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Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in $\mathbb{R}^n$

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 24, nº 1 (1997), p. 133-164
<http://www.numdam.org/item?id=ASNSP_1997_4_24_1_133_0>

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1. – Introduction

We consider the differential operator

$$(1.1)\quad \mathcal{A}u = \sum_{i,j=1}^{n} q_{ij} D_{ij} u + \sum_{i,j=1}^{n} b_{ij} x_j D_i u = \mathrm{Tr}\,(QD^2 u) + \langle Bx, Du \rangle, \quad x \in \mathbb{R}^n,$$

where B is any nonzero matrix, $Q = [q_{ij}]_{i,j=1,\dots,n}$ is any symmetric matrix such that

$$(1.2)\qquad \sum_{i,j=1}^{n} q_{ij} \xi_i \xi_j \geq 0, \quad \xi \in \mathbb{R}^n.$$

Moreover we assume that setting

$$(1.3)\qquad K_t = \frac{1}{t} \int_0^t e^{sB} Q e^{sB^*} ds, \quad t > 0,$$

then

$$(1.4)\qquad \det K_t > 0, \quad t > 0.$$

Condition (1.4) is equivalent to the fact that the operator $\mathcal{A}$ is hypoelliptic in the sense of Hörmander ([5]). So, if $f \in C^\infty(\mathbb{R}^n)$ and u is a distributional solution of

$$(1.5)\qquad \mathcal{A}u = f,$$

This work was partially supported by Italian National Project 40% "Equazioni di evoluzione e applicazioni ... etc."
Pervenuto alla Redazione il 30 settembre 1995.

then $u \in C^\infty(\mathbb{R}^n)$. In this paper we shall define a suitable distance d in $\mathbb{R}^n$ such that if $0 < \theta < 1$ and $f \in C_d^\theta(\mathbb{R}^n)$ then $u \in C_d^{2+\theta}(\mathbb{R}^n)$. $C_d^\theta(\mathbb{R}^n)$ is the Hölder space with exponent θ with respect to the distance d. $C_d^{2+\theta}(\mathbb{R}^n)$ will be defined later.

The distance d is equivalent to the usual euclidean distance if and only if $\det Q > 0$, i.e. the equation is nondegenerate. In such a case, assumption (1.4) is satisfied for any B, $C_d^{2+\theta}(\mathbb{R}^n)$ coincides with the usual Hölder space $C^{2+\theta}(\mathbb{R}^n)$ and the result has been proved in the previous paper [4].

To define the distance d we need to introduce another condition equivalent to (1.4), namely

$$\text{Rank}\,[Q^{1/2}, BQ^{1/2}, \dots, B^{n-1}Q^{1/2}] = n\,. \tag{1.6}$$

Such a condition is well known in control theory. It is called *Kalman rank condition.* See e.g. [15, Ch. 1] for a proof of the equivalence.

Let $k \in \{0, \dots, n-1\}$ be the smallest integer such that

$$\text{Rank}\,[Q^{1/2}, BQ^{1/2}, \dots, B^{k}Q^{1/2}] = n\,.$$

Note that the matrix Q is nonsingular if and only if $k = 0$. Set $V_0 =$ Range $Q^{1/2}$, $V_h =$ Range $Q^{1/2}$ + Range $BQ^{1/2}$ +...+ Range $B^hQ^{1/2}$. Of course, $V_h \subset V_{h+1}$ for every h, and $V_k = \mathbb{R}^n$. Define the orthogonal projections E_h as

$$\begin{cases} E_0 & = \text{projection on} \quad V_0\,, \\ E_{h+1} & = \text{projection on} \quad (V_h)^\perp \text{ in } V_{h+1}, \quad h = 1, \dots, k-1\,. \end{cases} \tag{1.7}$$

Then $\mathbb{R}^n = \bigoplus_{h=0}^k E_h(\mathbb{R}^n)$. By possibly changing coordinates in $\mathbb{R}^n$ we will consider an orthogonal basis $\{e_1, \dots, e_n\}$ consisting of generators of the subspaces $E_h(\mathbb{R}^n)$. For every $h = 0, \dots, k$ we define I_h as the set of indices i such that the vectors e_i with $i \in I_h$ span $E_h(\mathbb{R}^n)$. After the changement of coordinates the second order derivatives which appear in (1.1) are only the $D_{ij}u$ with $i, j \in I_0$.

The distance d is defined by

$$d(x, y) = \sum_{h=0}^k |E_h(x-y)|^{1/(2h+1)}\,,$$

where $|\cdot|$ is the usual euclidean norm. For $\theta > 0$ such that $\theta/(2h+1) \notin \mathbb{N}$ for $h = 0, \dots, k$, the space $C_d^\theta(\mathbb{R}^n)$ is the set of all the bounded functions $f : \mathbb{R}^n \mapsto \mathbb{R}$ such that for every $x_0 \in \mathbb{R}^n$, $0 \le h \le k$, the mapping $E_h(\mathbb{R}^n) \mapsto \mathbb{R}$, $x \mapsto f(x_0 + x)$ belongs to $C^{\theta/(2h+1)}(E_h(\mathbb{R}^n))$, with norm bounded by a constant independent of x_0. It is endowed with the norm

$$\|f\|_{C_d^\theta(\mathbb{R}^n)} = \sum_{h=0}^k \sup_{x_0 \in \mathbb{R}^n} \|f(x_0 + \cdot)\|_{C^{\theta/(2h+1)}(E_h(\mathbb{R}^n))}\,.$$

In particular, for $0 < \theta < 1$ it is the space of the bounded functions f such that

$$[f]_{C^\theta_d(\mathbb{R}^n)} = \sup_{x,y\in\mathbb{R}^n,\, x\neq y} |f(x) - f(y)| \Big(\sum_{h=0}^{k} |E_h(x-y)|^{\theta/(2h+1)} \Big)^{-1} < \infty .$$

The definition of $C^\theta_d(\mathbb{R}^n)$ in the case where $\theta/(2h+1)$ is integer for some h will be given in Section 2.

The precise statements of our main results are the following.

THEOREM 1.1. *Let $0 < \theta < 1$, $\lambda > 0$. Then for every $f \in C^\theta_d(\mathbb{R}^n)$ the problem*

$$\lambda u - \mathcal{A}u = f$$

has a unique distributional solution u in the space of the uniformly continuous and bounded functions. Moreover, u is a strong solution, it belongs to $C^{2+\theta}_d(\mathbb{R}^n)$, and there is $C > 0$, independent of f, such that

$$\|u\|_{C^{2+\theta}_d(\mathbb{R}^n)} \le C \|f\|_{C^\theta_d(\mathbb{R}^n)} . \tag{1.8}$$

THEOREM 1.2. *Let $0<\theta<1$, $T>0$, $u_0 \in C^{2+\theta}_d(\mathbb{R}^n)$, and let $f : [0,T]\times\mathbb{R}^n \mapsto \mathbb{R}$ be a continuous function such that $f(t,\cdot) \in C^\theta_d(\mathbb{R}^n)$ for every $t \in [0,T]$ and $\sup_{0\le t\le T} \|f(t,\cdot)\|_{C^\theta_d(\mathbb{R}^n)} < \infty$. Then the problem*

$$\begin{cases} u_t(t,x) = \mathcal{A}u(t,x) + f(t,x), & t>0,\ x\in\mathbb{R}^n , \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n , \end{cases} \tag{1.9}$$

has a unique distributional solution $u \in C([0,T]\times\mathbb{R}^n)$ such that $u(t,\cdot) \in C^{2+\theta}_d(\mathbb{R}^n)$ for every $t \in [0,T]$. Moreover, u is also a strong solution, and there is $C > 0$, independent of f, u_0, such that

$$\sup_{0\le t\le T} \|u(t,\cdot)\|_{C^{2+\theta}_d(\mathbb{R}^n)} \le C(\|u_0\|_{C^\theta_d(\mathbb{R}^n)} + \sup_{0\le t\le T} \|f(t,\cdot)\|_{C^\theta_d(\mathbb{R}^n)}) . \tag{1.10}$$

The proofs rely on (a) an explicit reprentation formula for the semigroup $T(t)$ associated to the operator $\mathcal{A}$, (b) the Laplace transform formula for the elliptic case, the variation of constants formula for the parabolic case, (c) interpolation techniques.

The representation formula for $T(t)$,

$$\begin{cases} (T(t)\varphi)(x) = \dfrac{1}{(4\pi t)^{n/2}(\mathrm{Det}\, K_t)^{1/2}} \\ \qquad\qquad \cdot \displaystyle\int_{\mathbb{R}^n} e^{-\frac{1}{4t}\langle K_t^{-1}y,y\rangle} \varphi(e^{tB}x - y)\,dy, \quad t>0 , \\ T(0)\varphi = \varphi , \end{cases} \tag{1.11}$$

with $e^{tB} = \sum_{n=0}^{\infty} t^n B^n/n!$, gives the solution of the evolution problem

$$(1.12)\qquad \begin{cases} u_t(t,x) = \mathcal{A}u(t,x), & t>0,\ x\in\mathbb{R}^n, \\ u(0,x) = \varphi(x), & x\in\mathbb{R}^n, \end{cases}$$

for a wide class of initial data φ. It is due to Kolmogoroff, see [6]. It lets us give precise estimates on the sup norm of the derivatives of $T(t)\varphi$ when φ belongs to $L^\infty(\mathbb{R}^n)$ or to $C_d^\theta(\mathbb{R}^n)$.

The Laplace transform formula lets us represent the solution of the elliptic equation $\lambda u - \mathcal{A}u = f$ for $\lambda > 0$ in terms of the semigroup $T(t)$:

$$(1.13)\qquad u = \int_0^{+\infty} e^{-\lambda t} T(t) f\, dt$$

whenever f is continuous and bounded. The variation of constants formula lets us represent the solution of the parabolic problem (1.9) in terms of the semigroup $T(t)$:

$$(1.14)\qquad u = T(t)u_0 + \int_0^t T(t-s) f(s,\cdot)\, ds, \quad 0 \le t \le T.$$

The space $C_d^\theta(\mathbb{R}^n)$ arises interpolating between the space X of the uniformly continuous and bounded functions in $\mathbb{R}^n$ and the domain of the realization A of $\mathcal{A}$ in X. To be precise, we prove that if $0<\theta<1$ then

$$(X, D(A))_{\theta,\infty} = C_d^{2\theta}(\mathbb{R}^n) \cap Y_\theta,$$

where Y_θ is defined by

$$(1.15)\qquad Y_\theta = \left\{ f \in X : [f]_{Y_\theta} = \sup_{0<t\le 1} \frac{|f(e^{tB}x) - f(x)|}{t^\theta} < \infty \right\}.$$

Once the interpolation spaces $(X, D(A))_{\theta,\infty}$ have been characterized, we get sharp Hölder regularity results up to $t=0$ for the solution of the evolution problem (1.12). See Corollary 6.5.

Moreover, $T(t)$ has a good behavior in the spaces $C_d^\theta(\mathbb{R}^n)$: indeed, there are C, ω such that

$$(1.16)\qquad \|T(t)f\|_{C_d^\alpha(\mathbb{R}^n)} \le \frac{Ce^{\omega t}}{t^{(\alpha-\theta)/2}} \|f\|_{C_d^\theta(\mathbb{R}^n)},\ t>0,\ 0\le\theta\le\alpha<3,\ f\in C_d^\theta(\mathbb{R}^n),$$

which is the same behavior of the semigroups associated to nondegenerate elliptic operators with bounded smooth coefficients in the usual Hölder spaces.

Coming back to the representation formulas (1.13) and (1.14), since we have good estimates for $T(t)$, in both cases we get good estimates for u. However, as one can expect, estimates (1.16) do not give immediately the optimal estimates (1.8) and (1.10). For instance, applying (1.16) to (1.13) with $\alpha = \theta + 2$ we get a nonintegrable singularity near $t = 0$. To prove Theorem 1.1 we use an interpolation procedure, showing that if $f \in C^\theta_d(\mathbb{R}^n)$ with $0 < \theta < 1$ then the function u defined by (1.13) belongs to the interpolation space

$$(C^\alpha_d(\mathbb{R}^n), C^{\alpha+2}_d(\mathbb{R}^n))_{1-\alpha/2+\theta/2,\infty} = C^{\theta+2}_d(\mathbb{R}^n), \quad \theta < \alpha < 1 .$$

Similar arguments are used to prove Theorem 1.2.

Techniques close to the present one have been used previously in [8, Ch. 3] and in [4], [10] where we considered the cases $B = 0$, $B \neq 0$, $B = B(x)$ respectively, in the nondegenerate case $\det Q \neq 0$.

Estimates (1.8) and (1.10) are used in the last section to extend the results of Theorems 1.1 and 1.2 to a case in which the coefficients of Q depend on x. We assume that there exists the limit $\lim_{|x|\to\infty} Q(x) = Q_\infty$, and that the matrices $Q(x)$, B have a certain block structure such that the projections $E_h(x)$ defined in (1.7) with Q replaced by $Q(x)$ are independent of x. Then by localization and perturbation methods we prove that the results of Theorems 1.1 and 1.2 hold also for the operator

$$\widetilde{\mathcal{A}} = \sum_{i,j=1}^{n} q_{ij}(x) D_{ij} + \sum_{i,j=1}^{n} b_{ij} x_j D_i .$$

Other optimal Hölder regularity results for the parabolic problem (1.9) in a bounded domain have been recently obtained in [11] (also for operators with variable q_{ij}), by different techniques.

2. – The spaces $C^\alpha_d(\mathbb{R}^n)$ and $C^\alpha_d(\mathbb{R}^n)$

For $m \in \mathbb{N}$ we denote by $UC^m(\mathbb{R}^n)$ the space of the functions $f : \mathbb{R}^n \mapsto \mathbb{R}$ uniformly continuous and bounded together with their derivatives up to the order m. To define the spaces $C^\alpha_d(\mathbb{R}^n)$ for general $\alpha > 0$ we introduce the Zygmund spaces $C^s(\mathbb{R}^n)$, s, $m \in \mathbb{N}$, defined by

$$C^s(\mathbb{R}^n) = \Bigg\{ f \in UC^{s-1}(\mathbb{R}^n) :$$

$$[f]_{C^s} = \sum_{|\beta|=s-1} \sup_{x,y\in\mathbb{R}^m, x\neq y} \frac{|D^\beta f(x) + D^\beta f(y) - 2D^\beta f((x+y)/2)}{|x-y|} < \infty \Bigg\},$$

$$\|f\|_{C^s(\mathbb{R}^n)} = [f]_{C^s} + \sum_{|\beta|<s} \|D^\beta f\|_\infty .$$

Now we define $C^\alpha_d(\mathbb{R}^n)$ as the set of all the bounded functions $f : \mathbb{R}^n \mapsto \mathbb{R}$ such that for every $x_0 \in \mathbb{R}^n$, $x \mapsto f(x_0 + E_h x)$ belongs to $C^{\alpha/(2h+1)}(E_h(\mathbb{R}^n))$, with norm bounded by a constant independent of x_0, for every h such that $\alpha/(2h+1)$ is not integer, and $x \mapsto f(x_0 + E_h x)$ belongs to $\mathcal{C}^{\alpha/(2h+1)}(E_h(\mathbb{R}^n))$, with norm bounded by a constant independent of x_0, for every h such that $\alpha/(2h+1)$ is integer. The space $C^\alpha_d(\mathbb{R}^n)$ is endowed with the norm

$$\|f\|_{C^\alpha_d(\mathbb{R}^n)} = \sum_{h:\,\alpha/(2h+1)\notin\mathbb{N}} \sup_{x_0\in\mathbb{R}^n} \|f(x_0 + E_h\cdot)\|_{C^{\alpha/(2h+1)}(E_h(\mathbb{R}^n))}$$

$$+ \sum_{h:\,\alpha/(2h+1)\in\mathbb{N}} \sup_{x_0\in\mathbb{R}^n} \|f(x_0 + E_h\cdot)\|_{\mathcal{C}^{\alpha/(2h+1)}(E_h(\mathbb{R}^n))}\,.$$

We shall use also the spaces $\mathcal{C}^\alpha_d(\mathbb{R}^n)$, defined as the set of all the bounded functions $f : \mathbb{R}^n \mapsto \mathbb{R}$ such that for every $x_0 \in \mathbb{R}^n$, $0 \le h \le k$, the mapping $E_h(\mathbb{R}^n) \mapsto \mathbb{R}$, $x \mapsto f(x_0+x)$ belongs to $\mathcal{C}^{\theta/(2h+1)}(E_h(\mathbb{R}^n))$, with norm bounded by a constant independent of x_0. It is endowed with the norm

$$\|f\|_{\mathcal{C}^\theta_d(\mathbb{R}^n)} = \sum_{h=0}^{k} \sup_{x_0\in\mathbb{R}^n} \|f(x_0 + E_h\cdot)\|_{\mathcal{C}^{\theta/(2h+1)}(E_h(\mathbb{R}^n))}\,.$$

Of course, $C^\alpha_d(\mathbb{R}^n) = \mathcal{C}^\alpha_d(\mathbb{R}^n)$ if $\alpha/(2h+1) \notin \mathbb{N}$ for every $h = 0, \dots, k$. Moreover it is not hard to see that if $\alpha/(2h+1) \in \mathbb{N}$ for some h, then $C^\alpha_d(\mathbb{R}^n) \subset \mathcal{C}^\alpha_d(\mathbb{R}^n)$ with continuous embedding.

To simplify notation for every $f \in C^\alpha_d(\mathbb{R}^n)$ we shall write $\|f\|_{C^{\alpha/(2h+1)}(E_h(\mathbb{R}^n))}$ instead of $\sup_{x_0\in\mathbb{R}^n} \|f(x_0 + \cdot)\|_{C^{\alpha/(2h+1)}(E_h(\mathbb{R}^n))}$. The same convention will be used if C^α is replaced by $\mathcal{C}^\alpha$.

We shall use throughout the interpolation inequalities

$$\|g\|_{\mathcal{C}^{\theta\theta_1+(1-\theta)\theta_0}(E_h(\mathbb{R}^n))} \le C\|g\|^{1-\theta}_{\mathcal{C}^{\theta_0}(E_h(\mathbb{R}^n))}\|g\|^{\theta}_{\mathcal{C}^{\theta_1}(E_h(\mathbb{R}^n))} \tag{2.1}$$

which hold for $0 \le \theta_0 < \theta_1$, $0 < \theta < 1$, $g \in \mathcal{C}^{\theta_1}(E_h(\mathbb{R}^n))$. See [14, § 2.7.2].

In the following lemma we prove further regularity properties of the functions in $\mathcal{C}^\alpha_d(\mathbb{R}^n)$. They will be used in Section 8.

LEMMA 2.1. *Let $f \in \mathcal{C}^\alpha_d(\mathbb{R}^n)$ with $\alpha > 0$. Assume that for some $r = 0, \dots, k$, $\alpha/(2r+1) > s$, with s integer. Then the derivative $D^\beta f$ with respect to the variables x_i, $i \in I_r$, belongs to $\mathcal{C}_d^{\alpha-s(2r+1)}(\mathbb{R}^n)$ for every multi-index β with $|\beta| = s$. There is $C > 0$, independent of f, such that*

$$\|D^\beta f\|_{\mathcal{C}_d^{\alpha-s(2r+1)}(\mathbb{R}^n)} \le C\|f\|_{\mathcal{C}^\alpha_d(\mathbb{R}^n)}\,.$$

PROOF. It is convenient to use an equivalent norm in the Hölder and Zygmund spaces. For every $h = 0, \dots, k$ and $\theta > 0$, m integer $> \theta$, the space $\mathcal{C}^\theta(E_h(\mathbb{R}^n))$ is the set of the bounded continuous functions g such that

$$[g]_{\mathcal{C}^\theta(E_h(\mathbb{R}^n))} = \sup_{x_0\in\mathbb{R}^n,\, y\in E_h(\mathbb{R}^n),\, y\neq 0} \frac{1}{|y|^\theta}\left|\sum_{l=0}^{m}\binom{m}{l}(-1)^l g(x_0+ly)\right| < \infty\,, \tag{2.2}$$

and the norm

$$g \mapsto \|g\|_\infty + [g]_{C^\theta(E_h(\mathbb{R}^n))}$$

is equivalent to the Hölder or to the Zygmund norm. For a proof see e.g. [14, § 2.7.2].

Let $f \in C_d^\alpha(\mathbb{R}^n)$. We shall prove that for every $x_0 \in \mathbb{R}^n$ the restriction to $E_h(\mathbb{R}^n)$ of $g = D^\beta f(x_0 + \cdot)$ belongs to $C^{(\alpha - s(2r+1))/(2h+1)}(E_h(\mathbb{R}^n))$. We shall use (2.1) with $\theta_0 = 0$, $\theta_1 = \alpha/(2r+1)$, $\theta = s(2r+1)/\alpha$.

Fix any integer $m > \alpha$. For every $h = 0, \dots, k$ and $x_0 \in \mathbb{R}^n$, $y \in E_h(\mathbb{R}^n)$ we have

$$\begin{aligned}
\left|\sum_{l=0}^{m}\binom{m}{l}(-1)^l D^\beta f(x_0 + ly)\right| &\le \left\|\sum_{l=0}^{m}\binom{m}{l}(-1)^l f(\cdot + ly)\right\|_{C^s(E_r(\mathbb{R}^n))} \\
&\le C\left\|\sum_{l=0}^{m}\binom{m}{l}(-1)^l f(\cdot + ly)\right\|_{C^{\alpha/(2r+1)}(E_r(\mathbb{R}^n))}^{s(2r+1)/\alpha} \\
&\quad\cdot \left\|\sum_{l=0}^{m}\binom{m}{l}(-1)^l f(\cdot + ly)\right\|_{L^\infty(\mathbb{R}^n)}^{1-s(2r+1)/\alpha} \\
&\le C(2^m\|f\|_{C_d^\alpha(\mathbb{R}^n)})^{s(2r+1)/\alpha}\left([f]_{C^{\alpha/(2h+1)}(E_h(\mathbb{R}^n))}|y|^{\alpha/(2h+1)}\right)^{1-s(2r+1)/\alpha} \\
&\le C\|f\|_{C_d^\alpha(\mathbb{R}^n)}|y|^{(\alpha - s(2r+1))/(2h+1)}.
\end{aligned}$$

The statement follows. □

Now we characterize some interpolation spaces between the space $X = UC(\mathbb{R}^n)$ of the uniformly continuous and bounded functions and $C_d^\alpha(\mathbb{R}^n)$. X is endowed with the sup norm.

THEOREM 2.2. *Let $\alpha > 0$, $0 < \gamma < 1$. Then*

$$(X, C_d^\alpha(\mathbb{R}^n))_{\gamma,\infty} = C_d^{\alpha\gamma}(\mathbb{R}^n).$$

PROOF. Let $f \in (X, C_d^\alpha(\mathbb{R}^n))_{\gamma,\infty}$. Then $f = u(0)$, where $t \mapsto t^{1-\gamma}u(t) \in L^\infty(0,1; C_d^\alpha(\mathbb{R}^n))$, $t \mapsto t^{1-\gamma}u'(t) \in L^\infty(0,1; X)$. For $x_0 \in \mathbb{R}^n$ and $h = 0, \dots, k$ define

$$u_h(t)(x) = u(t)(x_0 + x), \quad x \in E_h(\mathbb{R}^n).$$

Then $t \mapsto t^{1-\gamma}u_h(t) \in L^\infty(0,1; C^{\alpha/(2h+1)}(E_h(\mathbb{R}^n)))$, $t \mapsto t^{1-\gamma}u_h'(t) \in L^\infty(0,1; UC(E_h(\mathbb{R}^n)))$, so that $f(x_0 + E_h\cdot) = u_h(0)$ belongs to

$$(UC(E_h(\mathbb{R}^n)), C^{\alpha/(2h+1)}(E_h(\mathbb{R}^n)))_{\gamma,\infty} = C^{\alpha\gamma/(2h+1)}(E_h(\mathbb{R}^n))$$

([14, Thm. 1, § 2.7.2]).

Therefore, $f \in C_d^{\alpha\gamma}(\mathbb{R}^n)$ and $\|f\|_{C_d^{\alpha\gamma}(\mathbb{R}^n)} \le C\|f\|_{(X, C_d^\alpha(\mathbb{R}^n))_{\gamma,\infty}}$.

To prove the other inclusion we show preliminarily that if X is any Banach space and $A_h\colon D_{A_h} \mapsto X$, $h = 0, \dots, k$, are infinitesimal generators of commuting analytic semigroups $T_h(t)$ in X, $0 < \theta_h < 1$, then for every $\gamma \in (0, 1)$

$$\left(X, \bigcap_{h=0}^{k} (X, D_{A_h})_{\theta_h,\infty}\right)_{\gamma,\infty} = \bigcap_{h=0}^{k} (X, D_{A_h})_{\gamma\theta_h,\infty},$$

with equivalence of the respective norms.

Set $Y = \cap_{h=0}^{k} (X, D_{A_h})_{\theta_h,\infty}$, $Y_\gamma = \cap_{h=0}^{k} (X, D_{A_h})_{\gamma\theta_h,\infty}$. The embedding $(X, Y)_{\gamma,\infty} \subset Y_\gamma$ is obvious. To prove the converse, for $f \in Y_\gamma$ set

$$u(t) = \prod_{h=0}^{k} T_h(t^{1/\theta_h}) f, \quad 0 < t \le 1 .$$

Then for $r = 0, \dots, k$

$$\|u(t)\|_{(X,D_{A_r})_{\theta_r,\infty}} \le C\big(\|u(t)\|_X + \sup_{0<\xi\le 1} \|\xi^{1-\theta_r} A_r T_r(\xi) u(t)\|_X\big)$$

$$\le C(\|f\|_X + \sup_{0<\xi\le 1} \|\xi^{1-\theta_r} A_r T_r(\xi + t^{1/\theta_r}) f\|_X) \le C t^{\gamma-1} \|f\|_{(X,D_{A_r})_{\theta_r,\infty}} .$$

Moreover,

$$\|u(t) - f\|_X \le \sum_{h=0}^{k} \prod_{i=h+1}^{k} \|T_i(t^{1/\theta_i})(T_h(t^{1/\theta_h}) - 1) f\|_X \le C \sum_{h=0}^{k} t^\gamma \|f\|_{(X,D_{A_h})_{\gamma\theta_h,\infty}} .$$

Therefore, $f \in (X, Y)_{\gamma,\infty}$, and

$$\|f\|_{(X,Y)_{\gamma,\infty}} = \sup_{0<t\le 1} \frac{1}{t^\gamma} \inf_{f=a+b} (t\|a\|_Y + \|b\|_X)$$

$$\le \sup_{0<t\le 1} \frac{1}{t^\gamma} (\|u(t)\|_Y + \|u(t) - f\|_X) \le C\|f\|_{Y_\gamma} .$$

Now we apply this result taking $A_h = (-1)^{r+1}\Delta_h^r$, where $r \in \mathbb{N}$ is such that $2r - 2 < \alpha(2h+1) < 2r$, and $\theta_h = \alpha/2r(2h+1)$. Here Δ_h is the realization of the Laplace operator with respect to the variables x_i, $i \in I_h$, in X. It is easy to see that A_h generates an analytic semigroup in X, and by [14, § 1.14.3] we have

$$(X, D_{A_h})_{\gamma,\infty} = D_{\Delta_h}(r\gamma, \infty) = \{f \in D((\Delta_h)^s) : (\Delta_h)^s f \in D_{\Delta_h}(\sigma, \infty)\}$$

if $r\gamma = s + \sigma$, s integer, $0 < \sigma < 1$. On the other hand, if $0 < r\gamma < 1$ it is well known (see e.g. [13, Thm. 4*]) that

$$D_{\Delta_h}(r\gamma, \infty) = \{f \in X : f(x_0 + \cdot)_{|E_h(\mathbb{R}^n)} \in C^{2\gamma r}(E_h(\mathbb{R}^n)), \quad \sup_{x_0 \in \mathbb{R}^n} \|f(x_0 + \cdot)\|_{C^{2\gamma r}(E_h(\mathbb{R}^n))} < \infty\}. \tag{2.3}$$

Using the Schauder theorem we get easily that (2.3) holds also for $r\gamma > 1$, $2r\gamma$ not integer. By interpolation it follows that (2.3) holds also if $r\gamma$ is integer: indeed,

$$(X, D_{A_h})_{\gamma,\infty} = ((X, D_{A_h})_{\gamma-\varepsilon,\infty}, (X, D_{A_h})_{\gamma+\varepsilon,\infty})_{1/2,\infty}, \quad 0 < \varepsilon < \gamma .$$

Choosing ε such that $2r(\gamma - \varepsilon)$, $2r(\gamma + \varepsilon)$ are not integers and using the equalities

$$(C^{2(\gamma-\varepsilon)r}(E_h(\mathbb{R}^n)), C^{2(\gamma+\varepsilon)r}(E_h(\mathbb{R}^n)))_{1/2,\infty} = C^{2\gamma r}(E_h(\mathbb{R}^n))$$

we get (2.3). The statement follows. □

Using the Reiteration theorem ([14, § 1.10]) we get

COROLLARY 2.3. *For* $0 < \theta < \alpha$, $0 < \gamma < 1$

$$(C_d^{\theta}(\mathbb{R}^n), C_d^{\alpha}(\mathbb{R}^n))_{\gamma,\infty} = C_d^{\gamma\alpha+(1-\gamma)\theta}(\mathbb{R}^n).$$

3. – Estimates for $T(t)$

The representation formula (1.11) for $T(t)$ is written in terms of the operators K_t, e^{tB}. To give estimates on $T(t)f$ in several noms we need to know the behavior of such operators for $t \to 0$ and $t \to \infty$.

Set

$$\omega_0 = \sup_{\lambda \in \sigma(B)} \operatorname{Re} \lambda .$$

LEMMA 3.1. *Let* $\omega > \omega_0$. *Then there exists* $C > 0$ *such that for* $0 \le s, j \le k$, $t \ge 0$

$$\|E_s e^{tB} E_j\| = \|E_j e^{tB^*} E_s\| \le \begin{cases} Ct^{s-j} e^{\omega t}, & s \ge j, \\ Cte^{\omega t}, & s < j. \end{cases} \tag{3.1}$$

Moreover there exists $C > 0$ such that for $0 \le h \le k$

$$\|K_t^{-1/2}E_h\| \le \frac{C}{t^h}, \qquad \|E_hK_t^{1/2}\| \le Ct^h, \qquad 0 < t \le 1; \tag{3.2}$$

$$\|t^{-1/2}K_t^{-1/2}\| \le C, \qquad \|t^{1/2}K_t^{1/2}\| \le C\max(1, e^{\omega t}), \quad t \ge 1; \tag{3.3}$$

$$\|E_he^{tB^*}K_t^{-1/2}\| \le \frac{Ce^{\omega t}}{t^h}, \qquad \|K_t^{1/2}e^{tB}E_h\| \le Ce^{\omega t}t^h, \qquad t > 0. \tag{3.4}$$

PROOF. The norm of $E_je^{tB^*}E_s$ is equal to the norm of $E_se^{tB}E_j = \sum_{n=0}^{\infty} E_sB^nE_jt^n/n!$. For $s > n + j$, $E_sB^nE_j$ vanishes. (3.1) follows.
Let us prove estimates (3.2). Let the operators Γ_t be defined by

$$\Gamma_t = \sum_{h=0}^{k} t^{k-h}E_h, \quad t > 0.$$

It has been proved in [12] that

$$\exists \lim_{t\to 0} t^{-2k}\Gamma_tK_t\Gamma_t = R, \tag{3.5}$$

the matrix R being nonsingular. We get now an explicit expression of R which is not needed at this moment but will be useful in the case of coefficients depending on x. We have

$$K_t = \frac{1}{t}\int_0^t e^{(s-t)B}Qe^{(s-t)B^*}ds$$

$$= \frac{1}{t}\int_0^t \sum_{k,l=0}^{+\infty} \frac{1}{l!k!}s^ls^kds\, B^kQ(B^*)^l = \sum_{j=0}^{\infty} t^j \sum_{l=0}^{j} c_{j,l}B^{j-l}Q(B^*)^l,$$

with $c_{j,l} = 1/(j-l)!l!(j+1)$. Recalling that $E_hB^{j-l}Q(B^*)^lE_r = 0$ if $h > j-l$ or if $r > l$, we get

$$\begin{aligned} R &= \lim_{t\to 0} t^{-2k}\Gamma_tK_t\Gamma_t \\ &= \lim_{t\to 0} t^{-2k}\sum_{j=0}^{2k} t^j \sum_{h=0}^{k} t^{k-h}E_h \sum_{l=0}^{j} c_{j,l}B^{j-l}Q(B^*)^l \sum_{r=0}^{k} t^{k-r}E_r \\ &= \sum_{h,r=0}^{k} c_{h+r,r}E_hB^hQ(B^*)^rE_r. \end{aligned} \tag{3.6}$$

Since R is nonsingular, (3.5) implies that

$$\exists \lim_{t\to 0} t^{2k}\Gamma_t^{-1}K_t^{-1}\Gamma_t^{-1} = R^{-1}.$$

Therefore for every $z \in \mathbb{R}^n$ we have

$$|K_t^{-1/2} E_h z|^2 = \langle z, E_h K_t^{-1} E_h z\rangle = t^{2(k-h)} \langle z, E_h \Gamma_t^{-1} K_t^{-1} \Gamma_t^{-1} E_h z\rangle$$

where $t^{2k}\Gamma_t^{-1}K_t^{-1}\Gamma_t^{-1}$ goes to R^{-1} as t goes to 0, so that it is bounded near $t = 0$. The first estimate in (3.2) follows, and the second one may be proved similarly.

To prove (3.4) for $0 < t \leq 1$ we make a computation similar to the one above. We write $e^{-tB}K_t e^{-tB^*}$ as

$$e^{-tB}K_t e^{-tB^*} = \frac{1}{t}\int_0^t e^{(s-t)B} Q e^{(s-t)B^*} ds = \sum_{j=0}^{\infty} t^j \sum_{l=0}^{j} k_{j,l} B^{j-l} Q (B^*)^l ,$$

with $k_{j,l} = (-1)^j/(j-l)!l!(j+1)$. Arguing as we did in the computation of R we get

$$\lim_{t\to 0} t^{-2k}\Gamma_t e^{-tB} K_t e^{-tB^*}\Gamma_t = \sum_{h,r=0}^{k} k_{h+r,r} E_h B^h Q (B^*)^r E_r = S ,$$

where

$$\det S = \lim_{t\to 0} \det(t^{-2k}\Gamma_t K_t^{-1}\Gamma_t)(\det e^{-tB})^2 = \det R .$$

Therefore, there exists the limit

$$\lim_{t\to 0} t^{2k}\Gamma_t^{-1} e^{tB^*} K_t^{-1} e^{tB}\Gamma_t^{-1} = S^{-1} .$$

Arguing as above we get

$$\|K_t^{-1/2} e^{tB} E_h\| \leq C t^{-h}, \quad 0 < t \leq 1 ,$$

and (3.4) follows for $0 < t \leq 1$, recalling that $\|E_h e^{tB^*} K_t^{-1/2}\| = \|K_t^{-1/2} e^{tB} E_h\|$.

Now we prove the estimates for $t \geq 1$. For every $z \in \mathbb{R}^n$ we have

$$|K_t^{1/2} z|^2 = \langle K_t z, z\rangle = \frac{1}{t}\int_0^t |Q^{1/2} e^{sB^*} z|^2 ds \leq \frac{1}{t}\frac{C(e^{2\omega t}-1)}{2\omega}|z|^2 ,$$

and the second estimate of (3.3) follows. To prove the first one we note that $t \mapsto tK_t$ is nondecreasing in $(0, +\infty)$: indeed, for $t \geq t_0$ and $x \in \mathbb{R}^n$

$$\langle tK_t x, x\rangle - \langle t_0 K_{t_0} x, x\rangle = \int_{t_0}^t |Q^{1/2} e^{sB^*} x|^2 ds \geq 0 .$$

Therefore, $t \mapsto (tK_t)^{-1}$ is nonincreasing, so that for $t \geq 1$ we have $(tK_t)^{-1} \leq K_1^{-1}$. This means that for every $z \in \mathbb{R}^n$

$$|t^{-1/2} K_t^{-1/2} z|^2 = \langle (tK_t)^{-1} z, z\rangle \leq |K_1^{-1} z|^2 ,$$

and the first of (3.3) follows. Estimates (3.4) for $t \geq 1$ follow easily from (3.3). □

From the representation formula (1.11) it follows immediately that

$$\|T(t)f\|_\infty \le \|f\|_\infty, \qquad \forall f \in X,\ t \ge 0 \tag{3.7}$$

and that $T(t)f \in C^\infty(\mathbb{R}^n)$ for every $f \in X$ and $t > 0$. Estimates on the first, second, and third order derivatives of $T(t)f$ are provided by the next proposition.

PROPOSITION 3.2. *Let $\omega > \omega_0$. Then for $0 \le h \le k$ and $i \in I_h$*

$$\|D_i T(t)f\|_\infty \le \frac{Ce^{\omega t}}{t^{h+1/2}}\|f\|_\infty, \quad f \in X,\ t > 0, \tag{3.8}$$

for $0 \le h,\ r \le k$ and $i \in I_h$, $j \in I_r$

$$\|D_{ij} T(t)f\|_\infty \le \frac{Ce^{2\omega t}}{t^{h+r+1}}\|f\|_\infty, \qquad f \in X,\ t > 0; \tag{3.9}$$

for $0 \le h,\ r,\ s \le k$ and $i \in I_h$, $j \in I_r$, $l \in I_s$

$$\|D_{ijl} T(t)f\|_\infty \le \frac{Ce^{3\omega t}}{t^{h+r+s+3/2}}\|f\|_\infty, \qquad f \in X,\ t > 0. \tag{3.10}$$

PROOF. For every $t > 0$ we have (D denotes the gradient)

$$DT(t)f = -\frac{t^{-1/2}}{2(4\pi)^{n/2}}\int_{\mathbb{R}^n} e^{tB^*}K_t^{-1/2}ze^{-|z|^2/4}f(e^{tB}x - t^{1/2}K_t^{1/2}z)dz. \tag{3.11}$$

Therefore, the sup norm of each component of $t^{1/2}K_t^{1/2}e^{-tB^*}DT(t)f$ is bounded by $\|f\|_\infty/\sqrt{\pi}$. By (3.4) we get

$$\|E_h DT(t)f\|_\infty = \|E_h e^{tB^*}K_t^{-1/2}K_t^{1/2}e^{-tB^*}DT(t)f\|_\infty \le \frac{Ce^{\omega t}}{t^{h+1/2}}\|f\|_\infty.$$

To estimate the second order derivatives, we note that if $\varphi \in UC^1(\mathbb{R}^n)$ then

$$T(t)D\varphi = e^{-tB^*}DT(t)\varphi, \qquad DT(t)\varphi = e^{tB^*}T(t)D\varphi, \quad t > 0. \tag{3.12}$$

It follows that for $i \in I_h$, $j \in I_r$ we have

$$\begin{aligned}\|D_{ij}T(t)f\|_\infty &= \|D_i(D_jT(t/2)T(t/2)f)\|_\infty \\ &= \|D_i(e^{tB^*}T(t/2)DT(t/2)f)_j\|_\infty \\ &= \left\|\sum_{l=1}^n (e^{tB^*})_{jl}D_iT(t/2)D_lT(t/2)f\right\|_\infty \\ &\le \frac{Ce^{\omega t}}{t^{h+1/2}}\sum_{l=1}^n |(e^{tB^*})_{jl}|\,\|D_lT(t/2)f\|_\infty.\end{aligned} \tag{3.13}$$

Using (3.1) we get

$$\begin{aligned}&\|D_{ij}T(t)f\|_\infty \\ &\le \frac{C}{t^{h+1/2}}\left(\sum_{l=0}^{r-1} Ct\|E_lDT(t/2)f\|_\infty + \sum_{l=r}^{k} Ct^{l-r}\|E_lDT(t/2)\varphi\|_\infty\right).\end{aligned} \tag{3.14}$$

Recalling (3.8) we obtain (3.9). The proof of (3.10) is similar and it is left to the reader. □

COROLLARY 3.3. *Let $\alpha \in (0,3]$, and let $\omega > \omega_0$. Then for every $t > 0$ we have*

$$\|T(t)\|_{L(X,C_d^\alpha(\mathbb{R}^n))} \le C\left(1+\frac{e^{\beta t}}{t^{\alpha/2}}\right). \tag{3.15}$$

with $\beta = \alpha\omega/(2k+1)$ if $\omega < 0$, $\beta = \alpha\omega$ if $\omega \ge 0$.

PROOF. For every $f \in X$, $x_0 \in \mathbb{R}^n$ and $t > 0$, $T(t)f(x_0+\cdot)_{|E_h(\mathbb{R}^n)}$ belongs to $C^1(E_h(\mathbb{R}^n))$. By estimates (3.7) and (3.8) its sup norm is bounded by $\|f\|_\infty$, and its C^1 norm is bounded by $C(1+t^{-h-1/2}e^{\omega t})\|f\|_\infty$. By (2.1), with $\theta_0 = 0$, $\theta_1 = 1$, $\theta = \alpha/(2h+1)$, we get

$$\begin{aligned}\|T(t)f\|_{C^{\alpha/(2h+1)}(E_h(\mathbb{R}^n))} &\le C\big(\|T(t)f\|_\infty\big)^{1-\alpha/(2h+1)}\big(\|T(t)f\|_{C^1(E_h(\mathbb{R}^n))}\big)^{\alpha/(2h+1)}\\ &\le C\left(1+\frac{e^{\omega\alpha t/(2h+1)}}{t^{\alpha/2}}\right)\|f\|_\infty,\end{aligned}$$

for every h if $0 < \alpha < 1$, for $h \ge 1$ if $\alpha \ge 1$. (3.15) follows for $0 < \alpha \le 1$.

For $1 < \alpha < 3$ estimates (3.7) and (3.10) imply, for $i \in I_0$ and for every $\omega > \omega_0$

$$\|T(t)f\|_{C^3(E_0(\mathbb{R}^n))} \le C\left(1+\frac{e^{3\omega t}}{t^{3/2}}\right)\|f\|_\infty$$

By (2.1), with $\theta_0 = 0$, $\theta_1 = 3$, $\theta = \alpha/3$, we get

$$\begin{aligned}\|T(t)f\|_{C^\alpha(E_0(\mathbb{R}^n))} &\le C(\|T(t)f\|_\infty)^{1-\alpha/3}\big(\|T(t)f\|_{C^3(E_0(\mathbb{R}^n))}\big)^{\alpha/3}\\ &\le C\left(1+\frac{e^{\omega\alpha t}}{t^{\alpha/2}}\right)\|f\|_\infty,\end{aligned}$$

and (3.15) follows for $1 < \alpha \le 3$. □

THEOREM 3.4. *Let $0 < \theta \le \alpha < 3$. For every $t > 0$ we have*

$$\|T(t)\|_{L(C_d^\theta(\mathbb{R}^n),C_d^\alpha(\mathbb{R}^n))} \le C\left(1+\frac{e^{\beta t}}{t^{(\alpha-\theta)/2}}\right). \tag{3.16}$$

with $\beta = \omega\alpha/(2k+1)$ if $\omega \le 0$, $\beta = \omega(\alpha+\theta([\alpha]+1)/\alpha)$ if $\omega > 0$.

PROOF. It is sufficient to prove (3.16) for $\theta = \alpha$ non integer. The general case will follow from this one and from estimates (3.15) by interpolation. Indeed, since by Theorem 2.2 we have $(X, C_d^\alpha(\mathbb{R}^n))_{\theta/\alpha,\infty} = C_d^\theta(\mathbb{R}^n)$ for $0 < \theta < \alpha \le 3$, choosing any noninteger $\alpha \in (\theta, 3)$ we get

$$\|T(t)\|_{L(C_d^\theta(\mathbb{R}^n),C_d^\alpha(\mathbb{R}^n))} \le C\big(\|T(t)\|_{L(X,C_d^\alpha(\mathbb{R}^n))}\big)^{1-\theta/\alpha}\big(\|T(t)\|_{L(C_d^\alpha(\mathbb{R}^n))}\big)^{\theta/\alpha},$$

and (3.16) follows.

We shall show preliminarly that for $\omega > \omega_0$ and for $i \in I_h$, $0 \le h \le k$, $f \in C_d^\theta(\mathbb{R}^n)$ with $0 < \theta < 1$ we have

$$\|D_i T(t) f\|_\infty \le C e^{\omega\theta t} \max\{1, e^{\omega t}\} t^{-h-1/2+\theta/2} \|f\|_{C_d^\theta(\mathbb{R}^n)}, \quad t > 0; \tag{3.17}$$

for $i,\ j \in I_0$, $f \in C_d^\theta(\mathbb{R}^n)$ with $1 < \theta < 2$ we have

$$\begin{aligned} &\|D_{ij} T(t) f\|_\infty \le C e^{2\omega\theta t} \max\{1, e^{\omega t}\} t^{-1+\theta/2} \|f\|_{C_d^\theta(\mathbb{R}^n)}, \\ &\qquad f \in C_d^\theta(\mathbb{R}^n),\ 1 < \theta < 2,\ t > 0; \end{aligned} \tag{3.18}$$

for $i,\ j,\ l \in I_0$, $f \in C_d^\theta(\mathbb{R}^n)$ with $2 < \theta < 3$ we have

$$\begin{aligned} &\|D_{ijl} T(t) f\|_\infty \le C e^{3\omega\theta t} \max\{1, e^{\omega t}\} t^{-3/2+\theta/2} \|f\|_{C_d^\theta(\mathbb{R}^n)}, \\ &\qquad f \in C_d^\theta(\mathbb{R}^n),\ 2 < \theta < 3,\ t > 0. \end{aligned} \tag{3.19}$$

In fact for $i \in I_h$ we get, using first the equality $\int_{\mathbb{R}^n} z_r e^{-|z|^2/4} dz = 0$ and then estimates (3.3) and (3.4),

$$\begin{aligned} &|D_i T(t) f(x)| \\ &\le \frac{t^{-1/2}}{2(4\pi)^{n/2}} \int_{\mathbb{R}^n} |E_h e^{tB^*} K_t^{-1/2} z| \cdot |f(e^{tB} x - t^{1/2} K_t^{1/2} z) \\ &\quad - f(e^{tB} x)| e^{-|z|^2/4} dz \\ &\le \frac{C e^{\omega t}}{t^{h+1/2}} \int_{\mathbb{R}^n} \sum_{r=0}^{k} \left| E_r (t^{1/2} K_t^{1/2} z) \right|^{\theta/(2r+1)} e^{-|z|^2/4} dz [f]_{C_d^\theta(\mathbb{R}^n)} \\ &\le \frac{C e^{\omega t}}{t^{h+1/2}} \int_{\mathbb{R}^n} \sum_{r=0}^{k} \left(\max\{1, e^{\omega t}\} t^{r+1/2} |z| \right)^{\theta/(2h+1)} e^{-|z|^2/4} dz [f]_{C_d^\theta(\mathbb{R}^n)}, \end{aligned} \tag{3.20}$$

and (3.17) follows.

To prove (3.18) we use the identities $\int_{\mathbb{R}^n} z_{hlr} e^{-|z|^2/4} dz = 0$ and

$$\int_{\mathbb{R}^n} \left[(e^{tB^*} K_t^{-1/2} z)_i (e^{tB^*} K_t^{-1/2} z)_j - 2(e^{tB^*} K_t^{-1} e^{tB})_{ij} \right] e^{-|z|^2/4} dz = 0,$$

from which it follows

$$\begin{aligned} D_{ij} T(t) f(x) &= \frac{1}{4t(4\pi)^{n/2}} \int_{\mathbb{R}^n} \\ &\quad \cdot \left[(e^{tB^*} K_t^{-1/2} z)_i (e^{tB^*} K_t^{-1/2} z)_j - 2(e^{tB^*} K_t^{-1} e^{tB})_{ij} \right] \\ &\quad \cdot f(e^{tB} x - t^{1/2} K_t^{1/2} z) e^{-|z|^2/4} dz \\ &= \frac{1}{4t(4\pi)^{n/2}} \int_{\mathbb{R}^n} \left[(e^{tB^*} K_t^{-1/2} z)_i (e^{tB^*} K_t^{-1/2} z)_j - 2(e^{tB^*} K_t^{-1} e^{tB})_{ij} \right] \\ &\quad \cdot \big(f(e^{tB} x + t^{1/2} K_t^{1/2} z) - f(e^{tB} x) \\ &\quad - \sum_{l \in I_0} D_l f(e^{tB} x)(-t^{1/2} K_t^{1/2} z)_l \big) e^{-|z|^2/4} dz \end{aligned}$$

and using the inequality

$$\begin{aligned}
&\left|f(x-t^{1/2}K_t^{1/2}z)-f(x)-\sum_{l\in I_0}D_lf(x)(-t^{1/2}K_t^{1/2}z)_l\right|\\
&\le\left|f(x-t^{1/2}K_t^{1/2}z)-f(x-t^{1/2}E_0K_t^{1/2}z)\right|\\
&\quad+\left|f(x+t^{1/2}E_0K_t^{1/2}z)-f(x)-\sum_{l\in I_0}D_lf(x)(-t^{1/2}K_t^{1/2}z)_l\right|\\
&\le C\left(t^{\theta/2}|E_0K_t^{1/2}z|^\theta+\sum_{h=1}^k|t^{1/2}E_hK_t^{1/2}z|^{\theta/(2h+1)}\right)\|f\|_{C_d^\theta(\mathbb{R}^n)}\\
&\le C\sum_{h=0}^k t^{\theta/2}\left(\max\{1,e^{\omega t}\}|z|\right)^{\theta/(2h+1)}\|f\|_{C_d^\theta(\mathbb{R}^n)},
\end{aligned}$$

together with (3.3) we get (3.18).

To prove (3.19) we remark that for every continuous g with polynomial growth at infinity we have

$$(D_{ijl}T(t)g)(x)=\int_{\mathbb{R}^n}G_{ijl}(t,z)g(e^{tB}x-t^{1/2}K_t^{1/2}z)e^{-|z|^2/4}dz\,, \tag{3.22}$$

where

$$\begin{aligned}
G_{ijl}(t,z)=-\frac{t^{-3/2}}{8(4\pi)^{n/2}}\Big[&(e^{tB^*}K_t^{-1/2}z)_i(e^{tB^*}K_t^{-1/2}z)_j(e^{tB^*}K_t^{-1/2}z)_l\\
&+2(e^{tB^*}K_t^{-1}e^{tB})_{il}(e^{tB^*}K_t^{-1/2}z)_j+2(e^{tB^*}K_t^{-1}e^{tB})_{lj}(e^{tB^*}K_t^{-1/2}z)_i\\
&+2(e^{tB^*}K_t^{-1}e^{tB})_{ji}(e^{tB^*}K_t^{-1/2}z)_l\Big]\,.
\end{aligned}$$

Moreover we use the identity (which holds for every i, j, l, $r=1,\dots,n$)

$$\int_{\mathbb{R}^n}G_{ijl}(t,z)e^{-|z|^2/4}(t^{1/2}K_t^{1/2}z)_rdz=0\,.$$

It can be proved as follows: the function $g_r(x)=x_r$ is such that $T(t)g_r$ is affine with respect to x for every $t\ge 0$, so that its space derivatives of any order bigger than 1 vanish. In particular, the third order derivatives vanish. By formula (3.22) we have

$$0=(D_{ijl}T(t)g_r)(0)=-\int_{\mathbb{R}^n}G_{ijl}(t,z)e^{-|z|^2/4}t^{1/2}(K_t^{1/2}z)_rdz\,.$$

Let $f \in C^\theta_d(\mathbb{R}^n)$, with $2 < \theta < 3$. Then

$$D_{ijl}T(t)f(x) = \frac{1}{8t^{3/2}(4\pi)^{n/2}} \int_{\mathbb{R}^n} G_{ijl}(t,z)e^{-|z|^2/4}$$

$$(3.23) \qquad \cdot \Big[f(e^{tB}x - t^{1/2}K_t^{1/2}z) - f(e^{tB}x) - \sum_{r\in I_0} D_r f(e^{tB}x)(-t^{1/2}K_t^{1/2}z)_r$$

$$- \frac{1}{2}\sum_{r,s\in I_0} D_{rs}f(e^{tB}x)(-t^{1/2}K_t^{1/2}z)_r(-t^{1/2}K_t^{1/2}z)_r \Big] dz\,.$$

Arguing as in estimate (3.21) we see that

$$\Big| f(e^{tB}x - t^{1/2}K_t^{1/2}z) - f(e^{tB}x) - \sum_{r\in I_0} D_r f(e^{tB}x)(-t^{1/2}K_t^{1/2}z)_r$$

$$- \frac{1}{2}\sum_{r,s\in I_0} D_{rs}f(e^{tB}x)(-t^{1/2}K_t^{1/2}z)_r(-t^{1/2}K_t^{1/2}z)_r \Big|$$

$$\le C\sum_{h=0}^{k} t^{\theta/2}(\max\{1,e^{\omega t}\}|z|)^{\theta/(2h+1)}\|f\|_{C^\theta_d(\mathbb{R}^n)}\,.$$

Replacing in (3.23) and using (3.4), (3.19) follows.

We are ready now to prove estimates (3.16).

Let $0 < \theta < 1$, $f \in C^\theta_d(\mathbb{R}^n)$. For x_0, $y \in \mathbb{R}^n$, $|y| \ge 1$, we have obviously

$$|T(t)f(x_0 + E_h y) - T(t)f(x_0)| \le 2\|f\|_\infty \le 2\|f\|_\infty |y|^{\theta/(2h+1)}$$

To estimate $|T(t)f(x_0+E_h y) - T(t)f(x_0)|$ for $|y| < 1$ we distinguish two cases: $|y| \le t^{(2h+1)/2}$ and $|y| \ge t^{(2h+1)/2}$. In the first case we use (3.17), getting

$$|T(t)f(x_0 + E_h y) - T(t)f(x_0)| \le \frac{Ce^{\omega t}\max\{1,e^{\omega\theta t}\}}{t^{h+1/2-\theta/2}}|y|\;\|f\|_{C^\theta_d(\mathbb{R}^n)}$$

$$\le Ce^{\omega t}\max\{1,e^{\omega\theta t}\}|y|^{\theta/(2h+1)}\|f\|_{C^\theta_d(\mathbb{R}^n)}\,.$$

In the second case we use the representation formula (1.11) for $T(t)$ and estimates (3.1), getting

$$|T(t)f(x_0+E_hy)-T(t)f(x_0)|\le \frac{1}{(4\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-|z|^2/4}$$

$$\cdot\left|f(e^{tB}(x_0+E_hy)-t^{1/2}K_t^{1/2}z)-f(e^{tB}x_0-t^{1/2}K_t^{1/2}z)\right|dz$$

$$\le \sum_{r=0}^{k}|E_re^{tB}E_hy|^{\theta/(2r+1)}[f]_{C_d^\theta(\mathbb{R}^n)}$$

$$\le C\left(\sum_{r=0}^{h-1}(te^{\omega t}|y|)^{\theta/(2r+1)}+\sum_{r=h}^{k}t^{\theta(r-h)/(2r+1)}|e^{\omega t}y|^{\theta/(2r+1)}\right)[f]_{C_d^\theta(\mathbb{R}^n)}$$

$$\le C(k+1)\max\{e^{\omega\theta t},e^{\omega\theta t/(2k+1)}\}|y|^{\theta/(2h+1)}[f]_{C_d^\theta(\mathbb{R}^n)},$$

and (3.16) follows.

Let now $f\in C_d^\theta(\mathbb{R}^n)$ with $1<\theta<2$. It is convenient to use the seminorm (2.2) with $m=2$. For $x_0,\ y\in\mathbb{R}^n$ $|y|\le t^{1/2}$ and $h=0$, we use (3.18) to get

$$|(T(t)f)(x_0)-2(T(t)f)(x_0+E_0y)+(T(t)f)(x_0+2E_0y)|$$

$$\le\frac{1}{2}\sum_{i,j\in I_0}\|D_{ij}T(t)f\|_\infty|y|^2\le Ce^{2\omega t}\max\{1,e^{\omega\theta t}\}t^{-1+\theta/2}\|f\|_{C_d^\theta(\mathbb{R}^n)}|y|^2$$

$$\le Ce^{2\omega t}\max\{1,e^{\omega\theta t}\}|y|^\theta\|f\|_{C_d^\theta(\mathbb{R}^n)}.$$

For $h=1,\dots,k$, $|y|\le t^{(2h+1)/2}$ we use (3.17) to get

$$|(T(t)f)(x_0)-2(T(t)f)(x_0+E_hy)+(T(t)f)(x_0+2E_hy)|$$

$$\le 2\sum_{i\in I_0}\|D_iT(t)f\|_\infty|E_hy|\le Ce^{\omega t}\max\{1,e^{\omega\theta t}\}t^{-h-1/2+\theta/2}\|f\|_{C_d^\theta(\mathbb{R}^n)}|y|$$

$$\le Ce^{\omega t}\max\{1,e^{\omega\theta t}\}|y|^\theta\|f\|_{C_d^\theta(\mathbb{R}^n)}.$$

For $h=0,\dots,k$, $t^{(2h+1)/2}\le|y|\le 1$ we use the representation formula for

$T(t)$, getting

$$
\begin{aligned}
&|(T(t)f)(x_0) - 2(T(t)f)(x_0+E_0y) + (T(t)f)(x_0+2E_0y)| \\
&\le \frac{1}{(4\pi)^{n/2}} \int_{\mathbb{R}^n} |f(e^{tB}x_0 - t^{1/2}K_t^{1/2}z) - 2f(e^{tB}(x_0+E_hy) - t^{1/2}K_t^{1/2}z) \\
&\quad + f(e^{tB}(x_0+2E_hy) - t^{1/2}K_t^{1/2}z)|e^{-|z|^2/4}dz \\
&\le C\sum_{r=0}^{k} |E_r e^{tB}E_h y|^{\theta/(2r+1)}[f]_{C_d^\theta(\mathbb{R}^n)} \\
&\le C\left(\sum_{r=0}^{h-1} |te^{\omega t}y|^{\theta/(2r+1)} + \sum_{r=h}^{k} t^{\theta(r-h)/(2r+1)}|e^{\omega t}y|^{\theta/(2r+1)}\right)[f]_{C_d^\theta(\mathbb{R}^n)} \\
&\le C(k+1)\max\{e^{\omega\theta t}, e^{\omega\theta t/(2k+1)}\}e^{\omega t}|y|^{\theta/(2h+1)}[f]_{C_d^\theta(\mathbb{R}^n)}\,.
\end{aligned}
$$

For $|y| \ge 1$ we have obviously

$$
\begin{aligned}
&|(T(t)f)(x_0) - 2(T(t)f)(x_0+E_0y) \\
&\qquad + (T(t)f)(x_0+2E_0y)| \le 4\|f\|_\infty \le 4\|f\|_\infty |y|^{\theta/(2h+1)},
\end{aligned}
$$

and (3.16) follows for $1 < \theta < 2$. The proof for $2 < \theta < 3$ is similar and it is left to the reader; one has to use estimate (3.19) instead of (3.18) and the seminorm (2.2) with $m = 3$ instead of $m = 2$. □

4. – The generator A of $T(t)$

As a semigroup in X, $T(t)$ is generated by $\mathcal{A}$ in a weak sense, which we explain below.

For every $\varphi \in X$ and $x \in \mathbb{R}^n$ the function $t \mapsto (T(t)\varphi)(x)$ is continuous in $[0, +\infty)$ and it is bounded thanks to (3.7). Therefore for $\mathrm{Re}\,\lambda > 0$ the integral

$$
R(\lambda)\varphi(x) = \int_0^{+\infty} e^{-\lambda t}(T(t)\varphi)(x)dt, \quad x \in \mathbb{R}^n, \tag{4.1}
$$

makes sense, and it is easy to check that it defines a uniformly continuous and bounded function. So, $R(\lambda) \in L(X)$, and thanks to (3.7)

$$
\|R(\lambda)\|_{L(X)} \le \lambda^{-1}.
$$

Moreover $R(\lambda)$ satisfies the resolvent identity because $T(t)$ is a semigroup, and it is one to one because for every $x \in \mathbb{R}^n$ $(R(\lambda)\varphi)(x)$ is the anti-Laplace transform of the real continuous function $t \mapsto (T(t)\varphi)(x)$, which takes the value $\varphi(x)$ at $t=0$. Therefore there exists a closed operator

$$A : D(A) \mapsto X, \quad D(A) = \text{Range } R(\lambda) \quad \text{for Re} \quad \lambda > 0\,,$$

such that $R(\lambda) = R(\lambda, A)$ for $\mathrm{Re}\,\lambda > 0$. In the notation of [1], [3], A is the infinitesimal generator of the weakly continuous semigroup $T(t)$ in X. See next section and [1, § 6].

To characterize the domain of A and the domain of the part of A in $C^\theta_d(\mathbb{R}^n)$ we shall use some interpolation results which we collect in the next section.

5. – Some abstract interpolation results

We recall here some results proved in [9]. They will be crucial in the characterization of $D(A)$ and in the proofs of Theorems 1.1 and 1.2.

Let X be the space of the real uniformly continuous and bounded functions defined in a Hilbert space H, and let $T(t)$ be a weakly continuous semigroup. This means that $T(t)$ is a semigroup of linear operators in $L(X)$ such that

(a) for every $T > 0$ and $f \in X$ the family of functions $\{T(t)f : 0 \le t \le T\}$ are equi-uniformly continuous;

(b) for every $f \in X$ we have $\lim_{t\to 0} \|T(t)f - f\|_{L^\infty(K)} = 0$ for every compact set $K \subset H$;

(c) for every $f \in X$ and for every bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ converging to f uniformly on each compact set $K \subset H$, $\{T(t)f_n\}_{n\in\mathbb{N}}$ converges to $T(t)f$ uniformly on each compact set, uniformly with respect to $t \in [0,T]$ for every $T > 0$.

Assume that there are Banach spaces X_0, X_1, X_2, such that $X_2 \subset X_1 \subset X_0 \subset X$ with continuous embeddings, and there are $0 \le \gamma_1 < 1 < \gamma_2$, $c_1, c_2 > 0$ such that $T(t) \in L(X_0, X_2)$ and

$$t^{\gamma_i} e^{-\omega t} \|T(t)\|_{L(X_0,X_i)} \le c_i, \quad t > 0, \qquad i = 1,2\,. \tag{5.1}$$

Assume also that for $i = 1,2$ and for every interval $I \subset \mathbb{R}$, the following holds. If $\varphi : I \mapsto X_0$ is such that for every $x \in H$ the real function $t \mapsto \varphi(t)(x)$ is continuous in I, and $\|\varphi(t)\|_{X_i} \le c(t)$ with $c \in L^1(I)$, then the function

$$f(x) = \int_I \varphi(t)(x)dt, \qquad x \in H\,,$$

belongs to X_i, and

$$\|f\|_{X_i} \le \|c\|_{L^1(I)}\,.$$

Then the following results hold.

THEOREM 5.1. *Under the above assumptions let A be the linear operator in X defined by*

$$(R(\lambda, A)f)(x) = \int_0^\infty e^{-\lambda t}(T(t)f)(x)dt, \quad f \in X, \ x \in H, \ \lambda > \omega .$$

Then the domain $D(A_0)$ of the part of A in X_0 is contained in $(X_1, X_2)_{(1-\gamma_1)/(\gamma_2-\gamma_1),\infty}$, with continuous embedding.

Let us consider now evolution problems. For $T > 0$ we introduce the functional space $\widetilde{C}([0, T]; X)$, consisting of the functions $f : [0, T] \mapsto X$ such that $(t, x) \mapsto f(t)(x)$ is continuous and bounded in $[0, T] \times H$, and $f(t)$ is uniformly continuous in H, uniformly with respect to t. Moreover if Y is any Banach space we denote by $B([0, T]; Y)$ the space of all the bounded functions $f : [0, T] \mapsto Y$.

For every $f \in \widetilde{C}([0, T]; X)$ the function

$$u(t)(x) = \int_0^t T(t-s)f(s)(x)ds, \quad 0 \le t \le T, \ x \in H, \tag{5.2}$$

is said to be the mild solution of

$$\begin{cases} u'(t) &= Au(t) + f(t), \quad 0 \le t \le T, \\ u(0) &= 0. \end{cases} \tag{5.3}$$

See [3] for several properties of the mild solutions.

THEOREM 5.2. *Under the above assumptions let $f \in \widetilde{C}([0, T]; X) \cap B([0, T]; X_0)$ and let u be defined by* (5.2). *Then $u \in B([0, T]; (X_1, X_2)_{\theta,\infty})$, with $\theta = (1-\gamma_1)/(\gamma_2-\gamma_1)$, and there is $C > 0$ independent of f such that*

$$\sup_{0\le t\le T} \|u(t)\|_{(X_1,X_2)_{\theta,\infty}} \le C \sup_{0\le t\le T} \|f(t)\|_{X_0} .$$

6. – Characterizations of $D(A)$, $(X, D(A))_{\theta,\infty}$, $\overline{D(A)}$

Estimates (3.15) and Theorem 5.1 let us prove some regularity properties of the functions in $D(A)$.

THEOREM 6.1. *$D(A)$ is continuously embedded in $C^2_d(\mathbb{R}^n)$.*

PROOF. We use Theorem 5.1, with $X_0 = X = UC(\mathbb{R}^n)$, $X_1 = C^\alpha_d(\mathbb{R}^n)$, $X_2 = C^{2+\alpha}_d(\mathbb{R}^n)$, α being any number in $(0,1)$, and $T(t)$ defined in (1.11). We have already remarked that $T(t)$ is a weakly continuous semigroup in X. Estimates (5.1) are satisfied with $\gamma_1 = \alpha/2$, $\gamma_2 = 1+\alpha/2$, thanks to Corollary 3.3. The assumption about the integrals is easily seen to be satisfied: indeed, if I is any interval and $\varphi : I \mapsto X$ is such that $t \mapsto \varphi(t)(x)$ is continuous for every $x \in \mathbb{R}^n$ and $\|\varphi(t)\|_{C^\alpha_d(\mathbb{R}^n)} \le c(t)$ with $c \in L^1(I)$, then for every $x,\ y \in \mathbb{R}^n$, $h = 0, \dots, k$ we have

$$\left|\int_I \varphi(t)(x)dt\right| \le \int_I c(t)dt\,,$$

$$\left|\int_I \varphi(t)(x+E_h y)dt - \int_I \varphi(t)(x)dt\right| \le \int_I |\varphi(t)(x+E_h y) - \varphi(t)(x)|dt$$

$$\le \int_I c(t)dt\,|y|^{\alpha/(2h+1)}\,,$$

so that $f(x) = \int_I \varphi(t)(x)$ belongs to X_1. To prove that a similar property holds with X_1 replaced by X_2 we use the seminorm (2.2) with $m = 3$. If $\varphi : I \mapsto X$ is such that $t \mapsto \varphi(t)(x)$ is continuous for every $x \in \mathbb{R}^n$ and $\|\varphi(t)\|_{C^{2+\alpha}(\mathbb{R}^n)} \le c(t)$ with $c \in L^1(I)$, then for every $x,\ y \in \mathbb{R}^n$, $h = 0, \dots, k$ we have

$$\left|\sum_{l=0}^{3}(-1)^l \int_I \varphi(t)(x + lE_h y)dt\right|$$

$$\le \int_I \left|\sum_{l=0}^{3}(-1)^l \varphi(t)(x+lE_h y)\right|dt \le \int_I Kc(t)dt\,|h|^{(2+\alpha)/(2h+1)}\,,$$

so that $f(x) = \int_I \varphi(t)(x)dt$ belongs to X_2.

We apply Theorem 5.1 and Corollary 2.3, which give

$$D(A) \subset (C^\alpha_d(\mathbb{R}^n), C^{2+\alpha}_d(\mathbb{R}^n))_{1-\alpha/2,\infty} = C^2_d(\mathbb{R}^n). \qquad \square$$

Characterizations of $D(A)$ are proved in the following theorem.

THEOREM 6.2. *The operator*

$$A_0 \colon D(A_0) = \{\varphi \in UC^2(\mathbb{R}^n) : \mathcal{A}\varphi \in X\}\,, \quad A_0\varphi = \mathcal{A}\varphi\,, \quad \varphi \in D(A_0)\,,$$

is preclosed, and $A = \overline{A}_0$. *In particular,*

$$D(A) = \{\varphi \in X : \exists\, \{\varphi_n\}_{n\in\mathbb{N}} \subset D(A_0) \text{ such that } \varphi_n \to \varphi,\ \mathcal{A}\varphi_n \text{ converges in } X\}\,,$$

$$A\varphi = \lim_{n\to\infty} \mathcal{A}\varphi_n, \qquad \varphi \in D(A)\,.$$

Moreover,

$$D(A) = \{\varphi \in X : \mathcal{A}\varphi \in X\}\,, \quad A\varphi = \mathcal{A}\varphi\,, \quad \varphi \in D(A)\,,$$

where $\mathcal{A}$ *is in the sense of the distributions.*

PROOF. To prove the first statement we use the arguments of [3, Lemma 5.7]. First we show that $D(A_0) \subset D(A)$ and that for every $\varphi \in D(A_0)$, $\lambda > 0$ we have

$$\varphi = R(\lambda, A)(\lambda\varphi - \mathcal{A}\varphi).$$

Indeed, since $T(t)$ commutes with $\mathcal{A}$ on $D(A_0)$, then for each $x \in \mathbb{R}^n$

$$R(\lambda, A)(\lambda\varphi - \mathcal{A}\varphi)(x) = \int_0^\infty e^{-\lambda t}[\lambda(T(t)\varphi)(x) - (T(t)\mathcal{A}\varphi)(x)]dt$$

$$= \int_0^\infty e^{-\lambda t}[\lambda(T(t)\varphi)(x) - (\mathcal{A}T(t)\varphi)(x)]dt = \int_0^\infty -\frac{\partial}{\partial t}(e^{-\lambda t}T(t)\varphi)(x)dt = \varphi(x).$$

This implies easily that A_0 is preclosed and that $D(\overline{A}_0) \subset D(A)$. Let us prove that $A \subset \overline{A}_0$. Let $\varphi \in D(A)$ and let $\lambda > 2\omega > 2\omega_0$. Then $\varphi = R(\lambda, A)f$ for some $f \in X$. Let $\{f_m\}_{m\in\mathbb{N}}$ be a sequence of functions in $UC^2(\mathbb{R}^n)$ such that $\|f_m - f\|_\infty$ goes to 0 as m goes to ∞. It is easy to see that there exists $C > 0$ such that for every $g \in UC^2(\mathbb{R}^n)$ we have $\|T(t)g\|_{UC^2(\mathbb{R}^n)} \le C(1 + e^{2\omega t})\|g\|_{UC^2(\mathbb{R}^n)}$. It follows that $R(\lambda, A)g \in UC^2(\mathbb{R}^n)$ and

$$\mathcal{A}R(\lambda, A)g(x) = \int_0^{+\infty} e^{-\lambda t}\partial/\partial t(T(t)g)(x)dt = \lambda R(\lambda, A)g(x) - g(x), \quad x \in \mathbb{R}^n.$$

Therefore, $\mathcal{A}R(\lambda, A)g = AR(\lambda, A)g$, so that $R(\lambda, A)g \in D(A_0)$. Taking $g = f_m$ and $\varphi_m = R(\lambda, A)f_m$ we get $\varphi_m \in D(A_0)$, $\varphi_m \to \varphi$, $\mathcal{A}\varphi_m = A\varphi_m \to \lambda\varphi - f$ as $m \to \infty$, so that $A \subset \overline{A}_0$.

It is easy now to see that $D(\overline{A}_0) \subset \{\varphi \in X : \mathcal{A}\varphi \in X\}$, where $\mathcal{A}$ is in the distributional sense. Indeed, let $\mathcal{A}^*$ be the formal adjoint of $\mathcal{A}$, namely

$$\mathcal{A}^*u = \sum_{i,j=1}^n q_{ij}D_{ij}u - \sum_{i,j=1}^n b_{ij}x_jD_iu - \sum_{i=1}^n b_{ii}u = \mathrm{Tr}(QD^2u) - \langle Bx, Du\rangle - (\mathrm{Tr}B)u.$$

For $\varphi \in D(\overline{A}_0)$ let $\varphi_m \in UC^2(\mathbb{R}^n)$ be a sequence of functions converging uniformly to φ, such that $\mathcal{A}\varphi_n$ converges uniformly. Then for every smooth ξ with compact support we have

$$\int_{\mathbb{R}^n} \overline{A}_0\varphi\,\xi dx = \int_{\mathbb{R}^n} \lim_{m\to\infty} \mathcal{A}\varphi_m\xi\,dx = \lim_{m\to\infty}\int_{\mathbb{R}^n} \mathcal{A}\varphi_m\xi\,dx$$

$$= \lim_{m\to\infty}\int_{\mathbb{R}^n} \varphi_m\mathcal{A}^*\xi\,dx = \int_{\mathbb{R}^n} \varphi\,\mathcal{A}^*\xi dx.$$

This means that $\overline{A}_0\varphi = \mathcal{A}\varphi$ in the distributional sense, so that $D(\overline{A}_0) \subset \{\varphi \in X : \mathcal{A}\varphi \in X\}$.

Let us show that $\{\varphi \in X : \mathcal{A}\varphi \in X\} \subset D(A)$. Let $\varphi \in X$ be such that $\mathcal{A}\varphi$ (in the sense of the distributions) is a function in X. Fix $\lambda > 0$ and set

$\lambda\varphi - \mathcal{A}\varphi = f$. We shall prove that $\varphi = R(\lambda, A)f$, so that $\varphi \in D(A)$ and $\mathcal{A}\varphi = A\varphi$. Since $R(\lambda, A)f \in D(A)$, then $AR(\lambda, A)f = \mathcal{A}R(\lambda, A)f$ in the distributional sense. Therefore, $\lambda R(\lambda, A)f - (\lambda\varphi - \mathcal{A}\varphi) = \lambda R(\lambda, A)f - f = AR(\lambda, A)f = \mathcal{A}R(\lambda, A)f$, so that

$$\lambda(R(\lambda, A)f - \varphi) - \mathcal{A}(R(\lambda, A)f - \varphi) = 0.$$

The null function is the unique distributional solution of the equation $\lambda v - \mathcal{A}v = 0$ belonging to X. Indeed, if v is a solution, then $\mathcal{A}v$ is continuous, and it is nonpositive (respectively, nonnegative) at any maximum (respectively, minimum) point for v. The classical maximum principle may be therefore adapted to the present situation, again as in [2, Lemma 7.4], to get $v = 0$. Therefore, $R(\lambda, A)f = \varphi$ and this finishes the proof. □

The domain of A is not dense in X. Indeed, the semigroup $T(t)$ is not strongly continuous in X, as the following proposition shows.

PROPOSITION 6.3. *Set*

$$Y_0 = \{\varphi \in X : \lim_{t\to 0^+} \varphi(e^{tB}x) - \varphi(x) = 0, \textit{ uniformly for } x \in \mathbb{R}^n\}. \tag{6.1}$$

For $\varphi \in X$ we have

$$\lim_{t\to 0^+} \|T(t)\varphi - \varphi\|_\infty = 0 \iff \varphi \in Y_0.$$

PROOF. Set

$$(G(t)\varphi)(x) = \frac{1}{(4\pi t)^{n/2}(\det K_t)^{1/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{4t}\langle K_t^{-1}y, y\rangle} \varphi(x-y)dy, \quad t > 0. \tag{6.2}$$

It is not hard to check that for every $\varphi \in X$ we have

$$\lim_{t\to 0^+} \|G(t)\varphi - \varphi\|_\infty = 0.$$

Splitting $(T(t)\varphi - \varphi)(x)$ as

$$(T(t)\varphi - \varphi)(x) = \left[(G(t,0)\varphi)(e^{tB}x) - \varphi(e^{tB}x)\right] + \left[\varphi(e^{tB}x) - \varphi(x)\right]$$

the statement follows. □

Proposition 6.3 may be rephrased as follows: given a uniformly continuous and bounded initial datum u_0, the solution of problem (1.12) goes to u_0 as $t \to 0$ uniformly for $x \in \mathbb{R}^n$ if and only if $u_0 \in Y_0$.

The restriction of $T(t)$ to Y_0 is a strongly continuous semigroup. Its infinitesimal generator is the realization A_0 of $\mathcal{A}$ in Y_0, defined by

$$D(A_0) = \{f \in D(A) : Af \in Y_0\}, \quad A_0 f = Af.$$

To know the rate of convergence of $T(t)u_0$ to u_0 for $u_0 \in Y_0$ we have to characterize the interpolation spaces $(X, D(A))_{\theta,\infty}$. By [4, Lemma 3.6], we have $(X, D(A))_{\theta,\infty} = (Y_0, D(A_0))_{\theta,\infty}$ for every $\theta \in (0, 1)$. Therefore, as in the case of strongly continuous semigroups,

$$(X, D(A))_{\theta,\infty} = \left\{ f \in X : [f]_\theta = \sup_{0<t\le 1} \frac{\|T(t)f - f\|_\infty}{t^\theta} < \infty \right\}$$

and the norm $f \mapsto \|f\|_\infty + [f]_\theta$ is equivalent to the norm of $(X, D(A))_{\theta,\infty}$.

THEOREM 6.4. *For* $0 < \theta < 1$ *we have*

$$(X, D(A))_{\theta,\infty} = C_d^{2\theta}(\mathbb{R}^n) \cap Y_\theta\,,$$

with equivalence of the respective norms.

PROOF. We show preliminarly that if $f \in C_d^{2\theta}(\mathbb{R}^n)$ and $G(t)$ is defined by (6.2) then

$$\|G(t)f - f\|_\infty \le Ct^\theta \|f\|_{C_d^{2\theta}(\mathbb{R}^n)}, \quad 0 < t \le 1\,. \tag{6.3}$$

If $0 < \theta < 1/2$, using the equality $\int_{\mathbb{R}^n} e^{-|z|^2/4}dz = (4\pi)^{n/2}$ we get

$$(G(t)f - f)(x) = \frac{1}{(4\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-|z|^2/4}(f(x + t^{1/2}K_t^{1/2}z) - f(x))dz\,,$$

where, by (3.3),

$$\begin{aligned} |f(x + t^{1/2}K_t^{1/2}z) - f(x)| &\le \sum_{h=0}^{k} |t^{1/2}E_hK_t^{1/2}z|^{2\theta/(2h+1)}\|f\|_{C_d^{2\theta}(\mathbb{R}^n)} \\ &\le C\sum_{h=0}^{k} t^\theta |z|^{2\theta/(2h+1)}\|f\|_{C_d^{2\theta}(\mathbb{R}^n)}\,. \end{aligned} \tag{6.4}$$

Therefore,

$$\|G(t)f - f\|_\infty \le \frac{Ct^\theta}{(4\pi)^{n/2}} \cdot \int_{\mathbb{R}^n} e^{-|z|^2/4}\sum_{h=0}^{k}|z|^{2\theta/(2h+1)}dz\|f\|_{C_d^{2\theta}(\mathbb{R}^n)} = Ct^\theta\|f\|_{C_d^{2\theta}(\mathbb{R}^n)}\,.$$

If $1/2 < \theta < 1$, f is differentiable with respect to the variables x_i for $i \in I_0$. Using the equality $\int_{\mathbb{R}^n} e^{-|z|^2/4}z_i dz = 0$ for $i = 1, \dots, n$ we get

$$(G(t)f - f)(x) = \frac{1}{(4\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-|z|^2/4}\Big(f(x + t^{1/2}K_t^{1/2}z) - f(x) - \sum_{i\in I_0} D_i f(x)(t^{1/2}K_t^{1/2}z)_i\Big)dz\,,$$

Using now estimate (3.21) we get (6.3).

If $\theta = 1/2$ we argue by interpolation: we know that $\|G(t)-I\|_{L(C_d^{1/2}(\mathbb{R}^n),X)} \le Ct^{1/4}$, $\|G(t)-I\|_{L(C_d^{3/2}(\mathbb{R}^n),X)} \le Ct^{3/4}$. Since $C_d^1(\mathbb{R}^n) = (C_d^{1/2}(\mathbb{R}^n), C_d^{3/2}(\mathbb{R}^n))_{1/2,\infty}$ by Corollary 2.3, then $\|G(t) - I\|_{L(C_d^1(\mathbb{R}^n),X)} \le Ct^{1/2}$.

Let us prove that $C_d^{2\theta}(\mathbb{R}^n) \cap Y_\theta$ is continuously embedded in $(X, D(A))_{\theta,\infty}$. For $f \in C_d^{2\theta}(\mathbb{R}^n) \cap Y_\theta$ and $x \in \mathbb{R}^n$ we have

$$|T(t)f(x) - f(x)| \leq |(G(t)f - f)(e^{tB}x)| + |f(e^{tB}x) - f(x)|$$
$$\leq Ct^\theta \left(\|f\|_{C_d^{2\theta}(\mathbb{R}^n)} + \|f\|_{Y_\theta}\right),$$

so that $C_d^{2\theta}(\mathbb{R}^n) \cap Y_\theta$ is continuously embedded in $(X, D(A))_{\theta,\infty}$.

Let us prove that $(X, D(A))_{\theta,\infty}$ is continuously embedded in $C_d^{2\theta}(\mathbb{R}^n)$. We know from Theorem 6.1 that $D(A)$ is continuously embedded in $C_d^2(\mathbb{R}^n)$. It follows that $(X, D(A))_{\theta,\infty}$ is continuously embedded in $(C_d^2(\mathbb{R}^n), X)_{\theta,\infty} = C_d^{2\theta}(\mathbb{R}^n)$.

Let us prove that $(X, D(A))_{\theta,\infty}$ is continuously embedded in Y_θ. For $f \in (X, D(A))_{\theta,\infty}$ and $x \in \mathbb{R}^n$ we have, due to (6.3),

$$|f(e^{tB}x) - f(x)| \leq |f(e^{tB}x) - (G(t)f)(e^{tB}x)|$$
$$+ |T(t)f(x) - f(x)| \leq Ct^\theta \|f\|_{C_d^{2\theta}(\mathbb{R}^n)}.$$

This finishes the proof. □

COROLLARY 6.5. *Let $u_0 \in X$, and let u be the solution of problem* (1.12). *For $0 < \theta < 1$ and for every $T > 0$ the following conditions are equivalent.*

(i) *u is uniformly continuous and bounded in $[0, T] \times \mathbb{R}^n$, it is θ-Hölder continuous with respect to t, and $\sup_{x\in\mathbb{R}^n} \|u(\cdot, x)\|_{C^\theta([0,T])} < \infty$;*

(ii) *$u(\cdot, x) \in C^\theta([0, T])$ for every $x \in \mathbb{R}^n$, $u(t, \cdot) \in C_d^{2\theta}(\mathbb{R}^n)$ for every $t \in [0, T]$, and $\sup_{x\in\mathbb{R}^n} \|u(\cdot, x)\|_{C^\theta([0,T])} < \infty$, $\sup_{0\leq t\leq T} \|u(\cdot, x)\|_{C_d^{2\theta}(\mathbb{R}^n)} < \infty$;*

(iii) *$u_0 \in C_d^{2\theta}(\mathbb{R}^n) \cap Y_\theta$.*

PROOF. Condition (i) means that the function $t \mapsto u(t, \cdot) = T(t)u_0$ belongs to $C^\theta([0, T]; X)$. Since $T(t)$ is a semigroup, then $t \mapsto T(t)u_0$ is θ-Hölder continuous in $[0, T]$ if and only if it is θ-Hölder continuous near $t = 0$. Thanks to Theorem 6.4, (i) and (iii) are equivalent. On the other hand, if $u_0 \in (X, D(A))_{\theta,\infty}$ then $t \mapsto T(t)u_0$ is bounded in $[0, T]$ with values in $(X, D(A))_{\theta,\infty}$, so that (iii) implies (ii). In its turn, (ii) implies obviously (i). □

7. – Proof of Theorems 1.1 and 1.2

We use the abstract results of Theorems 5.1 and 5.2 and the procedure of Theorem 6.1, which works thanks to estimates (3.16).

Let $X = UC(\mathbb{R}^n)$, $X_0 = C_d^\theta(\mathbb{R}^n)$, $X_1 = C_d^\alpha(\mathbb{R}^n)$, with $0 < \theta < \alpha < 1$, and $X_2 = C_d^\alpha(\mathbb{R}^n)$. Let $T(t)$ be the weakly continuous semigroup given by (1.11). By estimates (3.16) $T(t)$ satisfies (5.1), with $\gamma_1 = (\alpha-\theta)/2$, $\gamma_2 = 1+(\alpha-\theta)/2$.

The assumptions on the integrals in the spaces X_i has been verified in the proof of Theorem 6.1. Therefore, all the hypotheses of Section 6 are satisfied.

PROOF OF THEOREM 1.1. For every $\lambda > 0$ and $f \in C_d^\theta(\mathbb{R}^n)$ the equation $\lambda u - Au = f$ has unique solution $u \in D(A)$. Thanks to Theorem 6.2, this means that u is the unique distributional solution of the equation $\lambda u - \mathcal{A}u = f$ in X. By estimate (3.7), $\|u\|_\infty \le \|f\|_\infty/\lambda$ and therefore $\|Au\|_\infty \le 2\|f\|_\infty$. Since $D(A) \subset (X, D(A))_{\theta/2,\infty} \subset X_1$ with continuous embeddings (see Theorem 6.4), then $u \in C_d^\theta(\mathbb{R}^n)$ and $\|u\|_{X_1} \le C\|f\|_\infty$. Therefore, $\mathcal{A}u = \lambda u - f \in X_1$, and $\|\mathcal{A}u\|_{X_1} \le C'\|f\|_{X_1}$, with $C' = C'(\lambda)$. We apply now Theorem 5.1, which given $u \in (X_1, X_2)_{1-(\alpha-\theta)/2,\infty}$ and $\|u\|_{(X_1,X_2)_{1-(\alpha-\theta)/2,\infty}} \le C''\|f\|_{X_1}$. On the other hand, by Corollary 2.3 $(X_1, X_2)_{1-(\alpha-\theta)/2,\infty} = C_d^{2+\theta}(\mathbb{R}^n)$, with equivalence of the norms, and the statement follows. □

PROOF OF THEOREM 1.2. Let

$$u(t,\cdot) = T(t)u_0 + \int_0^t T(t-s)f(s,\cdot)ds = T(t)u_0 + v(t,\cdot),\ 0 \le t \le T.$$

By Theorem 5.2 applied to the function v such as in the proof of Theorem 1.1 we get $v(t,\cdot) \in C_d^{2+\theta}(\mathbb{R}^n)$ for every $t \in [0, T]$, and

$$\sup_{0\le t\le T} \|v(t,\cdot)\|_{C_d^{2+\theta}(\mathbb{R}^n)} \le C \sup_{0\le t\le T} \|f(t,\cdot)\|_{C_d^\theta(\mathbb{R}^n)}.$$

By Theorem 3.4 we have

$$\sup_{0\le t\le T} \|T(t)u_0\|_{C_d^{2+\theta}(\mathbb{R}^n)} \le C\|u_0\|_{C_d^{2+\theta}(\mathbb{R}^n)},$$

and estimate (1.10) follows.

It remains to show that u is a strong (and hence, distributional) solution, and that the distributional solution is unique in $C([0, T] \times \mathbb{R}^n)$. Let $\{f_m\}_{m\in\mathbb{N}}$ be a sequence of bounded smooth functions with bounded derivatives converging uniformly to f on $[0, T] \times K$ for every compact set $K \subset \mathbb{R}^n$. Then for every compact set $K \subset \mathbb{R}^n$ the sequence $\{u_m\}_{m\in\mathbb{N}} \subset C^{1,2}([0, T] \times \mathbb{R}^n)$ defined by $u_m(t,\cdot) = T(t)u_0 + \int_0^t T(t-s)f_m(s,\cdot)ds$ converges uniformly to u on $[0, T] \times K$, and $\partial u_m/\partial t - \mathcal{A}u_m = f_m$. Therefore u is a strong solution.

Concerning uniqueness, let $v \in C([0, T] \times \mathbb{R}^n)$ be a distributional solution of (1.9) with $f = 0$, $u_0 = 0$. For $\lambda > 0$ the function $w = e^{\lambda t}v$ is a distributional solution of $w_t - \mathcal{A}w = \lambda w$, $w(0,\cdot) = 0$. Then $w_t - \mathcal{A}w$ is continuous, and it is nonnegative (respectively, nonpositive) at any relative maximum (respectively, minimum) point for w belonging to $(0, T] \times \mathbb{R}^n$. Then the classical maximum principle may be adapted to our situation such as in [10, Lemma 2.4], to get $w = 0$ and hence $v = 0$. □

8. – A case of x-dependent coefficients

We consider here the case in which the matrix Q depends continuously on x, $Q(x) = [q_{ij}(x)]_{i,j=1,\dots,n}$, while the matrix B is constant. We assume that $Q(x)$ and B have a particular structure,

$$Q(x) = \begin{pmatrix} Q_0(x) & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{8.1}$$

where $Q_0(x)$ is a $r_0 \times r_0$ symmetric matrix for every x, $r_0 \in \{1, \dots, n-1\}$, and there is $\nu > 0$ such that

$$\langle Q_0(x)\xi, \xi\rangle \geq \nu|\xi|^2, \quad \xi \in \mathbb{R}^{r_0}; \tag{8.2}$$

Moreover we assume that there exists the limit in $L(\mathbb{R}^{r_0})$

$$\lim_{|x|\mapsto\infty} Q_0(x) = Q_0^\infty. \tag{8.3}$$

By (8.2) Q_0^∞ is nonsingular, and it satisfies (8.2) too.

The matrix B is of the type

$$B = \begin{pmatrix} \star & \star & \star & \dots & \star & \star \\ B_1 & \star & \star & \dots & \star & \star \\ 0 & B_2 & \star & \dots & \star & \star \\ 0 & 0 & B_3 & \dots & \star & \star \\ 0 & 0 & 0 & \dots & B_k & \star \end{pmatrix} \tag{8.4}$$

where the $r_i \times r_{i-1}$ blocks B_i have maximum rank $= r_i$, and $r_0 \geq r_1 \geq \dots r_k$, $\sum_{i=0}^k r_i = n$.

One can see easily that for every fixed $x_0 \in \mathbb{R}^n$ the matrices $Q(x_0)$, B satisfy the Kalman rank condition (1.6). (We remark that the converse is also true: it has been shown in [7] that if $Q(x_0)$, B satisfy (1.6) then there exists a basis in $\mathbb{R}^n$, possibly depending on x_0, such that $Q(x_0)$, B are given by (8.1), (8.4) respectively).

Therefore for every $x_0 \in \mathbb{R}^n$ the operator with frozen second order coefficients

$$\mathcal{A}(x_0) = \sum_{i,j=1}^{r_0} q_{ij}(x_0)D_{ij} + \sum_{i,j=1}^{n} b_{ij}x_jD_i = \mathrm{Tr}\,(Q(x_0)D^2\cdot) + \langle Bx, D\cdot\rangle \tag{8.5}$$

is hypoelliptic. The structure assumptions (8.1), (8.4) guarantee that the projections E_h defined in (1.7) with Q replaced by $Q(x_0)$ are independent of x_0.

So, the results of Theorems 1.1 and 1.2 hold for the operator $\mathcal{A}(x_0)$, as well as for the operator $\mathcal{A}_\infty$, defined in an obvious way as

$$\mathcal{A}_\infty = \mathrm{Tr}\,(Q_\infty D^2\cdot) + \langle Bx, D\cdot\rangle\,, \tag{8.6}$$

where Q_∞ is the $n \times n$ matrix having Q_0^∞ in the first $r_0 \times r_0$ block, and zero entries in the other blocks.

We claim that the constants C of estimates (1.8) and (1.10) may be taken independent of x_0: indeed, they depend on x_0 through the estimates on $K_t(x_0)$ given by Lemma 3.1, which in their turn depend on x_0 through $\|Q_0(x_0)\|$ and $\det(R(x_0))^{-1}$, R being defined in (3.5). The representation formula (3.6) implies that R depends continuously on x_0, and that $\lim_{|x|\to\infty} R(x) = R_\infty$, the one associated to the matrix Q_∞. Since the couple (Q_∞, B) satisfy (1.6), then $\det R_\infty > 0$. Therefore, $\inf_{x\in\mathbb{R}^n} \det R(x) > 0$, and this proves the claim.

We are able now to state extensions of Theorems 1.1 and 1.2. We set

$$\tilde{\mathcal{A}} = \sum_{i,j=1}^{r_0} q_{ij}(x) D_{ij} + \sum_{i,j=1}^{n} b_{ij} x_j D_i = \mathrm{Tr}(Q(x)D^2\cdot) + \langle Bx, D\cdot\rangle \tag{8.7}$$

THEOREM 8.1. *Assume that $Q(x)$, B satisfy* (8.1), (8.2), (8.3), (8.4), *and that the coefficients q_{ij} belong to $C_d^\theta(\mathbb{R}^n)$, with $0 < \theta < 1$. Then for every $\lambda > 0$ and $f \in C_d^\theta(\mathbb{R}^n)$ the problem*

$$\lambda u - \tilde{\mathcal{A}}(x)u = f \tag{8.8}$$

has a unique solution $u \in C_d^{2+\theta}(\mathbb{R}^n)$, such that $x \mapsto \langle Bx, Du(x)\rangle$ (in the sense of the tempered distributions) belongs to $C_d^\theta(\mathbb{R}^n)$. Moreover there is $C > 0$, independent of f, such that

$$\|u\|_{C_d^{2+\theta}(\mathbb{R}^n)} \le C\|f\|_{C_d^\theta(\mathbb{R}^n)}. \tag{8.9}$$

PROOF. The main point is to prove the *apriori* estimates (8.9). Then existence of the solution will be shown in a standard way by the continuation argument.

Let D_θ be the subset of $C_d^{2+\theta}(\mathbb{R}^n)$ consisting of the functions u such that $x \mapsto \langle Bx, Du(x)\rangle$ (in the sense of the distributions) belongs to $C_d^\theta(\mathbb{R}^n)$. D_θ is a Banach space endowed with the norm

$$\|u\|_{D_\theta} = \|u\|_{C_d^{2+\theta}(\mathbb{R}^n)} + \|\langle B\cdot, Du\rangle\|_{C_d^\theta(\mathbb{R}^n)}\,.$$

By Theorem 1.1 and Proposition 6.2, D_θ is the domain of the realization of $\mathcal{A}(x_0)$ in $C_d^\theta(\mathbb{R}^n)$, for every $x_0 \in \mathbb{R}^n$, and the norm of D_θ is equivalent to the graph norm.

Fix $\varepsilon > 0$. Then there exists $R > 1$ be such that

$$|q_{ij}(x) - q_{ij}^\infty| \le \varepsilon \;\text{ for } |x| \ge R - 1\,.$$

Let η be a smooth cutoff function such that

$$\begin{cases} \eta \equiv 0 \text{ in } B(0, R-1), \quad \eta \equiv 1 \text{ outside } B(0, R), \\ |D_i \eta| \leq 1, \ |D_{ij}\eta| \leq 1, \qquad i, j = 1, \dots, n. \end{cases}$$

Let $u \in D_\theta$. Then ηu satisfies (in the distributional sense)

$$\begin{aligned} &\lambda(\eta u)(x) - \mathcal{A}_\infty(\eta u)(x) \\ &= \eta(x)(\lambda u(x) - \tilde{A}u(x)) - \eta(x) \sum_{i,j=1}^{r_0} (q_{ij}^\infty - q_{ij}(x)) D_{ij}u(x) \\ &- \sum_{i,j=1}^{n} b_{ij} x_j u(x) D_i \eta(x) - \sum_{i,j=1}^{r_0} q_{ij}^\infty (u(x) D_{ij}\eta(x) + 2 D_i \eta(x) D_j u(x)), \end{aligned} \tag{8.10}$$

so that by estimate (1.8) and by Lemma 2.1 we get

$$\begin{aligned} \|\eta u\|_{C_d^{2+\theta}(\mathbb{R}^n)} &\leq C\Big(\|\eta(\lambda u - \tilde{A}u)\|_{C_d^\theta(\mathbb{R}^n)} \\ &\quad + \varepsilon \sum_{i,j=1}^{r_0} [D_{ij}u]_{C_d^\theta(\mathbb{R}^n)} + C_1(R)\|u\|_{C_d^2(\mathbb{R}^n)} \Big) \\ &\leq C\Big(K\|\lambda u - \tilde{A}u\|_{C_d^\theta(\mathbb{R}^n)} + \varepsilon C_0 \|u\|_{C^{2+\theta}(\mathbb{R}^n)} + C_1(R)\|u\|_{C_d^2(\mathbb{R}^n)} \Big), \end{aligned} \tag{8.11}$$

where $K = 1 + [\eta]_{C_d^2(\mathbb{R}^n)}$.

Let now $\delta > 0$ be so small that

$$|q_{ij}(x) - q_{ij}(y)| \leq \varepsilon \ \text{ for } |x - y| \leq 4\delta.$$

Let η be a smooth cutoff function such that

$$\begin{cases} \eta \equiv 1 \text{ in } B(0, 1), \quad \eta \equiv 0 \text{ outside } B(0, 2), \\ |D_i \eta| \leq 1, \ |D_{ij}\eta| \leq 1, \quad i, j = 1, \dots, n. \end{cases}$$

Fixed any $x_0 \in B(0, R+1)$ let η_{x_0} be the function defined by

$$\eta_{x_0}(x) = \eta\left(\frac{x - x_0}{\delta}\right), \qquad x \in \mathbb{R}^n.$$

The function $\eta_{x_0}u$ satisfies equation (8.10), with η replaced by η_{x_0}, q_{ij}^{∞} replaced by $q_{ij}(x_0)$, $\mathcal{A}_\infty$ replaced by $\mathcal{A}(x_0)$. By estimate (1.8) and Lemma 2.1, we get again

$$\begin{aligned}\|\eta_{x_0}u\|_{C_d^{2+\theta}(\mathbb{R}^n)} &\le C(\|\eta_{x_0}(\lambda u - \tilde{A}u)\|_{C_d^\theta(\mathbb{R}^n)} \\ &\quad + \varepsilon \sum_{i,j=1}^{r_0} [D_{ij}u]_{C_d^\theta(\mathbb{R}^n)} + C_2(R)\|u\|_{C_d^2(\mathbb{R}^n)}) \\ &\le C(K(\delta)\|\lambda u - \tilde{A}u\|_{C_d^\theta(\mathbb{R}^n)} + \varepsilon C_0\|u\|_{C_d^{2+\theta}(\mathbb{R}^n)} + C_2(R)\|u\|_{C_d^2(\mathbb{R}^n)}),\end{aligned} \tag{8.12}$$

where $K(\delta) = 1 + \|\eta_{x_0}\|_{C_d^\theta(\mathbb{R}^n)}$.

We recall now that for every $g \in C^\theta(\mathbb{R}^m)$ we have

$$[g]_{C^\theta(\mathbb{R}^m)} \le \max\left\{\frac{2}{\delta^\theta}\|g\|_\infty, \sup_{x_0\in\mathbb{R}^m}[g]_{C^\theta(B(x_0,\delta))}\right\}.$$

It follows that

$$\|u\|_{C_d^{2+\theta}(\mathbb{R}^n)} \le C_3\big(\sup_{x_0\in\mathbb{R}^n}\|u\|_{C_d^{2+\theta}(B(x_0,\delta))} + \frac{1}{\delta^\theta}\|u\|_{C_d^2(\mathbb{R}^n)}\big).$$

Using estimates (8.11) and (8.12) we get

$$\|u\|_{C_d^{2+\theta}(\mathbb{R}^n)} \le C_3C\big(\varepsilon C_0\|u\|_{C_d^{2+\theta}(\mathbb{R}^n)} + C_4(\delta, R)\|u\|_{C_d^2(\mathbb{R}^n)} + C_5(\delta)\|\lambda u - \tilde{A}u\|_{C_d^\theta(\mathbb{R}^n)}\big).$$

Taking $\varepsilon = (2C_3C)^{-1}$ we get

$$\|u\|_{C_d^{2+\theta}(\mathbb{R}^n)} \le 2C_3C\big(C_4(\delta, R)\|u\|_{C_d^2(\mathbb{R}^n)} + C_5(\delta)\|\lambda u - \tilde{A}u\|_{C_d^\theta(\mathbb{R}^n)}\big).$$

It is not hard to see that for every $\sigma \in (0,1)$ there is $C(\sigma) > 0$ such that

$$\|g\|_{C_d^2(\mathbb{R}^n)} \le \sigma\|g\|_{C_d^{2+\theta}(\mathbb{R}^n)} + C(\sigma)\|g\|_\infty, \quad \forall g \in C_d^{2+\theta}(\mathbb{R}^n). \tag{8.13}$$

Taking $\sigma = (4C_3CC_4(\delta, R))^{-1}$ we get

$$\|u\|_{C_d^{2+\theta}(\mathbb{R}^n)} \le C_6(\|\lambda u - \tilde{A}u\|_{C_d^\theta(\mathbb{R}^n)} + \|u\|_\infty) \tag{8.14}$$

To finish the proof of (8.9) we estimate $\|u\|_\infty$ in terms of $\|\lambda u - \tilde{A}u\|_\infty$. This can be done adapting the maximum principle to our case. We get

$$\|u\|_\infty \le \frac{1}{\lambda}\|\lambda u - \tilde{A}u\|_\infty,$$

which, added to (8.14), gives (8.9).

To conclude, we remark that for every $\lambda > 0$ and $f \in C_d^\theta(\mathbb{R}^n)$, the equation

$$\lambda u - \tilde{A}u = f$$

has a unique solution $u \in D_\theta$. This can be seen using the continuity method: for every $\varepsilon \in [0, 1]$ consider the problem

$$\lambda u - (1-\varepsilon)\mathcal{A}_\infty u - \varepsilon\tilde{\mathcal{A}}u = f\,. \tag{8.15}$$

The operator $(1-\varepsilon)\mathcal{A}_\infty + \varepsilon\tilde{\mathcal{A}}$ satisfies (8.1), ..., (8.4). Using the *a priori* estimate (8.9) it is not hard to see that the set of all ε's such that (8.15) is uniquely solvable in D_θ is open and closed in [0, 1], so that it coincides with [0, 1]. Taking $\varepsilon = 1$ the statement follows. □

An analogous result concerning the parabolic initial value problem associated to the operator $\mathcal{A}$ holds. Since the proof is quite similar to the one of Theorem 1.3, we omit it. Of course the set D_θ has to be replaced by

$$P_\theta = \{u \in B([0,T]; C_d^{2+\theta}(\mathbb{R}^n)) \cap C([0,T]\times\mathbb{R}^n) : u_t - \langle B\cdot, Du\rangle$$
$$\in B([0,T]; C_d^{\theta}(\mathbb{R}^n)) \cap C([0,T]\times\mathbb{R}^n)\}\,,$$

where $u_t - \langle B\cdot, Du\rangle$ is to be understood in the sense of distributions.

THEOREM 8.2. *Assume that $Q(x)$, B satisfy* (8.1), (8.2), (8.3), (8.4), *and that the coefficients q_{ij} belong to $C_d^\theta(\mathbb{R}^n)$, with $0 < \theta < 1$. Let $T > 0$, $u_0 \in C_d^{2+\theta}(\mathbb{R}^n)$, and let $f : [0,T]\times\mathbb{R}^n \mapsto \mathbb{R}$ be a continuous function such that $f(t,\cdot) \in C_d^\theta(\mathbb{R}^n)$ for every $t \in [0,T]$ and $\sup_{0\le t\le T}\|f(t,\cdot)\|_{C_d^\theta(\mathbb{R}^n)} < \infty$. Then the problem*

$$\begin{cases} u_t(t,x) = \tilde{A}(x)u(t,x) + f(t,x), & 0 < t < T,\ x \in \mathbb{R}^n\,, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n\,, \end{cases} \tag{8.16}$$

has a unique distributional solution $u \in P_\theta$, and there is $C > 0$, independent of f, such that

$$\sup_{0\le t\le T}\|u(t,\cdot)\|_{C_d^{2+\theta}(\mathbb{R}^n)} \le C(\|u_0\|_{C_d^\theta(\mathbb{R}^n)} + \sup_{0\le t\le T}\|f(t,\cdot)\|_{C_d^\theta(\mathbb{R}^n)})\,. \tag{8.17}$$

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