_0(\Omega)$ *is such that*

$$
\mathbf{v}(x,t) \in \bigcap_{p>1} L^p((0,T);\ J^{1,p}(\Omega) \cap W^{2,p}(\Omega))\,,
$$

$$
\nabla p(x,t),\, \mathbf{v}_t(x,t) \in \bigcap_{p>1} L^p((0,T);\ J^p(\Omega))\,.
$$

4. – Some consequences of Theorem 1.1

In this section we present some consequences of Theorem 1.1. Applying estimate (1.2) to $(t^\alpha \mathbf{v}(x,t), t^\alpha \pi(x,t))$ we arrive at the following elementary *a priori* estimate

LEMMA 4.1. *The solution of problem* (1.1) *satisfies the inequality*

$$\begin{aligned}(4.1)\qquad &\int_0^T t^{\alpha p}(|D^2\mathbf{v}(t)|_p^p + |\mathbf{v}_t(t)|_p^p + |\nabla\pi(t)|_p^p)dt \\ &\leq C\left(\int_0^T t^{\alpha p}|\mathbf{F}(t)|_p^p dt + \int_0^T (t^{\alpha p-p} + t^{\alpha p})|\mathbf{v}(t)|_p^p dt\right),\end{aligned}$$

where $\alpha > \frac{1}{p'}$, *and* C *is independent of* T.

COROLLARY 4.1. *If* $\mathbf{f}(x,t) = 0$ *and*

$$|\mathbf{v}(t)|_q \leq Ct^{-\gamma(q,p)}|\mathbf{v}_0|_p, \quad q \in [p,\infty],$$

then

$$\begin{aligned}(4.2)\qquad &\int_0^T t^{\alpha p}(|D^2\mathbf{v}(t)|_p^p + |\mathbf{v}_t(t)|_p^p + |\nabla\pi(t)|_p^p)dt \\ &\leq C|\mathbf{v}_0|_p^p \int_0^T (t^{(\alpha-1-\gamma(p,p))p} + t^{(\alpha-\gamma(q,p))p})dt, \ \forall T > 0,\end{aligned}$$

where $\alpha > \max\left\{\gamma(p,p) + \frac{1}{p'}, \gamma(q,p) - \frac{1}{p}\right\}$ *and* C *is independent of* T.

Next, we obtain inequalities (1.4)-(1.5) in a finite time interval.

THEOREM 4.1. *Let* $\mathbf{v}(x,t)$ *be a solution of system* (1.1) *with* $\mathbf{v}_0(x) \in \mathcal{C}_0(\Omega)$ *and* $\mathbf{F}(x,t) = 0$. *Then for arbitrary* $t \in (0,T)$, $T < \infty$,

$$\begin{aligned}&|\mathbf{v}(t)|_p \leq C(T)|\mathbf{v}_0|_p, \\ (4.3)\qquad &|\nabla\mathbf{v}(t)|_p \leq C(T)t^{-\frac{1}{2}}|\mathbf{v}_0|_p, \\ &|D^2\mathbf{v}(t)|_p \leq C(T)t^{-1}|\mathbf{v}_0|_p,\end{aligned}$$

where $C(T)$ *is a constant independent of the support of* $\mathbf{v}_0(x)$.

PROOF. The proof of $(4.3)_1$ is reproduced from [27]. The functions $\mathbf{u}(x,t) = \int_0^t \mathbf{v}(x,\tau)d\tau$, $q(x,t) = \int_0^t p(x,\tau)d\tau$ solve the problem

$$\begin{aligned}&\mathbf{u}_t(x,t) - \Delta\mathbf{u}(x,t) + \nabla q(x,t) = \mathbf{v}_0(x), \ \nabla\cdot\mathbf{u}(x,t) = 0, \\ &\mathbf{u}_{|\partial\Omega}(x,t) = 0, \quad \mathbf{u}(x,0) = 0,\end{aligned}$$

so, by virtue of (1.3) (we recall that this inequality has been already proved with $C_2(T) = C'e^{C''T}$) we have

$$\int_0^t |\mathbf{v}(\tau)|_p^p d\tau \leq \int_0^t |\mathbf{u}_\tau(\tau)|_p^p d\tau \leq C(T)t|\mathbf{v}_0|_p^p . \tag{4.4}$$

Multiplying (1.1) by the cut-off function $\eta(t)$ vanishing for $\tau \in [0, t/3]$, equal to 1 for $\tau \in [2t/3, t]$ and satisfying the inequalities $0 \leq \eta(\tau) \leq 1$, $|\eta'(\tau)| \leq Ct^{-1}$, we see that $(\mathbf{w}(x,\tau) = \eta(\tau)\mathbf{u}(x,\tau), p(x,\tau) = \eta(\tau)\pi(x,\tau))$ is a solution of the problem

$$\mathbf{w}_\tau(x,\tau) - \Delta\mathbf{w}(x,\tau) + \nabla p(x,\tau) = \eta'(\tau)\mathbf{v}(x,\tau),\ \nabla\cdot\mathbf{w}(x,\tau) = 0,$$
$$\mathbf{w}(x,\tau)_{|\partial\Omega} = 0,\ \mathbf{w}(x,0) = 0 .$$

Hence, in virtue of (1.3) and (4.4)

$$\int_0^t \left(|\mathbf{w}_\tau(\tau)|_p^p + |\mathbf{w}(\tau)|_{2,p}^p\right) d\tau$$
$$\leq C(T)t^{-p} \int_{t/3}^t |\mathbf{v}(\tau)|_p^p d\tau \leq C(T)t^{-p+1}|\mathbf{v}_0|_p^p .$$

Finally

$$|\mathbf{v}(t)|_p = |\mathbf{w}(t)|_p \leq \int_0^t |\mathbf{w}_\tau(\tau)|_p d\tau$$
$$\leq t^{\frac{1}{p'}} \left(\int_0^t |\mathbf{w}_\tau(\tau)|_p^p d\tau\right)^{\frac{1}{p}} \leq C(T)|\mathbf{v}_0|_p ,$$

and $(4.3)_1$ is proved.

The estimate $(4.3)_3$ will be deduced from the inequality

$$|\mathbf{v}_t(t)|_p \leq C(T)t^{-1}|\mathbf{v}_0|_p . \tag{4.5}$$

To prove it, we differentiate (1.1) with respect to t, multiply by $t^{\alpha+1}\varphi(x, t-\tau)$, $\alpha > 0$ and integrate over $\Omega \times (0, T)$. By $\varphi(x,s)$ we mean a solution of (1.1) in the interval $(0,t)$ with $\mathbf{F}(x,t) = 0$ and with initial data $\varphi(x,0) = \varphi_0(x) \in C_0(\Omega)$. After the integration by parts we arrive at

$$t^{\alpha+1}(\mathbf{v}_t(t), \varphi_0) = (\alpha+1)\int_0^t t^\alpha(\mathbf{v}_\tau(\tau), \hat{\varphi}(t-\tau))dt . \tag{4.6}$$

Making use of $(4.3)_1$ and of (4.2) (with $\gamma = 0$) we obtain

$$|(\mathbf{v}_t(t), \varphi_0)| \leq C(T)t^{-\alpha-\frac{1}{p}} \left(\int_0^t \tau^{\alpha p}|\mathbf{v}_\tau(\tau)|_p^p d\tau\right)^{\frac{1}{p}} |\varphi_0|_{p'} \leq C(T)t^{-1}|\mathbf{v}_0|_p|\varphi_0|_{p'} .$$

Since $\varphi_0(x)$ is an arbitrary divergence free vector field from $C_0(\Omega)$, this inequality implies (4.5). Now, in virtue of (2.20)

$$|D^2\mathbf{v}(t)|_p \le C(|\mathbf{v}_t(t)|_p + |\mathbf{v}(t)|_{L^p(\Omega')}),$$

therefore from $(4.3)_1$ and (4.5) we deduce $(4.3)_2$. Estimate $(4.3)_2$ is a consequence of $(4.3)_1$, $(4.3)_3$ and of a well known inequality

$$|\nabla\mathbf{v}(t)|_p \le C(|D^2\mathbf{v}(t)|_p^{1/2}|\mathbf{v}(t)|_p^{1/2} + |\mathbf{v}(t)|_p).$$

The theorem is proved.

As shown in [27], another important consequence of Theorem 1.1 is the estimate for the resolvent of the Stokes operator $A = -P\Delta$.

THEOREM 4.2. *There exist $\rho > 0$, $\varphi > 0$ depending on the constant C of inequality* (1.2), *such that the domain $\Sigma_{\rho,\varphi} = \{\lambda : |\lambda| > \rho,\ \varphi \le |\arg(\lambda)| < \varphi + \pi/2\}$ ($\varphi < \pi/2$) is contained in a resolving set of the operator A and $\forall\,\lambda \in \Sigma_{\rho,\varphi}$*

$$|D^2(\lambda I - A)^{-1}\mathbf{f}|_p + |\lambda||(\lambda I - A)^{-1}\mathbf{f}|_p \le C|\mathbf{f}|_p, \tag{4.7}$$

with C in (4.7) *independent of $\lambda \in \Sigma_{\rho,\varphi}$.*

We complete this theorem with the following proposition

THEOREM 4.3. *If* (1.3) *holds with C_2 independent of T, then the statement of Theorem 4.2 holds true for $\rho = 0$.*

PROOF. We write the initial boundary value problem (1.1) as Cauchy problem in Banach space:

$$\frac{d}{dt}\mathbf{v}(t) + A\mathbf{v}(t) = \mathbf{f}(t),\ \ \mathbf{v}(t)_{|t=0} = \mathbf{v}_0 \in J_p(\Omega).$$

Following the proof of Theorem 4.2 in [27], we observe that if $\mathbf{v}(x) \in W_p^2(\Omega)$ is the solution of equation

$$\lambda\mathbf{v}(x) + A\mathbf{v}(x) = \mathbf{f}(x),$$

then $\mathbf{u}(x,t) = \mathbf{v}(x)e^{\lambda t}(1-\eta(th)) \in W_p^{2,1}(\Omega_T)$ ($h > 0$, $\eta(s)$ is a smooth monotone function equal to 1 for $s < \frac{1}{2}$ and to zero for $s > 1$) is a solution of the Cauchy problem

$$\frac{d\mathbf{u}}{dt}(x,t) + A\mathbf{u}(x,t) = \mathbf{f}(x)e^{\lambda t}(1-\eta(th)) - h\eta'(th)\mathbf{v}(x)e^{-\lambda t} \equiv \mathbf{g}(x,t),$$
$$\mathbf{u}(x,0) = 0,$$

i.e. of problem (4.8) with $\mathbf{v}_0(x) = 0$, $\mathbf{f}(x,t) = \mathbf{g}(x,t)$. The inequality (1.3) for the solution of this problem implies

$$(|\lambda||\mathbf{v}|_p + |A\mathbf{v}|_p) \le C\left(|\mathbf{f}|_p + h|\mathbf{v}|_p\Psi^{1/p}(\lambda,h,T)\Phi^{-1/p}(\lambda,h,T)\right), \tag{4.8}$$

with

$$\Phi(\lambda, h, T) = \int_0^T (1 - \eta(th))^p e^{p\Re\lambda t} dt ,$$

$$\Psi(\lambda, h, T) = \int_0^T |\eta'(th)|^p e^{p\Re\lambda t} dt .$$

We choose the parameters T and $h > 1/T$ in such a way that

$$C\frac{h}{|\lambda|}\Psi^{1/p}(\lambda, h, T)\Phi^{-1/p}(\lambda, h, T) \leq \delta < 1 , \tag{4.9}$$

then (4.8)-(4.9) imply (4.7). In the case $\Re\lambda \neq 0$ we set $h = |\Re\lambda|$. Since

$$\Phi(\lambda, h, T) \geq \int_h^T e^{p\Re\lambda t} dt = (e^{p\Re\lambda T} - e^{p\Re\lambda/h})/p\Re\lambda ,$$

$$\Psi(\lambda, h, T) \leq \max|\eta'(s)| \int_{\frac{1}{2h}}^{1/h} e^{p\Re\lambda t} dt = (e^{p\Re\lambda/h} - e^{p\Re\lambda/2h})/p\Re\lambda \max|\eta'(s)| ,$$

coefficient (4.9) on the left hand side of (4.8) can be made small by choosing a large T (for $\Re\lambda > 0$) or small $\varphi_1 = \tan\frac{|\Re\lambda|}{|\Im\lambda|}$ (for $\Re\lambda < 0$). In the case $\Re\lambda = 0$ we set $h = |\lambda| \equiv |\Im\lambda|$ and we observe that

$$\Phi(\lambda, h, T) \geq T - 1/h, \quad \Psi(\lambda, h, T) \leq \frac{2}{h}\max|\eta'(s)| .$$

So in this case coefficient (4.9) on the left-hand side of (4.8) also can be made small by the choise of a large T. The theorem is proved.

Theorems 4.2 and 4.3 show that the Stokes operator generates an analytic semigroup e^{-tA} and

$$\mathbf{v}(t) = e^{-tA}\mathbf{v}_0 = \frac{1}{2\pi i}\int_\ell e^{t\lambda}(\lambda I + A)^{-1}\mathbf{v}_0\, d\lambda \tag{4.10}$$

where $\ell = \partial\Sigma_{\rho,\varphi}$ or $\ell = \partial\Sigma_{\varepsilon,\varphi}$ with arbitrary small $\varepsilon > 0$, if the hypothesis of Theorem 4.3 holds.

As the third consequence of Theorem 1.1, we present $L_{p,r}$-estimates for the problem (1.1), that is we prove the first part of Theorem 1.4. We restrict ourselves here with the case of zero initial data.

LEMMA 4.2. *For arbitrary* $\mathbf{f}(x, t) \in L_{p,r}(Q_T)$ *problem* (1.1) *with* $\mathbf{v}_0(x) = 0$ *has a unique solution with* $\mathbf{v}(x, t) \in W^{2,1}_{p,r}(Q_T)$ $\nabla\pi(x, t) \in L_{p,r}(Q_T)$ *and the following estimate holds*

$$\int_0^T \Big(|\mathbf{v}_t(t)|_p^r + |\mathbf{v}(t)|^r_{W^2_p(\Omega)} + |\nabla\pi(t)|_p^r\Big) dt \leq C_1(T)\int_0^T |\mathbf{f}(t)|_p^r dt . \tag{4.11}$$

If $n > 2$, $p \in (1, n/2)$, *then there holds the inequality*

$$\int_0^T \Big(|\mathbf{v}_t(t)|_p^r + |D^2\mathbf{v}(t)|_p^r + |\nabla\pi(t)|_p^r\Big) dt \leq C_2\int_0^T |\mathbf{f}(t)|_p^r dt \tag{4.12}$$

with the constant C_2 *independent of* T.

PROOF. We follow here the arguments of V. I. Yudovich [31]. From the formula

$$\mathbf{v}(t) = \int_0^t e^{-(t-\tau)A}\mathbf{f}(\tau)d\tau\,,$$

we obtain

$$(4.13)\qquad A\mathbf{v}(t) = \int_0^t Ae^{-(t-\tau)A}\mathbf{f}(\tau)d\tau \equiv \int_0^t K(t-\tau)\mathbf{f}(\tau)d\tau\,.$$

Here $K(s)$ is an operator valued kernel defined by $K(s) = Ae^{-sA}$ for $s > 0$, $K(s) = 0$ for $s < 0$. With the aid of formula (4.10) for Ae^{-tA} it can be easily verified that the kernel $K(s)$ satisfies the inequality

$$(4.14)\qquad \int_{\tau''}^T ||K(t-\tau) - K(t-\tau')||d\tau \le C(T)\,,$$

where $\tau'' = \max(\tau, \tau') + |\tau - \tau'| < T$ and $||\cdot||$ is the operator norm in $L_p(\Omega)$. Indeed, assuming that $\tau < \tau'$ and making use of (4.10), (4.7) we easily obtain

$$(4.15)\qquad \begin{aligned} ||K(t-\tau) - K(t-\tau')|| &\le \frac{C}{2\pi}\int_\ell |e^{(t-\tau)\lambda} - e^{(t-\tau')\lambda}|ds \\ &\le C(\tau-\tau')\int_{\partial\Sigma_{\rho,\varphi}} e^{-(t-\tau)|\Re\lambda|}|\lambda|ds \le C(T)(\tau-\tau')/(t-\tau)^2 \end{aligned}$$

Moreover, in the case $n > 2$, $p < n/2$ we can make use of (1.3) and of Theorem 4.3 and take $\ell = \partial\Sigma_{\varepsilon,\varphi}$ with arbitrary small $\varepsilon > 0$. After the passage to the limit as $\varepsilon \to 0$ we obtain in this case the same estimate (4.15) with the constant independent of T. We have seen that (4.13) defines a continuous operator from $L_p((0,T); L_p(\Omega))$ into $L_p((0,T); L_p(\Omega))$.

Now, from the "extrapolation theorem" (see [1], and [31] Theorem 1.5 and Corollary 1) we conclude that this operator is continuous from $L_r((0,T); L_p(\Omega))$ to $L_r((0,T); L_p(\Omega))$ for arbitrary $r > 1$. This leads immediately to the estimate

$$(4.16)\qquad \int_0^T\Big(|\mathbf{v}_t(t)|_p^r + |P\Delta\mathbf{v}(t)|_p^r\Big)dt \le C\int_0^T |\mathbf{f}(t)|_p^r dt\,,$$

with the constant C indipendent of T in the case $n > 2, p < \frac{n}{2}$. Making use of inequality (2.20) we obtain (4.12) from (4.16).

Again applying (2.20) and evaluating $\int_0^T |\mathbf{v}(t)|_p^r dt$ by the Gronwall lemma we easily arrive at (4.11). The lemma is proved.

Now we prove (1.12) with zero initial data:

LEMMA 4.3. *Let* $\Omega \subseteq \mathbb{R}^n$ *be* $(n \geq 3)$. *For arbitrary* $\mathbf{f}(x,t) \in L_{p,r}(Q_T)$ *problem* (1.1) *with* $\mathbf{v}_0(x) = 0$ *has a unique solution with* $\mathbf{v}(x,t) \in W^{2,1}_{p,r}(Q_T)$, $\nabla\pi(x,t) \in L_{p,r}(Q_T)$ *and*

$$\int_0^T \left(|\mathbf{v}_t(t)|_p^r + |P\Delta\mathbf{v}(t)|_p^r\right) dt \leq C \int_0^T |\mathbf{f}(t)|_p^r dt\,, \tag{4.17}$$

for any $r, p \in (1,\infty)$, *and* C *is independent of* T.

Moreover, for $p \geq \frac{n}{2}$

$$\int_0^T \left(|\mathbf{v}_t(t)|_p^r + |D^2\mathbf{v}(t)|_p^r + |\nabla\pi(t)|_p^r\right) dt \leq C(T) \int_0^T |\mathbf{f}(t)|_p^r dt\,, \tag{4.18}$$

where $C(T) = 1 + C_1 T^b$ *with* $b = 1 + \eta - \frac{n}{2p}, \forall \eta > 0$.

PROOF. The existence of the solution was proved in Lemma 4.2. To prove inequality (4.17) we can restrict our considerations to the data $\mathbf{f}(x,t) \in C_0^\infty((0,T); \mathcal{C}_0(\Omega))$. In what follows exponents p, p' and r, r' are complementary. Let $\mathbf{g}(x,s) \in L_{p',r'}(Q_T)$ with $\mathbf{g}(x,0) = 0$, $p' \in (1, \frac{n}{2})$ and $r' \in (1,\infty)$, and let $\varphi(x,s)$ be the corresponding solution of system (4.8) with $\varphi(x,0) = 0$. We differentiate (4.8) with respect to t :

$$\mathbf{v}_{tt}(x,t) - P\Delta\mathbf{v}_t(x,t) = \mathbf{f}(x)s_t(x,t)\,. \tag{4.19}$$

Multiplying (4.19) by $\varphi(x, T-\tau), \tau \in (0,T)$, and integrating by parts over $\Omega \times (0,T)$ we get

$$\begin{aligned}\int_0^T (\mathbf{v}_\tau(\tau), \mathbf{g}(T-\tau))d\tau = \int_0^T (\mathbf{f}(\tau), \varphi(T-\tau))d\tau = \\ -\int_0^T (\mathbf{f}(\tau), \frac{\partial}{\partial\tau}\varphi(T-\tau))d\tau\,.\end{aligned} \tag{4.20}$$

Applying the Hölder inequality to the right hand side of (4.20) and taking into account that under our hypotheses on (p', r') inequality (4.12) holds with a constant C independent of T, we obtain

$$\begin{aligned}&\left|\int_0^T (\mathbf{v}_\tau(\tau), \mathbf{g}(T-\tau))d\tau\right| \\ &\leq \left(\int_0^T |\mathbf{f}(\tau)|_p^r d\tau\right)^{\frac{1}{r}} \left(\int_0^T |\varphi_\tau(T-\tau)|_{p'}^{r'} d\tau\right)^{\frac{1}{r'}} \\ &\leq C\left(\int_0^T |\mathbf{f}(\tau)|_p^r d\tau\right)^{\frac{1}{r}} \left(\int_0^T |\mathbf{g}(T-\tau)|_{p'}^{r'} d\tau\right)^{\frac{1}{r'}}\end{aligned} \tag{4.21}$$

Since $\mathbf{g}(x, s)$ in (4.21) is arbitrary, we deduce

$$\int_0^T |\mathbf{v}_\tau(\tau)|_p^r d\tau \le C \int_0^T |\mathbf{f}(\tau)|_p^r d\tau. \tag{4.22}$$

Subsequently from (4.8) we obtain

$$\int_0^T |P\Delta\mathbf{v}(\tau)|_p^r d\tau \le (C^{\frac{1}{r}} + 1)^r \int_0^T |\mathbf{f}(\tau)|_p^r d\tau. \tag{4.23}$$

If $n > 4$, then inequalities (4.22)-(4.23) imply (4.16) for any $p, r \in (1, \infty)$. In the cases $n = 3, 4$, inequalities (4.22)-(4.23) yield (4.16) for $p \in (1, \frac{n}{2}) \cup (\frac{n}{n-2}, \infty)$ and $\forall r \in (1, \infty)$. In this case, to complete the proof it is sufficient to apply the Riesz-Thorin theorem [23]. Now we prove (4.18). From (2.20) and (4.17) it follows

$$\int_0^T \left(|\mathbf{v}_t(t)|_p^r + |D^2\mathbf{v}(t)|_p^r + |\nabla\pi(t)|_p^r\right) dt \le C \int_0^T \left(|\mathbf{f}(t)|_p^r + |\mathbf{v}(t)|_{L_p(\Omega')}^r\right) dt\,,$$

where C is independent of T and $\Omega' \subseteq \Omega$ is bounded. From the result of Remark 6.1, (4.18) follows.

To establish (1.10)-(1.11)-(1.12) with initial data $\mathbf{v}_0(x) \neq 0$ it is sufficient to repeat the same arguments which were already employed in Section 3 in the proof of estimate (1.3). Indeed, the solution $\mathbf{v}(x, t) = \mathbf{v}_1(x, t) + \mathbf{v}_2(x, t)$, where $\mathbf{v}_1(x, t)$ is a solution of (4.8) corresponding to $\mathbf{f}(x, t)$ and $\mathbf{v}_1(x, 0) = 0$, and $\mathbf{v}_2(x, t)$ is a solution of (1.1) with $\mathbf{v}_2(x, 0) = \mathbf{v}_0(x)$, $\mathbf{f}(x, t) = 0$. Now the desidered result is a consequence of the above estimate for $\mathbf{v}_1(x, t)$ and of the estimate given in Theorem 1.2 for $|\mathbf{v}_{2t}(t)|_p$. Of course this last estimate is employed for $t \ge 1$, while for $t \in (0, 1)$ we take into account the local estimate obtained in Theorem 2.1 of the paper [22].

REMARK 4.1. A direct proof of inequalities (4.12), (4.17) based on estimates of imaginary powers of the Stokes operator is given by Y. Giga and H. Sohr [13].

REMARK 4.2. We want to stress that in the further hypothesis $\mathbf{f}(x, t) = \nabla\cdot\mathbf{F}(x, t)$, then (4.12) can be extended to $p \in (1, n)$ (to this end see Remark 6.2) as it follows

$$\int_0^T \left(|\mathbf{v}_t(t)|_p^r + |D^2\mathbf{v}(t)|_p^r + |\nabla\pi(t)|_p^r\right) dt \le C_2 \int_0^T \left(|\mathbf{f}(t)|_p^r + |\mathbf{F}(t)|_p^r\right) dt\,.$$

This result is in accord with ones obtained in [14] where also the case of (4.12) with fractionar power of the operators are considered.

Now we complete the proof of Theorem 1.4 relative to inequality (4.12). We give a proof *ab absurdo*. We assume that (4.12) holds for some $p \ge \frac{n}{2}$ and $r \in (1, \infty)$ and we consider the solution $(\mathbf{v}(x, t), \pi(x, t))$ to nonstationary Stokes system corresponding to $\mathbf{F}(x) \in L^r((0, T); \mathcal{C}_0(\Omega))$, $\forall T > 0$, and $\mathbf{v}_0(x) = 0$.

Since $\mathbf{F}(x) \in C_0(\Omega)$, we have $\mathbf{v}(x,t) \in W_p^2(\Omega), \mathbf{v}_t(x,t) \in J_p(\Omega), \forall p > 1$. Moreover, as $\mathbf{F}(x)$ is independent of t, $\mathbf{v}_t(x,t)$ is a solution to the system

$$\mathbf{v}_{tt}(x,t) - \Delta \mathbf{v}_t(x,t) = \nabla \pi_t(x,t), \quad \nabla \cdot \mathbf{v}_t(x,t) = 0, \quad \mathbf{v}_t(x,t)_{|\partial\Omega} = 0. \tag{4.24}$$

Regarding (4.24) as an initial boundary value problem, we have in particular from Theorem 1.2 and Corollary 1.1

$$|\mathbf{v}_t(t)|_s \leq C|\mathbf{v}_t(\bar{t})|_q (t-\bar{t})^{\frac{n}{2}\left(\frac{1}{q}-\frac{1}{s}\right)} \tag{4.25}$$

and

$$\lim_{t\to\infty} |\mathbf{v}_t(t)|_p = 0. \tag{4.26}$$

For any sequence $\{t_n\}_{n\in N}$ such that $t_n \to \infty$, $\{\mathbf{v}(x,t_n)\}_{n\in N}$ is a Cauchy sequence in $J_{s_1}(\Omega)$ for any $s_1 > \frac{n}{n-2}$. Indeed, from (4.25) there follows the estimate

$$\begin{aligned} |\mathbf{v}(t_n) - \mathbf{v}(t_m)|_{s_1} &\leq \int_{t_m}^{t_n} |\mathbf{v}_\tau(\tau)|_{s_1} d\tau \\ &\leq C|\mathbf{v}_t(\bar{t})|_{1+\varepsilon} \int_{t_m}^{t_n} \tau^{-\frac{n}{2}\left(\frac{1}{1+\varepsilon}-\frac{1}{s_1}\right)} d\tau \leq C t_m^{1-\frac{n}{2}\left(\frac{1}{1+\varepsilon}-\frac{1}{s_1}\right)}, \end{aligned}$$

with $\varepsilon > 0$ and $s_1 > \frac{n}{\frac{n}{1+\varepsilon}-2}$. Therefore $\{\mathbf{v}(x,t_n)\}_{n\in N}$ is a Cauchy sequence in $J_{s_1}(\Omega)$. Now, also $\{D^2\mathbf{v}(x,t_n), \nabla\pi(x,t_n)\}_{n\in N}$ are Cauchy sequences in $L_p(\Omega)$, because in virtue of inequality (2.20) we have

$$\begin{aligned} &|D^2(\mathbf{v}(t_n) - \mathbf{v}(t_m))|_p + |\nabla(\pi(t_n) - \nabla\pi(t_m)|_p \\ &\quad \leq C|\mathbf{v}_t(t_n) - \mathbf{v}_t(t_m)|_p + |\mathbf{v}(t_n) - \mathbf{v}(t_m)|_{L_p(\Omega')}. \end{aligned} \tag{4.27}$$

The property (4.26) and the strong convergence of $\{\mathbf{v}(x,t_n)\}_{n\in N}$ ensure via (4.27) that $\{D^2\mathbf{v}(x,t_n), \nabla\pi(x,t_n)\}_{n\in N}$ admit limit $(D^2\mathbf{v}(x), \nabla\pi(x))$ in $L_p(\Omega)$. It is not difficult to verify that $(\mathbf{v}(x), \pi(x))$ is a solution to the Stokes problem

$$\begin{aligned} &\Delta\mathbf{v}(x) + \nabla\pi(x) = \mathbf{F}(x), \quad \nabla\cdot\mathbf{v}(x) = 0 \\ &\mathbf{v}(x)_{|\partial\Omega} = 0, \quad \mathbf{v}(x) \to 0 \text{ for } |x| \to \infty. \end{aligned} \tag{4.28}$$

From (4.12) we obtain in particular

$$\int_{\frac{t}{2}}^{t} \left(|D^2\mathbf{v}(\tau)|_p^r + |\nabla\pi(\tau)|_p^r\right) d\tau \leq Ct|\mathbf{F}|_p^r.$$

Hence, in virtue of the mean value theorem,

$$|D^2\mathbf{v}(\bar{t})|_p + |\nabla\pi(\bar{t})|_p \leq 2C|\mathbf{F}|_p, \quad \text{with } \bar{t} \in \left[\frac{t}{2}, t\right], \forall t > 0.$$

Making $t \to \infty$ we get

$$|D^2\mathbf{v}|_p + |\nabla\pi|_p \le C|\mathbf{F}|_p\,. \tag{4.29}$$

Consider $\Omega_1 \subset \Omega$, $\operatorname{dist}(\Omega - \Omega_1, \partial\Omega) > 0$. In virtue of Lemma 2.7 and (4.29) we have

$$|\mathbf{v}|_{W^1_p(\Omega_1)} + |D^2\mathbf{v}|_p + |\nabla\pi|_p \le C|\mathbf{F}|_p\,, \tag{4.30}$$

with C independent of $\mathbf{F}(x)$. On the other hand, in [21] it has been proved that inequality (4.30) is not true for solution to system (4.28). Therefore we have obtained an *absurdum*. The proof of Theorem 1.4 is completed.

5. – Estimates of the resolving operator

In this section we proceed to the proof of Theorem 1.3, in particular, we obtain estimates (1.4) for $1 < p \le q < \infty$. We start with the following auxiliary proposition

LEMMA 5.1. *For the solution of the Cauchy problem*

$$\frac{\partial\mathbf{V}}{\partial t}(x,t) - \Delta\mathbf{V}(x,t) = 0, \ \ on\ \mathbb{R}^n \times (0,T), \qquad \mathbf{V}(x,0) = \mathbf{v}_0(x)\,, \tag{5.1}$$

with $\mathbf{v}_0(x) \in C_0^\infty(\mathbb{R}^n)$*, the following estimates hold:*

$$\begin{aligned} |\mathbf{V}(t)|_q &\le C_1|\mathbf{v}_0|_p t^{-\mu}, \forall t > 0,\\ |\nabla\mathbf{V}(t)|_q &\le C_2|\mathbf{v}_0|_p t^{-\mu-\frac{1}{2}}, \forall t > 0,\\ |\mathbf{V}_t(t)|_q &\le C_3|\mathbf{v}_0|_p t^{-\mu-1}, \forall t > 0,\\ |\nabla\mathbf{V}_t(t)|_q &\le C_4|\mathbf{v}_0|_p t^{-\mu-\frac{3}{2}}, \forall t > 0\,. \end{aligned} \tag{5.2}$$

Here $\infty \ge q \ge p \ge 1$, $\mu = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$. *The constants* C_i *are independent of* t, *and they are also uniformly boumded for all* p, q *satisfying the above conditions.*

PROOF. All the estimates follow from the formula

$$\mathbf{V}(x,t) = \int_{\mathbb{R}^n} \Gamma(x-y,t)\mathbf{v}_0(y)dy$$

and from the Young inequality for the convolution. For instance,

$$|\mathbf{V}(t)|_q \le |\Gamma(t)|_r|\mathbf{v}_0|_p \le C_1 t^{-\frac{n}{2}\left(1-\frac{1}{r}\right)}|\mathbf{v}_0|_p\,,$$

where $\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}$ and $C_1 = |\Gamma(1)|_r = (4\pi)^{-\frac{n}{2r'}} r^{-\frac{n}{2r}}$. As $r \in [1,\infty]$, C_1 remains uniformly bounded. Other inequalities are proved in the same way. Of course the constants C_i have different values, that is $C_2 = |\nabla\Gamma(1)|_r$, $C_3 = |D^2\Gamma(1)|_r + |\Gamma_t(1)|_r$, $C_4 = |\nabla\Gamma_t(1)|_r$. The lemma is proved.

In the following lemma auxiliary L_2-estimates of $\mathbf{v}(x,t)$ are obtained

LEMMA 5.2. *For the solution of problem* (1.1) *with* $\mathbf{f}(x,t) = 0$ *the following estimates hold*

$$
\begin{aligned}
&|D_t^j \mathbf{v}(t)| \le C_1 |\mathbf{v}_0| t^{-j}, \ \forall t > 0, \\
&|\nabla D_t^j \mathbf{v}(t)| \le C_2 |\mathbf{v}_0| t^{-j-\frac{1}{2}}, \ \forall t > 0 .
\end{aligned} \tag{5.3}
$$

Here $|\cdot| = |\cdot|_2$ *and the constants* C_1, C_2 *are independent of* t.

PROOF. We observe that the derivatives $D_t^j \mathbf{v}(x,t)$, $\nabla D_t^j \mathbf{v}(x,t)$ exist for $t > 0$, and they satisfy the relations

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|D_t^j \mathbf{v}(t)|^2 + |\nabla D_t^j \mathbf{v}(t)|^2 = 0, \\
&\frac{1}{2}\frac{d}{dt}|\nabla D_t^j \mathbf{v}(t)|^2 + |D_t^{j+1} \mathbf{v}(t)|^2 = 0 ,
\end{aligned} \tag{5.4}
$$

which imply

$$|D_t^{j+1}\mathbf{v}(t)|^2 \le |\nabla D_t^{j+1}\mathbf{v}(t)||\nabla D_t^j \mathbf{v}(t)| \le |\nabla D_t^{j+1}\mathbf{v}(t)||D_t^{j+1}\mathbf{v}(t)|^{1/2}|D_t^j \mathbf{v}(t)|^{1/2} .$$

If $(5.3)_1$ holds, then

$$|D_t^{j+1}\mathbf{v}(t)|^3 \le C|\nabla D_t^{j+1}\mathbf{v}(t)|^2 |\mathbf{v}_0| t^{-j} .$$

Substuting this inequality into $(5.4)_1$ we obtain

$$\frac{d}{dt}|D_t^{j+1}\mathbf{v}(t)|^2 + C|\mathbf{v}_0|^{-1} t^j |D_t^{j+1}\mathbf{v}(t)|^3 \le 0$$

or

$$-\frac{d}{dt}\frac{1}{|D_t^{j+1}\mathbf{v}(t)|} + \frac{1}{C}\frac{t^j}{|\mathbf{v}_0|} \le 0 .$$

This yields

$$-\frac{1}{|D_t^{j+1}\mathbf{v}(t)|} + \frac{1}{C}\frac{t^{j+1}}{|\mathbf{v}_0|} \le 0$$

which is equivalent to $(5.3)_1$ with j replaced by $j+1$. Since for $j = 0$ $(5.3)_1$ obviously holds with $C = 1$, it is established for arbitrary j. Now, $(5.4)_1$ yields

$$|\nabla D_t^j \mathbf{v}(t)|^2 \le |D_t^{j+1}\mathbf{v}(t)||D_t^j \mathbf{v}(t)| ,$$

which proves $(5.3)_2$.

LEMMA 5.3. *Let* $\mathbf{v}(x,t)$ *be a solution of* (1.1) *with* $\mathbf{f}(x,t) = 0$. *Then for arbitrary* $t > 1$

$$|\mathbf{v}_t(t)| + |\nabla \mathbf{v}(t)| + |D^2\mathbf{v}(t)| \le C_1 |\mathbf{v}_0|_r \left(t - \frac{1}{2}\right)^{-\gamma_1(r)-\frac{1}{2}}, \ \textit{if}\ r \in \left[\frac{2n}{(n+2)}, 2\right];$$

$$|\mathbf{v}_t(t)| + |\nabla \mathbf{v}(t)| + |D^2\mathbf{v}(t)| \le C_2 |\mathbf{v}_0|_r \left(t - \frac{1}{2}\right)^{-1}, \qquad \textit{if}\ r \in \left(1, \frac{2n}{(n+2)}\right],$$

where $\gamma_1(r) = \frac{n}{2}(\frac{1}{r} - \frac{1}{2})$. *and* C_i *are independent of* t.

PROOF. In virtue of (2.23) and of the Hölder inequality,

$$|(\mathbf{v}(t), \varphi_0)| \le |\mathbf{v}_0|_r |\varphi(t)|_{r'}, \ r' = \frac{r}{(r-1)}.$$

Let $n > 2$. If $r \in [\frac{2n}{n+2}, 2]$, then $r' \in [2, \frac{2n}{n-2}]$, and $|\varphi(t)|_{r'}$ can be evaluated by the well known multiplicative inequality

$$|\varphi(t)|_{r'} \le C |\nabla \varphi(t)|^a |\varphi(t)|^{1-a}, \ a = \frac{n}{2}\frac{2-r}{r} \in (0,1).$$

According to $(5.3)_2$,

$$|\varphi(t)|_{r'} \le C|\varphi_0| t^{-\frac{a}{2}} = C|\varphi_0| t^{-\gamma_1}.$$

In the case $r \in (1, \frac{2n}{n+2}]$ we apply another multiplicative inequality:

$$\begin{aligned} |\varphi(t)|_{r'} &\le C\left(|D^m\varphi(t)|^b |\varphi(t)|^{1-b}_{\frac{2n}{(n-2)}} + |\varphi(t)|_{\frac{2n}{(n-2)}}\right) \\ &\le C\left(|D^{\frac{m}{2}}\varphi(t)| + |\nabla\varphi(t)|\right). \end{aligned}$$

Now, we consider $\varphi(x,t)$ as a solution of exterior stationary problem (2.15) with $\mathbf{f}(x) = \varphi_t(x,t)$ and apply inequality (2.20) (it is possible since $\partial\Omega \in C^m$). This leads to

$$|\varphi(t)|_{r'} \le C\left(|D_t^{\frac{m}{2}}\varphi(t)| + \sum_{j=0}^{m-1} |\nabla D_t^j \varphi(t)|\right) \le C|\varphi_0| t^{-\frac{1}{2}}.$$

Thus, we have proved that in the case $n > 2$

$$|(\mathbf{v}(t), \varphi_0)| \le C|\mathbf{v}_0|_r |\varphi_0| t^{-\beta}, \beta = \begin{cases} \gamma_1, & \text{if} \quad r \in \left[\frac{2n}{(n+2)}, 2\right] \\ \frac{1}{2}, & \text{if} \quad r \in \left(1, \frac{2n}{n+2}\right]. \end{cases}$$

Since $\varphi_0(x) \in \mathcal{C}_0(\Omega)$ is arbitrary, we conclude that in this case

$$|\mathbf{v}(t)| \le Ct^{-\beta}|\varphi_0|_r\,.$$

For $n = 2$ this inequality is proved in [20], Lemma 2.3, with $\beta = \frac{2-r}{2r}$, $r \in (1, 2]$. Now, in virtue of the semigroup property of e^{-tA} and $(5.3)_1$, we have

$$|\mathbf{v}_t(t)| \le Ct^{-1}|\mathbf{v}\left(\frac{t}{2}\right)| \le C|\mathbf{v}_0|_r t^{-1-\beta}$$

and

$$|\nabla\mathbf{v}(t)|^2 \le |\mathbf{v}(t)||\mathbf{v}_t(t)| \le C|\mathbf{v}_0|_r t^{-1-2\beta}\,.$$

Moreover, from (2.20), applying the Poincarè inequality, we deduce

$$|D^2\mathbf{v}(t)| \le C\left\{|\mathbf{v}_t(t)| + |\nabla\mathbf{v}(t)|\right\} \le C|\mathbf{v}_0|_r t^{-\frac{1}{2}-\beta}\,.$$

The lemma is proved.

Let us proceed to the proof of inequality (1.4) for arbitrary $t > 0$. We consider at first the case $q = r$. The solution of problem (1.1) with $\mathbf{v}_0(x) \in \mathcal{C}_0(\Omega)$ can be represented in the form $\mathbf{v}(x,t) = \mathbf{V}(x,t) + \mathbf{u}(x,t)$ where $\mathbf{V}(x,t)$ is a solution of (5.1). Then

$$\begin{aligned}
&\mathbf{u}_t(x,t) - \Delta\mathbf{u}(x,t) = \nabla\pi(x,t), \text{ in } \Omega\times(0,T),\\
&\nabla\cdot\mathbf{u}(x,t) = 0, \text{ in } \Omega\times\Omega,\\
&\mathbf{u}(x,t)_{|\partial\Omega} = -\mathbf{V}(x,t)_{|\partial\Omega},\ \mathbf{u}(x,t)\to 0 \text{ for } |x|\to\infty,\\
&\mathbf{u}(x,0) = 0\,.
\end{aligned}$$

Let $(\varphi(x,t), p(x,t))$ be a solution of the same problem with initial condition $\varphi_0(x) \in \mathcal{C}_0(\Omega)$. In virtue of (2.24),

$$\begin{aligned}
(\mathbf{u}(t), \varphi_0) = &\int_0^t\int_{\partial\Omega}\mathbf{V}(x,\tau)\vec{n}\cdot\nabla\varphi(x,t-\tau)d\sigma d\tau\\
&+\int_0^t\int_{\partial\Omega}\mathbf{V}(x,\tau)\cdot\vec{n}\,p(x,t-\tau)d\sigma d\tau\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\mathbf{V}(t),\varphi_0) - (\mathbf{V}(t-1),\varphi(1)) = &-\int_{t-1}^t\int_{\partial\Omega}\mathbf{V}(x,\tau)\vec{n}\cdot\nabla\varphi(x,t-\tau)d\sigma d\tau\\
&-\int_{t-1}^t\int_{\partial\Omega}\mathbf{V}(x,\tau)\cdot\vec{n}\,p(x,t-\tau)d\sigma d\tau\,,
\end{aligned}$$

hence,

$$\begin{aligned}(\mathbf{u}(t),\varphi_0) &= \int_0^{t-1}\int_{\partial\Omega}\mathbf{V}(x,\tau)\vec{n}\cdot\nabla\varphi(x,t-\tau)d\sigma d\tau \\ &+\int_0^{t-1}\int_{\partial\Omega}\mathbf{V}(x,\tau)\cdot\vec{n}\,p(x,t-\tau)d\sigma d\tau \\ &+(\mathbf{V}(t-1),\varphi(1))-(\mathbf{V}(t),\varphi_0)=\sum_{i=1}^{4}I_i(t)\,.\end{aligned}$$

We evaluate

$$\begin{aligned}|I_1(t)+I_2(t)| = \Big|&\int_0^{t-1}\int_{\partial\Omega}\mathbf{V}(x,\tau)\cdot\vec{n}\cdot\nabla\varphi(x,t-\tau)d\sigma d\tau \\ &+\int_0^{t-1}\int_{\partial\Omega}\mathbf{V}(x,\tau)\cdot\vec{n}\,(p(x,t-\tau)-m(t-\tau))\,d\sigma d\tau\Big|\end{aligned}$$

where $m(t-\tau)=|\Omega'|^{-1}\int_{\Omega'}p(x,t-\tau)dx$ for some bounded $\Omega'\subset\Omega$ such that $\mathrm{dist}(\Omega-\Omega',\partial\Omega)>0$. In virtue of the Hölder inequality, we have for $p>2$

$$\begin{aligned}&|I_1(t)+I_2(t)|\le|\partial\Omega|^{\frac{2-p}{2p}} \\ &\int_0^{t-1}\Big(|\nabla\varphi(t-\tau)|_{L^2(\partial\Omega)}+|p(t-\tau)-m(t-\tau)|_{L^2(\partial\Omega)}\Big)\,|\mathbf{V}(\tau)|_{L^{p'}(\partial\Omega)}d\tau\,.\end{aligned}$$

Making use of multiplicative estimate

$$|\mathbf{u}|_{L_2(\partial\Omega)}\le C\left(|\nabla\mathbf{u}|^{1/2}_{L_2(\Omega')}|\mathbf{u}|^{1/2}_{L_2(\Omega')}+|\mathbf{u}|_{L_2(\Omega')}\right),$$

of the Poincarè inequality for $p(x,t-\tau)-m(t-\tau)$, and of (5.5), we evaluate the sum, of the norms of $\nabla\varphi(x,t-\tau)$ and $p(x,t-\tau)-m(t-\tau)$ by

$$\begin{aligned}&C\left(|D^2\varphi\;(t-\tau)|_2^{1/2}|\nabla\varphi(t-\tau)|_2^{1/2}+|\nabla\varphi(t-\tau)|_2+|\nabla p(t-\tau)|_2\right) \\ &\quad\le C\left(|D^2\varphi(t-\tau)|_2^{1/2}|\nabla\varphi(t-\tau)|_2^{1/2}+|\nabla\varphi(t-\tau)|_2+|\varphi_\tau(t-\tau)+\Delta\varphi(t-\tau)|_2\right) \\ &\quad\le C(t-\tau)^{-\beta-\frac{1}{2}}|\varphi_0|_{p'}\,.\end{aligned}$$

Hence,

$$|I_1(t)+I_2(t)|\le C|\mathbf{v}_0|_p\int_0^{t-1}(t-\tau)^{-\frac{1}{2}-\beta}|\mathbf{V}(\tau)|_{L^{p'}(\partial\Omega)},$$

$$\text{with }\beta=\begin{cases}\dfrac{n}{2}\left(\dfrac{1}{p}-\dfrac{1}{2}\right), & \text{if } p\in\left[\dfrac{2n}{n+2},2\right],\\[2mm] \dfrac{1}{2}, & \text{if } p\in\left(1,\dfrac{2n}{n+2}\right].\end{cases}\tag{5.6}$$

The norm of $\mathbf{V}(x,\tau)$ can be estimated as follows:

$$|\mathbf{V}(\tau)|_{L^{p'}(\partial\Omega)} \le \begin{cases} |\partial\Omega|^{\frac{p-1}{p}}|\mathbf{V}(\tau)|_\infty \le C|\mathbf{v}_0|_{p'}\tau^{-n\frac{p-1}{2p}}, & \forall \tau \ge 1, \\ C\left(|\mathbf{V}(\tau)|_{p'}+|\mathbf{V}(\tau)|^{\frac{1}{p'}}_{p'}|\nabla\mathbf{V}(\tau)|^{\frac{1}{p'}}_{p'}\right)\le C|\mathbf{v}_0|_{p'}\tau^{-\frac{1}{2p'}}, & \forall\tau\in(0,1]. \end{cases}$$

Substituting this into (5.6), we see that

$$|I_1+I_2| \le C|\varphi_0|_{p'}|\mathbf{v}_0|_p\left(\int_0^1 \tau^{-\frac{1}{2p}}d\tau+\int_1^{t-1}(t-\tau)^{-\beta-\frac{1}{2}}\tau^{-\frac{n}{2p}}d\tau\right)\le C|\varphi_0|_{p'}|\mathbf{v}_0|_p$$

since $-\beta-\frac{1}{2}-\frac{n}{2p}=-\frac{n}{4}-\frac{1}{2}\le -1$. It is also clear that

$$|I_3+I_4|\le C|\mathbf{v}_0|_p|\varphi_0|_{p'}$$

in virtue of estimates $(5.2)_1$ and $(4.3)_1$. Hence, we have proved the inequality

$$|(\mathbf{u}(t),\varphi_0)|\le C|\mathbf{v}_0|_p|\varphi_0|_{p'}$$

which implies

$$|(\mathbf{v}(t),\varphi_0)|\le C|(\mathbf{V}(t),\varphi_0)|+|(\mathbf{u}(t),\varphi_0)|\le C|\mathbf{v}_0|_p|\varphi_0|_{p'}$$

and

$$|\mathbf{v}(t)|_p\le C|\mathbf{v}_0|_p$$

for $p\ge 2$. In the case $p<2$, we use the duality arguments. From equation (2.23) (with $\mathbf{f}(x,t)=0$) we obtain

$$|(\mathbf{v}(t),\varphi_0)|\le C|\mathbf{v}_0|_p|\varphi(t)|_{p'}\le C|\mathbf{v}_0|_p|\varphi_0|_{p'},$$

since $p'>2$. This implies $|\mathbf{v}(t)|_p\le C|\mathbf{v}_0|_p$. Inequality (1.4) in the case of $q=p$ is completely proved.

Let us turn to the case $q>p$. We assume first that $q=2$. Then

$$|\mathbf{v}(t)|_2\le C|\nabla\mathbf{v}(t)|_2^a|\mathbf{v}(t)|_p^{1-a},\ a=\frac{n(2-p)}{n(2-p)+2p}.$$

Substituting this estimate into

$$\frac{d}{dt}|\mathbf{v}(t)|^2+2|\nabla\mathbf{v}(t)|^2=0,$$

we obtain

$$\frac{d}{dt}|\mathbf{v}(t)|+C|\mathbf{v}_0|_p^{-2\frac{1-a}{a}}|\mathbf{v}(t)|^{-1+\frac{2}{a}}\le 0,$$

or

$$-\frac{d}{dt}|\mathbf{v}(t)|^{2-\frac{2}{a}} + C|\mathbf{v}_0|_p^{-2\frac{1-a}{a}} \leq .$$

Hence, $-|\mathbf{v}(t)|^{2-\frac{2}{a}} + C|\mathbf{v}_0|_p^{-2\frac{1-a}{a}} t \leq 0$, which is equivalent to (1.4) with $q = 2$. In the case $q > 2$, $p \in (1, 2]$ we apply the relation (2.23) written for the interval $(\frac{t}{2}, t)$, i.e.

$$(\mathbf{v}(t), \varphi_0) = \left(\mathbf{v}\left(\frac{t}{2}\right), \varphi\left(\frac{t}{2}\right)\right).$$

As $p, q' < 2$, we have

$$|(\mathbf{v}(t), \varphi_0)| \leq |\mathbf{v}\left(\frac{t}{2}\right)||\varphi\left(\frac{t}{2}\right)| \leq |\mathbf{v}_0|_p|\varphi_0|_{q'}t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)},$$

which implies

$$|\mathbf{v}(t)|_q \leq Ct^{-\mu}|\mathbf{v}_0|_p, \ q \geq 2, \ p \in (1, 2]. \tag{5.7}$$

Let us consider other cases: $q > p > 2$ and $2 > q > p$. We again make use of equation (2.23), which yields

$$|(\mathbf{v}(t), \varphi_0)| \leq |\mathbf{v}_0|_p|\varphi(t)|_{p'} \tag{5.8}$$

Assume first that $p > 2$. Then $q' < p' < 2$, and

$$|\varphi(t)|_{p'} \leq |\varphi(t)|^b|\varphi(t)|_{q'}^{1-b}, \ b = \left(\frac{1}{p} - \frac{1}{q}\right) / \left(\frac{1}{2} - \frac{1}{q}\right).$$

Thus, according to (5.7) and (1.4) for $q = q' = p$, the right hand side of (5.8) does not exceed $Ct^{-\frac{nb}{2}\left(\frac{1}{q'}-\frac{1}{2}\right)}|\mathbf{v}_0|_p|\varphi_0|_{q'}$, so

$$|(\mathbf{v}(t), \varphi_0)| \leq Ct^{-\mu}|\mathbf{v}_0|_p|\varphi_0|_{q'}, \tag{5.9}$$

which yields (1.4) in the case $q > p > 2$. It remains to consider the case $2 > q > p$. Then $p' > q' > 2$, and, as it has been just proved,

$$|\varphi(t)|_{p'} \leq Ct^{-\mu}|\varphi_0|_{q'}.$$

Hence, (5.9) holds also for $p < q < 2$, and the proof of (1.4) is completed.

6. – Proof of Theorem 1.3 and Corollary 1.1

In the proof of Theorem 1.3 a foundamental role is played by estimate (1.4). We make use of the formula (2.23), i.e.

$$(\mathbf{v}(t), \varphi_0) = \int_0^t (\mathbf{f}(x)s(\tau), \varphi(t-\tau))d\tau$$

where $\varphi(x,s)$ is a solution of problem (1.1) with $\mathbf{f}(x,t) = 0$ and with initial data $\varphi_0(x) \in \mathcal{C}_0(\Omega)$. In virtue of (1.4),

$$|(\mathbf{v}(t), \varphi_0)| \leq \int_0^t |\mathbf{f}(\tau)|_p |\varphi(t-\tau)|_{p'} d\tau \leq C|\varphi_0|_{q'} \int_0^t |\mathbf{f}(\tau)|_p (t-\tau)^{\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} d\tau ,$$

which implies

$$|\mathbf{v}(t)|_q \leq C \int_0^t |\mathbf{f}(\tau)|_p (t-\tau)^{\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} d\tau .$$

Now (1.9) follows from the Hardy-Littlewood or the Young inequality. Let us turn to latter part of Theorem 3.1 (that is inequality (1.9) on compact subdomain of Ω). Let us introduce the vector field $\psi(x) \in L_{q'}(\Omega)$ with $\text{supp}\{\psi(x)\} \subset \Omega \cap S_R$ and with $|\psi|_{q'} = 1$ such that

$$(\mathbf{v}(t), \psi) = |\mathbf{v}(t)|_{L_q(\Omega\cap S_R)} .$$

Further, we define the projection of ψ onto $J_{q'}(\Omega)$ which is denoted by $\varphi_0(x)$ (since $\psi(x)$ has a compact support, $\varphi_0(x) \in J_\sigma(\Omega)$ with arbitrary $\sigma \in (1, q')$) and we construct the sequence $\{\varphi_{0m}(x)\} \in \mathcal{C}_0(\Omega)$ of functions approximating $\varphi_0(x)$ in $J_{q'}(\Omega)$ and in $J_{\frac{1}{1-\varepsilon}}(\Omega')$ with a certain small $\varepsilon > 0$. The identity (2.23), i.e.

$$(6.1) \qquad (\mathbf{v}(t), \varphi_{0m}) = \int_0^t (\mathbf{f}(\tau), \varphi_m(t-\tau))d\tau ,$$

where $\varphi_m(x,t)$ is the solution of (1.1) with $\mathbf{f}(x,t) = 0$ and with initial data $\varphi_m(x,0) = \varphi_0(x)$, implies

$$|(\mathbf{v}(t), \varphi_{0m})| \leq \int_0^t |\mathbf{f}(\tau)|_p |\varphi_m(t-\tau)|_{p'} d\tau .$$

Let us assume that $t > 1$. Making use of inequality (1.4) with different exponents in the intervals $\tau \in (0, t-1)$ and $\tau \in (t-1, t)$, we obtain

$$\begin{aligned}|(\mathbf{v}(t), \varphi_{0m})| \leq\ & C|\varphi_{0m}|_{\frac{1}{1-\varepsilon}} \int_0^{t-1} |\mathbf{f}(\tau)|_p (t-\tau)^{-\frac{n}{2}\left(\frac{1}{p}-\varepsilon\right)} d\tau \\ & + C|\varphi_{0m}|_{q'} \int_{t-1}^t |\mathbf{f}(\tau)|_p (t-\tau)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} d\tau .\end{aligned}$$

Now, we pass to the limit as $m \to \infty$ and we take account of (1.4) and of the inequalities

$$|\varphi_0|_{q'} \le C|\psi|_{q'} = C, \; |\varphi_{0m}|_{\frac{1}{1-\varepsilon}} \le C'|\psi|_{\frac{1}{1-\varepsilon}} \le C'|\psi|_{q'} \equiv C''.$$

Since $(\mathbf{v}(t), \varphi_0) = (\mathbf{v}(t), \psi)$, this leads to

$$\begin{aligned} &|\mathbf{v}(t)|_{L_q(\Omega\cap S_R)} \\ (6.2)\qquad &\le C\left(\int_1^t h^{-\frac{n}{2}\left(\frac{1}{p}-\varepsilon\right)}|\mathbf{f}(t-h)|_p dh + \int_0^1 h^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}|\mathbf{f}(t-h)|_p dh\right). \end{aligned}$$

The limitation for q implies $\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right) < 1$; in addition, since $p < \frac{n}{2}$, we may choose ε so small that $\frac{n}{2}\left(\frac{1}{p}-\varepsilon\right) > 1$. Hence,

$$\begin{aligned} \left(\int_1^T |\mathbf{v}(t)|^r_{L_q(\Omega\cap S_R)} dt\right)^{\frac{1}{r}} &\le C\left[\int_1^T h^{-\frac{n}{2}\left(\frac{1}{p}-\varepsilon\right)} dh \left(\int_1^T |\mathbf{f}(t-h)|_p^r dt\right)^{\frac{1}{r}}\right. \\ &\left. + \int_0^1 h^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} dh \left(\int_1^T |\mathbf{f}(t-h)|_p^r dt\right)^{\frac{1}{r}}\right] \\ &\le C\left(\int_0^T |\mathbf{f}(\tau)|_p^r d\tau\right)^{\frac{1}{r}}. \end{aligned}$$

The integral $\int_0^1 |\mathbf{v}(t)|^r_{L_q(\Omega\cap S_R)} dt$ is easily estimated with the aid of (1.3). The Lemma is proved.

REMARK 6.1. If $p \ge \frac{n}{2}$, then inequality (6.2) implies

$$\left(\int_1^T |\mathbf{v}(t)|^r_{L_q(\Omega\cap S_R)} dt\right)^{\frac{1}{r}} \le C(R)(1+T^{1-\frac{n}{2p}+\eta})\left(\int_0^T |\mathbf{f}(\tau)|_p^r d\tau\right)^{\frac{1}{r}}$$

where η is an arbitrary small positive number.

REMARK 6.2. Assume that $\mathbf{f}(x,t) = \nabla \cdot \mathbf{F}(x,t)$. Then after integration by parts in (6.1), making use of (1.5) we show that

$$\left(\int_0^T |\mathbf{v}(t)|^r_{L_q(\Omega\cap S_R)} dt\right)^{\frac{1}{r}} \le C(R)\left(\int_0^T |\mathbf{F}(t)|_p^r dt\right)^{\frac{1}{r}},$$

for $p \in [\frac{n}{2}, n)$. As a consequence we can extend (1.11) to the cases of $p \in [\frac{n}{2}, n)$ with a constant C independent of T.

The proof of (1.7) is quite analogous to similar arguments in [19]-[20]. Since it is very simple and brief, we present it here for the sake of completeness. Suppose $\mathbf{v}_0(x) \in J_p(\Omega)$ and denote by $\{\mathbf{v}_0^n(x)\}_{n\in\mathbb{N}} \subset \mathcal{C}_0(\Omega)$ a sequence converging to $\mathbf{v}_0(x)$ in $J_p(\Omega)$. From (1.5) we have

$$|\mathbf{v}(t) - \mathbf{v}^n(t)|_p \le C|\mathbf{v}_0 - \mathbf{v}_0^n|_p, \forall\, p > 1, \forall\, t > 0, \tag{6.3}$$

with $\{\mathbf{v}^n(x,t)\}_{n\in\mathbb{N}}$ sequence of solutions corresponding to $\{\mathbf{v}_0^n(x)\}_{n\in\mathbb{N}}$. From (6.3) $\mathbf{v}^n(x,t)$ converges to $\mathbf{v}(x,t)$ in $J^p(\Omega)$ uniformly in $t > 0$. Therefore $\lim_{t\to\infty} |\mathbf{v}(t)|_p = \lim_{t\to\infty} \lim_n |\mathbf{v}_n(t)|_p = \lim_n \lim_{t\to\infty} |\mathbf{v}_n(t)|_p = 0$.

Inequalities (1.8) are a consequence of estimates (1.4)-(1.5), (1.7) and semigroup property. The corollary is completely proved.

7. – Estimate (1.4) in the limiting cases $q = \infty$ and $p = 1$

We start proving inequality (1.4) for $q = \infty$, and $p > 1$, i.e.

$$|\mathbf{v}(t)|_\infty \le Ct^{-\frac{n}{2p}}|\mathbf{v}_0|_p, \quad (n > 2)\,. \tag{7.1}$$

To this end, we consider $\mathbf{v}(x,t)$ as a sum

$$\mathbf{v}(x,t) = (1 - h(x))\mathbf{v}^1(x,t) + \mathbf{v}^2(x,t) + \mathbf{v}^3(x,t) \tag{7.2}$$

(this representation formula has been used in [15], but the case $q = \infty$ was not studied). Here $\mathbf{v}^1(x,t) = \mathbf{V}(x,t)$ is a solution of problem (5.1), $h(x)$ is a smooth cut-off function equal 1 for $|x| \le R+\varepsilon$ and to zero for $|x| > R+2-\varepsilon$, $\varepsilon > 0$, R being a fixed number such that the domain $\mathbb{R}^n - \Omega$ is contained in the ball $|x| < R$; $\mathbf{v}^2(x,t)$ satisfies the relations

$$\begin{gathered} \nabla\cdot\mathbf{v}^2(x,t) = \nabla\cdot(h(x)\mathbf{v}^1(x,t)) \text{ for } R \le |x| \le R+2, \\ \mathbf{v}^2(x,t)_{||x|=R} = \mathbf{v}^2(x,t)_{||x|=R+2} = 0. \end{gathered} \tag{7.3}$$

$\mathbf{v}^2(x,t) = 0$ for $|x| < R$ and for $|x| > R+2$, and, finally, $\mathbf{v}^3(x,t)$ is a solution of the exterior problem

$$\begin{gathered} \mathbf{v}_t^3(x,t) - \Delta\mathbf{v}^3(x,t) = \nabla\pi(x,t) + \mathbf{G}(x,t),\ \nabla\cdot\mathbf{v}^3(x,t) = 0, \\ \mathbf{v}^3(x,0) = h(x)\mathbf{v}_0(x) + \mathbf{v}^2(x,0), \\ \mathbf{v}^3(x,t)_{|\partial\Omega} = 0,\ \mathbf{v}^3(x,t) \to 0 \text{ for } |x| \to \infty, \end{gathered} \tag{7.4}$$

with $\mathbf{G}(x,t) = -\{2\nabla h(x)\cdot\nabla\mathbf{v}^1(x,t) + \Delta h(x)\mathbf{v}^1(x,t)\} + \mathbf{v}_t^2(x,t) - \Delta\mathbf{v}^2(x,t)$. Clearly, $\mathbf{v}^1(x,t) = \mathbf{V}(x,t)$ satisfies inequalities (1.4)-(1.5). In particular, in virtue of (5.2)

$$|\mathbf{v}^1(t)|_\infty \le Ct^{-\frac{n}{2p}}|\mathbf{v}_0|_p\,. \tag{7.5}$$

The vector field $\mathbf{v}^2(x,t)$ can be constructed as proposed by Bogovskii [3]. The domain $C_R : R < |x| < R+2$ can be represented as a union of domains $C_k, k = 1, \cdots, M$, star-shaped with respect to the balls B_k of a fixed radius; moreover, there exists a smooth partition of unity: $\sum_{k=1}^M \psi_k(x) = 1$ with $\operatorname{supp}\{\psi_k(x)\} \subset S_k$. According to [3], the vector field $\mathbf{v}^2(x,t)$ satisfying (7.3) can be written in the form

$$\mathbf{v}^2(x,t) = \sum_{k=1}^M \mathbf{v}_k^2(x),$$

where

$$\mathbf{v}_k^2(x,t) = \mathcal{Q}^k[\psi_k \nabla h \cdot \mathbf{v}^1] = \int_{S_k} \mathbf{Q}^k(x-y,y)\psi_k(y)\nabla h(y)\cdot \mathbf{v}^1(y,t)dy$$

$$\mathbf{Q}^k(z,y) = \frac{z}{|z|^n}\int_{|z|}^\infty q^k\left(y+\xi\frac{z}{|z|}\right)\xi^{n-1}d\xi,$$

$$q^k(x) \in C_0^\infty(B_k), \text{ and } \int_{B_k} q_k(x)dx = 1.$$

LEMMA 7.1. *For arbitrary* $t > 0$, $\mathbf{v}^2(x,t) \in W_p^2(\Omega)$, $\mathbf{v}_t^2(x,t) \in W_p^1(\Omega)$, $\operatorname{supp}\{\mathbf{v}^2(x,t)\} \subset \{R+\varepsilon < |x| < R+2-\varepsilon\}$, *and the following estimates hold:*

$$\begin{aligned} &|\nabla \mathbf{v}^2(t)|_p + |\mathbf{v}^2(t)|_p \le C|\mathbf{v}^1(t)|_{L_p(\operatorname{supp}\{\nabla h\})}\\ &|\nabla \mathbf{v}_t^2(t)|_p + |\mathbf{v}_t^2(t)|_p \le C|\mathbf{v}_t^1(t)|_{L_p(\operatorname{supp}\{\nabla h\})}\\ &|D^2\mathbf{v}^2(t)|_p \le C\big(|\nabla \mathbf{v}^1(t)|_{L_p(\operatorname{supp}\{\nabla h\})} + |\mathbf{v}^1(t)|_{L_p(\operatorname{supp}\{\nabla h\})}\big)\\ &|\mathbf{v}_t^2(t) - \Delta \mathbf{v}(t)|_p \le C\big(|\nabla \mathbf{v}^1(t)|_{L_p(\operatorname{supp}\{\nabla h\})} + |\mathbf{v}^1(t)|_{L_p(\operatorname{supp}\{\nabla h\})}\big). \end{aligned} \tag{7.6}$$

PROOF. The property $\operatorname{supp}\{\mathbf{v}^2(x,t)\} \subset \{R+\varepsilon < |x| < R+2-\varepsilon\}$ follows from the definition of $\mathbf{v}^2(x,t)$. Since $\mathbf{v}_t^2(x,t) = \sum_{k=1}^M \mathcal{Q}^k[\psi_k(x)\nabla h(x)\cdot \mathbf{v}_t^1(x,t)]$, the forst two inequalities (7.6) follow from the estimates of integral operators with weakly singular kernels and from the Calderon-Zygmund theorem applied to $\frac{\partial}{\partial x_j}\mathcal{Q}^k[\psi_k \nabla h \cdot \mathbf{v}^1(t)]$. To prove $(7.6)_3$ we observe that

$$\frac{\partial}{\partial x_j}\mathcal{Q}^k[\psi_k\nabla h\cdot \mathbf{v}^1(t)] = \mathcal{Q}^k\left[\frac{\partial}{\partial y_j}\psi_k\nabla h\cdot \mathbf{v}^1(t)\right] + \mathcal{Q}_j^k[\psi_k\nabla h\cdot\mathbf{v}^1(t)], \tag{7.7}$$

where $\mathcal{Q}_j^k(\cdot)$ is the integral operator with the kernel

$$\mathbf{Q}_j^k(z,y) = \frac{z}{|z|^n}\int_{|z|}^\infty \frac{\partial}{\partial y_j}q^k\left(y+\xi\frac{z}{|z|}\right)\xi^{n-1}d\xi.$$

When we differentiate (7.7) once more and apply again the Calderon-Zigmund theorem, we arrive at $(7.6)_3$. Finally, since $\mathbf{v}^1(x,t)$ is a solution of the heat equation, (7.7) yields

$$\left(\frac{\partial}{\partial t} - \Delta\right)\mathcal{Q}^k[\psi_k \nabla h \cdot \mathbf{v}^1(t)]$$
$$= -\sum_{j=1}^{n}\left\{\frac{\partial}{\partial x_j}\mathcal{Q}_j^k[\psi_k \nabla h \mathbf{v}^1(t)] + \mathcal{Q}_j^k\left[\frac{\partial}{\partial y_j}(\psi_k \nabla h \mathbf{v}^1)\right]\right\}$$
$$- \mathcal{Q}^k[2\nabla(\psi_k \nabla h)\nabla \mathbf{v}^1(t) + \Delta(\psi_k \nabla h)\mathbf{v}^1(t)]$$

which leads immediately to $(7.6)_4$. The lemma is proved.

COROLLARY 7.1. *There holds the estimate*

$$|\mathbf{v}^2(t)|_\infty \le C|\mathbf{v}_0|_p t^{-\frac{n}{2p}}, \quad p > n\,. \tag{7.8}$$

PROOF. Indeed, in virtue of the Sobolev inequality and (7.5)-$(7.6)_1$ inequalities

$$|\mathbf{v}^2(t)|_\infty \le C\left(|\nabla \mathbf{v}^2(t)|_p + |\mathbf{v}^2(t)|_p\right)$$
$$\le C|\mathbf{v}^1(t)|_{L_p(\mathrm{supp}\{\nabla h\})} \le C|\mathbf{v}^1(t)|_\infty \le C|\mathbf{v}_0|_p t^{-\frac{n}{2p}},$$

thus the proof is achieved.

Let us consider $\mathbf{v}^3(x,t)$.

LEMMA 7.2. *The solution of problem* (7.4) *satisfies the inequalities*

$$\begin{aligned} &|\mathbf{v}^3(t)|_q \le C_1|\mathbf{v}_0|_p t^{-\frac{n}{2p}},\ t \ge 2 \\ &|\mathbf{v}_t^3(t)|_q \le C_2|\mathbf{v}_0|_p t^{-\frac{n}{2p}},\ t \ge 2, \end{aligned} \tag{7.9}$$

where $q > n, q > p > \frac{n}{2}, n > 2$, C_i *are independent of* t.

PROOF. We set $\mathbf{v}^3(x,t) = \mathbf{v}_0^3(x,t) + \mathbf{v}_1^3(x,t)$ where $\mathbf{v}_0^3(x,t)$ is a solution of (1.1) with $\mathbf{v}_0^3(x,0) = h(x)\mathbf{v}_0(x) + \mathbf{v}^2(x,0)$ and $\mathbf{f}(x,t) = 0$, while $\mathbf{v}_1^3(x,t)$ is a solution of (1.1) with $\mathbf{v}_1^3(x,0) = 0$ and $\mathbf{f}(x,t) = \mathbf{G}(x,t)$. From (1.4) it follows that

$$|\mathbf{v}_0^3(t)|_q \le C|\mathbf{v}_0^3(0)|_{1+\varepsilon} t^{-\gamma_2(q)} \le C|\mathbf{v}_0^3(t)|_p t^{-\gamma_2(q)},$$

since $\mathbf{v}_0^3(x,0)$ has a compact support. Here $\gamma_2(q) = \frac{n}{2}\left(\frac{1}{1+\varepsilon} - \frac{1}{q}\right) = \frac{n}{2}\left(1 - \frac{1}{q}\right) - \delta$, ε and $\delta = \frac{n}{2}\frac{\varepsilon}{1+\varepsilon}$ are small positive numbers. Since $1 - \frac{1}{q} > 1 - \frac{1}{n} = \frac{n-1}{n} \ge \frac{2}{n} > \frac{1}{p}$, we conclude that

$$|\mathbf{v}_0^3(t)|_q \le C|\mathbf{v}_0^3(0)|_p t^{-\frac{n}{2p}} \le C|\mathbf{v}_0|_p t^{-\frac{n}{2p}}\,. \tag{7.10}$$

To estimate $\mathbf{v}^3_{0_t}(x,t)$, we multiply the equation $\mathbf{v}_{0_{tt}}(x,t) - \Delta \mathbf{v}^3_{0_t}(x,t) = \nabla \pi^3_{0_t}$ by $\tau^{\alpha+1}\varphi(x, t-\tau)$, where $\varphi(x,s)$ is a solution of (1.1) with $\mathbf{f}(x,s) = 0$ and $\varphi_0(x) \in C_0(\Omega)$, and integrate with respect to $x \in \Omega$ and $\tau \in (0,t)$. This gives an analogue of (2.23):

$$|t^{\alpha+1}(\mathbf{v}^3_{0_t}(t), \varphi_0)| = (\alpha+1)\left|\int_0^t \tau^\alpha(\mathbf{v}^3_{0_\tau}(\tau), \varphi(t-\tau))d\tau\right|, \tag{7.11}$$

which yields

$$|t^{\alpha+1}(\mathbf{v}^3_{0_t}(t), \varphi_0)| \le C|\varphi_0|_{q'} t^{\frac{1}{q'}} \left(\int_0^t \tau^{\alpha q}|\mathbf{v}^3_{0_\tau}(\tau)|^q_q d\tau\right)^{\frac{1}{q}}.$$

Since (7.10) holds, the integral in the right hand side can be evaluated by inequality (4.2) with $\gamma(p,p) = \gamma(p,q) = \gamma_2(q) = \frac{n}{2p}$, which leads to

$$|(\mathbf{v}^3_{0_t}(t), \varphi_0)| \le C\left(t^{-1-\gamma_2(q)} + t^{-\gamma_2(q)}\right)|\mathbf{v}_0|_p|\varphi_0|_{q'},$$

and, consequently,

$$|\mathbf{v}^3_{0_t}(t)|_q \le Ct^{-\frac{n}{2p}}|\mathbf{v}_0|_p. \tag{7.12}$$

Let us pass to the estimate of $\mathbf{v}^3_1(x,t)$. Since

$$|\mathbf{G}(t)|_r \le 2|\nabla h \cdot \nabla \mathbf{v}^1(t) + \Delta h \mathbf{v}^1(t)|_r + |\mathbf{v}^2_t(t) - \Delta \mathbf{v}^2(t)|_r,$$

Lemma 7.1 and estimate (5.2) for $\mathbf{v}^1(t)$ imply

$$|\mathbf{G}(t)|_r \le C\begin{cases} |\nabla \mathbf{v}^1(t)|_r + |\mathbf{v}^1(t)|_r \le Ct^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}|\mathbf{v}_0|_p, \\ |\nabla \mathbf{v}^1(t)|_\infty + |\mathbf{v}^1(t)|_\infty \le Ct^{-n/2p}|\mathbf{v}_0|_r, \qquad r > 1. \end{cases} \tag{7.13}$$

We evaluate $\mathbf{v}^3_1(x,t)$ making use of the formula (2.23)

$$(\mathbf{v}^3_1(t), \varphi_0) = \int_0^t (\mathbf{G}(\tau), \varphi(t-\tau))\,d\tau.$$

Since

$$|\varphi(t-\tau)|_m \le C|\varphi_0|_{q'}(t-\tau)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{m}\right)} = C(t-\tau)^{-\gamma_2(q)}|\varphi_0|_{q'},$$

if $\frac{n}{2m} = \delta$, we have for arbitrary $r > 1$

$$|\varphi(t-\tau)|_{L_r(\text{supp}\{\nabla h\})} \le C(t-\tau)^{-\gamma_2(q)}|\varphi_0|_{q'} \tag{7.14}$$

and

$$\begin{aligned}
|(\mathbf{G}(\tau), \varphi(t-\tau))| &\le C|\mathbf{G}(\tau)|_p|\varphi(t-\tau)|_{L^{p'}(\mathrm{supp}\{\nabla h\})} \\
&\le C|\mathbf{v}_0|_p|\varphi_0|_{q'}\tau^{-\frac{1}{2}}(t-\tau)^{-\gamma_2(q)}, \forall\,\tau\in(0,1], \\
|(\mathbf{G}(\tau), \varphi(t-\tau))| &\le C|\mathbf{G}(\tau)|_p|\varphi(t-\tau)|_{L^{p'}(\mathrm{supp}\{\nabla h\})} \\
&\le C|\mathbf{v}_0|_p|\varphi_0|_{q'}\tau^{-\frac{n}{2p}}(t-\tau)^{-\gamma_2(q)}, \forall\,\tau\in(1,t-1], \\
|(\mathbf{G}(\tau), \varphi(t-\tau))| &\le C|\mathbf{G}(\tau)|_p|\varphi(t-\tau)|_{L^{p'}(\mathrm{supp}\{\nabla h\})} \\
&\le C|\mathbf{v}_0|_p|\varphi_0|_{q'}\tau^{-\frac{n}{2p}}, \forall\,\tau\in[t-1,t].
\end{aligned}$$

Hence, for small δ

$$\begin{aligned}
|(\mathbf{v}_1^3(t), \varphi(t-\tau))| &\le C|\mathbf{v}_0|_p|\varphi_0|_{q'} \\
&\times\left(\int_0^1 \tau^{-\frac{1}{2}}(t-\tau)^{-\gamma_2(q)}d\tau + \int_1^{t-1}\tau^{-\frac{n}{2p}}(t-\tau)^{-\gamma_2(q)}d\tau + \int_{t-1}^t \tau^{-\frac{n}{2p}}d\tau\right) \\
&\le C|\mathbf{v}_0|_p|\varphi_0|_{q'}t^{-\frac{n}{2p}},
\end{aligned}$$

which implies

$$|\mathbf{v}_1^3(t)|_q \le Ct^{-\frac{n}{2p}}|\mathbf{v}_0|_p. \tag{7.15}$$

For estimate of $\mathbf{v}_{1_t}^3(x,t)$ we use an analogue of the relation (7.12)

$$\begin{aligned}
|t^{\alpha+1}(\mathbf{v}_{1_t}^3(t), \varphi_0)| = (\alpha+1)\Big|\int_0^t \tau^\alpha\Big[(\mathbf{v}_{1_\tau}^3(\tau), \varphi(t-\tau)) \\
+\tau^{\alpha+1}(\mathbf{G}_\tau(\tau), \varphi(t-\tau))\Big]d\tau\Big| = (\alpha+1)(I_1+I_2).
\end{aligned}$$

In virtue of (1.4), (7.15) and (4.2) (applied) with $\gamma(p,p)=\gamma(q,p)=-\frac{n}{2p}$,

$$\begin{aligned}
|I_1| &\le |\varphi_0|_{q'}t^{\frac{1}{q'}}\left(\int_0^t \tau^{\alpha q}|\mathbf{v}_{1_\tau}^3(\tau)|_q^q d\tau\right)^{\frac{1}{q}} \\
&\le |\varphi_0|_{q'}t^{\frac{1}{q'}}\left[\left(\int_0^t\left(\tau^{\left(\alpha-\frac{n}{2p}\right)q}+\tau^{\left(\alpha-1-\frac{n}{2p}\right)q}d\tau\right)\right)^{\frac{1}{q}}|\mathbf{v}_0|_p + \left(\int_0^t \tau^{\alpha q}|\mathbf{G}(\tau)|_q^q d\tau\right)^{\frac{1}{q}}\right] \\
&\le C|\varphi_0|_{q'}|\mathbf{v}_0|_p t^{1+\alpha-\frac{n}{2p}}.
\end{aligned}$$

Further, by Lemma 7.1,

$$\begin{aligned}
|\mathbf{G}_t(t)|_q &\le C(|\nabla\mathbf{v}_t^1(t)|_{L_q(\mathrm{supp}\{\nabla h\})} + |\mathbf{v}_t^1(t)|_{L_q(\mathrm{supp}\{\nabla h\})}) \\
&\le C(|\nabla\mathbf{v}_t^1(t)|_\infty + |\mathbf{v}_t^1(t)|_\infty) \le Ct^{-1-\frac{n}{2p}}|\mathbf{v}_0|_p
\end{aligned}$$

and

$$|I_2| \le C|\varphi_0|_{q'} \int_0^t \tau^{\tau+1}|\mathbf{G}(\tau)|_q d\tau \le C|\varphi_0|_{q'}|\mathbf{v}_0|_p \int_0^t \tau^{\alpha-\frac{n}{2p}} d\tau$$

$$\le Ct^{\alpha+1-\frac{n}{2p}}|\varphi_0|_{q'}|\mathbf{v}_0|_p \, .$$

Hence,

$$|(\mathbf{v}^3_{1_t}(t), \varphi_0)| \le Ct^{-\frac{n}{2p}}|\varphi_0|_{q'}|\mathbf{v}_0|_p \, ,$$

which implies

$$|\mathbf{v}^3_{1_t}(t)|_q \le Ct^{-\frac{n}{2p}}|\mathbf{v}_0|_p \, . \tag{7.16}$$

Estimates (7.9) follow from (7.10), (7.12) and (7.15)-(7.16). The lemma is proved.

Finally, we evaluate $|\mathbf{v}^3(t)|_\infty$ and complete the proof of (7.1).

LEMMA 7.3. *The vector field* $\mathbf{v}^3(t)$ *satisfies the inequality*

$$|\mathbf{v}^3(t)|_\infty \le Ct^{-\frac{n}{2p}}|\mathbf{v}_0|_p \, , \tag{7.17}$$

where $p > n$, $t > 1$.

PROOF. In virtue of Sobolev imbedding theorem, and of estimates (2.20) and (7.9), we have

$$|\mathbf{v}^3(t)|_\infty \le C\left(|D^2\mathbf{v}^3(t)|_q + |\mathbf{v}^3(t)|_q\right) \le C\left(|\mathbf{v}^3_t(t)|_q + |\mathbf{v}^3(t)|_q\right) \le Ct^{-\frac{n}{2p}}|\mathbf{v}_0|_p \, ,$$

which proves (7.17).

From representation (7.2) and estimates (7.5), (7.8) and (7.17) we deduce (7.1), provided that $p > n$, $t > 2$. In fact, (7.1) holds for arbitrary $p > 1$ and arbitrary $t > 0$. Indeed, if $p \le n$, then in virtue of (7.1) and (1.4)

$$|\mathbf{v}(t)|_\infty \le Ct^{-\frac{n}{2q}}|\mathbf{v}\left(\frac{t}{2}\right)|_q \le Ct^{-\frac{n}{2p}}|\mathbf{v}_0|_p \quad (q > n) \, .$$

Moreover, for $t \in (0, 2)$

$$|\mathbf{v}(t)|_\infty \le C|\nabla\mathbf{v}(t)|_q^{\frac{n}{q}}|\mathbf{v}(t)|_q^{1-\frac{n}{q}} \le Ct^{-\frac{n}{2p}}|\mathbf{v}_0|_p \quad (q > n) \, .$$

This completes the proof of (7.1).

At the conclusion we show that (1.4) holds in the case $p = 1$. This can be easily done by duality arguments. Let $\varphi(x, s)$ be a solution of (1.1) with $\mathbf{f}(x, t) = 0$ and with initial data $\varphi_0(x) \in \mathcal{C}_0(\Omega)$. Making use of (2.23), we obtain

$$|(\mathbf{v}(t), \varphi_0)| \le |(\mathbf{v}_0, \varphi(t))| \le |\mathbf{v}_0|_1|\varphi(t)|_\infty \le C|\mathbf{v}_0|_1|\varphi_0|_{q'}t^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}$$

which implies

$$|\mathbf{v}(t)|_q \le Ct^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}|\mathbf{v}_0|_1 \, .$$

Moreover, (7.1) also holds for $p = 1$, since

$$|\mathbf{v}(t)|_\infty \le Ct^{-\frac{n}{2p}}|\mathbf{v}(\frac{t}{2})|_p \le Ct^{-\frac{n}{2}}|\mathbf{v}_0|_1, \quad (p > 1) \, .$$

8. – Estimate of $\nabla \mathbf{v}(x,t)$: inequality (1.6) and its optimality

In this section we prove estimate (1.6). The principal part of the proof concerns the case $q = p > 1$:

$$|\nabla \mathbf{v}(t)|_p \le C t^{-\hat{\mu}} |\mathbf{v}_0|_p . \tag{8.1}$$

For $q > p \ge 1$, (1.6) follows easily from (8.1), (1.4) and from the semigroup property of e^{-tA}. We begin with the following lemma.

LEMMA 8.1. *The solution of* (1.1) *with* $\mathbf{f}(x,t) = 0$ *satisfies the estimates*

$$(8.2^n) \qquad |\nabla \pi(t)|_p + |D^2 \mathbf{v}(t)|_p + |\mathbf{v}_t(t)|_p \le C|\mathbf{v}_0|_p t^{-\gamma_3}, \ n \ge 3,$$

$$\begin{aligned} &|\nabla \pi(t)|_p + |D^2 \mathbf{v}(t)|_p + |\mathbf{v}_t(t)|_p \le C|\mathbf{v}_0|_p t^{-\gamma_4}, \ n = 2, \ p > 2,\\ (8.2^2) \qquad &|\nabla \pi(t)|_{L_p(\Omega')} + |D^2 \mathbf{v}(t)|_{L_p(\Omega')} + |\mathbf{v}_t(t)|_{L_p(\Omega')}\\ &\quad \le C|\mathbf{v}_0|_p t^{-\gamma_4}, \ n = 2, \ p \in (1,2], \end{aligned}$$

where

$$\gamma_3 = \begin{cases} 1, \ t \in (0,1], & p \in (1,\infty),\\ 1, \ t > 0, & p \in \left(1, \dfrac{n}{2}\right],\\ \dfrac{n}{2p}, \ t \ge 1, & p \in \left[\dfrac{n}{2}, \infty\right); \end{cases}$$

$$\gamma_4 = \begin{cases} 1, \ t \in (0,1), & p \in (1,\infty),\\ \dfrac{1}{p}, \ t \ge 1, & p \in (1,2],\\ \dfrac{1}{p} - \delta, \ t \ge 1, & p \in (2,\infty) \ \left(\delta \in \left(0, \dfrac{1}{p}\right)\right). \end{cases}$$

PROOF. The case $t \in (0,1]$ is considered in Theorem 4.1. Assune that $t \ge 1$. Let us prove (8.2) for $\mathbf{v}_t(x,t)$. We again make use of the relation (4.6) and of estimate (4.2) with $\gamma(q,p) = \gamma(\infty,p)$, if $n \ge 3$, and with q very large, if $n = 2$. This gives

$$\begin{aligned} t^{\alpha+1}|(\mathbf{v}_t(t), \varphi_0)| &\le (\alpha+1)\int_0^t \tau^\alpha (\mathbf{v}_\tau(\tau), \varphi(t-\tau)d\tau\\ &\le C|\varphi_0|_{p'} t^{\frac{1}{p'}} \left(\int_0^t \tau^{\alpha p}|\mathbf{v}_\tau(\tau)|_p^p d\tau\right)^{1/p}\\ &\le C\left(t^\alpha + t^{\alpha+1-\frac{n}{2p}}\right)|\mathbf{v}_0|_p|\varphi_0|_{p'} \ (n > 3);\\ t^{\alpha+1}|(\mathbf{v}_t(t), \varphi_0)| &\le \left(t^\alpha + t^{\alpha+1-\frac{1}{p}+\delta}\right)|\mathbf{v}_0|_p|\varphi_0|_{p'} \ (n = 2), \end{aligned}$$

with $\delta = \frac{1}{q}$. These inequalities imply (8.2) for $\mathbf{v}_t(x,t)$ in the cases $n > 2$ and $n = 2$, $p > 2$. Further, in virtue of the estimate (2.20), (where the last term may be omitted, if $p < \frac{n}{2}$), we easily arrive at (8.2) for $D^2\mathbf{v}(x,t)$ and $\nabla\pi(x,t)$ in the same cases. It remains to consider the case $n = 2$, $p \in (1,2]$, $t \geq 1$. Under these hypotheses, the following estimates were proved in the paper [20]:

$$|\mathbf{v}_t(t)|_2 \leq Ct^{-1-\frac{1}{p}}|\mathbf{v}_0|_p, \quad |\nabla\mathbf{v}(t)|_2 \leq Ct^{-\frac{1}{p}}|\mathbf{v}_0|_p .$$

Hence, for arbitrary bounded $\Omega' \subset \Omega$ such that $\text{dist}(\Omega - \Omega', \partial\Omega) > 0$, we have

$$|\mathbf{v}_t(t)|_{L_p(\Omega'')} \leq Ct^{-1-\frac{1}{p}}|\mathbf{v}_0|_2, \quad |\nabla\mathbf{v}(t)|_{L_p(\Omega'')} \leq Ct^{-\frac{1}{p}}|\mathbf{v}_0|_p .$$

Finally, assuming that $\Omega' \subset \Omega'$, $\text{dist}(\Omega' - \Omega', \partial\Omega) > 0$, from inequality (2.20*) we obtain

$$|D^2\mathbf{v}(t)|_{L_p(\Omega')} + |\nabla\pi(t)|_{L_p(\Omega')} \leq C\left(|\mathbf{v}_t(t)|_{L_p(\Omega'')} + |\mathbf{v}(t)|_{L_p(\Omega'')}\right) \leq Ct^{-\frac{1}{p}}|\mathbf{v}_0|_p ,$$

which completes the proof of (8.2^2) in the case $p \in (1,2]$.

Now, let us turn to the proof of (8.1) assuming again that $t \geq 1$. We consider separately the domain Ω'_R bounded by $\partial\Omega$ and $|x| = R+3$ and the domain $|x| > R+2$ (we assume, as above, that $\partial\Omega$ is located inside the ball $|x| < R$). Clearly,

$$\begin{aligned} |\nabla\mathbf{v}(t)|_{L_p(\Omega'_R)} &\leq C\left(|D^2\mathbf{v}(t)|_{L_p(\Omega'_R)} + |\mathbf{v}(t)|_{L_p(\Omega'_R)}\right) \\ &\leq C\left(|D^2\mathbf{v}(t)|_p + |\mathbf{v}(t)|_\infty\right) \leq Ct^{-\hat{\mu}}|\mathbf{v}_0|_p . \end{aligned}$$

It remains to evaluate $|\nabla\mathbf{v}(t)|_{L_p(\Omega'^{c}_R)}$. Let $h_1(x)$ be a smooth function equal 1 for $|x| > R+2$ and to zero for $|x| < R$. We write $(h_1(x)\mathbf{v}(x,t), h_1(x)\pi(x,t))$ in the form

$$h_1(x)\mathbf{v}(x,t) = \mathbf{w}^1(x,t)+\mathbf{w}^2(x,t)+\mathbf{w}^3(x,t), \quad h_1(x)\pi(x,t) = \pi^1(x,t)+\pi^2(x,t)$$

where

$$\begin{aligned} &\mathbf{w}^1_t(x,t) - \Delta\mathbf{w}^1(x,t) = \nabla\pi^1(x,t) + \mathbf{F}(x,t), \text{ in } \mathbb{R}^n \times (1,T), \\ &\nabla\cdot\mathbf{w}^1(x,t) = 0, \text{ in } \mathbb{R}^n \times (1,T), \\ &\mathbf{w}^1(x,t) \to 0 \text{ for } |x| \to +\infty, \ \mathbf{w}^1(x,1) = 0 , \end{aligned}$$

with $\mathbf{F}(x,t) = -2\nabla h_1(x)\cdot\nabla\mathbf{v}(x,t) - \Delta h_1(x)\mathbf{v}(x,t) - (\pi(x,t) - m(t))\nabla h_1(x)$, $m(t) = |C_R|^{-1}\int_{R\leq|x|\leq 2R}\pi(x,t)dx$,

$$\begin{aligned} &\mathbf{w}^2_t(x,t) - \Delta\mathbf{w}^2(x,t) = 0, \text{ in } \mathbb{R}^n \times (1,T) , \\ &\mathbf{w}^2(x,t) \to 0 \text{ for } |x| \to +\infty, \ \mathbf{w}^2(x,1) = h_1(x)\mathbf{v}(x,1) . \end{aligned}$$

and

(8.3)
$$\begin{aligned} &\mathbf{w}_t^3(x,t) - \Delta \mathbf{w}^3(x,t) = \nabla \pi^3(x,t), \text{ in } \mathbb{R}^n \times (1,T), \\ &\nabla \cdot \mathbf{w}^3(x,t) = -\nabla \cdot (\mathbf{w}^2(x,t) - h_1(x)\mathbf{v}(x,t)), \text{ in } \mathbb{R}^n \times (1,T), \\ &\mathbf{w}^3(x,t) \to 0 \text{ for } |x| \to +\infty, \ \mathbf{w}^3(x,1) = 0. \end{aligned}$$

The norms of $\nabla \mathbf{w}^2(x,t)$ and $\nabla \mathbf{w}^3(x,t)$ are easily estimated. In virtue of Lemma 5.1,

(8.4)
$$|\nabla \mathbf{w}^2(t)|_p \le C(t-1)^{-\frac{1}{2}} |\mathbf{v}(1)|_p \le C(t-1)^{-\frac{1}{2}} |\mathbf{v}_0|_p .$$

The solution of problem (8.3) is given by the formula (3.5), hence the Calderon-Zygmund theorem yields

(8.5)
$$\begin{aligned} |\nabla \mathbf{w}^3(t)|_p &\le C|\nabla \cdot \mathbf{w}^2(t) - \mathbf{v}(t) \cdot \nabla h_1|_p \\ &\le C\left(|\nabla \cdot \mathbf{w}^2(t)|_p + C|\mathbf{v}(t)|_\infty\right) \le Ct^{-\hat{\mu}}|\mathbf{v}_0|_p \end{aligned}$$

(in the two-dimensional case $|\mathbf{v}(t)|_\infty$ should be replaced by $|\mathbf{v}(t)|_q$ with a large $q : \frac{1}{q} = \delta$).

The vector field $\mathbf{w}^1(x,t)$ can be written explicitly by the formula (3.5) (we write it in a compact form):

$$\begin{aligned} \mathbf{w}^1(x,t) = \mathbf{T}(x,t) * \mathbf{F} = &-\mathbf{T}(x,t) * (\pi - m)\nabla h_1 \\ &+ \mathbf{T}(x,t) * \Delta h_1 \mathbf{v} - \nabla_x \mathbf{T}(x,t) * \nabla h_1 \otimes \mathbf{v} \\ = &\ \mathbf{w}_1^1(x,t) + \mathbf{w}_2^1(x,t) + \mathbf{w}_3^1(x,t) . \end{aligned}$$

The Oseen's tensor $\mathbf{T}(z,s)$ satisfies the inequalities (see for example [26])

$$|\mathbf{T}(z,s)| \le C\left(|z|^2 + t\right)^{-\frac{n}{2}}, \quad |D^k \mathbf{T}(z,s)| \le C\left(|z|^2 + t\right)^{-\frac{n+k}{2}},$$

in particular, if $|y| < R+2$ and $|x| > R+3$, then

$$|D^k \mathbf{T}(x-y, t-\tau)| \le C\left[|x|^2 + 1 + (t-\tau)\right]^{-\frac{n+k}{2}}, \ k = 0,1,2.$$

Hence, in virtue of the Minkowski inequality,

(8.6)
$$\begin{aligned} &|\nabla \mathbf{w}_2^1(t)|_{L_p(|x|\ge R+3)} + |\nabla \mathbf{w}_3^1(t)|_{L_p(|x|\ge R+3)} \\ &\le C \int_1^t \int_{R \le |y| \le R+2} |\mathbf{v}(y,\tau)| dy d\tau \left[\left(\int_{|x| \ge R+3} |D^2 \mathbf{T}(x-y), t-\tau)|^p dx \right)^{\frac{1}{p}} \right. \\ &\quad \left. + \left(\int_{|x|\ge R+3} |\nabla \mathbf{T}(x-y,t-\tau)|^p dx \right)^{\frac{1}{p}} \right] \\ &\le C \int_1^t |\mathbf{v}(\tau)|_{L_p(C_R)} [(t-\tau) + R^2]^{-\frac{1}{2}(n+1-\frac{n}{p})} d\tau . \end{aligned}$$

Since $\mathbf{v}(x,t)$ satisfies the inequality

$$|\mathbf{v}(t)|_{L_p(C_R)} \leq C|\mathbf{v}_0|_p t^{-\frac{n}{2p}+\delta_1}$$

($\delta_1 = 0$ for $n \geq 3$ and $\delta_1 > 0$ arbitrarily small for $n = 2$), the right hand side of (8.6) is less then $Ct^{-\hat{\mu}}|\mathbf{v}_0|_p$. In a similar way, applying the Poincarè inequality and making use of (8.2), we obtain

$$\begin{aligned}|\nabla \mathbf{w}_1^1(t)|_{L_p(|x|\geq R+3)} &\leq C\int_1^t |\pi(\tau) - m(\tau)|_{L_p(R\leq|y|\leq R+2}[t-\tau+1]^{-(n+1-\frac{n}{p})/2}d\tau \\ &\leq C\int_1^t |\nabla\pi(\tau)|_{L_p(R\leq|y|\leq R+2}[t-\tau+1]^{-(n+1-\frac{n}{p})/2}d\tau \leq C|\mathbf{v}_0|_p t^{-\hat{\mu}}.\end{aligned}$$

Hence, for $t > 2$

$$\begin{aligned}|\nabla\mathbf{v}(t)|_p &\leq \left(|\nabla\mathbf{v}(t)|^p_{L_p(\Omega\cap S_{R+3})} + |\nabla\mathbf{v}(t)|^p_{L_p(|x|\geq R+3)}\right)^{\frac{1}{p}} \\ &\leq |\nabla\mathbf{v}(t)|_{L_p(\Omega\cap S_{R+3})} + \sum_{i=1}^3 |\nabla\mathbf{w}^i(t)|_p \leq C|\mathbf{v}_0|_p t^{-\hat{\mu}}.\end{aligned}$$

For $t < 2$, this inequality follows from (4.3). The proof of (8.1) is now complete.

We conclude this section with the proof of the fact that estimate (1.5) is sharp, namely, that the inequality

$$|\nabla\mathbf{v}(t)|_q \leq C|\mathbf{v}_0|_p t^{-\hat{\mu}-\eta}, \forall\, \mathbf{v}_0(x) \in \mathcal{C}_0(\Omega), \tag{8.7}$$

with $\eta > 0$ and with C independent of t is not true. Assume that (8.7) holds for some $q \geq p > \frac{n}{2}$, $n \geq 2$. Then

$$|\nabla\mathbf{v}(t)|_q \leq C\left|\mathbf{v}\left(\frac{t}{2}\right)\right|_p t^{-\hat{\mu}-\eta} \leq |\mathbf{v}_0|_{\frac{n}{2}} t^{-\hat{\mu}-\eta-\mu(p,\frac{n}{2})} = |\mathbf{v}_0|_{\frac{n}{2}} t^{-1-\eta}. \tag{8.8}$$

Let $(\mathbf{\Phi}(x), P(x))$ be a solution of exterior stationary problem

$$\begin{aligned}&\Delta\mathbf{\Phi}(x) = \nabla P(x),\ \nabla\cdot\mathbf{\Phi}(x) = 0,\ x\in\Omega \\ &\mathbf{\Phi}(x)_{|\partial\Omega} = \mathbf{a}(x) \in C^2(\partial\Omega), \quad \mathbf{\Phi}(x) \to 0 \text{ for } |x| \to \infty.\end{aligned} \tag{8.9}$$

It is well known that $\mathbf{\Phi}(x) = O(|x|^{-n+2})$ at infinity, so $\mathbf{\Phi}(x) \in L_q(\Omega)$ with arbitrary $q > n/(n-2)$. We are going to show that under hypothesis (8.7)

$$|(\mathbf{\Phi}, \mathbf{v}_0)| \leq C|\mathbf{v}_0|_{\frac{n}{2}}, \quad \mathbf{v}_0(x) \in \mathcal{C}_0(\Omega). \tag{8.10}$$

To this end, we make use of the formula

$$
(8.11)\quad \begin{aligned}(\mathbf{\Phi}, \mathbf{v}_0) = (\mathbf{\Phi}, \mathbf{v}(t)) - \int_0^t \int_0^t \int_{\partial\Omega} \mathbf{a}(x)\vec{n} \cdot \nabla \mathbf{v}(x, t-\tau) d\sigma\, d\tau \\ - \int_0^t \int_{\partial\Omega} \mathbf{a}(x) \cdot \vec{n} \pi(x, t-\tau) d\sigma d\tau = I_1 + I_2 + I_3 ,\end{aligned}
$$

where $\mathbf{v}(x,t)$ is a solution of problem (1.1) with $\mathbf{f}(x,t) = 0$ and $\mathbf{v}(x,0) = \mathbf{v}_0(x)$, and estimate integrals I_k. In virtue of (1.4),

$$
(8.12)\quad |I_1| \le |\mathbf{\Phi}|_r |\mathbf{v}(t)|_{r'} \le C |\mathbf{\Phi}|_r |\mathbf{v}_0|_s t^{-\frac{n}{2}\left(\frac{1}{s} - \frac{1}{r'}\right)}
$$

where $r > n/(n-1)$, $r' > s$. Further we have

$$
\begin{aligned}
|I_2(t)| &\le C \left(\int_0^t |\nabla \mathbf{v}(\tau)|_{L_r(\partial\Omega)} d\tau + \int_1^t |\nabla \mathbf{v}(\tau)|_{L_q(\partial\Omega)} d\tau \right) \\
&\le C \left[\int_0^1 \left(|\nabla \mathbf{v}(\tau)|_r + |\nabla \mathbf{v}(\tau)|_r^{\frac{1}{r'}} |D^2 \mathbf{v}(\tau)|_r^{\frac{1}{r}} \right) d\tau \right. \\
&\quad \left. + \int_1^t \left(|\nabla \mathbf{v}(\tau)|_q + |\nabla \mathbf{v}(\tau)|_q^{1/q'} |D^2 \mathbf{v}(\tau)|_q^{1/q} \right) d\tau, \right] .
\end{aligned}
$$

Choosing $p = \frac{n}{2}$ and $q > n$ and making use of $(4.3)_2$-$(4.3)_3$, (8.2) and (8.7), we obtain

$$
(8.13)\quad \begin{aligned}|I_2(t)| \le& C |\mathbf{v}_0|_{n/2} \left[\int_0^1 \left(\tau^{\frac{1}{2}} + \tau^{\frac{1}{2} + \frac{1}{n}} \right) d\tau \right. \\ &\left. + \int_1^t \left(\tau^{-1-\eta} + \tau^{-(1+\eta)/q' - 1/q} \right) d\tau \right] \le |\mathbf{v}_0|_{\frac{n}{2}} .\end{aligned}
$$

Finally,

$$
(8.14)\quad \begin{aligned}
|I_3(t)| &\le C \left[\int_0^1 |\pi(\tau)|_{L_r(\partial\Omega)} d\tau + \int_1^t |\pi(\tau)|_{L_q(\partial\Omega)} d\tau \right] \\
&\le C \left[\int_0^1 \left(|\pi(\tau)|_{L_r(\Omega')} + |\pi(\tau)|_{L_r(\Omega')}^{\frac{1}{r'}} |\nabla \pi(\tau)|_{L_r(\Omega')}^{\frac{1}{r}} \right) d\tau \right. \\
&\quad \left. + \int_1^t \left(|\pi(\tau)|_{L_q(\Omega')} + |\pi(\tau)|_{L_q(\Omega')}^{\frac{1}{q'}} |\nabla \pi(\tau)|_{L_q(\Omega')}^{\frac{1}{q}} \right) d\tau \right]
\end{aligned}
$$

The function $\pi(x,t)$ is a solution to the Neumann problem of the type (2.14), hence, according to Lemma 2.3

$$
|\pi(t)|_{L_p(\Omega')} \le C |\nabla \mathbf{v}(t)|_{B_p^\lambda(\partial\Omega)}, \ |\nabla \pi(t)|_{L_p(\Omega')} \le C |\nabla \mathbf{v}(t)|_{B_p^{1-\frac{1}{p}}(\partial\Omega)} ,
$$

If $\lambda = \lambda(p) < 1 - \frac{1}{p} \equiv \frac{1}{p'}$, then

$$|\nabla \mathbf{v}(t)|_{B_p^\lambda(\partial\Omega)} \leq C \left(|\nabla \mathbf{v}(t)|_{L_p(\partial\Omega)} + |\nabla \mathbf{v}(t)|_{L_p(\partial\Omega)}^{1-\lambda p'} |\nabla \mathbf{v}(t)|_{B_p^{\frac{1}{p'}}(\partial\Omega)} \right).$$

Hence

$$\begin{aligned} (8.15) \qquad & |\pi(t)|_{L_p(\Omega')} + |\pi(t)|_{L_p(\Omega')}^{\frac{1}{p'}} |\nabla\pi(t)|_{L_p(\Omega')}^{\frac{1}{p}} \\ & \leq C \left(|\nabla \mathbf{v}(t)|_{L_p(\partial\Omega)} + |\nabla \mathbf{v}(t)|_{L_p(\partial\Omega)}^{\frac{1}{p'}-\lambda} |\nabla \mathbf{v}(t)|^{\frac{1}{p}}_{B_p^{\frac{1}{p'}}(\partial\Omega)} \right) \\ & \leq C \left(|\nabla \mathbf{v}(t)|_{L_p(\Omega')} + |\nabla \mathbf{v}(t)|_{L_p(\Omega')}^{\frac{1}{p'}\left(\frac{1}{p'}-\lambda\right)} |D^2 \mathbf{v}(t)|_{L_p(\Omega')}^{1-\frac{1}{p'}\left(\frac{1}{p'}-\lambda\right)} \right) \end{aligned}$$

In particualr, taking $p = \frac{n}{2}$, and making use of (4.3), we obtain

$$\begin{aligned} (8.16) \qquad & |\pi(t)|_{L_{\frac{n}{2}}(\Omega')} + |\pi(t)|_{L_{\frac{n}{2}}(\Omega')}^{\frac{n-2}{n}} |\nabla\pi(t)|_{L_p(\Omega')}^{\frac{2}{n}} \\ & \leq C |\mathbf{v}_0|_{\frac{n}{2}} \tau^{1-\frac{2}{p'}\left(\frac{2}{p'}-\lambda\right)}, \quad \lambda = \lambda_1 < \frac{n-2}{n}. \end{aligned}$$

This inequality implies

$$(8.17) \qquad \int_0^1 \left(|\pi(\tau)|_{L_{\frac{n}{2}}(\Omega')} + |\pi(\tau)|_{L_{\frac{n}{2}}(\Omega')}^{\frac{n-2}{n}} |\nabla\pi(\tau)|_{L_{\frac{n}{2}}(\Omega')}^{\frac{2}{n}} \right) d\tau \leq C |\mathbf{v}_0|_{\frac{n}{2}}$$

We evaluate the last integral in (8.14) by inequality (8.15) with $p = q > n$. In virtue of (8.2) and (1.4),

$$|D^2 \mathbf{v}(t)|_q \leq C t^{-\frac{n}{2q}} \left|\mathbf{v}\left(\frac{t}{2}\right)\right|_q \leq C |\mathbf{v}_0|_{\frac{n}{2}},$$

hence, choosing $\lambda = \lambda_2 < \frac{1}{q'}$, we obtain

$$\begin{aligned} (8.18) \qquad & \int_1^t \left(|\pi(\tau)|_{L_q(\Omega')} + |\pi(\tau)|_{L_q(\Omega')}^{\frac{1}{q'}} |\nabla\pi(\tau)|_{L_q(\Omega')}^{\frac{1}{q}} \right) d\tau \\ & \leq C |\mathbf{v}_0|_{\frac{n}{2}} \int_1^t \left(\tau^{-1-\eta} + \tau^{-1-\frac{n}{q'}\left(\frac{1}{q'}-\lambda_2\right)} \right) d\tau \leq C |\mathbf{v}_0|_{\frac{n}{2}}. \end{aligned}$$

According to (8.11),

$$|(\mathbf{\Phi}, \mathbf{v}_0)| \leq |I_1| + |I_2| + |I_3|.$$

As $t \to \infty$, the contribution of I_1 vanishes, and in virtue of (8.17)-(8.18) we arrive at (8.10). In virtue of Lemma 2.6, inequality (8.10) implies $\mathbf{\Phi}(x) \in L_{\frac{n}{n-2}}(\Omega)$ for arbitrary boundary data $\mathbf{a}(x)$, which is impossible. This shows that inequality (8.7) with $\eta > 0$ can not be true.

9. – Estimate of $\mathbf{v}_t(x,t)$

We must prove

$$|\mathbf{v}_t(t)|_q \le M|\mathbf{v}_0|_p t^{-\mu'}, \tag{9.1}$$

with $q \ge p$ and

$$\mu' = \begin{cases} 1+\mu, & \text{if } n>2, q\in[p,\infty], t-s>0, p\ge 1;\\ 1+\mu, & \text{if } n=2, q\in[p,\infty), 1\ge t-s>0, p>1;\\ 1+\mu, & \text{if } n=2, q=2, t-s>0, p\in(1,2];\\ \dfrac{1}{2}+\dfrac{1}{p}-\delta, & \text{if } n=2, q\in[p,\infty)\cap(2,\infty), t-s\ge 1, p>1;\\ \dfrac{3}{2}+\dfrac{1}{p}-\dfrac{2}{q}-\delta, & \text{if } n=2, q\in[p,2], t-s\ge 1, p>1,\end{cases}$$

which improves inequality (8.2) of Lemma 8.1. Let us consider another solution $\psi(x,s)$ to system (1.1). Solution $\psi(x,t)$ corresponds to the initial data $\psi_0(x)\in\mathcal{C}_0(\Omega)$ and $\mathbf{f}(x,t)=0$. We define $\psi(x,t-\tau)$ for any $\tau\in[0,t]$. We differentiate $(1.1)_1$ with respect to t and multiply by $\psi(x,t-\tau)$. Integrating on $\Omega\times(\frac{t}{2},t)$, we obtain

$$\begin{aligned}(\mathbf{v}_t(t),\psi_0) &= \left(\mathbf{v}_t\left(\frac{t}{2}\right),\psi\left(\frac{t}{2}\right)\right) = \left(P\Delta\mathbf{v}\left(\frac{t}{2}\right),\psi\left(\frac{t}{2}\right)\right)\\ &= \left(\mathbf{v}\left(\frac{t}{2}\right),P\Delta\psi\left(\frac{t}{2}\right)\right) = -\left(\mathbf{v}\left(\frac{t}{2}\right),\psi_t\left(\frac{t}{2}\right)\right).\end{aligned} \tag{9.2}$$

Applying the Hölder inequality and inequalities (1.4) and (8.2), we have

$$|(\mathbf{v}_t(t),\psi_0)| \le |\mathbf{v}\left(\frac{t}{2}\right)|_p|\psi_t\left(\frac{t}{2}\right)|_{p'} \le C|\mathbf{v}_0|_p|\psi_0|_{p'}t^{-1},\ p\in\left[\frac{n}{n-2},\infty\right).$$

Because of the arbitrariness of $\psi_0(x)\in\mathcal{C}_0(\Omega)$ we have proved

$$|\mathbf{v}_t(t)|_p \le C|\mathbf{v}_0|_p t^{-1},\ p\in\left[\frac{n}{n-2},\infty\right). \tag{9.3}$$

If $n\ge 4$, then inequalities (8.2) and (9.3) prove the result for $q=p>1$. For $n=3$ we must recover the cases of $p\in(\frac{3}{2},3)$. To this end we modify (9.2) as it follows:

$$(\mathbf{v}_t(t),\psi_0) = \left(P\Delta\mathbf{v}\left(\frac{t}{2}\right),\psi\left(\frac{t}{2}\right)\right) = -\left(\nabla\mathbf{v}\left(\frac{t}{2}\right),\nabla\psi\left(\frac{t}{2}\right)\right). \tag{9.4}$$

Applying the Hölder inequality, we obtain

$$|(\mathbf{v}_t(t), \psi_0)| \leq |\nabla \mathbf{v}\left(\frac{t}{2}\right)|_p |\nabla \psi\left(\frac{t}{2}\right)|_{p'}, \ p \in \left(\frac{3}{2}, 3\right).$$

Therefore from (1.5) we deduce

$$|(\mathbf{v}_t(t), \psi_0)| \leq C|\mathbf{v}_0|_p |\psi_0|_{p'} t^{-1}, \ p \in \left(\frac{3}{2}, 3\right),$$

which implies again

$$|\mathbf{v}_t(t)|_p \leq C|\mathbf{v}_0|_p t^{-1}, \ p \in \left(\frac{3}{2}, 3\right),$$

and completes the proof for $q = p > 1$ in the three-dimensional case. In the case of $q \in [p, \infty)$, we apply the semigroup property of the solutions and (1.4):

$$|\mathbf{v}_t(t)|_q \leq Ct^{-1}|\mathbf{v}\left(\frac{t}{2}\right)|_q \leq C|\mathbf{v}_0|_p t^{-1-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

For $q = \infty$ we argue as follows

$$|\mathbf{v}_t(t)|_\infty \leq Ct^{-\frac{n}{2q}}|\mathbf{v}_t\left(\frac{t}{2}\right)|_q \leq Ct^{-1-\frac{n}{2p}}|\mathbf{v}_0|_p.$$

It remains to consider the case of $n = 2$. The case of $q = 2$ has been already treated in [20]. Thus we restrict our attention to the case $q \neq 2$ and $t \geq 2$. We consider again estimate (9.4). Applying Hölder inequality and inequality (1.5) we obtain

$$|(\mathbf{v}_t(t), \psi_0)| \leq |\nabla \mathbf{v}\left(\frac{t}{2}\right)|_p |\nabla \psi\left(\frac{t}{2}\right)|_{p'} \leq C|\mathbf{v}_0|_p |\psi_0|_{p'} t^{-\gamma_6},$$

where

$$\gamma_6 = \begin{cases} -\dfrac{3}{2} + \dfrac{1}{p} + \delta & \text{if} \quad p \in (1, 2) \\[2ex] -\dfrac{1}{2} - \dfrac{1}{p} + \delta & \text{if} \quad p \in (2, \infty). \end{cases}$$

Therefore we have proved (9.1) for $q = p > 1$. Taking into account the semigroup property of the solutions and inequality (1.4), one easily completes the proof of (9.1).

10. – Appendix

a) *Proof of inequalities* (2.18)

The proof of inequality (2.18) given in [27] for the three-dimensional case is based on the estimate

$$|\nabla\phi'|_p^p \leq C \sum_{i,k=1}^{n} \int_{\partial\Omega}\int_{\partial\Omega} \frac{|a_{ik}(x) - a_{ik}(y)|^p}{|x-y|^{n-2+p}} d\sigma_x d\sigma_y \,, \tag{10.1}$$

for the single layer potential

$$\phi'(x) = \sum_{ik=1}^{n} \int_{\partial\Omega} \mathcal{E}(x-y)\left(n_i \frac{\partial}{\partial y_k} - n_k \frac{\partial}{\partial y_i}\right) a_{ik}(y) d\sigma_y \,,$$

which, in its turn, was deduced from a coercive estimate for the Neumann problem in $W_p^2(\Omega)$. Here a direct proof of (10.1) is given.

In virtue of the Stokes formula,

$$\phi'(x) = \sum_{i,k=1}^{n} \int_{\partial\Omega} [a_{ik}(y) - \alpha_{ik}]\left(n_k(y)\frac{\partial}{\partial y_i} - n_i(y)\frac{\partial}{\partial y_k}\right)\mathcal{E}(x-y) d\sigma_y \,,$$

with arbitrary $\alpha_{ik} = \text{const}$. Let $\omega = \{x \in \Omega : \text{dist}(x, \partial\Omega) > d\}$ with a certain small $d > 0$. It is evident that

$$|\nabla\phi'|_{L^p(\Omega-\omega)} \leq C(d) \sum_{i,k=1}^{n} \int_{\partial\Omega} |a_{ik}(y) - \alpha_{ik}| d\sigma_y \,,$$

and, setting $\alpha_{ik} = [\text{meas}\{\partial\Omega\}]^{-1} \int_{\partial\Omega} a_{ik}(z) d\sigma_z$, we easily obtain

$$\begin{aligned} |\nabla\phi'|_{L^p(\Omega-\omega)} &\leq C(d) \sum_{i,k=1}^{n} \int_{\partial\Omega} |a_{ik}(y) - \alpha_{ik}| d\sigma_y \\ &\leq C_2 \left(\sum_{i,k=1}^{n} \int_{\partial\Omega}\int_{\partial\Omega} \frac{|a_{ik}(y) - a_{ik}(z)|^p}{|z-y|^{n-2+p}} d\sigma_y d\sigma_z\right)^{\frac{1}{p}} . \end{aligned} \tag{10.2}$$

Further we set $\alpha_{ik} = a_{ik}(\bar{x})$ where $\bar{x}$ is the closest point of $\partial\Omega$ to x and we obtain for $x \in \omega$

$$\begin{aligned} |\nabla\phi'(x)|^p &\leq C_3 \sum_{i,k=1}^{n} \int_{\partial\Omega} \frac{|a_{ik}(y) - a_{ik}(\bar{x})|^p}{|x-y|^{n+p-1-\varepsilon p}} d\sigma_y \left(\int_{\partial\Omega} \frac{1}{|x-y|^{n-1+\varepsilon p'}} d\sigma_y\right)^{p-1} \\ &\leq \frac{C}{|x-\bar{x}|^{\varepsilon p}} \sum_{i,k=1}^{n} \int_{\partial\Omega} \frac{|a_{ik}(y) - a_{ik}(\bar{x})|^p}{|x-y|^{n+p-1-\varepsilon p}} d\sigma_y \end{aligned}$$

and

$$(10.3)\qquad \int_\omega |\nabla\phi'(x)|^p dx \le C \sum_{i,k=1}^n \int_{\partial\Omega}\int_\omega \frac{|a_{ik}(y)-a_{ik}(\bar{x})|^p}{|x-y|^{n+p-1-\varepsilon p}|\bar{x}-x|^{\varepsilon p}} dx\, d\sigma_y\,.$$

If the number d is small enough, then to each point $x \in \omega$ there corrisponds a unique $\bar{x} \in \partial\Omega$, and

$$x = \bar{x} + \vec{n}(x)\rho, \quad |x-\bar{x}|\,.$$

Moreover, for arbitrary $x \in \omega$, $y \in \partial\Omega$ we have

$$|x-y|^2 = |\bar{x}-y|^2 + \rho^2 - 2\vec{n}(x)\cdot(\bar{x}-y)\rho\,,$$

so, since $|\vec{n}(x)\cdot(\bar{x}-y)| \le C_5|\bar{x}-y|^2$, there exist such constants C_6 and C_7 that

$$C_6|x-y|^2 \le \left(|\bar{x}-y|^2+\rho^2\right) \le C_7|x-y|^2\,.$$

If we split $\partial\Omega$ into small pieces and introduce on every piece local coordinates, we show easily that

$$\int_\omega \frac{|a_{ik}(y)-a_{ik}(\bar{x})|^p}{|x-y|^{n+p-1-p\varepsilon}|x-\bar{x}|^{\varepsilon p}}dx$$
$$\le C_8 \int_{\partial\Omega} |a_{ik}(y)-a_{ik}(\bar{x})|^p d\sigma_{\bar{x}} \int_0^d [\rho^{\varepsilon p}\left(|\bar{x}-y|^2+\rho^2\right)^{\frac{1}{2}(n+p-1-\varepsilon p)}]^{-1} d\rho\,.$$

Since the last integral does not exceed $C_9|\bar{x}-y|^{-n+2-q}$, this inequality together with (10.2) and (10.3) yields (10.1).

b) *Justification of formula* (2.6)

We show that the vector field $\mathbf{w}(x,t)$ (2.6) solves problem (2.3). The initial and boundary conditions for $\mathbf{w}(x,t)$ are evident. Further, the formula

$$\sum_{k=1}^n \frac{\partial}{\partial x_n}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_k}\mathcal{E}(x-z)\mathbf{a}(z)dz$$
$$= \lim_{\varepsilon\to 0}\sum_{k=1}^n \frac{\partial}{\partial x_n}\int_0^{x_n-\varepsilon}\int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_k}\mathcal{E}(x-z)\mathbf{a}(z)dz$$
$$= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_k}\mathcal{E}(x'-z',\varepsilon)\mathbf{a}(z',x_n-\varepsilon)dz = -\frac{1}{2}\mathbf{a}(x)$$

implies

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}G_{ij}(x,y,s) = \frac{\partial}{\partial x_j}(\Gamma(x-y,s)$$
$$-\Gamma(x-y^*,s)) + 2(1-\delta_{jn})\frac{\partial}{\partial x_j}\Gamma(x-y^*,s)$$
$$= -\frac{\partial}{\partial y_j}\left(\Gamma(x-y,s)+\Gamma(x-y^*,s)\right),$$

hence,

$$\nabla\cdot\mathbf{w}(x,t) = -\int_0^t\int_{\mathbb{R}^{n-1}} \nabla_y\big(\Gamma(x-y,t-\tau)+\Gamma(x-y^*,t-\tau)\big)P\mathbf{g}(y,\tau)dyd\tau = 0\,.$$

Finally, let us calculate $\frac{\partial}{\partial t}\mathbf{w}(x,t) - \Delta\mathbf{w}(x,t)$. Consider

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) G_{ij}(x-y,t) = 4(1-\delta_{jn})\frac{\partial}{\partial x_j}\int_0^{x_n} &\\ \cdot\int_{\mathbb{R}^{n-1}} \mathcal{E}_i(x-z)\frac{\partial}{\partial t}\Gamma(z-y^*,t)dz &\\ - 4(1-\delta_{jn})\frac{\partial}{\partial x_j}\Delta\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \mathcal{E}_i(x-z)\Gamma^*(z,t)dz\,, &\end{aligned}$$

where $\mathcal{E}_i(x-y) = \frac{\partial}{\partial x_i}\mathcal{E}(x-y)$, $\Gamma^*(z,t) = \Gamma(x-y^*,t)$. Since

$$\begin{aligned}&\Delta\int_0^{x_n}\int_{\mathbb{R}^{n-1}} \mathcal{E}_i(x-z)\Gamma^*(z,t)dz\\ &= \frac{\partial}{\partial x_n}\int_{\mathbb{R}^{n-1}} \mathcal{E}_i(\xi',x_n)\Gamma^*(x'-\xi'-y',-y_n,t)d\xi'\\ &+ \int_{\mathbb{R}^{n-1}} \mathcal{E}_i(\xi',x_n)\Gamma^*(x'-\xi'-y',-y_n,t)d\xi'\\ &+ \int_0^{x_n}\int_{\mathbb{R}^{n-1}} \mathcal{E}_i(\xi)\Delta'\Gamma(x-\xi-y^*,t)d\xi\,,\end{aligned}$$

we easily obtain

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \Delta\right) G_{ij}(x-y,t)\\ &= -4(1-\delta_{ij})\frac{\partial}{\partial x_j}\left\{\frac{\partial}{\partial x_n}\int_{\mathbb{R}^{n-1}} \mathcal{E}_i(\xi',x_n)\Gamma^*(x'-\xi'-y',-y_n,t)d\xi'\right.\\ &\left.+ \int_{\mathbb{R}^{n-1}} \mathcal{E}_i(\xi',x_n)\frac{\partial}{\partial x_n}\Gamma^*(x'-\xi'-y',-y_n,t)d\xi'\right\}\\ &= \frac{\partial}{\partial x_i}\left[\int_{\mathbb{R}^{n-1}} \mathcal{E}_n(x'-\xi',x_n)\Gamma^*(\xi'-y',y_n,t)d\xi'\right.\\ &\left.+ \int_{\mathbb{R}^{n-1}} \mathcal{E}(x'-\xi',x_n)\frac{\partial}{\partial x_n}\Gamma(\xi'-y',y_n,t)d\xi'\right].\ (t>0)\,.\end{aligned}$$

This shows that $\mathbf{w}(x,t)$ satisfies the relations (2.6) with the pressure

$$s(x,t) = -4\sum_{j=1}^{n-1}\frac{\partial}{\partial x_j}\int_0^t\int_{\mathbb{R}^n_+} \mathcal{Q}(x,y,t-\tau)(P\mathbf{g})_j(y,\tau)dy\,d\tau\,,$$

where

$$Q(x,y,t)=\int_{\mathbb{R}^{n-1}}\mathcal{E}_n(x'-y',x_n)\Gamma(\xi'-y',y_n,t)d\xi' + \int_{\mathbb{R}^{n-1}}\mathcal{E}(x'-\xi';x_n)\frac{\partial}{\partial x_n}\Gamma(\xi'-y',y_n,t)d\xi\,.$$

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