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Effective Diophantine Approximation on $\mathbb{G}_M$, II

E. BOMBIERI - P. B. COHEN

1. – Introduction

In [Bo], Theorem 1 a result on effective approximations at archimedean places to roots of high order of algebraic numbers was obtained and applied to give an effective result on archimedean diophantine approximation in a number field by a finitely generated multiplicative subgroup. The result obtained there was strong enough to derive a new effective solution of Thue's equation and a new proof of the Baker–Feldman theorem on approximations of algebraic numbers by rationals.

In this article we derive analogous results in the non-archimedean case. We obtain analogues of Theorems 1 and 2 in [Bo], incorporating in particular the refinements announced there. In fact, we are able to improve on the previous treatment in several respects, by introducing new tools and ideas.

The new tool used is M. Laurent's determinantal method, which replaces the traditional Siegel lemma arguments along the lines followed in recent work of P. Corvaja [Cor1], [Cor2]. His strategy has to be adapted somewhat for our purposes and we found it convenient to use the results of [Bo-V], [S-V] here. The main purpose in substituting the determinantal method for Siegel's lemma is to obtain in the end better numerical constants and a more constructive treatment. The quality of the results so obtained is definitely comparable with what could be obtained using Siegel's lemma and successive minima.

For the proof of Theorem 1 we use the equivariant Thue-Siegel principle developed by E. Bombieri, A.J. van der Poorten and J. Vaaler. One wants to apply the Thue-Siegel principle using $(\alpha, 1)$ as anchor pair, where α is an r-th root of a non-zero algebraic number a, together with the pairs $(\varepsilon\alpha, \varepsilon)$ where ε runs over all the r-th roots of unity. This exploits the fact that the action of the group of r-th roots of unity on the original anchor pair produces new anchor pairs and a corresponding gain in Dyson's lemma. There is another way to produce new anchor pairs from $(\alpha, 1)$ which is especially easy to control in the p-adic case. Namely, if α is close to 1 in some ultrametric valuation then α^m

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is at least as close to 1 in that same valuation for any integer m. One can again consider the orbit of $(\alpha^m, 1)$ under the action of the r-th roots of unity and apply Dyson's lemma with the anchor pairs $(\varepsilon\alpha^m, \varepsilon)$ where $|m| \leq M$ for some large integer M to be chosen. This enables us to gain substantially in Dyson's lemma as we work with these $(2M+1)r$ points instead of $[K(\alpha) : K]$ points as in [Bo]. The results obtained in this way allow us to replace exponential bounds by polynomial bounds, obtaining at the same time much better numerical constants than before.

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2. – Statement of results

We follow the notation of [Bo]. Therefore, if K is a number field then the absolute values $|\ |_v$ in K are normalised by requiring that for $x \in K$

$$|x|_v = \|x\|_v^{[K_v:\mathbb{Q}_v]/[K:\mathbb{Q}]}$$

where $\|x\|_v$ is the unique extension to the completion K_v of the ordinary real or p-adic absolute value in $\mathbb{Q}_v$. With this normalisation we have the following formula for the absolute logarithmic Weil height of a non-zero $x \in K$:

$$h(x) = \sum_v \log^+ |x|_v$$

where $\log^+ x = \max(0, \log x)$ and the summation is over all places v of K. One can for convenience define $h(0) = 0$. We denote by $H(\ )$ the absolute Weil height so that $h(x) = \log H(x)$. We sometimes work in an extension L of K and with an extension $|\ |_{\tilde{v}}$ to L of the normalised absolute value $|\ |_v$ on K.

We let $K(v)$ be the residue field of K_v and as usual f_v, e_v are the residue class degree and ramification index of the extension $K_v/\mathbb{Q}_v$. Finally, throughout the paper we abbreviate

$$d_v^* = \frac{d}{f_v \log p}, \quad D_v^* = \max(1, d_v^*)\,.$$

We shall prove the following result on effective approximation to roots which is a p-adic analogue of an improved version of Theorem 1 of [Bo].

THEOREM 1. *Let K be a number field of degree d and $a \in K$, not equal to zero or a root of unity. Let p be a rational prime, and suppose that v is a place of K dividing p such that $|a-1|_v < 1$.*

Let r be a positive integer coprime with p. Then a has an r-th root α satisfying $0 < |\alpha - 1|_{\tilde{v}} < 1$ for a place $\tilde{v}$ of $K(\alpha)$ extending v, where $|\ |_{\tilde{v}}$ is normalised so as to agree with $|\ |_v$ on K.

Let $\alpha' = \alpha\gamma$ with $\gamma \in K$, $\gamma \neq 0$. Then the following holds.

For any $0 < \kappa \leq 1$ the inequalities

$$\text{(H1)} \qquad r \geq 3.4 \times 10^{11} (D_v^*)^{10} \frac{1}{\kappa} \left(\log \frac{1}{\kappa} + 1\right)^7 h(a)$$

and

$$\text{(H2)} \qquad h(\alpha') \geq 1.7 \times 10^6 (D_v^*)^4 \frac{1}{\kappa} \left(\log \frac{1}{\kappa} + 1\right)^4$$

imply that

$$|\alpha' - 1|_{\tilde{v}} \geq H(\alpha')^{-r\kappa}.$$

As in [Bo], Theorem 1 can be applied to diophantine approximation in a number field by a finitely generated multiplicative subgroup.

Given that we have a result in Theorem 1 with (H1) of the form

$$r \geq c_1 h(a)$$

and (H2) of the form

$$h(\alpha') \geq c_2$$

with $c_1 \geq D_v^*$ (and c_1 and c_2 independent of r, a and α') then we can obtain a result in Theorem 2 of the form:

THEOREM 2. *Let K be a number field of degree d and let v be an absolute value of K dividing a rational prime p.*

Let Γ be a finitely generated subgroup of K^* and let $\xi_1, \dots, \xi_t$ be generators of Γ/tors. Let $\xi \in \Gamma$, $A \in K^*$ and $\kappa > 0$ be such that

$$0 < |1 - A\xi|_v < H(A\xi)^{-\kappa} \leq 1.$$

Define $h'(\xi_i) = \max(h(\xi_i), 1/d_v^*)$ and

$$Q = (2c_1 t)^t (50 c_2) p^{f_v} \prod_{i=1}^{t} h'(\xi_i).$$

Then we have

$$h(A\xi) \leq c_1 Q \max(h'(A), Q).$$

The proof of Theorem 2 will be given in Section 6. The same proof goes through for the case of archimedean valuations, giving thereby an improvement of Theorem 2 of [Bo], and in particular a better dependence of the upper bound for $h(A\xi)$ on $\prod_{i=1}^{t} h'(\xi_i)$.

A comparison with Baker's method using linear forms in logarithms may be in order here. Results of similar quality to our Theorem 1 can be obtained directly by Baker's method, in fact using only linear forms in two logarithms, which can be treated with the simpler earlier methods of Gelfond and Schneider. Indeed, this leads to forms of Theorem 1 with quite good constants, far better than those given here. However, our method is essentially different and therefore it seems reasonable to explore how far one can go with it in proving results of this type.

The deduction of Theorem 2 from Theorem 1 rests on a simple argument based on Geometry of Numbers.

As an application, we point out here that Theorem 2 allows us to obtain an explicit extension of the Baker–Feldman theorem to the p-adic case, obtaining an effective improvement in the exponent in the Liouville bound by $c(D_v^*, r)/(p^{f_v}R)$, where R is the regulator of the field $K(\alpha)$ and r is the degree of $K(\alpha)/K$. This is comparable with the best bounds obtained using the theory of linear forms in p-adic logarithms.

In conclusion, the Thue-Siegel method at the present stage does not yield general estimates superior to those obtainable using Baker's theory. In any case, we hope that the novelty of our approach is of sufficiently independent interest to deserve investigation.

3. – Construction of the interpolation matrix

In this section we draw on ideas from [Cor1] and [Cor2] although there are some differences as we shall explain. We adopt the notations and hypotheses of Section 2 and the statement of Theorem 1. We can clearly assume for the proof of Theorem 1 that $\alpha \neq 1$ and $|\alpha' - 1|_{\tilde{v}} < 1$.

We shall work with the following interpolation matrix with coefficients in the field $L = K(\alpha, \zeta)$ where ζ is a primitive r-th root of unity. Let N_1, N_2 be positive integers and $N = \max(N_1, N_2)$. Throughout, our estimates will be valid for N_1, N_2 sufficiently large and the asymptotics for the limit when N_1, N_2 tend to ∞. We suppose that

$$N_1/N_2 \to z > 0 \tag{3.1}$$

in this limit. Let $0 < \theta_1, \theta_2 \leq 1$ and for any $0 < t \leq 1$ let $G(t)$ be the set

$$G(t) = \left\{ (i_1, i_2) \in \mathbb{N}^2 : \frac{i_1}{\theta_1 N_1} + \frac{i_2}{\theta_2 N_2} < t \right\}$$

and $T(t)$ the area of the region

$$\left\{0 < x_1,\ x_2 < 1 : \frac{x_1}{\theta_1} + \frac{x_2}{\theta_2} < t\right\}.$$

Let $G = G(1)$ and $T = T(1)$. We have $|G(t)|/N_1N_2 \sim T(t)$. Let $\tau \in \mathbb{R}$ with $0 < \tau < \min(1, \frac{1}{2}\theta_1^{-1})$. Let M be a positive integer and consider the following matrix partitioned into $(2M+1)r$ vertical blocks

$$\mathcal{A} = (\,\mathcal{A}_{\varepsilon,m}\,)$$

where ε runs over all r-th roots of unity, $|m| \le M$, and

$$\mathcal{A}_{\varepsilon,m} = \left(\binom{u}{p}\binom{v}{q}\left(\varepsilon\alpha^m\right)^{u-p}\left(\varepsilon\alpha'^m\right)^{v-q}\right).$$

Here the columns of $\mathcal{A}_{\varepsilon,m}$ are indexed by the set S of integer pairs (u, v) with $0 \le u < N_1$ and $0 \le v < N_2$. The rows of the $\mathcal{A}_{\varepsilon,m}$ are indexed by integer pairs $(p, q) \in G$, for $m = 0$ and $(p, q) \in G(\tau)$, for $m \ne 0$. Therefore the number of rows of $\mathcal{A}$ equals $r(|G| + 2M|G(\tau)|)$ and the number of columns equals N_1N_2. Let b denote a positive sufficiently large constant, independent of N_1, N_2. Suppose that

$$r(|G| + 2M|G(\tau)|) = N_1N_2 + \frac{(2M+1)r}{2}\cdot\frac{\theta_2}{\theta_1}N_2^2 + bN + o(N) \tag{3.2}$$

with, for suitable b,

$$rT(1 + 2M\tau^2) > 1 + \frac{(2M+1)r}{2}\cdot\frac{\theta_2}{\theta_1}\,\frac{N_2}{N_1} \tag{3.3}$$

Notice that (as $r, \theta_1, \theta_2, \tau$ are independent of N_1, N_2)

$$rT(1 + 2M\tau^2) = 1 + \frac{(2M+1)r}{2}\cdot\frac{\theta_2}{\theta_1 z}. \tag{3.4}$$

By (3.2) we see that, for large N, the number of rows of the matrix M exceeds the number of its columns.

LEMMA 3.1. *Let r be a positive integer coprime with p. Suppose that $a \in K$ satisfies $|a-1|_v < 1$. Then there are an r-th root α of a and an extension $|\ \ |_{\tilde v}$ of the absolute value $|\ \ |_v$ of K to $K(\alpha)$ satisfying*

$$|\alpha - 1|_{\tilde v} = |a - 1|_v\,.$$

PROOF. Since $|a-1|_v < 1$ and $a - 1 = \prod(\varepsilon\alpha - 1)$ with the product ranging over all r-th roots of unity ε, the existence of α and $\tilde v$ such that $|\alpha - 1|_{\tilde v} < 1$ is clear. Moreover, since $|r|_v = 1$ we have

$$|a-1|_v = |(1+\alpha-1)^r - 1|_{\tilde v} = |r(\alpha-1) + \binom{r}{2}(\alpha-1)^2 + \cdots|_{\tilde v} = |\alpha - 1|_{\tilde v}\,. \ \square$$

LEMMA 3.2. *The matrix $\mathcal{A}$ has maximal rank N_1N_2.*

PROOF. This is a direct application of Viola's theorem as in [Bo], p. 70. Indeed, if there is a non-trivial linear relation between the columns of $\mathcal{A}$ then there is a non-zero polynomial of bi-degree (N_1-1, N_2-1) with the following properties. At the $(2M+1)r$ points $(\varepsilon\alpha^m, \varepsilon\alpha'^m)$ the polynomial has index at least

$$c_{\varepsilon,m} = \begin{cases} \tau & m \neq 0, \\ 1 & m = 0. \end{cases}$$

Now as by assumption α is not a root of unity each $\varepsilon\alpha^m$ occurs only once as ε and m vary. By the corresponding assumption on α', the same conclusion holds for $\varepsilon\alpha'^m$. Hence, as we have $0 < \tau, \theta_1, \theta_2 \leq 1$, Viola's version of Dyson's lemma allows us to conclude that this is impossible because of (3.3). □

By Lemma 3.2, there exists a subset I of the row indices of $\mathcal{A}$ with $|I| = N_1N_2$ such that $\det(\mathcal{A}_I) \neq 0$. Here we use the notations of [Bo-V] and [S-V]. We shall make a particular choice of I and estimate $|\det(\mathcal{A}_I)|_w$ at each place w of L. For all the places we use the following observations given in Lemmas 3.3 and 3.4.

LEMMA 3.3. *Let $\mathcal{C}$ be a square matrix, with coefficients in L_w, partitioned into two vertical blocks:*

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{pmatrix}.$$

Then

$$|\det(\mathcal{C})|_w \leq H_w(\mathcal{C}_1)H_w(\mathcal{C}_2).$$

PROOF. The statement of the lemma is a special case of the fundamental inequality for heights of matrices (the height of a matrix which is not of maximal rank vanishes by definition). □

As a special case we have

LEMMA 3.4. *Let $\mathcal{C} = (c_{ij})$ be an $n \times n$ matrix with coefficients in L_w satisfying*

$$\max_{(i,j)} |c_{ij}|_w \leq C_w.$$

Then

$$\log^+ |\det(\mathcal{C})|_w \leq \begin{cases} n \log^+ C_w & w \nmid \infty \\ n \log^+ C_w + \log(n!) & w \mid \infty. \end{cases}$$

PROOF. The lemma is a direct consequence of the Laplace expansion

$$\det(\mathcal{C}) = \sum \pm c_{1\sigma(1)} \cdots c_{n\sigma(n)}. \qquad \square$$

We choose the set I in the following way. Let $\mathcal{V}$ be the vector subspace of $L^{N_1 N_2}$ generated by the rows of the matrix

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{\varepsilon_1,0} \\ \mathcal{A}_{\varepsilon_2,0} \\ \cdots \\ \mathcal{A}_{\varepsilon_r,0} \end{pmatrix}$$

with ε_h running over all r-th roots of unity.

Then $\mathcal{V}$ has dimension $R = \operatorname{rank}(\mathcal{A}) \leq r|G| \lesssim rTN_1N_2$. We require that

$$rT < 1\,. \tag{3.5}$$

Now let $\mathcal{A}'$ be an $R \times N_1N_2$ matrix whose rows are a maximal linearly independent set of rows of $\mathcal{A}$. By Lemma 3.2, there is an $(N_1N_2 - R) \times N_1N_2$ matrix $\mathcal{B}'$ whose rows are rows of

$$\mathcal{B} = (\,\mathcal{A}_{\varepsilon,m}\,)_{\varepsilon^r=1,\,0<|m|\leq M}$$

and such that the matrix

$$\mathcal{C}' = \begin{pmatrix} \mathcal{A}' \\ \mathcal{B}' \end{pmatrix}$$

is an $N_1N_2 \times N_1N_2$ matrix with non-zero determinant.

4. – The equivariant Thue-Siegel principle

The idea of the estimates that follow is to use Lemma 3.3 and apply the results of [Bo-V] and [S-V] to deal with the term $H_w(\mathcal{A}')$ which will appear in the estimate of $|\det(\mathcal{C}')|_w$ at each place w of L. By definition, we have

$$H(\mathcal{V}) =: \prod_w H_w(\mathcal{A}')$$

where the product is taken over all the places of L. The choice of $\mathcal{A}'$ to have rows generating $\mathcal{V}$ is essential here and differs from the choice in [Cor1,2]. Our choice is also crucial for the estimate of $H_w(\mathcal{A}')$ for $w|v$. We obtain estimates which are better than the estimates obtained following the recipe of [Cor1,2] and moreover our construction is equivariant in that all the r-th roots of unity ε appear in the original interpolation matrix $\mathcal{A}$. The estimate of other contributions to $|\det(\mathcal{C}')|_w$ coming from the formula in Lemma 3.3 is done using Lemma 3.4. The following identity leads to the estimate which is the counterpart to the lower bound for the index as in Lemma 1 of [Bo]: for any $\alpha_1, \alpha_2, \beta_1, \beta_2$ we have

$$\binom{u}{p}\binom{v}{q}\alpha_1^{u-p}\alpha_2^{v-q} = \binom{u}{p}\binom{v}{q}(\beta_1 + (\alpha_1 - \beta_1))^{u-p}(\beta_2 + (\alpha_2 - \beta_2))^{v-q}$$

$$= \sum_{(i,j)\in\mathbb{N}^2} \binom{p+i}{i}\binom{q+j}{j}(\alpha_1-\beta_1)^i(\alpha_2-\beta_2)^j \tag{4.1}$$

$$\times \binom{u}{p+i}\binom{v}{q+j}\beta_1^{u-p-i}\beta_2^{v-q-j}\,.$$

Now, for any place w of L we have by Lemma 3.3

$$\log|\det(\mathcal{C}')|_w \le \log H_w(\mathcal{A}') + \log H_w(\mathcal{B}')\,. \tag{4.2}$$

Suppose first that $w \nmid v$. In this case we estimate the terms in (4.2) trivially. Since $\mathcal{B}'$ has $N_1N_2 - R$ rows, by the definition of the matrix height we see that

$$\log H_w(\mathcal{B}') \begin{cases} \le (N_1N_2-R)(N_1M\log^+|\alpha|_w + N_2M\log^+|\alpha'|_w) \\ \qquad\qquad\qquad\qquad \text{if} \quad w\nmid\infty,\ w\nmid v \\ \lesssim (N_1N_2-R)(N_1(M\log^+|\alpha|_w + \delta_w\phi(\theta_1\tau)) \\ \quad +N_2(M\log^+|\alpha'|_w + \delta_w\phi(\theta_2\tau))) \\ \qquad\qquad\qquad\qquad \text{if} \quad w|\infty,\ w\nmid v \end{cases} \tag{4.3}$$

where $\delta_w = [L_w : \mathbb{Q}_w]/[L:\mathbb{Q}]$ and $\phi(t) = t\log\frac{1}{t} + (1-t)\log\frac{1}{1-t}$.

If instead $w|v$, we proceed using a Taylor expansion as follows. As v is a finite place, there is a unique r-th root of unity ε_w such that

$$\begin{cases} |\alpha^m \ - \varepsilon_w^m|_w \le |\alpha \ - \varepsilon_w|_w = |\alpha-1|_{\tilde v}^{\tilde\delta_w} < 1, \\ |\alpha'^m - \varepsilon_w^m|_w \le |\alpha' - \varepsilon_w|_w = |\alpha'-1|_{\tilde v}^{\tilde\delta_w} < 1 \end{cases} \tag{4.4}$$

If for $(p,q)\in G(\tau)$, ε an r-th root of unity and $0<|m|\le M$ the vector in $L^{N_1N_2}$ given by

$$r(p,q,m,\varepsilon) = \left(\binom{u}{p}\binom{v}{q}(\varepsilon\alpha^m)^{u-p}(\varepsilon\alpha'^m)^{v-q}\right)_{(u,v)\in S}$$

occurs in $\mathcal{B}'$, let $\mathcal{C}^{(i,j)}_{(p,q,m,\varepsilon)}$ be the matrix obtained from $\mathcal{C}'$ by replacing the row $r(p,q,m,\varepsilon)$ by the row

$$r^{(i,j)}(p,q,m,\varepsilon) = \left(\binom{u}{p+i}\binom{v}{q+j}(\varepsilon\varepsilon_w^m)^{u-p-i}(\varepsilon\varepsilon_w^m)^{v-q-j}\right)_{(u,v)\in S}$$

Then, as the determinant of a matrix is a linear function of any one of its rows, we have, using (4.1) with the substitution $(\alpha_1,\alpha_2) = (\varepsilon\alpha^m, \varepsilon\alpha'^m)$, $(\beta_1,\beta_2) = (\varepsilon\varepsilon_w^m, \varepsilon\varepsilon_w^m)$, that

$$\det(\mathcal{C}') = \sum_{(i,j)} \binom{p+i}{i}\binom{q+j}{j}(\varepsilon\alpha^m - \varepsilon\varepsilon_w^m)^i(\varepsilon\alpha'^m - \varepsilon\varepsilon_w^m)^j \det\left(\mathcal{C}^{(i,j)}_{(p,q,m,\varepsilon)}\right).$$

For $(i, j) \in G(1-\tau)$, the row vector $r^{(i,j)}(p, q, m, \varepsilon)$ is in $\mathcal{V}$ so that the rows of $\mathcal{C}^{(i,j)}_{(p,q,m,\varepsilon)}$ are linearly dependent. Therefore,

$$\det(\mathcal{C}') = \sum_{(i,j)\notin G(1-\tau)} \binom{p+i}{i}\binom{q+j}{j}(\varepsilon\alpha^m - \varepsilon\varepsilon_w^m)^i(\varepsilon\alpha'^m - \varepsilon\varepsilon_w^m)^j \det\left(\mathcal{C}^{(i,j)}_{(p,q,m,\varepsilon)}\right).$$

Let $\mathcal{E}_w$ be the matrix obtained from $\mathcal{B}'$ by replacing the row $r(p, q, m, \varepsilon)$ of $\mathcal{B}'$ by the row with (u, v)-th entry

$$\sum_{(i,j)\notin G(1-\tau)} \binom{p+i}{i}\binom{q+j}{j}(\varepsilon\alpha^m - \varepsilon\varepsilon_w^m)^i(\varepsilon\alpha'^m - \varepsilon\varepsilon_w^m)^j$$
$$\times \binom{u}{p+i}\binom{v}{q+j}(\varepsilon\varepsilon_w^m)^{u-p-i}(\varepsilon\varepsilon_w^m)^{v-q-j}.$$

Then by the above discussion,

$$\det(\mathcal{C}') = \det\begin{pmatrix}\mathcal{A}'\\ \mathcal{E}_w\end{pmatrix}$$

and by Lemma 3.3,

$$\log|\det(\mathcal{C}')|_w \le \log H_w(\mathcal{A}') + \log H_w(\mathcal{E}_w). \tag{4.5}$$

As $w|v$, and in view of (4.4), the logarithm of the $|\ |_w$-valuations of the entries of $\mathcal{E}_w$ are bounded above by

$$\tilde{\delta}_w \max_{(i,j)\notin G(1-\tau)} \{i \log|\alpha - 1|_{\tilde{v}} + j \log|\alpha' - 1|_{\tilde{v}}\}$$
$$\le -\tilde{\delta}_w(1-\tau)\min(N_1\theta_1 \log|\alpha - 1|_{\tilde{v}}^{-1}, N_2\theta_2 \log|\alpha' - 1|_{\tilde{v}}^{-1}).$$

By Lemma 3.4 and $\sum_{w|v} \tilde{\delta}_w = 1$ it follows that

$$\sum_{w|v} \log H_w(\mathcal{E}_w) \tag{4.6}$$
$$\le -(N_1N_2 - R)(1-\tau)\min(N_1\theta_1 \log|\alpha - 1|_{\tilde{v}}^{-1}, N_2\theta_2 \log|\alpha' - 1|_{\tilde{v}}^{-1}).$$

Now by the product formula and $\det(\mathcal{C}') \neq 0$ we have

$$0 = \sum_w \log|\det(\mathcal{C}')|_w. \tag{4.7}$$

From (4.2), (4.3), (4.5), (4.6) and (4.7),

$$
\begin{aligned}
&(N_1N_2 - R)(1-\tau)\min(N_1\theta_1 \log|\alpha - 1|_{\tilde v}^{-1}, N_2\theta_2 \log|\alpha' - 1|_{\tilde v}^{-1}) \\
&\lesssim \log H(\mathcal{V}) + (N_1N_2 - R)(N_1Mh(\alpha) \\
&\quad + N_2Mh(\alpha') + \phi(\theta_1\tau)N_1 + \phi(\theta_2\tau)N_2).
\end{aligned} \tag{4.8}
$$

Equation 1.11 of Theorem 2 and Corollary 6 in [S-V] give directly

$$
\begin{aligned}
&\log H(\mathcal{V}) \le rTN_1N_2 \\
&\quad \cdot \left\{ N_1\frac{1}{3}\theta_1\Big(\log\Big(\frac{1}{4\theta_1}\Big) + \frac{11}{18}\Big) + N_2\frac{1}{3}\theta_2\Big(\log\Big(\frac{1}{4\theta_2}\Big) + \frac{11}{18}\Big)\right\}.
\end{aligned} \tag{4.9}
$$

Dividing both sides of (4.8) by $N_1N_2^2$ and passing to the limit as $N_1, N_2 \to \infty$ we have from (4.8) and (4.9) the following result.

LEMMA 4.1. *Let $a, \gamma \in K$ and let v be a place of K dividing a rational prime p. Let r be an integer coprime with p, and suppose that a is not a root of unity and has an r-th root α satisfying $|\alpha - 1|_{\tilde v} < 1$ for a place $\tilde v$ of $K(\alpha)$ extending v. Let $\alpha' = \gamma\alpha$, suppose that α' is not a root of unity and that $|\alpha' - 1|_{\tilde v} < 1$.*

Let $0 < \theta_1, \theta_2 \le 1$ and $0 < \tau \le \min(1, \frac{1}{2}\theta_1^{-1})$. Suppose that $T = \frac{1}{2}\theta_1\theta_2$ satisfies $rT < 1$. Let $z > 0$ and let M be a positive integer such that

$$
\tau = \sqrt{\frac{1 - rT}{2MrT} + \frac{2M+1}{4M} \cdot \frac{1}{T} \cdot \frac{\theta_2}{\theta_1 z}}\,.
$$

Then we have

$$
\begin{aligned}
(1-\tau)\min(z\theta_1 \log|\alpha - 1|_{\tilde v}^{-1}, \theta_2 \log|\alpha' - 1|_{\tilde v}^{-1}) \\
\le (Mh(\alpha) + V(\theta_1))z + Mh(\alpha') + V(\theta_2)
\end{aligned}
$$

where

$$
V(\theta) = \phi(\theta\tau) + \frac{rT}{(1 - rT)}\Big(\frac{1}{3}\theta \log\Big(\frac{1}{4\theta}\Big) + \frac{11}{18}\theta\Big)
$$

and

$$
\phi(t) = t\log\frac{1}{t} + (1-t)\log\frac{1}{1-t}\,.
$$

5. – Proof of Theorem 1

As in the statement of Theorem 1, let $a, \gamma \in K$, with a not a root of unity and $\gamma \neq 0$, and let v be a place of K dividing a rational prime p. Let r be an integer coprime with p and suppose that a has an r-th root α which we assume satisfies $|\alpha - 1|_{\tilde{v}} < 1$ for a place $\tilde{v}$ of $K(\alpha)$ extending v. We write $\alpha' = \gamma\alpha$ and suppose that α' is not a root of unity.

Let $1 \geq \kappa > 0$ and suppose that

$$|\alpha' - 1|_{\tilde{v}} < H(\alpha')^{-r\kappa} \leq 1$$

so that

$$\log \frac{1}{|\alpha' - 1|_{\tilde{v}}} > r\,\kappa\, h(\alpha') \geq 0\,. \tag{A5.1}$$

Let $0 < \theta_1, \theta_2 \leq 1$ and $0 < \tau \leq \min(1, 1/(2\theta_1))$. Let $T = \frac{1}{2}\theta_1\theta_2$ with $rT < 1$, let $z > 0$ and let M be a positive integer with

$$\tau = \sqrt{\frac{1 - rT}{2MrT} + \frac{2M+1}{4M} \cdot \frac{1}{T} \cdot \frac{\theta_2}{\theta_1 z}}\,.$$

Then by Lemma 4.1 and with the notation of that same lemma we have

$$\begin{aligned}(1 - \tau)\min(\theta_1 z \log|\alpha - 1|_{\tilde{v}}^{-1}, \theta_2 \log|\alpha' - 1|_{\tilde{v}}^{-1}) \\ \leq (Mh(\alpha) + V(\theta_1))z + Mh(\alpha') + V(\theta_2).\end{aligned} \tag{5.2}$$

We abbreviate

$$X = rT, \quad A = \frac{Mh(a)}{r} + V(\theta_1), \quad \Lambda = \log \frac{1}{|\alpha - 1|_{\tilde{v}}}$$

and choose $z = h(\alpha')\theta_2 r\kappa/(\theta_1 l)$ with

$$0 < l \leq \min(1, \Lambda)\,. \tag{A5.3}$$

Then (5.2), using (A5.1) and dividing both sides by $\theta_2 r\kappa h(\alpha')$, simplifies to

$$1 - \tau \leq \frac{A}{\theta_1 l} + \frac{M+1}{\theta_2 r\kappa}\,, \tag{5.4}$$

provided we make the additional assumption

$$h(\alpha') \geq V(\theta_2)\,. \tag{A5.5}$$

With our new notation, τ becomes

$$\tau = \sqrt{\frac{1-X}{2MX} + \frac{2M+1}{4M} \cdot \frac{1}{X} \cdot \frac{l}{\kappa h(\alpha')}}\,. \tag{5.6}$$

We have, using $(\frac{1-u}{u}) \log \frac{1}{1-u} < 1$ for $0 < u < 1$ and $\frac{1}{3} \log \frac{1}{4} < -\frac{1}{3}$:

$$\begin{aligned}\frac{A}{\theta_1} = \frac{Mh(a)}{\theta_1 r} &+ \frac{X}{1-X}\left(\frac{1}{3}\log\frac{1}{4\theta_1} + \frac{11}{18}\right)\\ &+ \tau \log\frac{1}{\theta_1\tau} + \tau\,\frac{1-\theta_1\tau}{\theta_1\tau}\,\log\frac{1}{1-\theta_1\tau}\\ &\le \frac{Mh(a)}{\theta_1 r} + \frac{X}{3(1-X)}\left(\log\frac{1}{\theta_1} + 1\right) + \tau\log\frac{1}{\theta_1\tau} + \tau\,.\end{aligned}$$

We substitute this inequality into (5.4) and find *a fortiori*

$$\begin{aligned}1 \le \frac{Mh(a)}{\theta_1 lr} &+ \left(\frac{X}{3l(1-X)} + \frac{\tau}{l}\right)\left(\log\frac{1}{\theta_1} + 1\right)\\ &+ \frac{\tau}{l}\left(\log\frac{1}{\tau} + 1\right) + \frac{M+1}{\theta_2 r\kappa}\,.\end{aligned} \tag{5.7}$$

With

$$0 < uX \le \frac{1}{4} \tag{A5.8}$$

we choose

$$M = \left[\frac{1}{2u^2X^3}\right] - 1 \tag{5.9}$$

and assume

$$h(\alpha') \ge \frac{2l}{u^2X^4}\,\frac{1}{\kappa}\,, \tag{A5.10}$$

thus guaranteeing that $\tau \le uX$. Since the right-hand side of (5.7) increases with τ we may replace τ with uX in (5.7) obtaining

$$\begin{aligned}1 \le \frac{Mh(a)}{\theta_1 lr} &+ \left(u + \frac{1}{3(1-X)}\right)\frac{X}{l}\left(\log\frac{1}{\theta_1} + 1\right)\\ &+ \frac{uX}{l}\left(\log\frac{1}{uX} + 1\right) + \frac{M+1}{\theta_2 r\kappa}\,.\end{aligned} \tag{5.11}$$

Next, we have

$$\frac{M+1}{\theta_2 r\kappa} = \frac{(M+1)\theta_1}{2X\kappa} < \frac{\theta_1}{4u^2X^4\kappa}$$

so that (5.11) simplifies to

$$1 \le \frac{h(a)}{2u^2X^3\theta_1 lr} + \left(u + \frac{1}{3(1-X)}\right)\frac{X}{l}\left(\log\frac{1}{\theta_1}+1\right) + \frac{uX}{l}\left(\log\frac{1}{uX}+1\right) + \frac{\theta_1}{4u^2X^4\kappa}. \tag{5.12}$$

We choose $u = \frac{4}{9}$, $\theta_1 = (\frac{4}{9})^3X^4\kappa$ and assume

$$r \ge 2\left(\frac{9}{4}\right)^6 \frac{1}{X^7 l}\frac{1}{\kappa}h(a). \tag{A5.13}$$

Our choices for u and θ_1 together with (A5.13) transform (5.12) into

$$1 \le \frac{1}{9} + \left(\frac{4}{9} + \frac{1}{3(1-X)}\right)\frac{X}{l}\left(\log\frac{1}{\kappa} + 4\log\frac{1}{X} + 4\log\frac{9}{4}+1\right) + \frac{4}{9}\frac{X}{l}\left(\log\frac{1}{X}+\log\frac{9}{4}+1\right) + \frac{1}{9}. \tag{5.14}$$

If we choose

$$X = \frac{1}{20}l^{1.25}\left(\log\frac{1}{\kappa}+1\right)^{-1}$$

then we verify that the maximum of the right-hand side of (5.14) in the square $0 < \kappa \le 1$, $0 < l \le 1$ does not exceed 0.98, a contradiction.

We conclude that one of our assumptions (A5.1), (A5.3), (A5.5), (A5.8), (A5.10), (A5.13) does not hold. Our choice of X shows that (A5.8) is automatically verified, also (A5.10) implies (A5.5). Moreover, by Lemma 1 we have

$$\Lambda = \log\frac{1}{|1-a|_v} \ge \frac{f_v \log p}{d} = \frac{1}{d_v^*}$$

and choosing $l = \min(1, 1/d_v^*) = 1/D_v^*$ we see that (A5.3) is satisfied. It follows that, with the above choices of X and l, (A5.10) and (A5.13) imply the negation of (A5.1), which is the conclusion of Theorem 1. □

REMARK. If $|a-1|_v = 1$ and $|a|_v = 1$ we raise a to its exponent $e = e_v(a)$ and work with α^e instead of α. As $h(\alpha^e) = eh(\alpha)$, the lower bound in (H1) is multiplied by $e_v(a)$. This also allows for the assumption $\theta_2 \le e^{-1}$ which is required in Dyson's lemma if one works with the points $(\varepsilon\alpha^{em}, \varepsilon\alpha'^{m})$ as there may then be at most e repetitions in the first coordinate.

6. – Proof of Theorem 2

To simplify notations, we rewrite the hypotheses (H1) and (H2) of Theorem 1 in the form

$$\text{(H1)} \qquad r \geq c_1 h(a),$$

$$\text{(H2)} \qquad h(\alpha') \geq c_2\,,$$

where $c_1 \geq D_v^*$ and c_2 are given by that same theorem.

LEMMA 6.1. *Let n_i, d_i, g_i, $i = 1, \dots, t$ be rational integers, let*

$$B = \Big|\sum_{i=1}^{t} d_i n_i\Big|$$

and let λ_i, $i = 1 \dots, t$ be positive real numbers with $\prod_{i=1}^{t} \lambda_i = 1$.

Define

$$\Delta = \sqrt{1 + \sum_{i=1}^{t} (d_i\lambda_i)^2 Q^{-2/t}}\,.$$

Let D, N, Q be positive integers with $Q > \max \lambda_i^t$ and $N \geq 2DBQ$. Let p be a rational prime. Then there are natural integers r, q with $r \geq 2$ coprime to p, rational integers p_i, $i = 1, \dots, t$ and a rational number u with $|u| \leq 1$ such that

$$n_i - rp_i = r\Big(q\frac{n_i}{N} - p_i\Big) + u\,\frac{n_i}{r+u}\,,$$

$$\sum_{i=1}^{t} d_i p_i = 0\,,$$

$$\sum_{i=1}^{t} g_i p_i \equiv 0 \pmod{D}\,,$$

$$N/(2D\Delta Q) - 1 \leq r \leq N + 1$$

and

$$\Big|q\frac{n_i}{N} - p_i\Big| \leq \lambda_i Q^{-1/t}\,.$$

PROOF. . Let $x_i = n_i/N$ and let $\mathcal{S}$ be the region of points $(v, u_0, u_1, \dots, u_t) \in \mathbb{R}^{t+2}$ defined by

$$\Big|Dv - \sum_{i=1}^{t} g_i u_i\Big| < 1\,,$$

$$|u_0| < 2D\Delta Q\,,$$

$$\Big|\sum_{i=1}^{t} d_i u_i\Big| < 1\,,$$

$$|x_i u_0 - u_i| < \lambda_i Q^{-1/t}, \quad i = 1, \dots, t\,.$$

The region $\mathcal{S}$ is a convex symmetric open set about the origin. A lower bound for its volume can be obtained as follows.

Let us abbreviate $L_i = \lambda_i^{-1} Q^{1/t}$ and let $\mathcal{M}$ be the $(t+3)\times(t+2)$ matrix

$$\mathcal{M} = \begin{pmatrix} D & 0 & -g_1 & -g_2 & \cdots & -g_t \\ 0 & \frac{1}{2D\Delta Q} & 0 & 0 & \cdots & 0 \\ 0 & 0 & d_1 & d_2 & \cdots & d_t \\ 0 & x_1 L_1 & -L_1 & 0 & \cdots & 0 \\ 0 & x_2 L_2 & 0 & -L_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & x_t L_t & 0 & 0 & \cdots & -L_t \end{pmatrix}.$$

Then we have (see [Bo-V]):

$$\text{Vol}(\mathcal{S}) \geq 2^{t+2}\left(\det(\mathcal{M}^t\cdot\mathcal{M})\right)^{-\frac{1}{2}} \tag{6.1}$$

where $\mathcal{M}^t$ is the transpose. By the Cauchy-Binet theorem we get

$$\begin{aligned}\det(\mathcal{M}^t\cdot\mathcal{M}) = {} & D^2\,(L_1L_2\cdots L_t)^2 \\ & \cdot\left\{(2D\Delta Q)^{-2}\Delta^2 + (d_1x_1 + d_2x_2 + \cdots + d_tx_t)^2\right\}.\end{aligned}$$

Since $L_1L_2\cdots L_t = Q$ and $|\sum d_i x_i| = B/N \leq 1/(2DQ)$, we see that

$$\det(\mathcal{M}^t\cdot\mathcal{M}) \leq (\frac{1}{2})^2 + (\frac{1}{2})^2 < 1$$

and, using (6.1), we infer that

$$\text{vol}(\mathcal{S}) > 2^{t+2}.$$

By Minkowski's theorem, the set $\mathcal{S}$ contains a lattice point $(v, q, p_1, \ldots, p_t)$ of $\mathbb{Z}^{t+2}$ other than the origin. Clearly we must have

$$Dv - \sum_{i=1}^{t} g_i u_i = 0$$

and

$$\sum_{i=1}^{t} d_i u_i = 0$$

since they are integers less than 1 in absolute value. Moreover, we cannot have $q = 0$, because otherwise $|p_i| < \lambda_i Q^{-1/t} \leq 1$ for $i = 1, \ldots, t$, giving $p_1 = \cdots = p_t = 0$ and $v = 0$, a contradiction. Thus, changing the sign of the coordinates if needed, we may suppose $1 \leq q < 2D\Delta Q$.

To complete the proof of Lemma 6.1 we observe that

$$qx_i = q\frac{n_i}{N} = \frac{n_i}{r+u}$$

with $r = N/q - u$. The closed interval $[(N/q) - 1, (N/q) + 1]$ contains two consecutive integers, hence it contains an integer r coprime with p. The identity

$$\frac{n_i}{r} - p_i = qx_i - p_i + \frac{u}{r} \cdot \frac{n_i}{r+u}$$

completes the proof. □

LEMMA 6.2. *Let K be a number field of degree d, let Γ be a finitely generated multiplicative subgroup of K^* and let $\xi_1, \dots, \xi_t$ be a set of generators of Γ/tors. Let also $A \in K$, $A \neq 0$, let v be an absolute value of K dividing a rational prime p and let Q, N be positive real numbers such that*

$$Q \geq (td_v^*)^t \prod_{i=1}^{t} h'(\xi_i)$$

and

$$N \geq 6p^{f_v} d_v^* h'(A) Q$$

where f_v is the residue class degree of $K_v/\mathbb{Q}_v$. Finally, let $\xi \in \Gamma$ be such that $A\xi$ is not a root of unity and

$$0 < |1 - A\xi|_v < H(A\xi)^{-\kappa} \leq 1\,.$$

Then we can find an element $a \in A\Gamma$ not a root of unity, an element $\gamma \in \Gamma$, an r-th root α of a with r coprime to p and an extension $|\ |_{\tilde{v}}$ of $|\ |_v$ to $K(\alpha)$ such that:

$$\begin{aligned}
&N/(2\sqrt{2}p^{f_v}Q) - 1 \leq r \leq N + 3,\\
&(\alpha\gamma)^r = A\xi\,,\\
&h(a) \leq h(A) + rt\Big(Q^{-1}\prod_{i=1}^{t} h'(\xi_i)\Big)^{1/t} + \frac{4}{r}h(\xi)
\end{aligned}$$

and

$$0 < |1-\alpha|_{\tilde{v}} < 1, \qquad 0 < |1-\alpha\gamma|_{\tilde{v}} < H(\alpha\gamma)^{-\kappa r}\,.$$

PROOF. Write $\xi = \zeta\xi_1^{n_1}\dots\xi_t^{n_t}$ where $\zeta \in \mathrm{tors}(\Gamma)$ and let $\mathrm{ord}_v(\xi_i) = d_i/e_v$, $\mathrm{ord}_v(A) = \delta/e_v$. Notice that $|1 - A\xi|_v < 1$ implies

$$\delta = -\sum_{i=1}^{t} d_i n_i\,. \tag{6.2}$$

Let π be a uniformising parameter in K_v, let ψ be a generator of the multiplicative group of the residue field $K(v)$ (hence of order $D = p^{f_v} - 1$) and let ψ^{g_i} be the class of $\pi^{-d_i}\xi_i$ in $K(v)^*$.

We apply Lemma 6.1 to $n_1, \ldots, n_t$ with the above definition of D, d_i and g_i and

$$\lambda_i = h'(\xi_i)^{-1}\left(\prod_{i=1}^{t} h'(\xi_i)\right)^{1/t}.$$

In this instance, by (6.2) we have

$$\begin{aligned} B = \left|\sum_{i=1}^{t} d_i n_i\right| &= e_v|\mathrm{ord}_v(A)| \\ &= \left(\frac{e_v d}{[K_v : \mathbb{Q}_v]\log p}\right) \big|\log|A|_v\big| \\ &= d_v^* \cdot \big|\log|A|_v\big| \le d_v^*\, h'(A)\,. \end{aligned}$$

In particular, we see that the hypothesis of Lemma 6.2 implies the condition $N \ge 2DBQ$ postulated in Lemma 6.1.

In a similar way, we have $|d_i| \le d_v^* h'(\xi_i)$ and

$$\Delta^2 \le 1 + t(d_v^*)^2\left(Q/\prod h'(\xi_i)\right)^{-2/t} \le 2\,.$$

Write $q_i = n_i - rp_i$, $\gamma = \xi_1^{p_1}\cdots\xi_t^{p_t}$ and $\xi_0 = \zeta\xi_1^{q_1}\ldots\xi_t^{q_t}$. Then

$$\mathrm{ord}_v(\gamma) = \frac{1}{e_v}\sum_{i=1}^{t} d_i p_i = 0$$

and the class of γ in the residue field $K(v)$ is ψ^g with

$$g = \sum_{i=1}^{t} g_i p_i \equiv 0 \,(\mathrm{mod}\ p^{f_v} - 1)\,,$$

giving $\psi^g = 1$ and

$$|\gamma - 1|_v < 1\,. \tag{6.3}$$

Moreover, since $\Delta \le \sqrt{2}$ and $N \ge 6p^{f_v}d_v^* h'(A)Q > 4\sqrt{2}p^{f_v}Q$, we have

$$1 < N/(2\sqrt{2}DQ) - 1 \le r \le N + 1$$

and $r \geq 2$ coprime to p. Now, abbreviating $x_i = n_i/N$, we have

$$\begin{aligned} h(\xi_0) &= h(\xi_1^{n_1 - rp_1} \cdots \xi_t^{n_t - rp_t}) \\ &= h\left(\prod_{i=1}^{t} \xi_i^{r(qx_i - p_i)} \prod_{i=1}^{t} \xi_i^{\frac{u}{r+u} \cdot n_i}\right) \\ &\leq r \sum_{i=1}^{t} |qx_i - p_i| h(\xi_i) + \frac{|u|}{r+u} h\left(\prod_{i=1}^{t} \xi_i^{n_i}\right). \end{aligned}$$

Hence on using Lemma 6.1 and the previous inequality we find

$$\begin{aligned} h(\xi_0) &\leq r \sum_{i=1}^{t} \lambda_i Q^{-1/t} h(\xi_i) + \frac{|u|}{r+u} h(\xi) \\ &\leq r \sum_{i=1}^{t} \lambda_i Q^{-1/t} h(\xi_i) + \frac{1}{r-1} h(\xi). \end{aligned}$$

Set $a = A\xi_0$ so that $a \in A\Gamma$ and $a\gamma^r = A\xi$. Then

$$h(a) \leq h(A) + h(\xi_0) \leq h(A) + rt\left(Q^{-1} \prod_{i=1}^{t} h'(\xi_i)\right)^{1/t} + \frac{1}{r-1} h(\xi).$$

As r is coprime to p we have by Lemma 3.1

$$|1 - \alpha\gamma|_{\tilde{v}} = |1 - A\xi|_v < 1$$

for some choice of an r-th root α of a and a suitable extension $\tilde{v}$ of v to $K(\alpha)$. By (6.3), we thus obtain $|\alpha - 1|_{\tilde{v}} < 1$, as wanted. Therefore, if for $\kappa > 0$

$$0 < |1 - A\xi|_v < H(A\xi)^{-\kappa}$$

then

$$0 < |1 - \alpha\gamma|_{\tilde{v}} < H(\alpha\gamma)^{-\kappa r}.$$

This concludes the proof of Lemma 6.2, with something to spare, if a is not a root of unity.

Now suppose that $a = A\xi_0$ is a root of unity. This implies $h(A) = h(\xi_0)$. Also we have $A\xi = a\gamma^r$, giving $h(\gamma) = \frac{1}{r} h(\xi/\xi_0)$. Now for $\varepsilon = 1$ or $\varepsilon = 2$ we see that $r + \varepsilon$ is coprime with p. Set $\widetilde{r} = r + \varepsilon$, $\widetilde{a} = a\gamma^{-\varepsilon}$; $\widetilde{a}$ is not a root of unity because $A\xi$ is not a root of unity by hypothesis.

Since $A\xi = a\gamma^r$, we have $A\xi = \widetilde{a}\gamma^{\widetilde{r}}$. Since a is a root of unity, we also have

$$h(\widetilde{a}) = h(a\gamma^{-\varepsilon}) = \frac{\varepsilon}{r} h(\xi/\xi_0) \leq \frac{\varepsilon}{r} h(\xi_0) + \frac{\varepsilon}{r} h(\xi) \leq h(A) + \frac{4}{\widetilde{r}} h(\xi).$$

This proves what we wanted, with $\widetilde{r}$, $\widetilde{a}$ in place of r, a. □

We now apply Lemma 6.2 to prove Theorem 2.

We may assume that $A\xi$ is not a root of unity, otherwise Theorem 2 is trivial. Suppose that the hypotheses of Lemma 6.2 are satisfied, namely

$$0 < |1 - A\xi|_v < H(A\xi)^{-\kappa} \leq 1 , \tag{6.4}$$

$$Q \geq (td_v^*)^t \prod_{i=1}^{t} h'(\xi_i) \tag{6.5}$$

and

$$N \geq 6p^{f_v} d_v^* h'(A) Q . \tag{6.6}$$

Then by Lemma 6.2 we have

$$|1 - \alpha|_{\tilde{v}} < 1, \qquad |1 - \alpha\gamma|_{\tilde{v}} < H(\alpha\gamma)^{-\kappa r} .$$

Hence, with α and γ as in Lemma 6.2, not all the hypotheses of Theorem 1 can be satisfied. In the notation of (H1) and (H2) given at the beginning of the present section, assume that (H1) holds, namely

$$r \geq c_1 h(a) . \tag{6.7}$$

Then (H2) cannot hold and we must have $h(\alpha\gamma) \leq c_2$, yielding

$$h(A\xi) \leq c_2 r . \tag{6.8}$$

It follows that (6.4), (6.5), (6.6) and (6.7) imply (6.8).

It remains to verify (6.7) and obtain a nontrivial upper bound for r. We begin with the verification of (6.7). By Lemma 6.2 we have

$$h(a) \leq h(A) + rt\left(Q^{-1} \prod_{i=1}^{t} h'(\xi_i) \right)^{1/t} + \frac{4}{r} h(\xi) .$$

We choose

$$Q = (2c_1 t)^t \prod_{i=1}^{t} h'(\xi_i) . \tag{6.9}$$

Then we deduce the inequality

$$h'(a) \leq h'(A) + \frac{r}{2c_1} + \frac{4}{r} h(\xi)$$

and for (6.7) to be true it suffices to have

$$r \geq c_1 \cdot \left(h'(A) + \frac{r}{2c_1} + \frac{4}{r}\, h(\xi)\right)$$

which may be rewritten as

$$r \geq 2c_1 h'(A) + \frac{8c_1}{r}\, h(\xi)\,.$$

For this, and hence for (6.7) to hold, we note that $x \geq 2u + v/x$ if $x \geq u + \sqrt{u^2+v}$, thus it suffices to have

$$r \geq (\sqrt{2}+1)\max(c_1 h'(A),\, \sqrt{8c_1 h(\xi)})\,. \tag{6.10}$$

Now Lemma 6.2 provides upper and lower bounds for r:

$$N/(2\sqrt{2}p^{f_v}Q) - 1 \leq r \leq N + 3\,.$$

The lower bound shows that (6.10), and hence (6.7), will hold as soon as N is large enough, namely

$$N = 2\sqrt{2}p^{f_v}Q(1 + (\sqrt{2}+1)\max(c_1 h'(A),\, \sqrt{8c_1 h(\xi)}))\,. \tag{6.11}$$

Note that this lower bound implies (6.6). Similarly, (6.5) follows from (6.9) (recall that we are assuming $c_1 \geq D_v^*$), therefore (6.9) and (6.11) imply (6.8).

The upper bound and (6.8) now show that (use $c_1 h'(A) \geq c_1/d_v^* \geq 1$ and $Q \geq 1$):

$$\begin{aligned} h(A\xi) &\leq 2\sqrt{2}p^{f_v}c_2 Q(1 + (\sqrt{2}+1)\max(c_1 h'(A),\, \sqrt{8c_1 h(\xi)})) + 3c_2 \\ &\leq 12p^{f_v}c_2 Q\max(c_1 h'(A),\, \sqrt{8c_1 h(\xi)})\,. \end{aligned}$$

Since $h(\xi) \leq 2\max(h'(A), h(A\xi))$, we may simplify the last inequality to

$$h(A\xi) \leq 50p^{f_v}c_2 Q\max\left(c_1 h'(A),\, \sqrt{c_1 h(A\xi)}\right)\,.$$

The first choice for the maximum yields

$$h(A\xi) \leq 50p^{f_v}c_1 c_2 h'(A)Q\,,$$

which is one of the possible conclusions in Theorem 2.

The second choice for the maximum yields

$$h(A\xi) \leq 50p^{f_v}c_2 Q\sqrt{c_1 h(A\xi)}\,,$$

whence

$$h(A\xi) \leq c_1(50p^{f_v}c_2 Q)^2\,,$$

which is the other possible conclusion in Theorem 2. □

REFERENCES

[B-W] A. BAKER - G. WÜSTHOLZ, *Logarithmic forms and group varieties*, J. reine angew. Math. **442** (1993), 19–62.

[Bo] E. BOMBIERI, *Effective Diophantine Approximation on* $\mathbf{G}_m$, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **20** (1993), 61–89.

[Bo-V] E. BOMBIERI - J. VAALER, *On Siegel's lemma*, Invent. Math. **73** (1983), 11-32; Addendum, *ibid* **75** (1984), 177.

[S-V] T. STRUPPECK - J. D. VAALER, *Inequalities for heights of algebraic subspaces and the Thue-Siegel Principle*, Analytic Number Theory, Proceedings of a Conference in Honor of Paul T. Bateman, B.C. Berndt, H.G. Diamond, H. Halberstam, A. Hildebrand eds., Progr. Math. Birkhäuser, Boston **85** (1990), 493–528.

[Cor 1] P. CORVAJA, *Autour du théorème de Roth*, preprint.

[Cor 2] P. CORVAJA, *Une application nouvelle de la méthode de Thue*, Ann. Inst. Fourier, Grenoble **45** (1995), 1177-1203.

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