ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

RAJA DZIRI
JEAN-PAUL ZOLÉSIO

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1. – Introduction

We consider the minimization with respect to the domain of a stationary viscous flow energy. Let D be a given smooth domain in $\mathbb{R}^3$, Ω a Cacciopoli set in D (cf. [8]) and u_Ω the solution of the following Navier-Stokes equations

$$\begin{aligned} -\operatorname{div}\big((k_1\chi_\Omega + k_2\chi_{\Omega^c})Du\big) + Du\cdot u + \nabla p = f \quad & \text{in} \quad D \\ \operatorname{div} u = 0 \quad & \text{in} \quad D \\ u = 0 \quad & \text{on} \quad \partial D\,. \end{aligned}$$

with "in some sense" on the interface $u\cdot n = 0$ and $[Du\cdot n]_\Gamma = [\langle Du\cdot n, n\rangle]_\Gamma n$, $[\ \ ]_\Gamma$ being the jump at the interface $\Gamma = \partial\Omega$. The right-hand side f will be depending on the temperature in the fluid. The energy is chosen in the form

$$e_\theta(\Omega) = E(\Omega, u_\Omega, y_\Omega) + \theta P_D(\Omega)$$

where y_Ω is the temperature of the fluid, $E(\ )$ is the system free energy, θ the surface tension and $P_D(\Omega)$ the perimeter of Ω relative to D (cf. [8]). We study the minimization of e_θ under the volume constraint for the set Ω and give the first order necessary optimality condition. This question arises from the analogous situation in hydrodynamics involving the Bernoulli condition. A well known problem is the water wave equilibrium for which several linearized approaches exist, see Stoker [10]. Nevertheless, the non-linearized approach is an important issue. If we denote by u the velocity of a given fluid occupying at time t a given volume Ω_t (with a given initial data), the evolution of the fluid is described by the Navier-Stokes equations:

$$\begin{aligned} u_t - \Delta u + Du\cdot u + \nabla p = \rho\vec{g} \quad & \text{in} \quad \Omega_t \\ \operatorname{div} u = 0 \quad & \text{in} \quad \Omega_t \\ u = 0 \quad & \text{on} \quad \Gamma_t = \partial\Omega_t\,. \end{aligned}$$

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The classical approach from hydrodynamic consists in looking for irrotational solution $u = \nabla\varphi, \varphi$ being the potential which satisfies the incompressibility condition: $\Delta\varphi = 0$. The previous nonlinear equation takes the form $\nabla\{\varphi_t + \frac{1}{2}|\nabla\varphi|^2 + p + \rho g z\} = 0$ and the Bernoulli condition in the fluid can be derived.

As on the free boundary of the wave the pressure is $p = p_a$ (the atmospheric pressure), we get $\varphi_t + \frac{1}{2}|\nabla\varphi|^2 + \rho g z = c$, c being constant on $\Gamma = \partial\Omega$.

In the case of a stationary free boundary, we have the condition $\nabla\varphi\cdot n = u_0$ on Γ and $\Delta\varphi = 0$ in Ω. We can rewrite this problem with a distributed right-hand term f and an homogeneous Neumann boundary condition. This Neumann BVP is solved by the minimization of the energy term

$$E(\Omega;\varphi) = \int_\Omega \left(\frac{1}{2}|\nabla\varphi|^2 - f\varphi\right) dx\,,$$

when φ ranges over the Sobolev space $H^1(\Omega)/\mathbb{R}$.

The problem of minimization with respect to the domain Ω of the following energy

$$e_0(\Omega) = \min_{\varphi\in H^1(\Omega)/\mathbb{R}} E(\Omega;\varphi) + \int_\Omega G\,dx$$

involves the Bernoulli condition as a necessary optimality condition, see [11], [12] or [5]. To obtain existence results in several similar situations, one has to improve the modelling of the Bernoulli flow by adding the surface energy $\theta P_D(\Omega)$. We are, thus, led to minimize with respect to the domain Ω an energy term in the following form:

$$e_\theta(\Omega) = e_0(\Omega) + \theta P_D(\Omega)\,.$$

Concerning Navier-Stokes flow, in three dimensions, there is no hope to get the minimum of the fluid energy only through a variational principle, at least since the model is not variational, cf. [6]. In the previous variational Bernoulli modelling the functional $e_0(\Omega)$ represented the system energy. It was the sum of the kinetic energy and, $\int_\Omega G\,dx$, the potential energy. In the same way, we hall consider the whole energy in a steady viscous flow with heat convection:

$$e_0(\Omega) = \int_\Omega \left(\delta_1 k|\varepsilon(u_\Omega)|^2 + \delta_2\frac{\mu}{2}|\nabla y\Omega|^2 + \frac{1}{2}|u_\Omega|^2 + g x_3\right) dx$$

where y is the temperature of the fluid, $|\varepsilon(u)|^2 = \sum \varepsilon(u)_{ij}^2$ and δ_i, $i = 1,2$ are fixed positive constants. We shall consider the energy $e_\theta(\Omega) = e_0(\Omega) + \theta P_D(\Omega)$. In the correspondant shape minimization problem the state is the solution $(y_\Omega, (u_\Omega, p_\Omega))$ of the stationary Navier-Stokes problem coupled with the heat equation:

$$\begin{aligned} -k\Delta u + Du\cdot u + \nabla p &= f(y) \\ \operatorname{div} u &= 0 \\ -\mu\Delta y + u\cdot\nabla y &= h \quad \text{in} \quad \Omega\,, \end{aligned}$$

where f is linear in y and h is a given function independent of u. With the boundary conditions $u \cdot n = 0$, $\varepsilon(u) \cdot n = \langle \varepsilon(u)n, n \rangle n$ and $\partial y/\partial n = 0$ on Γ. In order to get existence of solution in the family of measurable subsets with finite perimeter, we make the standard physical assumption that in the outer domain, $\Omega^c = D \setminus \Omega$, there is another fluid following the same rheological law but with eventually arbitrarily small viscosity and/or density. The two fluids will be assumed to be immiscible. When the set Ω is an open subset in D and when its boundary in D has a zero 3-dimensional measure, the outer domain could be understood in the sequel as the smooth open set $D \setminus \overline{\Omega}$. Denote by $k(\Omega) = k_1 \chi_\Omega + k_2 \chi_{\Omega^c}$ (resp. $\mu = \mu_1 \chi_\Omega + \mu_2 \chi_{\Omega^c}$) the function characterizing the viscosity (resp. the conductivity) parameters of the two fluids occupying respectively the domains Ω and Ω^c. The pointwise non-penetration condition at the interface, $u \cdot n = 0$ on Γ, and the incompressibility condition, $\operatorname{div} u = 0$ in Ω, $\operatorname{div} u = 0$ in Ω^c, when Ω is a non smooth measurable subset of D turn out to be the $L^2(D)^3$-orthogonality to all functions of the form: $\chi_\Omega \nabla p + \chi_{\Omega^c} \nabla q$, with $p, q \in H_0^1(D)$. Using that relaxed formulation, we are able to get existence and uniqueness of solution for the state equations and then existence of solution for the minimization of the energy on the family of measurable subsets having a given volume:

$$e_\theta(\Omega) = \int_D \left(\frac{1}{2}|u|^2 + \delta_1 k |\varepsilon(u)|^2 + \delta_2 \frac{\mu}{2} |\nabla y|^2 + \rho g x_3 \right) dx$$
$$+ \theta P_D(\Omega), \rho \quad \text{is the density}.$$

As usual in Control Theory, the Eulerian derivative $de_0(\Omega; V)$ will be characterized through and "adjoint" problem which turns out to be associated with a linearization of the problem in the neighborhood of the optimal solution (y_Ω, u_Ω) and having a forcing term that arises from the chosen energy equation. If $(Y, \mathbb{U})$ is the solution to that linear problem, the necessary condition for the optimality of $e_\theta(\Omega)$ is the solution to that linear problem, the necessary condition for the optimality of $e_\theta(\Omega)$ leads to a boundary condition which is quadratic in the variables (y, u) and $(Y, \mathbb{U})$. In the same framework, the previous Bernoulli condition was also quadratic in the potential φ. If we denote the jump at the boundary Γ by $[\cdot]_\Gamma$ and by R(resp. r) the jump at the interface of the normal stress associated to the adjoint problem (resp. to the state equations), the optimality condition takes the following form (when sufficient smoothness of the boundary Γ is assumed)

$$\Big[\delta_1 \frac{\nu}{2} |\varepsilon(u)|^2 + \delta_2 \frac{\mu}{2} |\nabla y|^2 - \nu \varepsilon(u)..\varepsilon(\mathbb{U}) - \beta y g \mathbb{U} - \nabla_\Gamma r \cdot \mathbb{U} - \nabla_\Gamma R \cdot u$$
$$- \mu \nabla_\Gamma y \cdot \nabla_\Gamma Y + hY + \rho g x_3\Big]_\Gamma + \theta H = \text{cst on} \quad \Gamma,$$

where H is the mean curvature of Γ and $\nu = 2k$.

Formally the minimization of e_θ is similar to the problem considered by L. Ambrosio and G. Buttazzo [1] and under smoothness result on the solution u, which are not available for the Navier-Stokes equations, the optimal set Ω would be open with finite perimeter.

2. – Preliminaries

2.1. – Main notations

Let Ω be a measurable subset contained in a bounded smooth domain $D \subset \mathbb{R}^3$. We begin by introducing the main spaces and continuous forms that will be used:

$$a_{0,\Omega}(\cdot,\cdot) : \left[H_0^1(D,\mathbb{R}^3)\right]^2 \longrightarrow \mathbb{R};\ (u,v)$$
$$\longrightarrow \int_D k(\Omega) Du \ldots Dv\, dx = \sum_{i,j=1}^{3} \int_D k(\Omega)\partial_i u_j \cdot \partial_i v_j\, dx\,,$$
$$a_1(\cdot;\cdot,\cdot) : \left[H_0^1(D,\mathbb{R}^3)\right]^3 \longrightarrow \mathbb{R};\ (u,v,w)$$
$$\longrightarrow \int_D \langle Dv \cdot u, w\rangle dx = \sum_{i,j=1}^{3} \int_D (\partial_j v_i) u_j w_i\, dx\,,$$
$$c_{0,\Omega}(\cdot,\cdot) : (H_0^1(D))^2 \longrightarrow \mathbb{R};\ (y,z) \longrightarrow \int_D \mu(\Omega)\nabla y \cdot \nabla z\, dx\,,$$
$$c_1(\cdot;\cdot,\cdot) : H_0^1(D,\mathbb{R}^3) \times (H_0^1(D))^2 \longrightarrow \mathbb{R};\ (u;y,z) \longrightarrow \int_D (u\cdot\nabla y) z\, dx$$

where $k(\Omega) = k_1\chi_\Omega + k_2\chi_{\Omega^c}$, $0 < k_2 \le k_1$, and $\mu(\Omega) = \mu_1\chi_\Omega + \mu_2\chi_{\Omega^c}$, $0 < \mu_2 \le \mu_1$. Denoting by $\mathcal{E}(\Omega)$ the closure of the linear space

$$E(\Omega) = \left\{ l = \chi_\Omega \nabla p + \chi_{\Omega^c}\nabla q;\ p,q \in H_0^1(D)\right\} \quad \text{in} \quad H^{-1}(D,\mathbb{R}^3)$$

and by $\langle\cdot,\cdot\rangle$ the duality product, we introduce the bilinear form

$$b(\cdot,\cdot) : H_0^1(D,\mathbb{R}^3) \times \mathcal{E}(\Omega) \longrightarrow \mathbb{R};\ (v,l) \longrightarrow \langle l, v\rangle_{H^{-1}(D,\mathbb{R}^3), H_0^1(D,\mathbb{R}^3)}\,.$$

Let $\mathcal{E}(\Omega)'$ be the dual space of $\mathcal{E}(\Omega)$. Define the operator $B \in \mathcal{L}\big(H_0^1(D,\mathbb{R}^3), \mathcal{E}(\Omega)'\big)$ as:

$$\langle Bv, l\rangle_{\mathcal{E}(\Omega)',\mathcal{E}(\Omega)} = b(v,l) \quad \forall\, l \in \mathcal{E}(\Omega), \quad \forall\, v \in H_0^1(D,\mathbb{R}^3)\,.$$

Denote by B^* the adjoint operator of B and by $X(\Omega)$ the kernel of B.

Finally, $(\cdot,\cdot)$ and $((\cdot,\cdot))$ denote the inner products in $L^2(D)$ and $H_0^1(D,\mathbb{R}^3)$ respectively.

LEMMA 2.1. *When the boundary Γ is a smooth manifold, the linear space $\mathcal{E}(\Omega)$ can be characterized as follows:*

$$\mathcal{E}(\Omega) = \Bigg\{ \zeta \in H^{-1}(D,\mathbb{R}^3) | \forall\varphi \in H_0^1(D,\mathbb{R}^3),\ \langle\zeta,\varphi\rangle$$
$$= \int_\Omega p \operatorname{div}\varphi\, dx + \int_{\Omega^c} q \operatorname{div}\varphi\, dx + \langle r, \varphi_{|\Gamma}\cdot n\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}$$
$$\text{where} \quad p \in L^2(\Omega), q \in L^2(\Omega^c), r \in H^{-1/2}(\Gamma) \Bigg\}$$

PROOF. Let ζ be given in $\mathcal{E}(\Omega)$ and φ in $H_0^1(D, \mathbb{R}^3)$, by definition

$$\langle \zeta, \varphi \rangle = \lim_k \left[\int_\Omega - \operatorname{div} \varphi p_k \, dx + \int_{\Omega^c} - \operatorname{div} \varphi q_k \, dx + \int_\Gamma (p_k - q_k) \varphi \cdot n \, d\Gamma \right]$$

As a result we get that ∇p_k (rep. ∇q_k) is bounded in $H^{-1}(\Omega)$ (resp. $H^{-1}(D \setminus \overline{\Omega})$). We make use of the following inequality: there exists a constant $c > 0$ such that,

$$\forall \psi \in L^2(\Omega), \; \|\psi\|_{[L^2(\Omega)]/\mathbb{R}} \leq c \|\nabla \psi\|_{H^{-1}(\Omega)} .$$

Let $\bar{p}_k = (\text{meas } \Omega)^{-1} \int_\Omega p_k \, dx$. From the previous inequality we deduce the boundedness of $p_k - \bar{p}_k$ (resp. $q_k - \bar{q}_k$) in $L^2(\Omega)$ (resp. in $L^2(\Omega^c)$). Finally we obtain the boundedness in $H^{-1/2}(\Gamma)$ of the term $p_k - \bar{p}_k - (q_k - \bar{q}_k)$ and we get the existence of weak limiting corresponding elements p, q and r. Conversely, let be given three such elements $(p, q, r) \in L^2(\Omega) \times L^2(\Omega^c) \times H^{-1/2}(\Gamma)$. Let be given $r_n \in H^{1/2}(\Gamma)$, $r_n \longrightarrow r$ in $H^{-1/2}(\Gamma)$, p_n (resp. q_n) in $H_0^1(\Omega)$ (resp. $H_0^1(D \setminus \overline{\Omega})$), with, $p_n \longrightarrow p$ in $L^2(\Omega)$ (resp. $q_n \longrightarrow q$ in $L^2(D \setminus \overline{\Omega})$).

Let P be a linear continuous extension mapping $P \in \mathcal{L}(H^{1/2}(\Gamma), H^1(D))$ such that $P \cdot \varphi_{|\Gamma} = \varphi, \forall \varphi \in H^{1/2}(\Gamma)$. We set $R_n = P \cdot r_n$ and denoting by d_Γ the distance function to the smooth boundary Γ, element in $W^{1,\infty}(D)$, we consider

$$\beta_n^m = \left(\frac{1}{1 + d_\Gamma} \right)^m R_n .$$

For all m, $|\beta_n^m| \leq |R_n|$, $R_n \in L^1(D)$ and we have the pointwise convergence: $\beta_n^m(x) \longrightarrow 0 (m \longrightarrow \infty)$ for almost every x in D. Then from the Lebesgue convergence theorem we get that β_n^m converges to zero in $L^2(D)$ as $m \longrightarrow \infty$. Let $m(n)$ denotes the first integer for which $\|\beta_n^m\|_{L^2(D)} \leq \frac{1}{n}$. Set $\theta_n = \beta_n^{m(n)}$. By construction we get $\theta_{n|\Gamma} = r_n$ and $\theta_n \longrightarrow 0$ in $L^2(D)$. Finally we consider the element $\tilde{p}_n = p_n + \theta_{n|\Omega}$. We get $\tilde{p}_n \longrightarrow p$ in $L^2(\Omega)$ while $\tilde{p}_{n|\Gamma} = r_n \longrightarrow r$ in $H^{-1/2}(\Gamma)$. □

2.2. – Equations

Consider the following problem:

For a given $h \in L^2(D)$, find $(y; (u, L)) \in H_0^1(D) \times (H_0^1(D, \mathbb{R}^3) \times \mathcal{E}(\Omega))$ such that

(1) $$a_\Omega(u; u, v) + b(v, L) = -(\beta(\Omega) y \vec{g}, v) \quad \forall v \in H_0^1(D, \mathbb{R}^3)$$

(2) $$b(u, l) = 0 \quad \forall l \in \mathcal{E}(\Omega)$$

(3) $$c_\Omega(u; y, z) = \int_D h z \, dx \quad \forall z \in H_0^1(D)$$

where $a_\omega(u; v, w) = a_{0,\Omega}(u, v) + a_1(u; v, w)$, $c_\Omega(u; y, z) = c_{0,\Omega}(y, z) + c_1(u; y, z)$ and $\beta(\Omega) = \beta_1 \chi_\Omega + \beta_2 \chi_{\Omega^c}$, $0 < \beta_1 \leq \beta_2$.

Equations (1)-(2) can be formulated differently (see [7]). Then, system (1)-(3) is equivalent to the following one: for a given $h \in L^2(D)$, find $(y, u) \in H_0^1(D) \times X(\Omega)$ such that

$$a_\Omega(u; u, v) = -(\beta(\Omega)u\vec{g}, v) \quad \forall\, v \in X(\Omega) \tag{4}$$

$$c_\Omega(u; y, z) = \int_D hz\, dx \qquad \forall z \in H_0^1(D)\,. \tag{5}$$

Set $\alpha = k_2(1 - c_1 k_2^{-2}\mu_2^{-1}\beta_2 g_0\|h\|_{H^{-1}})$ and $C = c_1\beta_2 g_0 + c_2\mu_2^{-1}$ where c_1 is the Poincare's constant, c_2 is the norm of the canonical embedding $H^1(D) \hookrightarrow L^4(D)$ and $g_0 = \|g\|_{L^\infty(D,\mathbb{R}^3)}$.

PROPOSITION 2.1. *Assume that the following holds:*

$$\alpha > 0 \quad \textit{and} \quad \frac{C^2}{4\mu_2\alpha} < 1\,. \tag{6}$$

Then there exists a unique solution $(y; (u, L)) \in H_0^1(D) \times (H_0^1(D, \mathbb{R}^3) \times \mathcal{E}(\Omega))$ *of system* (1)-(3)*. In the specific case where the boundary* Γ *is a smooth manifold, we have solved the following problem:*

$$\big(y, (u, (p, q, r))\big) \in H_0^1(D) \times \big(H_0^1(D, \mathbb{R}^3) \times (L^2(\Omega) \times L^2(\Omega^c) \times H^{-1/2}(\Gamma))\big)$$

such that $\operatorname{div} u = 0$ *in* D*,* $u \cdot n = 0$ *on* Γ*, and:*

$$\begin{aligned} -\operatorname{div}(k_1 Du) + Du \cdot u + \nabla p &= \beta_1 y\vec{g} \quad \textit{in} \quad \Omega\,,\\ -\operatorname{div}(k_2 Du) + Du \cdot u + \nabla q &= \beta_2 y\vec{g} \quad \textit{in} \quad \Omega^c\,. \end{aligned}$$

The tangential component of the jump of the normal stress is zero:

$$\big[(k\varepsilon(u) \cdot n)_\Gamma\big]_\Gamma = 0\,, \tag{7}$$

and the following regularity result for the normal component of the previous jump across Γ*:*

$$r = (p - q) - \big[\langle 2k\varepsilon(u) \cdot n, n\rangle\big]_\Gamma\,. \tag{8}$$

PROOF. Given $u_0 \in X(\Omega)$, consider the sequence of $(y_n; (u_n, L_n))$, $n \in \mathbb{N}^*$, is the unique solution of

$$\begin{aligned} a_\Omega(u_n; u_n, v) + b(v, L_n) &= -(\beta(\Omega)y_n g, v) \quad \forall\, v \in H_0^1(D, \mathbb{R}^3)\\ b(u_n, l) &= 0 \quad \forall\, l \in \mathcal{E}(\Omega)\\ c_\Omega(u_{n-1}; y_n, z) &= \int_D hz\, dx \quad \forall\, z \in H_0^1(D)\,. \end{aligned}$$

Since $c_\Omega(u; y, y) = (\mu(\Omega)\nabla_y, \nabla_y)$ for $u \in H_0^1(D, \mathbb{R}^3)$, $\operatorname{div} u = 0$ and $y \in H_0^1(D)$, it is easy to verify that the sequence $\{((u_n, L_n), y_n)\}_{\mathbb{N}^*}$ is bounded in $(H_0^1(D, \mathbb{R}^3) \times \mathcal{E}(\Omega)) \times H_0^1(D)$ and its weak limit $((u, L); y)$ is a solution of system (1)-(3). As for the uniqueness of solution, we assume that the problem has two different solutions $((u_i, L_i); y_i)$, $i = 1, 2$. Then, the functions $\bar{u} = u_1 - u_2$ and $\bar{u} = y_1 - y_2$ verify:

$$a_\Omega(u_1; \bar{u}, v) + a_1(\bar{u}; u_2, v) + (\beta(\Omega)\bar{y}g, v) = 0 \quad \forall v \in X(\Omega) \tag{9}$$

$$c_\Omega(u_1; \bar{y}, z) + c_1(\bar{u}; y_2, z) = 0 \quad \forall z \in H_0^1(D) \tag{10}$$

In particular for $v = \bar{u}$ and $z = \bar{y}$, we obtain

$$\left(k_2 - c_1 c_2 k_2^{-1} \mu_2^{-1} \beta_2 g_0 \|h\|_{H^{-1}}\right) |\nabla \bar{u}|^2 \leq c_1 \beta_2 g_0 |\nabla \bar{y}| |\nabla \bar{u}|$$

$$\mu_2 |\nabla \bar{y}|^2 \leq c_2 \mu_2^{-1} \|h\|_{H^{-1}} |\nabla \bar{u}| |\nabla \bar{y}|$$

$$\alpha |\nabla \bar{u}|^2 + \mu_2 |\nabla \bar{y}|^2 - C |\nabla \bar{u}| |\nabla \bar{y}| \leq 0 .$$

Condition (6) implies the existence of an $\varepsilon_0 > 0$ such that $\alpha - C\varepsilon_0/2 > 0$ and $\mu_2 - C/2\varepsilon_0 > 0$. Then

$$(\alpha - C\varepsilon_0/2)|\nabla \bar{u}|^2 + (\mu_2 - C/2\varepsilon_0)|\nabla \bar{y}|^2 \leq 0$$

implies that $|\nabla \bar{u}| = |\nabla \bar{y}| = 0$. □

3. – Continuity with respect to the Domain

The existence of solution for the minimization problem under consideration requires some continuity properties. In this section, we give a shape continuity result for the solution of system (4)-(5). A sequence of measurable subsets $\{\Omega_n\}$ is said to converge to Ω in the char (D)-topology if $\exists \Omega$ a measurable subset such that

$$\chi_{\Omega_n} \longmapsto \chi_\Omega \quad \text{in} \quad L^2(D), \quad (\text{we denote} \quad \Omega_n \overset{\text{char(D)}}{\longmapsto} \Omega) .$$

We shall consider a compact family of measurable subsets Ω of D in that topology and prove that $\Omega_n \overset{\text{char(D)}}{\longmapsto} \Omega$ implies

$$(u_{\Omega_n}, y_{\Omega_n}) \rightharpoonup (u_\Omega, y_\Omega)(= (u, y)) \quad \text{in} \quad H_0^1(D, \mathbb{R}^3) \times H_0^1(D) ,$$

where $H_0^1(D, \mathbb{R}^3)$ and $H_0^1(D)$ are endowed with their weak topologies and where $(u_{\Omega_n}, y_{\Omega_n})$ denote the solution of problem (4)-(5) in Ω_n.

We shall consider a compact family of finite perimeter sets in D.

3.1. – Finite perimeter sets

Denote by $BPS(D)$ the family of finite perimeter sets of D:

$$BPS(D) = \left\{ \Omega \subset D; \Omega \text{ measurable} \mid \sup_{\|g\|_{C^0(D)} \leq 1} \left\{ \int_\Omega \operatorname{div} g \, dx | g \in C_c^\infty(D; \mathbb{R}^3) \right\} < \infty \right\}.$$

It is immediate to see that $\{\chi_\Omega | \Omega \in BPS(D)\}$ is contained in $BV(D)$ so the norm of Ω in $BPS(D)$ is given by

$$\text{meas } \Omega + \|\nabla \chi_\Omega\|_{M^0(D)} = \|\chi_\Omega\|_{BV(D)} .$$

The perimeter of a subset Ω in $BPS(D)$ (relative to D) is given by

$$(11) \qquad P_D(\Omega) = \sup \left\{ \int_\Omega \operatorname{div} g \, dx | g \in C_c^\infty(D; \mathbb{R}^3), \max_D \|g(x)\| \leq 1 \right\}.$$

We have the following compactness result

LEMMA 3.1. *Let $\{\Omega_n\}$ be a sequence in $BPS(D)$ such that*

$$P_D(\Omega_n) \leq c .$$

Then, there exists a subsequence $\{\Omega_{n_k}\}$ and $\Omega \in BPS(D)$ such that

$$\chi_{\Omega_{n_k}} \longrightarrow \chi_\Omega \quad in \quad L^1(D) .$$

Moreover, for any g in $C_c(D; \mathbb{R}^3)$, we have

$$\langle \nabla \chi_{\Omega_{n_k}}, g \rangle \longrightarrow \langle \nabla \chi_\Omega, g \rangle$$

and $P_D(\Omega) \leq \liminf P_D(\Omega_{n_k})$.

REMARK 3.1. The perimeter for any measurable subset Ω of D could be defined by (11) as being an element of $\mathbb{R}^+ \cup \{\infty\}$.

For more details see for example [8], [3]. Finally, recall

LEMMA 3.2. *Let Ω in $BPS(D)$, $\overline{\Omega} \subset D$, denoted $\Omega \subset\subset D$. There exists a smooth open set O_Ω such that*

$$O_\Omega \subset\subset D \quad and \quad \partial^*\Omega \subset O_\Omega,$$

$\partial^*\Omega$ is the reduced boundary of Ω.

3.2. – Kuratowski limit

Denote $F^{\perp}$ the orthogonal of a closed subspace F of $H_0^1(D, \mathbb{R}^3)$ and by P_F the projection operator on F. Recall the characterization of elements in the linear space of divergence-free functions (denoted X_0) by means of the curl operator (see [7]).

LEMMA 3.3. *Every function v of X_0 has the following form:*

$$v = \operatorname{curl} \Phi$$

where $\Phi \in H^2(D, \mathbb{R}^3)$ with $\operatorname{div} \Phi = 0$ *is the unique solution of*

$$(12) \qquad \begin{aligned} (-\Delta\Phi, \operatorname{curl} w) &= (\operatorname{curl} v, \operatorname{curl} w) \quad \forall\, w \in X_0\,, \\ \Phi \cdot n_D &= 0 \quad on \quad \partial D \end{aligned}$$

where n_D is the unit normal vector field on ∂D, outward to D.

In the following, we hall prove that for all v in $X(\Omega)$, there exists a sequence v_n in $X(\Omega_n)$ such that $v_n \longrightarrow v$ strongly in $H_0^1(D)$ if Ω_n converges in char(D)-topology to Ω.

More precisely, let $\Omega_n \subset\subset D$, $n \in \mathbb{N}^*$, be a sequence of sets in BPS(D). From Lemma 3.2, we get the existence of a smooth function ζ_n solution of $(-\Delta)^2\zeta_n = \varepsilon_n$ in $D \setminus \overline{O}_{\Omega_n}$, $\zeta_n = 0$ on ∂O_{Ω_n}, $\zeta_n = 1$ on ∂D and $\partial\zeta_n/\partial n = 0$ on $\partial(D \setminus \overline{O}_{\Omega_n})$, where ε_n belongs to $L^2(D)$ and $|\varepsilon_n|_{L^2} \leq \varepsilon$. We also define the following continuous forms

$$\begin{aligned} c_{\zeta_n} : X_0 &\longrightarrow H_0^1(D, \mathbb{R}^3) \\ v = \operatorname{curl}\Phi &\longmapsto \operatorname{curl}(\zeta_n \Phi) \\ b_0(\cdot,\cdot) : H_0^1(D, \mathbb{R}^3) \times L_0^2(D) &\longrightarrow \mathbb{R} \\ (\varphi, p) &\longmapsto -\int_D p \operatorname{div} \varphi\, dx = \langle \nabla p, \varphi \rangle\,. \end{aligned}$$

By the generalized Gauss-Green formula for finite perimeter sets, we obtain that

$$c_{\zeta_n}(X_0) \subset X(\Omega_n)\,.$$

On the other hand, for any given g in $H^{-1}(D, \mathbb{R}^3)$, we can state the following problem: To find $(\varphi_n, p_n) \in H_0^1(D, \mathbb{R}^3) \times L^2(D)/\mathbb{R}$ such that $\forall (\psi, l_n) \in H_0^1(D, \mathbb{R}^3) \times \mathcal{E}(\Omega_n)$

$$(13) \qquad \begin{aligned} ((\varphi_n, P_{X_n}\psi)) + \frac{1}{n}((\varphi, c_{\zeta_n}(P_{X_n^{\perp}\cap X_0}\psi))) + b_0(\psi, p_n) &= \langle g, \psi \rangle\,, \\ b_n(\varphi_n, l_n) &= 0\,. \end{aligned}$$

LEMMA 3.4. *Problem* (13) *is well-posed.*

PROOF. It is easily seen that

$$\forall \varphi \in X_n, \sup_{\psi \in X_0} ((\varphi, \psi)) + \frac{1}{n}((\varphi, c_{\zeta_n}(P_{X_n^{\perp} \cap X_0} \psi))) \geq \|\varphi\|^2_{H_0^1(D,\mathbb{R}^3)}$$

(it is sufficient to choose $\psi = \varphi$). It remains to prove that

$$\forall \psi \in X_0, \psi \neq 0, \sup_{\varphi \in X_n} ((\varphi, \psi)) + \frac{1}{n}((\varphi, c_{\zeta_n}(P_{X_n^{\perp} \cap X_0} \psi))) > 0 .$$

The worst situation would be when $P_{X_n}\psi = -c_{\zeta_n}(\frac{1}{n}P_{X_n^{\perp}}\psi)$, $\psi \in X_0$. This would imply that $\tilde{\psi} = P_{X_n}\psi + \frac{1}{n}P_{X_n^{\perp}}\psi = (1 - c_{\zeta_n})\frac{1}{n}P_{X_n^{\perp}}\psi = c_1 - \zeta_n P_{X_n^{\perp}}\psi$.

$$\begin{aligned} \|\tilde{\psi}\|_{H_0^1} &\leq \|\mathrm{curl}\|_{\mathcal{L}(H^2,H^1)}\|(1-\zeta_n)\Phi_{2,n}\|_{H^2} \leq c_0(D)\|(1-\zeta_n)\|_{H^2}\|\Phi_{2,n}\|_{H^2} \\ &\leq c_1(D)\|1-\zeta_n\|_{H^2}\|\tilde{\psi}\|_{H_0^1} . \end{aligned}$$

Thus $\tilde{\psi} = 0$ and so $\psi = 0$. Thus, we can state that for $\psi \in X_0$, $\psi \neq 0$

$$\sup_{\varphi \in X_n} ((\varphi, \psi)) + \frac{1}{n}((\varphi, c_{\zeta_n}(P_{X_n^{\perp} \cap X_0} \psi))) > 0 .$$

Moreover, the following "*inf-sup*" conditions:

$$\sup_{\varphi \in H_0^1(D,\mathbb{R}^3)} \frac{b_n(\varphi, l)}{\|\varphi\|} = \| l \|_{H^{-1}(D,\mathbb{R}^3)} \quad \text{and} \quad \sup_{\psi \in H_0^1(D,\mathbb{R}^3)} \frac{b_0(\psi, p)}{|p|_{L^2}\|\psi\|} \geq \beta_0 > 0$$

are obviously satisfied for all $l \in \mathcal{E}(\Omega_n)$ and $p \in L^2(D)/\mathbb{R}$. Then (see for instance [2]) problem (13) has a unique solution. □

THEOREM 3.1. *Let $\Omega_n \subset\subset D$ be a sequence in $BPS(D)$.*

Assume that $\exists \Omega \in BPS(D)$ such that $\Omega_n \overset{\mathrm{char}(D)}{\longrightarrow} \Omega$. Then, the linear space $X(\Omega)$ is contained in the Kuratowski Limit of $X(\Omega_n)$.

PROOF. Let v be a function in $X(\Omega)$. For each $n \in \mathbb{N}^*$, we know, from Lemma 3.4, that there exists a unique pair $(v_n, p_n) \in X(\Omega_n) \times L_0^2(D)$ verifying (14)

$$((v_n, \psi)) + \frac{1}{n}((v_n, c_{\zeta_n} P_{X_n^{\perp} \cap X_0} \psi)) + b_0(\psi, p_n) = ((v, \psi)) \quad \forall \psi \in H_0^1(D, \mathbb{R}^3) .$$

For $\psi = v_n - v$, we obtain $\|v_n - v\| \leq \frac{c_0}{n}\|v\|$($c_0$ is a constant). Then, $v_n \longrightarrow v$ in $H_0^1(D, \mathbb{R}^3)$-strong. □

3.3. – Continuity

The fact that $X(\Omega_n)$ is contained in the Kuratowski limit of $X(\Omega)$ when $\Omega_n \overset{\text{char}(D)}{\longrightarrow} \Omega$ allows us to characterize the weak limit u_Ω as the unique solution of problem (4)-(5) relative to Ω.

THEOREM 3.2. *Let $\Omega_n \subset\subset D$ be a sequence in $BPS(D)$.*
Assume that $\Omega_n \overset{\text{char}(D)}{\longrightarrow} \Omega$. Then, there exists a subsequence $\{(u_{\Omega_{n_k}}, y_{\Omega_{n_k}})\}$ such that

$$(u_{\Omega_{n_k}}, y_{\Omega_{n_k}}) \rightharpoonup (u_\Omega, y_\Omega) \quad \textit{weakly in} \quad H_0^1(D, \mathbb{R}^3)\,.$$

PROOF. Let $v \in X(\Omega)$. There exists a sequence $\{v_n \in X(\Omega_n)\}_{n\in\mathbb{N}^*}$ such that

$$v_n \longrightarrow v \quad \text{strongly in} \quad H_0^1(D, \mathbb{R}^3)\,.$$

As $\chi_{\Omega_n} \longrightarrow \chi_\Omega$ in $L^2(D)$, we can extract a subsequence (still denoted χ_{Ω_n}) converging to χ_Ω almost everywhere in D. Thus it is easy to show that

$$k(\Omega_n)Dv_n \longrightarrow k(\Omega)Dv \quad \text{strong in} \quad L^2(D, \mathbb{R}^3)\,.$$

On the other hand, we know that

$$a_{\Omega_n}(u_n; u_n, v_n) + \int_D \beta(\Omega_n) y_n g \cdot v_n \, dx = 0 \quad \forall n \in \mathbb{N}^*\,.$$

Moreover the heat equation (3) implies that

$$|\nabla y_n| \le \mu_2^{-1} \|h\|_{H^{-1}(D)}\,.$$

Then, $\{y_n\}$ converges weakly, in $H_0^1(D)$, to a function y. Therefore, $u_n \rightharpoonup u$, as $n \longrightarrow \infty$, and we get:

$$a(u; u, v) = -\int_D \beta(\Omega) y g \cdot v \, dx\,.$$

In the same sense we get equation (5).

This proves the continuity of the solution of Problem (4)-(5) with respect to the domain. □

3.4. – Existence

The minimization problem we consider is the following

$$\min \left\{ e_\theta(\Omega) | \Omega \subset D, \ \text{measurable}, \ \text{meas } \Omega = m_0 \right\} \tag{15}$$

where

$$e_\theta(\Omega) = \frac{1}{2}|u_\Omega|^2_{L^2(D)} + \int_D \rho g x_3\, dx + \delta_1 \int_D k(\Omega)\varepsilon(u_\Omega)..\varepsilon(u_\Omega)dx + \frac{\delta_2}{2}\int_D \mu(\Omega)|\nabla y|^2 dx + \theta P_D(\Omega)\,,$$

δ_1 and δ_2 are positive constants.

We denote by $e_0(\Omega)$ the term $e_\theta(\Omega) - \theta P_D(\Omega)$ and by $e_1(\Omega)$ the term

$$\frac{1}{2}|u_\Omega|^2_{L^2(D)} + \delta_1 \int_D k(\Omega)\varepsilon(u_\Omega)..\varepsilon(u_\Omega)dx + \frac{\delta_2}{2}\int_D \mu(\Omega)|\nabla y|^2 dx\,.$$

PROPOSITION 3.1. *There exists at least a measurable subset Ω in $BPS(D)$ solution of the minimization problem* (15).

PROOF. First consider the following problem $\min\{e_\theta(\Omega)|\Omega \subset\subset D_\delta$, measurable, meas $\Omega = m_0\}$, D_δ is the δ-contracted of $D, \delta > 0$ small.

Let $\Omega_{n,\delta}$ be a minimizing sequence. The sequence $\{e_\theta(\Omega_{n,\delta})\}_n$ being bounded, there exists a constant c_δ such that:

$$P_D(\Omega_{n,\delta}) \le c_\delta\,.$$

Therefore according to Lemma 3.1, $\exists \Omega_\delta \subset D_\delta$ and a subsequence $\{\Omega_{n_k,\delta}\}$ such that $\Omega_{n_k,\delta} \overset{\text{char}(D)}{\longrightarrow} \Omega_\delta$, meas $\Omega_\delta = m_0$ and $P_D(\Omega_\delta) \le \liminf P_D(\Omega_{n_k,\delta})$.

From Theorem 3.2. there exists a sequence $(u_{\Omega_{n_k,\delta}}, y_{\Omega_{n_k,\delta}})$ weakly convergent (in $H^1_0(D,\mathbb{R}^3)\times H^1_0(D)$) to $(u_{\Omega_\delta}, y_{\Omega_\delta})$. Then, using the lower continuity of the norm in $H^1_0(D,\mathbb{R}^3)$ and of the perimeter, we conclude that Ω_δ is a solution of our problem. In the other hand, we have

$$\min\{e_\theta(\Omega)|\Omega \subset D, \text{ measurable, meas } \Omega = m_0\} = \liminf_{\delta\searrow 0} e_\theta(\Omega_\delta)\,.$$

Since the value of $\min\{e_\theta(\Omega)|\Omega \subset\subset D_\delta$, measurable, meas $\Omega = m_0\}$ decreases with δ, we conclude that $P_D(\Omega_\delta)$ is bounded. Then there exists a subsequence $\{\Omega_{\delta_k}\}$ and a set $\Omega \subset D$, meas $\Omega = m_0$ such that

$$\Omega_{\delta_k} \overset{\text{char}(D)}{\longrightarrow} \Omega\,.$$

Moreover, using the same arguments as in the first part of the proof, we get the existence of a sequence $(u_{\Omega_{\delta_k}}, y_{\Omega_{\delta_k}})$ weakly convergent (in $H^1_0(D,\mathbb{R}^3)\times H^1_0(D)$) to (u_Ω, y_Ω) and hence that Ω is solution of (15). □

4. – Differentiability

In this section we shall study the differentiability of the mapping $t \longmapsto (u_t \circ T_t, y_t \circ T_t)$ with respect to the domain. T_t is a given transformation defined in $\overline{D}$.

4.1. – Material derivative

We apply the *Velocity Method* cf. [9]. Consider a family of vector fields $V \in \mathcal{C}^0([0,\tau); \mathcal{C}^2(\overline{D}, \mathbb{R}^3))$ verifying:

$$V(t,x) \cdot n_D(x) = 0 \quad \text{for a.e.} \quad x \in \partial D$$

and

$$\exists c > 0, \forall x, y \in \mathbb{R}^3, \|V(\cdot, y) - V(\cdot, x)\|_{\mathcal{C}^0([0,\tau);\mathbb{R}^3)} \leq c|y - x|$$

We know from [9] that there exists an interval $I, 0 \in I$, and a family of one-to-one transformations $\{T_t(V), t \in I\}$ mapping $\overline{D}$ onto $\overline{D}$ verifying:

$$V = \frac{\partial}{\partial t} T_t(V) \circ T_t^{-1}(V).$$

As we are interested in incompressible fluids, transformations $\{T_t\}$ should satisfy $\det DT_t = 1$. Then the associated vector fields V will be of divergence-free.

DEFINITION 4.1. *For any transformation $T_t, t \in I$, we define the following isomorphisms:*

1. $H_0^1(D, \mathbb{R}^3) \longrightarrow H_0^1(D, \mathbb{R}^3)$; $w \longmapsto DT_t w \circ T_t^{-1}$
2. $\mathcal{E}(\Omega_t) \longrightarrow \mathcal{E}(\Omega)$; $l_t \longmapsto l_t \star T_t = l^t$, *where*

$$\langle l^t, w \rangle = \langle l_t, DT_t w \circ T_t^{-1} \rangle \quad \forall w \in H_0^1(D, \mathbb{R}^3).$$

For all $t \in I = [0, \tau)$ and $(v, z) \in H_0^1(D, \mathbb{R}^3) \times H_0^1(D)$, we have

$$\text{(16)} \qquad \int_D k(\Omega_t) Du_t .. Dv \, dx + \int_D \langle Du_t \cdot u_t, v \rangle dx + \langle L_t, v \rangle = -\int_D \beta(\Omega_t) y_t g \cdot v \, dx$$

$$\text{(17)} \qquad \langle l_t, u_t \rangle_{H^{-1}(D,\mathbb{R}^3), H_0^1(D,\mathbb{R}^3)} = 0 \quad \forall l_t \in \mathcal{E}(\Omega_t).$$

$$\text{(18)} \qquad \int_D \mu(\Omega_t) \nabla y_t \cdot \nabla z \, dx + \int_D (u_t \cdot \nabla y_t) z \, dx = \int_D hz \, dx.$$

We make the change of coordinates defined by the transformation $T_t(t \in I)$, and get:

$$(19)\quad \begin{cases} \int_D k(\Omega) D(u_t \circ T_t) \cdot DT_t^{-1} .. D(v \circ T_t) \cdot DT_t^{-1} dx \\ + \int_D \left\langle D(u_t \circ T_t) \cdot DT_t^{-1} \cdot (u_t \circ T_t), v \circ T_t \right\rangle dx \\ + \langle L^t, DT_t^{-1} v \circ T_t \rangle = - \int_D \beta(\Omega) y_t \circ T_t g \circ T_t \cdot v \circ T_t dx \end{cases}$$

$$(20)\quad \left\langle l^t, DT_t^{-1}(u_t \circ T_t) \right\rangle = 0,$$

and

$$(21)\quad \begin{aligned} &\int_D \mu(\Omega) \left\langle DT_t^{-1} .^{*} DT_t^{-1} \nabla(y_t \circ T_t), \nabla z \right\rangle dx \\ &+ \int_D \left\langle u_t \circ T_t^{*}, DT_t^{-1} \nabla(y_t \circ T_t) \right\rangle z \, dx = \int_D h \circ T_t z \, dx . \end{aligned}$$

Setting $w = DT_t^{-1}(v \circ T_t)$, $u^t = DT_t^{-1}(u_t \circ T_t)$ and $y^t = y_t \circ T_t$, we obtain:

$$(22)\quad \begin{cases} \int_D k(\Omega) D[DT_t u^t] \cdot DT_t^{-1} .. D[DT_t w] \cdot DT_t^{-1} dx \\ + \int_D \langle D[DT_t u^t] \cdot u^t, DT_t w \rangle \, dx \\ + \langle L^t, w \rangle = - \int_D \beta(\Omega) y^t g \circ T_t \cdot DT_t w \, dx \quad \forall w \in H_0^1(D, \mathbb{R}^3) \end{cases}$$

$$(23)\quad \langle l_t \star T_t, u^t \rangle = 0 \quad \forall l_t \in \mathcal{E}(\Omega_t)$$

and for all z in $H_0^1(D)$,

$$(24)\quad \int_D \mu(\Omega) \left\langle DT_t^{-1} .^{*} DT_t^{-1} \nabla y^t, \nabla z \right\rangle dx + \int_D \langle u_t, \nabla y^t \rangle z \, dx = \int_D h \circ T_t z \, dx ,$$

REMARK 4.1. Equation (23) can be replaced by $\langle l, u^t \rangle = 0 \; \forall l \in \mathcal{E}(\Omega)$.

We shall prove the differentiability at the origin of the mapping $t \longmapsto (u^t, y^t)$ using a weak form of the implicit function theorem:

THEOREM 4.1. *Let E and F be two Banach spaces and*

$$\Phi : I \times E \longrightarrow F\,, \quad I \text{ is an open set in } \mathbb{R}\,.$$

Assume that

$$(25)\qquad \begin{cases} \forall g' \in F' \quad (\text{dual of } F) \\ s \longmapsto \langle \Phi(s, f), g' \rangle_{F \times F'} \quad \text{is continuously differentiable} \\ \dfrac{\partial}{\partial s}\Phi(s, f) \quad \text{denotes its weak derivative}\,. \end{cases}$$

$$(26)\qquad (s, f) \longmapsto \frac{\partial}{\partial s}\Phi(s, f) \text{ is continuous from } I \times E \text{ into } F\text{-weak}\,,$$

there exists a function U such that

$$(27)\qquad U \in Lip(I, E)$$

$$\Phi(s, U(s)) = 0 \quad \forall s \in I\,,$$

the mapping $f \longrightarrow \Phi(s, f)$ is differentiable and

$$(28)\qquad (s, f) \longmapsto \frac{\partial}{\partial f}\Phi(s, f) \quad \text{is continuous}\,.$$

Moreover at $(s_0, U(s_0))$,

$$\frac{\partial}{\partial f}\Phi(s_0, U(s_0)) \quad \text{is an isomorphism from } E \text{ onto } F\,.$$

Then the mapping $s \longmapsto U(s)$ is differentiable in E-weak on $s = s_0$ and

$$U'(s_0) = -\frac{\partial}{\partial f}\Phi(s_0, U(s_0))^{-1} \cdot \frac{\partial}{\partial s}\Phi(s_0, U(s_0))\,.$$

PROOF. Set $t = U(s_0 + \varepsilon) - U(s_0) \in E$. It is obvious that t goes to 0 as ε goes to 0 that $\Phi(s_0 + \varepsilon, U(s_0) + t) = \Phi(s_0, U(s_0)) = 0$. For any $g' \in F'$, the mapping

$$(s, f) \longmapsto \langle \Phi(s, f), g' \rangle$$

is continuously differentiable. Then, for $k > 0$, there exists $r > 0$ such that

$$\left| \left\langle \Phi(s_0+\varepsilon, U(s_0)+t) - \Phi(s_0, U(s_0)) - \varepsilon \frac{\partial \Phi(s_0, U(s_0))}{\partial s} - \frac{\partial \Phi(s_0, U(s_0))}{\partial f} \cdot t, g' \right\rangle \right| \\ \leq k(|\varepsilon| + \|t\|)$$

which is equivalent to

$$\left|\left\langle \varepsilon T^{-1} \cdot \frac{\partial}{\partial s}\Phi(s_0, U(s_0)) + t \cdot T^* g' \right\rangle\right| \leq k(|\varepsilon| + \|t\|)$$

where $T = \frac{\partial}{\partial f}\Phi(s_0, U(s_0))$.
Since u is Lipschitz-continuous, one can find a constant K such that $\|t\| \leq K\varepsilon$. Hence, $\forall k > 0$, $\exists r > 0$ such that

$$\left|\left\langle T^{-1} \cdot \frac{\partial}{\partial s}\Phi(s_0, U(s_0)) + \frac{t}{\varepsilon}, T^* g' \right\rangle\right| \leq k(1 + K) \quad \text{for} \quad |\varepsilon| \leq r\,. \tag{29}$$

Besides $T^*(F') = E'$. Thus

$$\frac{t}{\varepsilon} \rightharpoonup -T^{-1} \cdot \frac{\partial}{\partial s}\Phi(s_0, U(s_0)) \quad (\text{as} \ \ \varepsilon \longrightarrow 0)\,.$$

in E-weak. □

Thanks to the previous theorem it is possible to show the existence of the material derivative of (u_Ω, y_Ω) solution of problem (4)-(5). First, we prove the following result

PROPOSITION 4.1. *The mapping*

$$t \longmapsto (u^t, y^t)$$

is weakly differentiable at the origin.

PROOF. Apply Theorem 4.1 with $I = [0, \tau)$, $E = X(\Omega) \times H_0^1(D)$, $F = X(\Omega)' \times H^{-1}(D)$ and

$$\begin{aligned}
&\big\langle \Phi(s; (v, \varphi)), (w, \psi) \big\rangle \\
&\quad = \Big(\big\langle \Phi_1(s; (v, \varphi)), (w, \psi) \big\rangle; \big\langle \Phi_2(s; (v, \varphi)), (w, \psi) \big\rangle \Big) \\
&\quad + \bigg(\int_D k(\Omega) D[DT_s v] \cdot DT_s^{-1} ..D[DT_s w] \cdot DT_s^{-1}\, dx \\
&\quad + \int_D \big\langle D[DT_s v] \cdot v, DT_s w \big\rangle\, dx + \int_D \beta(\Omega)\varphi g \circ T_s \cdot DT_s w\, dx; \\
&\quad \int_D \mu(\Omega) \big\langle A(s)\nabla\varphi, \nabla\psi \big\rangle\, dx \\
&\quad + \int_D (v \cdot \nabla\varphi)\psi\, dx - \int_D h \circ T_s \psi\, dx \bigg)\,,
\end{aligned} \tag{30}$$

where $A(s) = DT_s^{-1} \cdot^* DT_s^{-1}$. Since $T \in \mathcal{C}^1([0,\tau); \mathcal{C}^2(\overline{D}, \mathbb{R}^3))$, conditions (25) and (26) are obviously verified and we have

$$
\begin{aligned}
&\frac{\partial}{\partial s}\langle \Phi_1(s,(v,\varphi)),(w,\psi)\rangle \\
&\quad = \int_D k(\Omega)(D[DV(s)v]\cdot DT_s^{-1}..D[DT_s w]\cdot DT_s^{-1} \\
&\qquad - D[DT_s v]\cdot DV(s)..D[DT_s w]\cdot DT_s^{-1} \\
&\qquad + D[DT_s v]\cdot DT_s^{-1}..D[DV(s)w]\cdot DT_s^{-1} \\
&\qquad - D[DT_s v]\cdot DT_s^{-1}..D[DT_s w]\cdot DV(s)dx \\
&\qquad + \int_D \langle D[DV(s)v]\cdot v, DT_s w\rangle dx \\
&\qquad + \int_D \langle D[DT_s v]\cdot v, DV(s)w\rangle dx \\
&\qquad + \int_D \beta(\Omega)\varphi\big(g\circ T_s\cdot DV(s)w + Dg\cdot V(s)w\big)dx
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial s}\langle \Phi_2(s,(v,\varphi)),(w,\psi)\rangle \\
&\quad = \int_D \mu(\Omega)\langle A'(s)\nabla\varphi, \nabla\psi\rangle dx - \frac{\partial}{\partial s}\int_D h\circ T_s\psi\, dx\,.
\end{aligned} \tag{32}
$$

The mapping $s(\in I) \longmapsto U(s) = (u^s, y^s)$ satisfies $\Phi(s, U(s)) = 0$. To prove condition (27), we need to compute the difference between equations (3) and (23) and between (1) and (22). So (3)-(23)=

$$
\begin{aligned}
&\int_D \mu(\Omega)\nabla(y^t - y)\nabla z\, dx + \int_D u\cdot\nabla(y^t - y)z\, dx \\
&\quad = -\int_D \mu(\Omega)\langle (A(t) - I)\nabla y^t, \nabla z\rangle\, dx \\
&\qquad - \int_D (u^t - u)\cdot\nabla y^t z\, dx + \int_D (h\circ T_t - h)z\, dx\,,
\end{aligned}
$$

and the first term in (1)-(22) is

$$
\begin{aligned}
&\int_D k(\Omega)D[DT_t u^t]DT_t^{-1}..D[DT_t w]DT_t^{-1}\, dx - \int_D k(\Omega)Du..Dw\, dx \qquad (33)\\
&= \int_D k(\Omega)D[DT_t u^t - u]DT_t^{-1}..D[DT_t w]DT_t^{-1}\, dx \\
&\quad + \int_D k(\Omega)Du\cdot(DT_t^{-1} - I)..D[DT_t w]DT_t^{-1}\, dx \\
&\quad + \int_D k(\Omega)Du..D[(DT_t - I)w]DT_t^{-1}\, dx + \int_D k(\Omega)Du..Dw(DT_t^{-1} - I)dx
\end{aligned}
$$

$$= \int_D k(\Omega)D(u^t - u)..Dw\,dx + \int_D k(\Omega)D(u^t - u)(DT_t^{-1} - I)..Dw\,dx$$
$$+ \int_D k(\Omega)D(u^t - u)DT_t^{-1}..Dw(DT_t^{-1} - I)dx$$
$$+ \int_D k(\Omega)D(DT_t - I)(u^t - u)DT_t^{-1}..D[DT_t w]DT_t^{-1}\,dx$$
$$+ \int_D k(\Omega)D(u^t - u)DT_t^{-1}..D[(DT_t - I)w]DT_t^{-1}\,dx$$
$$+ \int_D k(\Omega)D[DT_t u]DT_t^{-1}..D[DT_t w]DT_t^{-1}\,dx$$
$$+ \int_D k(\Omega)Du(DT_t^{-1} - I)..D[DT_t w]DT_t^{-1}\,dx$$
$$+ \int_D k(\Omega)Du..D[(DT_t - I)w]DT_t^{-1}\,dx + \int_D k(\Omega)Du..Dw(DT_t^{-1} - I)dx$$

The second one is:

$$(34)\quad \int_D \langle D[DT_t u^t]\cdot u^t, DT_t w\rangle dx - \int_D \langle Du\cdot u, w\rangle dx$$
$$= \int_D \langle D[DT_t u^t]\cdot u^t - Du\cdot u, DT_t w\rangle dx + \int_D \langle Du\cdot u, (DT_t - I)w\rangle dx$$
$$= \int_D \langle D[(DT_t, -I)u^t]\cdot u^t + D(u^t - u)\cdot u^t + Du(u^t - u), DT_t w\rangle dx$$
$$+ \int_D \langle Du\cdot u, (DT_t - I)w\rangle dx$$
$$= \int_D \langle D(u^t - u)\cdot u^t + Du(u^t - u), w\rangle dx$$
$$+ \int_D \langle D(u^t - u)\cdot u^t + Du(u^t - u), (DT_t - I)w\rangle dx$$
$$+ \int_D \langle D[(DT_t - I)u^t]\cdot u^t, DT_t w\rangle dx + \int_D \langle DU\cdot u, (DT_t - I)w\rangle dx$$

Therefore, $(u^t - u, y^t - y)$ appears as the unique solution of the linearized (on (u, y)) of problem (1)-(3) with as right-hand term: $\beta(\Omega)y^t(g\circ T_t - g) - \beta(\Omega)y^t g\circ T_t\cdot(DT_t - I)$ added to the remaining terms of (33) and (34). Using the regularity of the transformations $\{T_t\}$ and condition (6), we deduce (27). On the other-hand, the mapping $U = (v, \varphi) \longmapsto \Phi(s, U)$ is clearly differentiable and

$$(s, U) \longmapsto \frac{\partial}{\partial U}\Phi(s, U) \quad \text{is continuous}.$$

Indeed, for all $(s, \overline{U}) \in I \times (X(\Omega) \times H_0^1(D))$,

$$\Big\langle \frac{\partial}{\partial U}\Phi(s, \overline{U})(v, \varphi), (w, \psi)\Big\rangle$$
$$= \Big(\int_D k(\Omega) D[DT_s v] \cdot DT_s^{-1} .. D[DT_s w] \cdot DT_s^{-1}\, dx$$
$$+ \int_D \langle D[DT_s \bar{v}]v, DT_s w\rangle dx + \int_D \langle D[DT_s v]\bar{v}, DT_s w\rangle dx$$
$$+ \int_D \beta(\Omega)\varphi g \circ T_s \cdot DT_s w\, dx\,;$$
$$\int_D \mu(\Omega)\langle A(s)\nabla\varphi, \nabla\psi\rangle dx + \int_D (v \cdot \nabla\bar{\varphi} + \bar{v} \cdot \nabla\varphi)\psi\, dx\Big).$$

Finally, for any $F \in X' \times H^{-1}(D)$, there exists a unique function $(\bar{u}, \bar{y}) \in X \times H_0^1(D)$ (under hypothesis (6)) such that: $\forall (w, z) \in X \times H_0^1(D)$

$$\Big\langle \frac{\partial}{\partial U}\Phi(0, U)(\bar{u}, \bar{y}), (w, z)\Big\rangle = \langle F, (w, z)\rangle$$

or equivalently

$$\int_D k(\Omega) D\bar{u}..Dw\, dx + \int_D \langle D\bar{u} \cdot u + Du \cdot \bar{u}, w\rangle dx$$
$$+ \int_D \mu(\Omega)\nabla\bar{y} \cdot \nabla z\, dx + \int_D (u \cdot \nabla\bar{y} + \bar{u} \cdot \nabla y) z\, dx = \langle F, (w, z)\rangle$$

As a consequence, the mapping $t \longmapsto (u^t, y^t)$ is weakly differentiable in $X(\Omega) \times H_0^1(D)$ and its derivative (at the origin) is given by

$$(\tilde{u}, \tilde{y}) = -\frac{\partial}{\partial U}\Phi\big(0, (u, y)\big)^{-1} \cdot \frac{\partial}{\partial t}\Phi(0, U)$$

More explicitly, it means that, for all $(v, z) \in X(\Omega) \times H_0^1(D)$, $(\tilde{u}, \tilde{y})$ satisfies

$$\int_D k(\Omega) D\tilde{u}..Dv\, dx + \int_D \langle Du \cdot \tilde{u} + D\tilde{u} \cdot u, v\rangle dx$$
$$= -\int_D k(\Omega)\big(D[DV(0)u]..Dv - Du \cdot DV(0)..Dv$$
$$+ Du..D[DV(0)v] - Du..Dv \cdot DV(0)\big)dx$$
$$- \int_D \langle D[DV(0)u] \cdot u, v\rangle dx - \int_D \langle Du \cdot u, DV(0)v\rangle dx$$
$$- \int_D \beta(\Omega)\tilde{y}g \cdot v\, dx - \int_D \beta(\Omega)\langle Dg \cdot V(0), v\rangle dx - \int_D \beta(\Omega) y g \cdot DV(0)v\, dx$$

and

$$\int_D \mu(\Omega)\nabla\tilde{y}\nabla z\, dx + \int_D (u\nabla\tilde{y} + \tilde{u}\nabla y) z\, dx$$
$$= -\int_D \mu(\Omega)\langle A'(0)\nabla y, \nabla z\rangle dx + \langle \nabla h \cdot V(0), z\rangle$$

Besides, from equation (22), we conclude that the mapping $t \longmapsto L^t$ is weakly differentiable in $H^{-1}(D, \mathbb{R}^3)$. □

REMARK 4.2. We cannot use the classical Implicit Function Theorem since it requires more regularity specifically the strong differentiability of the mapping $t \longmapsto h \circ T_t$ in $H^{-1}(D, \mathbb{R}^3)$. It is proved (cf. [9]) that for any $F \in H^s(D)$, $s \geq 1$,

$$\frac{F \circ T_t - F}{t} \longrightarrow \nabla F \cdot V(0)(t \longrightarrow 0) \quad \text{strongly in} \quad H^{s-1}(D).$$

If $s - 1 < 0$, the convergence hold only in H^{s-1}-weak.

The derivative $\tilde{y}$ is called the material derivative and is generally denoted $\dot{y} = \lim_{t \searrow 0}(y_t \circ T_t - y)/T$.

COROLLARY 4.1. *The mapping* $t \longmapsto u_t \circ T_t$ *is weakly differentiable in* $H_0^1(D,$ $\mathbb{R}^3)$ *and its weak material derivative* $\dot{u}$ *verifies the variational equation:* $\forall\, v \in X(\Omega)$,

$$\begin{aligned}
&\int_D k(\Omega) D\dot{u}..Dv\, dx + \int_D \langle D\dot{u} \cdot u, v\rangle dx + \int_D \langle Du \cdot \dot{u}, v\rangle dx - \int_D \beta(\Omega \dot{y} g \cdot v\, dx \\
&= \int_D k(\Omega) Du \cdot DV(0)..Dv\, dx - \int_D k(\Omega) Du..D[DV(0)v] dx \\
&+ \int_D k(\Omega) Du..Dv \cdot DV(0) dx + \int_D \langle Du \cdot DV(0)u, v\rangle dx \\
&- \int_D \langle Du \cdot u, DV(0)v\rangle dx - \int_D \beta(\Omega) y \langle Dg \cdot V(0), v\rangle dx \\
&- \int_D \beta(\Omega) y g \cdot DV(0) v\, dx .
\end{aligned}$$

5. – Optimality condition

Let Ω be a solution of the considered minimization Problem (15). Our objective is to get the necessary optimality condition verified by the associated flow.

A distributed necessary optimality condition will be presented when Ω is not smooth. In the smooth case, the same condition can be expressed as a boundary equation. In both cases, we need to introduce the "adjoint-problem".

5.1. – Nonsmooth case

Let $(Y, (\mathbb{U}, \mathbb{L})) \in H_0^1(D) \times (H_0^1(D, \mathbb{R}^3) \times \mathcal{E}(\Omega))$ be the solution of the following adjoint problem: $\forall (y, (v, l)) \in H_0^1(D) \times (H_0^1(D, \mathbb{R}^3) \times \mathcal{E}(\Omega))$

$$\begin{aligned}(35)\quad & \int_D k(\Omega) D\mathbb{U}..Dv\,dx + \int_D \langle Dv\cdot u, \mathbb{U}\rangle dx + \int_D \langle Du \cdot v, \mathbb{U}\rangle dx \\ &+ \int_D \beta(\Omega)\psi g \mathbb{U}\,dx + \int_D \mu(\Omega)\nabla Y \nabla \psi\,dx \\ &+ \int_D (u\cdot\nabla\psi + v\cdot\nabla y) Y\,dx + \langle \mathbb{L}, v\rangle \\ &= \int_D uv\,dx + 2\delta_1 \int_D k(\Omega)\varepsilon(u)..\varepsilon(v)dx + \delta_2 \int_D \mu(\Omega)\nabla y \nabla\psi\,dx\end{aligned}$$

$$(36)\quad \langle l, \mathbb{U}\rangle = 0\,.$$

Our attention turnes to $e_1(\Omega)$ and its Eulerian derivative. The remaining terms of $e_\theta(\Omega)$ are easy to compute.

$$\begin{aligned}(37)\quad de_1(\Omega; V) &= \int_D u\dot{u}\,dx + 2\delta_1\int_D k(\Omega)\varepsilon(u)..\varepsilon(\dot{u})dx + \delta_2\int_D \mu(\Omega)\nabla y\nabla\dot{y}\,dx \\ &+ \frac{1}{2}\int_D \mu(\Omega)\langle A'(0)\nabla y, \nabla y\rangle dx - \int_D k(\Omega)\varepsilon(u)..S(0)(u)dx\,.\end{aligned}$$

where $S(0)(u) = Du\cdot DV(0) +^* (Du\cdot DV(0))$.

Thanks to equation (35), we can express differently the right-hand side term of (37):

$$\begin{aligned}de_1(\Omega; V) &= \int_D k(\Omega)D\mathbb{U}..D\dot{u}\,dx + \int_D \langle D\dot{u}\cdot u, \mathbb{U}\rangle dx \\ &+ \int_D \langle Du\cdot\dot{u}, \mathbb{U}\rangle dx + \langle \mathbb{L}, \dot{u}\rangle + \int_D \beta(\Omega)\dot{y}g\mathbb{U}\,dx \\ &+ \int_D \mu(\Omega)\nabla Y\nabla\dot{y}\,dx + \int_D (u\cdot\nabla\dot{y} + \dot{u}\cdot\nabla y)Y\,dx \\ &+ \frac{1}{2}\int_D \mu(\Omega)\langle A'(0)\nabla y, \nabla y\rangle dx - \int_D k(\Omega)\varepsilon(u)..S(0)(u)dx \\ &= \langle \mathbb{L}, DV(0)u\rangle + \int_D k(\Omega)Du\cdot DV(0)..D\mathbb{U}\,dx - \int_D k(\Omega)Du..D[DV(0)\mathbb{U}]dx \\ &+ \int_D k(\Omega)Du..D\mathbb{U}DV(0)dx - \int_D \langle Du\cdot u, DV(0)\mathbb{U}\rangle dx \\ &+ \int_D \langle Du\cdot DV(0)u, \mathbb{U}\rangle dx - \int_D \beta(\Omega)y\langle Dg\cdot V(0), \mathbb{U}\rangle dx \\ &- \int_D \beta(\Omega)yg\cdot DV(0)\mathbb{U}\,dx - \int_D \mu(\Omega)\langle A'(0)\nabla y, \nabla Y\rangle dx \\ &+ \frac{1}{2}\int_D \mu(\Omega)\langle A'(0)\nabla y, \nabla y\rangle dx \\ &- 2\int_D k(\Omega)\varepsilon(u)..S(0)(u)dx + \langle \nabla h\cdot V(0), Y\rangle\,.\end{aligned}$$

The "Eulerian semi-derivative" of $P_D(\Omega)$ at Ω in the direction V is defined by (cf. [12]):

$$\begin{aligned} d_-P_D(\Omega;V) &= \liminf_{t>0,t\mapsto 0} t^{-1}\big(P_D(\Omega_t)-P_D(\Omega)\big) \\ &= \inf\big\{ \liminf_{n\longrightarrow\infty} t_n^{-1}\big(P_D(\Omega_{t_n})-P_D(\Omega)\big) \,|\, \{t_n\}\in\mathbb{R}^{\mathbb{N}}, t_n>0, t_n\longrightarrow 0\big\} \end{aligned}$$

where $\Omega_t = T_t(\Omega)$.

PROPOSITION 5.1. *Let Ω be an optimal solution in $BPS(D)$ of problem* (15), *then for any admissible field $V=(V_1,V_2,V_3)$, we have*

$$\begin{aligned} &\int_D k(\Omega)Du\cdot DV(0)..D\mathbb{U}\,dx + \int_D k(\Omega)Du..D\mathbb{U}DV(0)dx \\ &+\int_D \langle Du\cdot DV(0)u,\mathbb{U}\rangle dx + \langle \mathbb{L}, DV(0)u\rangle - \langle L, DV(0)\mathbb{U}\rangle \\ &-\int_D \beta(\Omega)y\langle Dg\cdot V(0),\mathbb{U}\rangle dx - \int_D \beta(\Omega)yg\cdot DV(0)\mathbb{U}\,dx \\ &-\int_D \mu(\Omega)\langle A'(0)\nabla y,\nabla Y\rangle dx + \frac{1}{2}\int_D \mu(\Omega)\langle A'(0)\nabla y,\nabla y\rangle dx \\ &-\int_D k(\Omega)\varepsilon(u)..S(0)(u)dx + \langle \nabla h\cdot V(0), Y\rangle \\ &+\int_D gV_3(0)dx + \theta d_-P_D(\Omega;V)\ge 0 \end{aligned} \tag{38}$$

REMARK 5.1. The "Eulerian semi-derivative" at Ω in the direction V, $d_-P_D(\Omega;V) > -\infty$.

Indeed,

$$\begin{aligned} &e_0(\Omega_t)-e_0(\Omega)+\theta\big(P_D(\Omega_t)-P_D(\Omega)\big)\ge 0 \\ &\quad\theta\big(P_D(\Omega_t)-P_D(\Omega)\big)\ge -tde_0(\Omega;V)-to(t), \big(o(t)\longrightarrow 0; \quad \text{as} \quad t\searrow 0\big). \end{aligned}$$

5.2. – Smooth case

In this section, assume that Ω is a sufficiently smooth open set. To simplify, denote by Ω_i, $i=1,2$, respectively Ω and $D\setminus\overline{\Omega}$. We consider the adjoint states $(Y,(\mathbb{U},\mathbb{L}))$ in $H_0^1(D)\times(H_0^1(D,R^3)\times\mathcal{E}(\Omega))$ introduced in the previous section Condition (36) is equivalent to

$$\begin{cases} \operatorname{div}\mathbb{U}=0 & \text{in} \quad D \\ \mathbb{U}\cdot n=0 & \text{on} \quad \Gamma. \end{cases}$$

Let $u_i = u_{|\Omega_i}$, p_i the associated pressure, $y_i = y_{|\Omega_i}$, we have

$$-\nu_i \operatorname{div} \varepsilon(u_i) + Du_i \cdot u_i + \nabla p_i = -\beta_i y_i g \quad \text{in} \quad \Omega\,, \tag{39}$$

$$\operatorname{div} u_i = 0 \quad \text{in} \quad \Omega_i \tag{40}$$

$$-\mu_i \Delta y_i + u_i \cdot \nabla y_i = h \quad \text{in} \quad \Omega\,, \tag{41}$$

$$u_i \cdot n = 0, \quad \text{and} \quad y_i = y_{|\Gamma} \quad \text{on} \quad \Gamma\,. \tag{42}$$

From classical regularity results for elliptic problems, we conclude that, at least, $(y_i, (u_i, p_i)) \in H^2(\Omega_i) \times (H^2(\Omega_i, \mathbb{R}^3) \times H^1(\Omega_i))$. Moreover there exists P_i in $H^1(\Omega_i)$ and r, $R \in H^{1/2}(\Gamma)$ such that

$$\langle L, \varphi\rangle = \sum_{i=1}^{2} \int_{\Omega_i} p_i \operatorname{div} \varphi \, dx + \langle r, \varphi \cdot n\rangle\,,$$
$$\langle \mathbb{L}, \varphi\rangle = \sum_{i=1}^{2} \int_{\Omega_i} P_i \operatorname{div} \varphi \, dx + \langle R, \varphi \cdot n\rangle, \quad \forall \varphi \in H_0^1(D, \mathbb{R}^3)\,.$$

As stated in Proposition 2.1, we have at the interface the two conditions (7) and (8) and

$$\mu_1 \frac{\partial y_1}{\partial n} - \mu_2 \frac{\partial y_2}{\partial n} = 0 \tag{43}$$

To derive the optimality condition, we need to characterize the shape derivatives $(y_i', (u_i', p_i'))$ of $(y_i, (u_i, p_i))$. First, we provide some preliminary Tangential and Shape Calculus.

LEMMA 5.1. *Let E be any smooth vector field defined on Γ. Then,*

$$\nabla_\Gamma (V \cdot E) = {}^*D_\Gamma V \cdot E + {}^*D_\Gamma E \cdot V$$

PROOF. We denote by $\overline{E}$ any extension of E such that $\overline{E}_{|\Gamma} = E$. We know that $\nabla(V \cdot \overline{E}) = {}^*DV \cdot \overline{E} + {}^*D\overline{E} \cdot V$ and ${}^*D\overline{E} = {}^*D_\Gamma E + n \cdot {}^*(D\overline{E} \cdot n)$. Then,

$$\begin{aligned}
\nabla_\Gamma (V \cdot E) + \frac{\partial}{\partial n}(V \cdot \overline{E})\vec{n} &= \big({}^*D_\Gamma V + n \cdot {}^*(DV \cdot n)\big) \cdot E \\
&\quad + \big({}^*D_\Gamma E + n \cdot {}^*(D\overline{E} \cdot n)\big) \cdot V \\
&= {}^*D_\Gamma V \cdot E + {}^*D_\Gamma E \cdot V + \big({}^*(DV \cdot n) \cdot E\big) n + \big({}^*(D\overline{E} \cdot n) \cdot V\big) n
\end{aligned}$$

If we consider the tangential component of each vector, on both sides of the equality, we obtain the desired result. □

LEMMA 5.2. *The shape derivative of $u \cdot n = 0$ on Γ gives*

$$u'(\Omega, V)_{|\Gamma} \cdot n = \mathrm{div}_\Gamma \big((V \cdot n)u\big) \quad on \quad \Gamma .$$

PROOF. $\int_{\gamma_t} u_t \cdot n_t \varphi \, d\Gamma_t = 0$ for all $\varphi \in C^1(\Gamma_t)$.

$$\int_\Gamma u_t \circ T_t \cdot n_t \circ T_t \psi \omega(t) dt = 0 , \quad \forall \psi \in C^1(\Gamma), \quad \omega(t) = \det DT_t \|{}^*(DT_t)^{-1} \cdot n\|_{\mathbb{R}^3} .$$

Since $\omega(o) = 1$, we obtain $\dot{u} \cdot n + u \cdot \dot{n} = 0$.
Recall $u' = \dot{u} - Du \cdot V$ the shape derivative of u and $n'_\Gamma = \dot{n} - D_\Gamma n \cdot V_\Gamma = -{}^*D_\Gamma V \cdot n - D_\Gamma n \cdot V_\Gamma$ the boundary shape derivative of n (cf. [9]). Then,

$$u' \cdot n = -\big\langle Du \cdot V, n\big\rangle - u \cdot (n'_\Gamma + D_\Gamma n \cdot V_\Gamma)$$
$$u' \cdot n = -\big\langle Du \cdot V, n\big\rangle + \big\langle u, \nabla_\Gamma (V \cdot n)\big\rangle - \big\langle u, D_\Gamma n \cdot V_\Gamma\big\rangle$$

$$\big\langle {}^*Du \cdot n, V\big\rangle = \big\langle \big({}^*D_\Gamma u + n \cdot {}^*(Du \cdot n)\big)n, V\big\rangle = \big\langle {}^*D_\Gamma u \cdot n, V\big\rangle + \big\langle Du \cdot n, n\big\rangle V \cdot n$$
$$u' \cdot n = \big\langle u, \nabla_\Gamma (V \cdot n)\big\rangle - \big\langle D_\Gamma n \cdot u, V_\Gamma\big\rangle - \big\langle {}^*D_\Gamma u \cdot n, V\big\rangle - \big\langle Du \cdot n, n\big\rangle V \cdot n$$

Using the fact that $\nabla_\Gamma V \cdot E = {}^*D_\Gamma V \cdot E + {}^*D_\Gamma E \cdot V$ and ${}^*D_\Gamma n = D_\Gamma n$ ([4]), we get

$$\begin{aligned} u' \cdot n &= \big\langle u, \nabla_\Gamma (V \cdot n)\big\rangle - \big\langle \nabla_\Gamma (u \cdot n), V_\Gamma\big\rangle - \big\langle Du \cdot n, n\big\rangle V \cdot n \\ &= \big\langle u, \nabla_\Gamma (V \cdot n)\big\rangle - \big\langle Du \cdot n, n\big\rangle V \cdot n = \big\langle u, \nabla_\Gamma (V \cdot n)\big\rangle + \mathrm{div}_\Gamma u (V \cdot n) \\ &= \mathrm{div}_\Gamma \big((V \cdot n)u\big) \end{aligned}$$

(since $\mathrm{div}\, u = 0$ we can replace $-\langle Du \cdot n, n\rangle$ by $\mathrm{div}_\Gamma u$). □

LEMMA 5.3.

$$\int_\Gamma \varepsilon(u)..\varepsilon(v) V \cdot n \, d\Gamma = -\int_\Gamma \mathrm{div}_\Gamma \big((V \cdot n)\varepsilon(u)\big) \cdot v \, d\Gamma + \int_\Gamma \big\langle Dv \cdot n + Hv, \varepsilon(u) \cdot n\big\rangle V \cdot n \, d\Gamma$$

PROOF.

$$\int_\Gamma \varepsilon(u)..\varepsilon(v) V \cdot n \, d\Gamma = \int_\Gamma (V \cdot n) \varepsilon(u)..Dv \, d\Gamma$$

Indeed, $2\varepsilon(u)..\varepsilon(v) = \varepsilon(u)_{ij} \partial_j v_i + \varepsilon(u)_{ij} \partial_i v_j = 2\varepsilon(u) \cdot Dv$

$$\begin{aligned} \int_\Gamma V \cdot n \varepsilon(u)..Dv \, d\Gamma &= \int_\Gamma V \cdot n \big(\varepsilon(u)_i \cdot \nabla v_i\big) \, d\Gamma \\ &= -\int_\Gamma v_i \,\mathrm{div}_\Gamma (V \cdot n \varepsilon(u)_i) \, d\Gamma + \int_\Gamma \left(\frac{\partial}{\partial n} v_i + H v_i\right) (\varepsilon(u)_i \cdot n) V \cdot n \, d\Gamma \\ &= -\int_\Gamma \mathrm{div}_\Gamma \big((V \cdot n)\varepsilon(u)\big) \cdot v \, d\Gamma + \int_\Gamma \big\langle Dv \cdot n + Hv, \varepsilon(u) \cdot n\big\rangle V \cdot n \, d\Gamma \end{aligned}$$ □

LEMMA 5.4. *For any sufficiently smooth v such that $v \cdot n = 0$ on Γ, $\operatorname{div} v = 0$ in D, we have*

$$\int_\Gamma p\langle DV \cdot v - Dv \cdot V, n\rangle \, d\Gamma = \int_\Gamma p \operatorname{div}_\Gamma\left((V \cdot n)v\right) d\Gamma = -\int_\Gamma \nabla p \cdot v(V \cdot n)\, d\Gamma$$

PROOF. Recall ${}^*Dv ={}^* D_\Gamma v + n \cdot{}^* (Dv \cdot n)$, and $D_\Gamma n ={}^* D_\Gamma n$. Then,

$$\begin{aligned}
\langle DV \cdot v - Dv \cdot V, n\rangle &= \langle D_\Gamma V \cdot v, n\rangle - \langle Dv \cdot V, n\rangle \\
&= \langle D_\Gamma V \cdot v, n\rangle - \langle V, {}^*D_\Gamma v \cdot n\rangle - \langle V, (n \cdot {}^*(Dv \cdot n))n\rangle \\
&= \langle D_\Gamma V \cdot v, n\rangle + \langle V, D_\Gamma n \cdot v\rangle - \langle Dv \cdot n, n\rangle V \cdot n \\
&= \langle v, {}^* D_\Gamma V \cdot n + D_\Gamma n \cdot V\rangle + \operatorname{div}_\Gamma v(V \cdot n) \\
&= \langle v, \nabla_\Gamma(V \cdot n)\rangle + \operatorname{div}_\Gamma v(V \cdot n) = \operatorname{div}_\Gamma((V \cdot n)v)
\end{aligned}$$
□

PROPOSITION 5.2. *Let Ω be a sufficiently smooth optimal domain for problem* (15). *Then, we have*

$$\begin{aligned}
\Big[\delta_1 \frac{\nu}{2}|\varepsilon(u)|^2 + \delta_2 \frac{\mu}{2}|\nabla y|^2 - \nu\varepsilon(u)..\varepsilon(\mathbb{U}) - \beta y g \mathbb{U} - \nabla_\Gamma r \cdot \mathbb{U} - \nabla_\Gamma R \cdot u \\
- \mu \nabla_\Gamma y \cdot \nabla_\Gamma Y + hY + \rho g x_3\Big]_\Gamma + \theta H = cst \ \ on \ \ \Gamma.
\end{aligned} \tag{44}$$

where $[\cdot]_\Gamma$ designates the jump at the boundary Γ and H is the mean curvature of Γ.

LEMMA 5.5. *Let $v \in H_0^1(D, \mathbb{R}^3)$, $\operatorname{div} v = 0$ in D, $v \cdot n = 0$ on Γ.*

$$\begin{aligned}
\int_\Gamma \langle [\nu(\varepsilon(u')n)_\Gamma], v\rangle \, d\Gamma &= -\int_\Gamma \nabla_\Gamma r \cdot v V \cdot n \, d\Gamma \\
&- \int_\Gamma [\nu\varepsilon(u)..\varepsilon(v)]_\Gamma V \cdot n \, d\Gamma - \int_\Gamma [\beta]_\Gamma y g v V \cdot n \, d\Gamma
\end{aligned} \tag{45}$$

PROOF. For all $v \in H_0^1(D, \mathbb{R}^3)$, $\operatorname{div} v = 0$ in D, $v \cdot n = 0$ on Γ

$$\sum_{i=1}^2 \nu_i \int_{\Omega_i} \varepsilon(u)..\varepsilon(v) dx + \int_{\Omega_i} \langle Du \cdot u, v\rangle dx = -\int_{\Omega_i} \beta_i y g v \, dx$$

Differentiating with respect to the shape

$$\begin{aligned}
&\sum_{i=1}^2 \nu_i \int_{\Omega_i} \varepsilon(u')..\varepsilon(v) dx + \int_{\Omega_i} \langle Du' \cdot u + Du \cdot u', v\rangle dx \\
&= -\nu_i \int_{\Omega_i} \varepsilon(u)..\varepsilon(v') dx - \int_{\Omega_i} \langle Du \cdot u, v'\rangle dx \\
&\quad - \int_\Gamma \left(\nu_i \varepsilon(u)..\varepsilon(v) + \langle Du \cdot u, v\rangle\right) V \cdot n_{\Omega_i} \, d\Gamma \\
&\quad - \int_{\Omega_i} \beta_i y' g v + \beta_i y g v' \, dx - \beta_i \int_\Gamma y g v (V \cdot n_{\Omega_i}) \, d\Gamma
\end{aligned}$$

On the other hand,

$$\nu_i \int_{\Omega_i} \varepsilon(u')..\varepsilon(v)dx - \nu_i \int_\Gamma \big\langle (\varepsilon(u')n_{\Omega_i})_\Gamma, v\big\rangle\, d\Gamma$$
$$+ \int_{\Omega_i} \big\langle Du' \cdot u + Du \cdot u', v\big\rangle dx = -\beta_i \int_{\Omega_i} y' g v\, dx$$

So

$$\sum_{i=1}^{2} \nu_i \int_\Gamma \big\langle \big(\varepsilon(u')n\big)_\Gamma, v\big\rangle\, d\Gamma = \sum_{i=1}^{2} \nu_i \int_{\Omega_i} \varepsilon(u')..\varepsilon(v)dx$$
$$+ \int_{\Omega_i} \big\langle Du' \cdot u + Du \cdot u', v\big\rangle dx + \beta_i \int_{\Omega_i} y' g v\, dx$$
$$= \sum_{i=1}^{2} -\nu_i \int_\Omega \varepsilon(u)..\varepsilon(v')dx - \nu_i \int_\Gamma \varepsilon(u)..\varepsilon(v)(V \cdot n_{\Omega_i})\, d\Gamma$$
$$- \int_\Gamma \big\langle Du \cdot u, v\big\rangle (V \cdot n_{\Omega_i})\, d\Gamma$$
$$- \int_{\Omega_i} \big\langle Du \cdot u, v'\big\rangle dx - \beta_i \int_{\Omega_i} y g v'\, dx - \beta_i \int_\Gamma y g v (V \cdot n_{\Omega_i})\, d\Gamma$$
$$= \sum_{i=1}^{2} -\nu_i \int_\Gamma \big\langle \varepsilon(u)n, n\big\rangle (v' \cdot n)\, d\Gamma + \int_\Gamma p(v' \cdot n_{\Omega_i})\, d\Gamma$$
$$- \nu_i \int_\Gamma \varepsilon(u)..\varepsilon(v) V \cdot n_{\Omega_i}\, d\Gamma$$
$$- \int_\Gamma \big\langle Du \cdot u, v\big\rangle (V \cdot n_{\Omega_i})\, d\Gamma - \beta_i \int_\Gamma y g v (V \cdot n_{\Omega_i})\, d\Gamma$$
$$= \int_\Gamma r\, \mathrm{div}_\Gamma (V \cdot n v)\, d\Gamma - \int_\Gamma [\nu \varepsilon(u)..\varepsilon(v) + \langle Du \cdot u, v\rangle]_\Gamma V \cdot n\, d\Gamma$$
$$- \int_\Gamma [\beta]_\Gamma y g v V \cdot n\, d\Gamma$$
$$= - \int_\Gamma \nabla_\Gamma r \cdot v V \cdot n\, d\Gamma - \int_\Gamma [\nu \varepsilon(u)..\varepsilon(v)]_\Gamma V \cdot n\, d\Gamma - \int_\Gamma [\beta]_\Gamma y g v V \cdot n\; d\Gamma$$

Since $u \cdot n = 0$ on Γ, we can replace $Du \cdot u$ by $D_\Gamma u \cdot u$ then $[\langle D_\Gamma u \cdot u, v\rangle]_\Gamma = 0$.

LEMMA 5.6.

$$\int_\Gamma \left[\mu \frac{\partial y'}{\partial n}\right]_\Gamma z\, d\Gamma = - \int_\Gamma [\mu \nabla y \cdot \nabla z]_\Gamma\, V \cdot n\; d\Gamma + \int_\Gamma [h]_\Gamma z V \cdot n\, d\Gamma \tag{46}$$

PROOF. $\forall z \in H_0^1(D)$ we have

$$\sum_{i=1}^{2} \mu_i \int_{\Omega_i} \nabla y \cdot \nabla z\, dx + \int_{\Omega_i} (u \cdot \nabla y) z\, dx = \int_{\Omega_i} h z\, dx\,.$$

Taking the derivative with respect to the shape, we obtain

$$\sum_{i=1}^{2} \mu_i \int_{\Omega_i} \nabla y' \cdot \nabla z \, dx + \int_\Gamma [\mu \nabla y \cdot \nabla z]_\Gamma (V \cdot n) \, d\Gamma + \int_{\Omega_i} (u' \cdot \nabla y + u \cdot \nabla y') z \, dx = \int_\Gamma [h]_\Gamma z V \cdot n \, d\Gamma \, .$$

Indeed, $u \cdot n = 0$, then $[u \cdot \nabla y]_\Gamma = u \cdot [\nabla_\Gamma y]_\Gamma = 0$ on Γ and so $\int_\Gamma [u \cdot \nabla y]_\Gamma z V \cdot n \, d\Gamma = 0$. On the other hand,

$$\sum_{i=1}^{2} \mu_i \int_{\Omega_i} \nabla y' \cdot \nabla z \, dx - \mu_i \int_\Gamma \frac{\partial y'}{\partial n} z \, d\Gamma + \int_{\Omega_i} (u' \cdot \nabla y + u \cdot \nabla y') z \, dx = 0$$

By comparison, we easily deduce (46). □

PROOF OF PROPOSITION 5.2. Using the regularity of $(y, (u, L))$ on both side of Γ, we can introduce the shape derivatives $u' = \dot{u} - Du \cdot V(0)$ and $y' = \dot{y} - \nabla y \cdot V(0)$ which are in $H^1(\Omega_1)$ (resp. $H^1(\Omega_2)$) when restricted to Ω_1 (resp. Ω_2). For $i = 1, 2$, we have

$$-\nu_i \operatorname{div} \varepsilon(u_i') + Du_i' \cdot u_i + Du_i \cdot u_i' + \nabla p_i' = -\beta_i y' g \quad \text{in } \Omega_i \tag{47}$$

$$\operatorname{div} u_i' = 0 \quad \text{in } \Omega_i \tag{48}$$

$$-\mu_i \Delta y_i' + u_i' \nabla y_i + u_i \nabla y_i' = 0 \quad \text{in } \Omega_i \tag{49}$$

The Eulerian derivative of $e_1(\cdot)$ with respect to the domain in the direction V is given by

$$\begin{aligned} de_1(\Omega; V) = &\int_\Gamma \left[\delta_1 \frac{\nu}{2} |\varepsilon(u)|^2 + \delta_2 \frac{\mu}{2} |\nabla y|^2 \right]_\Gamma V \cdot n \, d\Gamma + \int_D u \cdot u' \, dx \\ &+ \delta_1 \left(\nu_1 \int_\Omega \varepsilon(u)..\varepsilon(u') dx + \nu_2 \int_{\Omega^c} \varepsilon(u)..\varepsilon(u') dx \right) \\ &+ \delta_2 \left(\mu_1 \int_\Omega \nabla y' \cdot \nabla y \, dx + \mu_2 \int_{\Omega^c} \nabla y' \cdot \nabla y \, dx \right) . \end{aligned}$$

At this stage and with the help of $(Y, (\mathbb{U}, \mathbb{L}))$ solution of Problem (35)-(36), it comes

$$\begin{aligned} de_1(\Omega; V) = &\int_\Gamma \left[\delta_1 \frac{\nu}{2} |\varepsilon(u)|^2 + \delta_2 \frac{\mu}{2} |\nabla y|^2 \right]_\Gamma V \cdot n \, d\Gamma + \int_\Gamma \langle [\nu \varepsilon(u') n]_\Gamma, \mathbb{U} \rangle \, d\Gamma \\ &+ \int_\Gamma R u' \cdot n \, d\Gamma + \int_\Gamma \left[\mu \frac{\partial}{\partial n} y' \right]_\Gamma Y \, d\Gamma + \int_\Gamma \left[\mu \frac{\partial Y}{\partial n} y' \right]_\Gamma d\Gamma \, . \end{aligned}$$

$$\begin{aligned} de_1(\Omega; V) = &\int_\Gamma \left[\delta_1 \frac{\nu}{2} |\varepsilon(u)|^2 + \delta_2 \frac{\mu}{2} |\nabla y|^2 \right]_\Gamma V \cdot n \, d\Gamma - \int_\Gamma [\nu \varepsilon(u)..\varepsilon(\mathbb{U})]_\Gamma V \cdot n \, d\Gamma \\ &- \int_\Gamma [\beta]_\Gamma y g \mathbb{U} V \cdot n \, d\Gamma - \int_\Gamma (\nabla_\Gamma r \cdot \mathbb{U} + \nabla_\Gamma R \cdot u) V \cdot n \, d\Gamma \\ &- \int_\Gamma [\mu \nabla_\Gamma y \cdot \nabla_\Gamma Y]_\Gamma V \cdot n \, d\Gamma + \int_\Gamma [h]_\Gamma Y V \cdot n \, d\Gamma \, . \end{aligned}$$

□

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Ecole des Mines de Paris
Centre de Mathématiques Appliquées
B.P. 207
06904 Sophia Antipolis
France

CNRS-INLN
1361 Route des Lucioles
06560 Valbonne
France