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Introduction

In the Classical Potential Theory (CPT) regular domains can be described as follows (see e.g. [Doo]): for a bounded domain Ω in $\mathbb{R}^m$ the following are equivalent:

i) Ω is regular with respect to the Laplace equation.
ii) Ω is regular with respect to the Poisson equation.
iii) Every boundary point of Ω admits a strong subharmonic barrier.
iv) Every boundary point of Ω admits a weak subharmonic barrier.

In this paper we take up a corresponding problem in the Pluripotential Theory where the situation is much more complicated.

A domain in $\mathbb{C}^n$ admitting a weak plurisubharmonic (psh) barrier at every boundary point is called hyperconvex. Kerzman and Rosay [KR] proved that in a hyperconvex domain $\Omega \Subset \mathbb{C}^n$ there exists an exhaustion function ψ (that is $\psi \in \mathrm{PSH}(\Omega)$, $\psi < 0$ and $\lim_{z\to\partial\Omega}\psi(z) = 0$) which is smooth (by smooth we always mean C^∞) and strictly psh. On the other hand the class of domains that admit strong psh barriers was investigated by Sibony [Sib] and following him we will call them B-regular. Sibony proved in particular that a domain is B-regular if and only if it admits a smooth exhaustion function ψ such that

$$(0.1) \qquad \sum_{j,k=1}^{n} \frac{\partial^2\psi}{\partial z_j \partial \bar{z}_k}\alpha_j\bar{\alpha}_k \geq |\alpha|^2, \quad \alpha \in \mathbb{C}^n$$

(that is every eigenvalue of the matrix $(\partial^2\psi/\partial z_j\partial\bar{z}_k)$ is ≥ 1). The proofs of the above results were based on a theorem of Richberg [Rich] concerning global approximation of continuous strictly psh functions.

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The main result of this paper is the following: any hyperconvex domain admits a smooth exhaustion function ψ such that

$$M\psi := \det\left(\frac{\partial^2\psi}{\partial z_j \partial \bar{z}_k}\right) \geq 1 \tag{0.2}$$

(that is the product of all eigenvalues of the matrix $(\partial^2\psi/\partial z_j\partial\bar{z}_k)$ is ≥ 1). This essentially strengthens the Kerzman-Rosay's result (what they got was in fact $M\psi > 0$).

The operator M is the complex Monge-Ampère operator and it plays a similar role in the Pluripotential Theory as the Laplacean in CPT (see [BT2], [Bed] and [Kli]). Bedford and Taylor [BT1], using an inequality proved earlier by Chern, Levine and Nirenberg [CLN], showed that Mu can be well defined as a non-negative Borel measure for any continuous psh u. They considered the following Dirichlet problem:

$$\begin{cases} u \in \mathrm{PSH}(\Omega) \cap C(\bar{\Omega}) \\ Mu = F \\ u|_{\partial\Omega} = f\,, \end{cases} \tag{0.3}$$

where $f \in C(\partial\Omega)$ and $F \in C(\bar{\Omega})$, $F \geq 0$. (0.3) is a counterpart of the Poisson equation in CPT. The main result from [BT1] is that (0.3) has a unique solution if Ω is strictly pseudoconvex. However, really essential is the existence of an exhaustion function satisfying (0.1), thus the problem (0.3) is solvable in B-regular domains for arbitrary boundary data. On the other hand, as shown in [Sib], Ω is B-regular if and only if every $f \in C(\partial\Omega)$ is a restriction of some $v \in \mathrm{PSH}(\Omega) \cap C(\bar{\Omega})$. This implies that if for some $F \in C(\bar{\Omega})$, $F \geq 0$, (0.3) has a solution for every $f \in C(\partial\Omega)$ then Ω must be B-regular.

In [Bło2] we considered the problem (0.3) when Ω is only hyperconvex. A necessary assumption on the data is that f must be a restriction of some $v \in \mathrm{PSH}(\Omega) \cap C(\bar{\Omega})$. It turns out also to be sufficient. For example (0.3) is solvable if $f \equiv 0$ and $F \equiv 1$. In this case for arbitrary hyperconvex domain we get a uniquely defined continuous exhaustion function u with $Mu = 1$. We do not know whether u is smooth in general. However, it can be used to construct a smooth exhaustion function satisfying (0.2) via approximation methods from [Rich].

The paper is assumed to be self-contained. We present (although rather briefly) proofs of all required results concerning hyperconvex and B-regular domains as well as the complex Monge-Ampère operator which are not available in well known monographs like [Doo], [GT], [Hör] or [Rud].

In Section 1 we generalize (Theorem 1.3) a result of Richberg (Theorem 1.1) concerning global approximation of continuous psh functions, so that we can use it in Section 6. It is convenient to use the terminology of sheaves. Next, we prove already mentioned characterizations of hyperconvex (Theorem 1.6) and B-regular (Theorem 1.7) domains. Very useful is a result of J.B. Walsh (Theorem 1.5).

Section 2 includes a few well known (but hard to find in the literature) elementary results concerning positive forms and matrices. Together with an estimate from [Bło1] they are used in Section 3 to define the complex Monge-Ampère operator for continuous psh functions (see also [Ceg], [Dem2] and [Kli] for slightly different approaches). We also prove those of its properties which will be needed later (they were established in [BT1]). Since considering only continuous psh functions is sufficient for our purposes, we do not discuss a wider class of psh functions for which the complex Monge-Ampère operator can be well defined (e.g. locally bounded psh functions). Also, generalized versions of results like Theorems 3.4, 3.7 and 3.8 are more difficult to prove (cf. [BT2] and [Dem2]).

Section 4 is devoted to the solution of the Dirichlet problem (0.3) in B-regular domains. For $F \equiv 0$ (then we have a homogeneous equation - it is a counterpart of the Laplace equation in CPT) the original arguments from [BT1] have been essentially shortened by Demailly [Dem2] but his improvements work also in the inhomogeneous case. The main Demailly's contribution made Step IV of the proof of Theorem 4.1 much shorter by applying Rademacher theorem. Moreover, Demailly's proof that the estimate (4.8) for a psh u implies that u is $C^{1,1}$ is much simpler than the original one. On the other hand, the application of Theorem 3.11 allowed to avoid introducing the operator Φ (essentially $M^{1/n}$) used in [BT1].

In Section 5 we consider the notion of stability for the complex Monge-Ampère operator. The main result, used later to solve the Dirichlet problem in hyperconvex domains, is Theorem 5.3 due to Cegrell and Persson [CP]. They used an idea of Cheng and Yau presented in [Bed] concerning a relation between real and complex Monge-Ampère operators (Lemma 5.5).

Finally, in Section 6, we solve the Dirichlet problem in hyperconvex domains (Theorem 6.1) and prove the existence of smooth subsolutions (Theorem 6.2). In particular, we get an exhaustion function satisfying (0.2). The corresponding result for convex domains in $\mathbb{R}^n$ and the real Monge-Ampère operators was proved in [Bło3].

This paper is an English version of the author's Ph.D. thesis. My thanks go to the supervisor, professor Józef Siciak, for his scientific advice and to professors Sławek Kołodziej and Marek Jarnicki for fruitful discussions.

1. – Global approximation of plurisubharmonic functions

Let Ω be an open subset of $\mathbb{C}^n$. If u is a psh function on Ω then for $\delta > 0$ we have the standard regularizations of u:

$$u_\delta(z) := (u * \rho_\delta)(z) = \int_B u(z - \delta w)\rho(w) d\lambda(w)\,,$$

where λ is the Lebesgue measure, B the unit ball in $\mathbb{C}^n$, $\delta > 0$ whereas

$\rho \in C_0^\infty(\mathbb{C}^n)$ is nonnegative, depends only on $|w|$, $\operatorname{supp}\rho = \bar{B}$, $\int_B \rho d\lambda = 1$ and $\rho_\delta(w) := \delta^{-2n}\rho(w/\delta)$. Then $u_\delta \in \mathrm{PSH}\cap C^\infty(\Omega_\delta)$, where $\Omega_\delta := \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) > \delta\}$, and $u_\delta \downarrow u$ as $\delta \downarrow 0$. If u is continuous then the convergence is uniform.

We say that a function u is *strictly psh* on Ω if for every $\varphi \in C_0^\infty(\Omega)$ there exists $\varepsilon_0 > 0$ such that $u + \varepsilon\varphi \in \mathrm{PSH}(\Omega)$ for $\varepsilon \in [0, \varepsilon_0]$. One can easily show that u is strictly psh on Ω if and only if for an open $\Omega' \Subset \Omega$ one can find $c > 0$ such that the function $u(z) - c|z|^2$ is psh on Ω'.

Concerning the global approximation of psh functions we have the following result due to Richberg:

THEOREM 1.1. ([Rich]). *Assume that Ω is open in $\mathbb{C}^n$ and u is continuous, strictly psh on Ω. Let $\varepsilon > 0$ be a continuous function on Ω. Then one can find a smooth, strictly psh function v on Ω such that $u \le v \le u + \varepsilon$.*

Using Richberg's methods we will generalize the above theorem to apply it in Section 6. One of the main ideas in the proqf of Theorem 1.1 in [Rich] was to consider functions of the following form:

$$u_\theta(z) := u_{\theta(z)}(z) = \int_B u(z - \theta(z)w)\rho(w)d\lambda(w), \qquad z \in \Omega_\delta\,,$$

where $\theta \in C^\infty(\Omega)$, $0 \le \theta \le \delta$. Observe that if u is smooth on an open $D \subset \Omega$ then so is u_δ on D_δ, for we can then differentiate under the sign of integration.

DEFINITION. *A subsheaf $\mathcal{S}$ of the sheaf of continuous psh functions over $\mathbb{C}^n$ will be called a* Richberg sheaf *if the following conditions are satisfied:*

(1.1) *For any $u \in \mathcal{S}(\Omega)$, $\varphi \in C_0^\infty(\Omega)$ and $c \in \mathbb{R}$ there exists $\varepsilon_0 > 0$ such that $u + \varepsilon\varphi + c \in \mathcal{S}(\Omega)$ for $\varepsilon \in [0, \varepsilon_0]$.*

(1.2) *If $u, v \in \mathcal{S}(\Omega)$ then* $\max\{u, v\} \in \mathcal{S}(\Omega)$.

(1.3) *If $\Omega' \Subset \Omega$, $\theta \in C^\infty(\Omega)$, $0 \le \theta \le 1$ and $u \in \mathcal{S}(\Omega)$ is smooth on a neighborhood of $\overline{\{\theta < 1\}} \cap \bar{\Omega}'$ then $u_{\delta\theta} \in \mathcal{S} \cap C^\infty(\Omega')$ for $\delta > 0$ small enough.*

The condition (1.3) implies in particular that if $\Omega' \Subset \Omega$ and $u \in \mathcal{S}(\Omega)$ then $u_\delta \in \mathcal{S} \cap C^\infty(\Omega')$ for δ small enough (we assume that the empty set is a neighborhood of itself).

PROPOSITION 1.2. *The sheaf of continuous strictly psh functions is a Richberg sheaf.*

In Section 6 we will construct another Richberg sheaf.

PROOF OF PROPOSITION 1.2. It is enough to show (1.3). We can find $D \Subset \Omega$, a neighborhood of $\overline{\{\theta < 1\}} \cap \bar{\Omega}'$, such that for δ small enough $u_{\delta\theta} = u_\delta$ on a neighborhood of $\Omega' \backslash D$ and $u_{\delta\theta}$ is smooth on D. We have

$$\frac{\partial^2 u(z - \delta\theta(z)w)}{\partial z_j \partial \bar{z}_k}(z_0) = \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z_0 - \delta\theta(z_0)w) + \delta\gamma_{jk}(z_0, w, \delta)\,,$$

where the function γ_{jk} is uniformly bounded for $z_0 \in \tilde{D}$, $w \in \bar{B}$ and $\delta \leq r$. We get uniform convergence of the partial derivatives $\partial^2 u_{\delta\theta}/\partial z_j \partial \bar{z}_k \to \partial^2 u/\partial z_j \partial \bar{z}_k$ on D. This implies that for δ sufficiently small $u_{\delta\theta}$ is strictly psh on Ω'. The proof is complete. □

The next result is therefore a generalization of Theorem 1.1.

THEOREM 1.3. *Assume that $\mathcal{S}$ is a Richberg sheaf and let Ω and ε be as in Theorem 1.1. Then for $u \in \mathcal{S}(\Omega)$ one can find $v \in \mathcal{S} \cap C^\infty(\Omega)$ such that $u \leq v \leq u + \varepsilon$.*

The proof of Theorem 1.3 relies on the following:

LEMMA 1.4. *Let $u \in \mathcal{S}(\Omega)$ where $\mathcal{S}$ is a Richberg sheaf and Ω an open subset of $\mathbb{C}^n$. Assume that u is smooth on a neighborhood of $\bar{D}$ where $D \Subset \Omega$ is open. Let V and W be open with $V \Subset W \Subset \Omega$ and let $\varepsilon > 0$ (constant). Then there exists $v \in \mathcal{S}(\Omega)$ such that*

i) *$v = u$ on $\Omega \setminus W$,*

ii) *$u \leq v \leq u + \varepsilon$ on Ω,*

iii) *v is smooth on a neighborhood of $\bar{D} \cup \bar{V}$.*

Lemma 1.4 easily implies Theorem 1.3:

PROOF OF THEOREM 1.3. Suppose $\Omega_k \uparrow \Omega$ where the sets $\Omega_k \Subset \Omega_{k+1} \Subset \Omega$, $k \geq 0$, are open, $\Omega_0 = \emptyset$. For $k \geq 1$ set $W_k := \Omega_{k+1} \setminus \bar{\Omega}_{k-2}$ ($W_1 := \Omega_2$) and let V_k be open such that $\Omega_k \setminus \Omega_{k-1} \Subset V_k \Subset W_k$. Let $\gamma_k > 0$; it will be specified later. From Lemma 1.4 we can get a sequence $\{u_k\} \subset \mathcal{S}(\Omega)$ such that $u_0 = u$ and

$u_k = u_{k-1}$ on $\Omega \setminus W_k$,

$u_{k-1} \leq u_k \leq u_{k-1} + \gamma_k$ on Ω,

u_k is smooth in a neighborhood of $\bigcup_{j=1}^{k} \bar{V}_j$

($D = \bigcup_{j=1}^{k-1} V_j$). The sequence $\{u_k\}$ is locally constant for k big enough, thus we may define $v := \lim u_k \in \mathcal{S} \cap C^\infty(\Omega)$. Then $u \leq v$ on Ω and for $z \in \Omega \setminus \Omega_k$ one has

$$v(z) = u(z) + \sum_{j=k}^{\infty} (u_j(z) - u_{j-1}(z)) \leq u(z) + \sum_{j=k}^{\infty} \gamma_j .$$

Now, if γ_k are such that

$$\sum_{j=k}^{\infty} \gamma_j \leq \min_{\bar{\Omega}_{k+1}} \varepsilon ,$$

then $u \leq v \leq u + \varepsilon$ on Ω. □

PROOF OF LEMMA 1.4. Let $\eta \in C_0^\infty(\Omega)$ be such that $0 \leq \eta \leq 1$ on Ω, $\operatorname{supp} \eta \subset W$ and $\eta = 1$ on a neighborhood of $\bar{V}$.

First assume that D is empty. From (1.1) it follows that there exists $c_0 \in (0, 2\varepsilon)$ such that $u + c_0\eta \in \mathcal{S}(\Omega)$. Regularization of $u + c_0\eta$ and (1.3) give a function $\psi_0 \in \mathcal{S} \cap C^\infty(W)$ such that $u + c_0\eta \le \psi_0 \le u + c_0\eta + \frac{c_0}{2}$ on W. Define

$$u := \begin{cases} \max\{u, \psi_0 - c_0\} & \text{on} \quad W, \\ u & \text{on} \quad \Omega \backslash W. \end{cases}$$

Then $v = u$ if $\eta = 0$, and $v = \psi_0 - c_0$ if $\eta = 1$. Hence by (1.2), $v \in \mathcal{S}(\Omega)$, v is smooth on a neighborhood of $\bar{V}$ and $u \le v \le u + \frac{c_0}{2} \le u + \varepsilon$ on Ω.

Let now D be arbitrary. Choose open sets G_j, D_j and $\theta \in C^\infty(\Omega)$, $j = 1, 2$, so that $0 \le \theta_j \le 1$, $\bar{G}_j = \{\theta_j = 0\}$, $D_j = \{\theta_j < 1\}$, $D \Subset G_1 \Subset D_1 \Subset G_2 \Subset D_2 \Subset \Omega$ and u is smooth on a neighborhood of $\bar{D}_2$. By (1.1) we can find $c \in (0, \varepsilon/2)$ such that

$$\tilde{u} := u + c\eta \in \mathcal{S}(\Omega), \quad u + c\eta - c\theta_1 \in \mathcal{S}(\Omega). \tag{1.4}$$

We claim that for $\delta > 0$ small enough the function $\tilde{\psi} := \tilde{u}_{\delta\theta_2}$, defined on W, satisfies the following conditions:

(1.5) $\tilde{\psi} = \tilde{u}$ on a neighborhood of $W \cap \bar{D}_1$,

(1.6) $\tilde{u} \le \tilde{\psi} \le \tilde{u} + c$ on W,

(1.7) $\tilde{\psi} \in \mathcal{S} \cap C^\infty(W)$.

To get (1.5) take δ such that $\bar{D}_1 \subset \{z \in \Omega : \mathrm{dist}(z, \partial G_2) > 2\delta\}$. We have $\tilde{u}_\delta \downarrow \tilde{u}$ and the convergence is locally uniform on Ω. Hence, if $\tilde{u}_\delta \le \tilde{u} + c$ on W then $\tilde{\psi}(z) = \tilde{u}_{\delta\theta_2(z)}(z) \le \tilde{u}_\delta(z)$ and we get (1.6). The condition (1.7) follows immediately from (1.3).

Now put $\psi := \tilde{\psi} - c\theta_1$ on W. We claim that

(1.8) $\psi \in \mathcal{S} \cap C^\infty(W)$,

(1.9) $\psi \le u$ if $\eta = 0$.

Indeed, on $W \backslash \bar{D}_1$ we have $\psi = \tilde{\psi} - c$ and on a neighborhood of $W \cap \bar{D}_1$, by (1.5), $\psi = \tilde{u} - c\theta_1$. Therefore from (1.4) and (1.7) we obtain (1.8); (1.9) follows from (1.5) and (1.6).

Define

$$v := \begin{cases} \max\{u, \psi\} & \text{on} \quad W, \\ u & \text{on} \quad \Omega \backslash W. \end{cases}$$

By (1.8), (1.9) and (1.2) we have $v \in \mathcal{S}(\Omega)$. Obviously i) is fulfilled and from (1.6) it follows that ii) is also satisfied. It remains to show iii). If $\eta = 1$ then by (1.6) we have $\psi \ge \tilde{u} - c\theta_1 = u + c - c\theta_1 \ge u$, hence by (1.8) v is smooth on a neighborhood of $\bar{V}$. Now it is enough to show that v is smooth on G_1. If $\eta = 0$ then by (1.9) $v = u$, therefore it is enough to prove that v is smooth on $G_1 \cap W$. There by (1.5) $\psi = \tilde{\psi} = u + c\eta$, thus $v = u + c\eta$ on $G_1 \cap W$. The proof of the lemma is complete. □

Having Theorem 1.1 at our disposal we will now characterize hyperconvex and B-regular domains. The following result of J.B. Walsh will also be useful:

THEOREM 1.5. ([Wal]). *Suppose Ω is a bounded domain in $\mathbb{C}^n$ and $f \in C(\partial\Omega)$. Set*

$$u := \sup\{v \in \mathrm{PSH}(\Omega) : v^*|_{\partial\Omega} \le f\}$$

(v^*, respectively v_*, denotes the upper, respectively lower, regularization of v; it is defined on $\bar{\Omega}$). Assume moreover that $u_* = u^* = f$ on $\partial\Omega$. Then u is continuous.

PROOF. The function u^* is psh on Ω by the hypothesis $u^*|_{\partial\Omega} \le f$. Thus $u = u^*$ and u is upper semicontinuous. To show that is lower semicontinuous take $z_0 \in \Omega$ and $\varepsilon > 0$. By the compactness of $\partial\Omega$ we may find $0 < \delta < \mathrm{dist}(z_0, \partial\Omega)$ such that

$$z \in \bar{\Omega},\, w \in \partial\Omega,\, |z - w| \le 2\delta \Rightarrow |u(z) - f(w)| < \varepsilon\,. \tag{1.10}$$

Take $\tilde{z} \in \Omega$ such that $|z_0 - \tilde{z}| < \delta$ and for $z \in \bar{\Omega}$ define

$$v(z) := \begin{cases} \max\{u(z), u(z + z_0 - \tilde{z}) - 2\varepsilon\} & \text{if } \; z + z_0 - \tilde{z} \in \bar{\Omega}\,, \\ u(z) & \text{if } \; z + z_0 - \tilde{z} \notin \bar{\Omega}\,. \end{cases}$$

If $z + z_0 - \tilde{z} \in \partial\Omega$ then by (1.10) $u(z + z_0 - \tilde{z}) \le u(z) - \varepsilon$, hence $v \in \mathrm{PSH}(\Omega)$. From (1.10) it also follows that $v = u$ on a neighborhood of $\partial\Omega$, therefore $v \le u$ on Ω. Eventually we have $u(\tilde{z}) \ge v(\tilde{z}) \ge u(z_0) - 2\varepsilon$, which shows that u is lower semicontinuous. □

THEOREM 1.6. ([KR]). *For a bounded domain Ω in $\mathbb{C}^n$ the following are equivalent:*

i) *Every boundary point of Ω admits a* weak psh barrier, *that is for every $z_0 \in \partial\Omega$ there exists $v \in \mathrm{PSH}(\Omega)$ such that $v < 0$ and* $\lim_{z\to z_0} v(z) = 0$.

ii) *There exists smooth, strictly psh function ψ in Ω such that* $\lim_{z\to\partial\Omega} \psi(z) = 0$.

If Ω satisfies the condition i) then it is called *hyperconvex.* Another alternative definition is that in Ω there exists a negative psh u such that $\lim_{z\to\partial\Omega} u(z) = 0$ (this means that u is a bounded *exhaustion function*).

By CPT hyperconvex domains are regular with respect to the Laplace equation, that is every continuous function on $\partial\Omega$ can be extended to a harmonic function on Ω, continuous on $\bar{\Omega}$ (see e.g. [Doo], p. 125). As proved in [Dem1], every bounded, pseudoconvex domain in $\mathbb{C}^n$ with Lipschitz boundary is hyperconvex. On the other hand, an example of the Hartogs triangle $T := \{(z_1, z_2) : |z_1| < |z_2| < 1\}$ shows that not every pseudoconvex, regular domain is hyperconvex. (It follows from the exterior cone condition that T is regular and since $T \cap \{z_1 = 0\}$ is a punctured disc, T is not hyperconvex).

A stronger version of Theorem 1.6 will be proved in Section 6. Namely, we shall show that the function ψ from ii) can have additional property

$$\det\left(\frac{\partial^2\psi}{\partial z_j \partial \bar{z}_k}\right) \ge 1\,.$$

PROOF OF THEOREM 1.6. The implication ii) $\Rightarrow$ i) is obvious, so assume that Ω satisfies i). First we are going to show that there is a continuous exhaustion function. Take any ball $K \Subset \Omega$ and put

$$u := \sup\{v \in \mathrm{PSH}(\Omega) : v \le 0, v|_K \le -1\}.$$

Then $-1 \le u \le 0$, $u^*|_{\partial K} = -1$ (by logarithmic convexity of psh functions) and $u_*|_{\partial\Omega} = 0$ (by i)). Applying Theorem 1.5 to $\Omega\setminus\bar{K}$ we have therefore $u \in \mathrm{PSH}(\Omega) \cap C(\bar{\Omega})$, $u|_{\partial\Omega} = 0$, $u|_K = -1$, thus u is a continuous exhaustion function.

Let $C > 0$ be such that $v(z) := |z|^2 - C \le 0$ on Ω. Put $\tilde{u} := -2\sqrt{uv}$. Then $\tilde{u}$ is continuous on $\bar{\Omega}$ and $\tilde{u}|_{\partial\Omega} = 0$. In case of smooth functions of one variable we have

$$\frac{\partial^2(-2\sqrt{uv})}{\partial z \partial \bar{z}} = (uv)^{-1/2}\left(-u\frac{\partial^2 v}{\partial z\partial\bar{z}} - v\frac{\partial^2 u}{\partial z \partial\bar{z}}\right) + \frac{1}{2}(uv)^{-3/2}\left|u\frac{\partial v}{\partial z} - v\frac{\partial u}{\partial z}\right|^2 \ge \sqrt{\frac{u}{v}}\,\frac{\partial^2 v}{\partial z\partial\bar{z}},$$

therefore in the general case $\tilde{u}$ is strictly psh. Now, if $\varepsilon > 0$ is continuous on Ω and such that $\lim_{z\to\partial\Omega}\varepsilon(z) = 0$ then from Theorem 1.1 we get the required ψ. □

THEOREM 1.7. ([Sib]). *For a bounded domain Ω in $\mathbb{C}^n$ the following are equivalent:*

i) *Every boundary point admits a* strong psh barrier, *that is for every $z_0 \in \partial\Omega$ there exists $v \in \mathrm{PSH}(\Omega)$ such that $\lim_{z\to z_0} v(z) = 0$ and $v^*|_{\bar{\Omega}\setminus\{z_0\}} < 0$.*

ii) *In Ω there exists a smooth exhaustion function ψ such that*

$$\sum_{j,k=1}^{n} \frac{\partial^2\psi}{\partial z_j \partial\bar{z}_k}\alpha_j\bar{\alpha}_k \ge |\alpha|^2, \quad \alpha \in \mathbb{C}^n.$$

iii) *Continuous functions on $\partial\Omega$ are extendable to psh function on Ω, continuous on $\bar{\Omega}$, that is for every $f \in C(\partial\Omega)$ there exists $v \in \mathrm{PSH}(\Omega)\cap C(\bar{\Omega})$ such that $v|_{\partial\Omega} = f$.*

A domain Ω satisfying any of the above conditions is called *B-regular*. In particular its boundary has no analytic structure (that is no analytic disc can be embedded in $\partial\Omega$). Of course B-regular implies hyperconvex but these notions are not equivalent; for example a polydisc is hyperconvex but not B-regular.

PROOF OF THEOREM 1.7. The implication iii) $\Rightarrow$ i) is obvious. To show the converse take $f \in C(\partial\Omega)$ and let u be as in Theorem 1.5. There is a function $h \in C(\overline{\Omega})$, harmonic on Ω and equal to f on $\partial\Omega$. By the definition of u we have $u \le h$, hence $u^* \le f$ on $\partial\Omega$. Take any $z_0 \in \partial\Omega$ and $\varepsilon > 0$. By i) there exists $v_0 \in \mathrm{PSH}(\Omega)\cap C(\bar{\Omega})$ with $\lim_{z\to z_0} v_0(z) = 0$ and $v_0^*|_{\bar{\Omega}\setminus\{z_0\}} < 0$. Now if

$v := f(z_0) + tv_0$ then for t big enough we have $v^*|_{\partial\Omega} \leq f + \varepsilon$. Thus $v - \varepsilon \leq u$ on Ω, therefore $f(z_0) - \varepsilon \leq u^*(z_0)$ and finally $f \leq u_*$ on $\partial\Omega$. So we have $u_* = u^* = f$ on $\partial\Omega$ and from Theorem 1.5 it follows that $u \in \mathrm{PSH}(\Omega) \cap C(\bar{\Omega})$.

Next we want to prove that iii) implies ii). By iii) there is $v \in \mathrm{PSH}(\Omega) \cap C(\bar{\Omega})$ such that for $z \in \partial\Omega$ one has $v(z) = -2|z|^2$. Set $u(z) := v(z) + |z|^2$. Theorem 1.1 gives $\tilde{u} \in \mathrm{PSH} \cap C^\infty(\Omega)$ such that $\lim_{z\to\partial\Omega}(u(z) - \tilde{u}(z)) = 0$. Now it suffices to put $\psi(z) := \tilde{u}(z) + |z|^2$.

It remains to show the implication ii) $\Rightarrow$ iii). Take $f \in C(\partial\Omega)$ and again let u be as in Theorem 1.5. Let $\varepsilon > 0$; we will then find $g \in C^\infty(\bar{\Omega})$ such that $f \leq g \leq f + \varepsilon$ on $\partial\Omega$. For A big enough the functions $g + A\psi$ and $-g + A\psi$ are psh on Ω. The function $g + A\psi - \varepsilon$ is $\leq f$ on $\partial\Omega$, thus is also $\leq u$ on Ω. On the other hand for $v \in \mathrm{PSH}(\Omega)$ such that $v^*|_{\partial\Omega} \leq f$ the function $v^* - g + A\psi$ is ≤ 0 on $\partial\Omega$, thus is also ≤ 0 on Ω, hence $u - g + A\psi \leq 0$ on Ω. Therefore, we have obtained $g + A\psi - \varepsilon \leq u \leq g - A\psi$ on Ω. If we now let $\varepsilon \to 0$ then we get $u_* = u^* = f$ on $\partial\Omega$. By Theorem 1.5 $u \in \mathrm{PSH}(\Omega) \cap C(\bar{\Omega})$. □

2. – Positive forms and matrices

Let $\alpha \in \mathbb{C}_{(p,p)}$ be a (p, p)-form with constant coefficients $(1 \leq p \leq n)$; that is α can be written in the form

$$\alpha = \sum_{|J|=|K|=p} a_{JK} dz_J \wedge d\bar{z}_K, \quad a_{JK} \in \mathbb{C}.$$

We say that α is *positive* if for any $\alpha_1 \dots, \alpha_{n-p} \in \mathbb{C}_{(1,0)}$ one has $\alpha \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \dots \wedge i\alpha_{n-p} \wedge \bar{\alpha}_{n-p} \geq 0$ (we identify $\mathbb{C}_{(n,n)}$ with $\mathbb{C}$).

PROPOSITION 2.1. *A $(1, 1)$-form $\alpha = \sum_{j,k=1}^n a_{jk} i\, dz_j \wedge d\bar{z}_k$ is positive if and only if the matrix (a_{jk}) is nonnegative.*

PROOF. If $\alpha_s = \sum_{t=1}^n b_{st}\, dz_t \in \mathbb{C}_{(1,0)}$, $s = 1, \dots, n-1$, then

$$\alpha \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \dots i\alpha_{n-1} \wedge \bar{\alpha}_{n-1} = 2^n \sum_{j,k=1}^n a_{jk} M_j \bar{M}_k,$$

where $M_j = \det(a_{st})_{\substack{s=1,\dots,n-1\\ t=1,\dots,n,t\neq j}}$. □

PROPOSITION 2.2. *If α is a positive (p, p)-form and β a positive $(1, 1)$-form then $\alpha \wedge \beta$ is positive.*

PROOF. After a change of variables we may write $\beta = \sum_{j=1}^n a_j i dz_j \wedge d\bar{z}_j$, where $a_j \geq 0$. Then for $\alpha_1, \dots, \alpha_{n-p-1} \in \mathbb{C}_{(1,0)}$ we have

$$\alpha \wedge \beta \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \dots \wedge i\alpha_{n-p-1} \wedge \bar{\alpha}_{n-p-1}$$
$$= \sum_{j=1}^n a_j \alpha \wedge i dz_j \wedge d\bar{z}_j \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \dots \wedge i\alpha_{n-p-1} \wedge \bar{\alpha}_{n-p-1} \geq 0.$$

□

We shall also need some simple properties of hermitian matrices:

LEMMA 2.3. *Suppose* $A \in \mathrm{gl}(n, \mathbb{C})$, *the set of all square matrices with elements from* $\mathbb{C}$. *By* $\tilde{}$ *denote the natural embedding of* $\mathrm{gl}(n, \mathbb{C})$ *into* $\mathrm{gl}(2n, \mathbb{R})$. *Then* $\det \tilde{A} = |\det A|^2$.

PROOF. Write $A = M + iN$, where $M, N \in \mathrm{gl}(n, \mathbb{R})$; then

$$\tilde{A} = \begin{pmatrix} M & -N \\ N & M \end{pmatrix} .$$

If $\lambda_1, \dots, \lambda_n$ are all the eigenvalues of A then $\lambda_1, \dots, \lambda_n, \bar{\lambda}_1, \dots, \bar{\lambda}_n$ are all the eigenvalues of $\tilde{A}$. □

LEMMA 2.4. ([Gav]). *By* $\mathcal{A}$ *denote the family of all hermitian matrices* $A \in \mathrm{gl}(n, \mathbb{C})$ *with* $\det A = 1$. *Then for a hermitian matrix B we have*

$$(\det B)^{1/n} = \frac{1}{n} \inf_{A \in \mathcal{A}} \mathrm{tr}(AB) .$$

PROOF. Take $A \in \mathcal{A}$. Then we can find an orthogonal matrix P such that the matrix $C := PABP^{-1}$ is diagonal. From the inequality between geometric and arithmetic means we get

$$(\det B)^{1/n} = (\det C)^{1/n} \leq \frac{1}{n} \mathrm{tr}\, C \leq \frac{1}{n} \mathrm{tr}(AB) .$$

It remains to show that the infimum is attained. This is straightforward if B is diagonal. Then general case can be obtained after diagonalization of B. □

COROLLARY 2.5. ([BT1]). *The mapping* $B \mapsto (\det B)^{1/n}$ *is concave on the set of hermitian matrices.*

PROOF. By Lemma 2.4 we have

$$\begin{aligned}(\det(B_1 + B_2))^{1/n} &= \frac{1}{n} \inf_{A \in \mathcal{A}} \mathrm{tr}(AB_1 + AB_2) \\ &\geq \frac{1}{n} \inf_{A \in \mathcal{A}} \mathrm{tr}(AB_1) + \frac{1}{n} \inf_{A \in \mathcal{A}} \mathrm{tr}(AB_2) = (\det B_1)^{1/n} + (\det B_2)^{1/n} .\end{aligned}$$

The concavity now follows from the homogeneity of the mapping. □

3. – The complex Monge-Ampère operator

For a smooth function u defined on an open subset of $\mathbb{C}^n$ we set

$$Mu = \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right).$$

M is called the *complex Monge-Ampère operator*. The aim of this section is to extend the definition of Mu to the class of continuous psh functions and list some of the basic properties.

The following estimate is a variation of a result from [Bło1] and will be crucial in our presentation:

THEOREM 3.1. *Let Ω be a bounded domain in $\mathbb{C}^n$ and suppose that $u, v, g, h \in$ PSH $\cap C^\infty(\Omega)$ are such that $u \leq v \leq 0$, $g \leq h$ and $g = h$ on a neighborhood of $\partial\Omega$. Then*

$$\int_\Omega (h-g)^n (Mu - Mv) \leq n! \|v-u\|_\Omega \sum_{j=0}^{n-1} \|u\|_\Omega^j \|v\|_\Omega^{n-1-j} \int_\Omega Mg .$$

In the proof of Theorem 3.1 we shall use the operators ∂ and $\bar\partial$. We have $d = \partial + \bar\partial$; set $d^c := i(\bar\partial - \partial)$. Then $dd^c = 2i\partial\bar\partial$ and, if u is smooth,

$$(dd^c u)^n = dd^c u \wedge \ldots \wedge dd^c u = n! 4^n \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right).$$

Moreover, from Proposition 2.1 and 2.2 it follows that if $u_1, \ldots, u_k \in$ PSH $\cap C^\infty$ then $dd^c u_1 \wedge \ldots \wedge dd^c u_k \geq 0$ (that is the form is positive at every point).

PROPOSITION 3.2. *If $\alpha \in C^\infty_{(p,p)}$, $\beta \in C^\infty_{(n-1-p,n-1-p)}$ then*

$$\alpha \wedge dd^c\beta - dd^c\alpha \wedge \beta = d(\alpha \wedge d^c\beta - d^c\alpha \wedge \beta).$$

PROOF. We have

$$d(\alpha \wedge d^c\beta - d^c\alpha \wedge \beta) = \alpha \wedge dd^c\beta - dd^c\alpha \wedge \beta + d\alpha \wedge d^c\beta + d^c\alpha \wedge d\beta$$

and

$$d\alpha \wedge d^c\beta = i(\partial\alpha \wedge \bar\partial\beta - \bar\partial\alpha \wedge \partial\beta) = -d^c\alpha \wedge d\beta . \qquad \square$$

PROOF OF THEOREM 3.1. Write

$$(dd^c u)^n - (dd^c v)^n = dd^c(u-v) \wedge T ,$$

where

$$T = \sum_{j=0}^{n-1} (dd^c u)^j \wedge (dd^c v)^{n-1-j} \geq 0 .$$

By Stokes Theorem and Proposition 3.2 we have

$$\int_\Omega (h-g)^n((dd^c u)^n - (dd^c v)^n) = -\int_\Omega (v-u)dd^c(h-g)^n \wedge T\,.$$

Further

$$\begin{aligned}-dd^c(h-g)^n &= -n(h-g)^{n-1}dd^c(h-g)\\ &\quad - n(n-1)(h-g)^{n-2}d(h-g)\wedge d^c(h-g)\\ &\le n(h-g)^{n-1}dd^c g\,,\end{aligned}$$

because for any smooth φ one has $d\varphi \wedge d^c\varphi = i\partial\varphi \wedge \bar\partial\varphi \geq 0$. Hence, by Proposition 2.2

$$\int_\Omega (h-g)^n((dd^c u)^n) - (dd^c v)^n) \le n\|v-u\|_\Omega \int_\Omega (h-g)^{n-1}dd^c g \wedge T\,.$$

Repeating the above arguments $n-1$ times we easily get

$$\int_\Omega (h-g)^{n-1}dd^c g \wedge (dd^c u)^j \wedge (dd^c v)^{n-1-j} \le (n-1)!\|u\|_\Omega^j \|v\|_\Omega^{n-1-j} \int_\Omega (dd^c g)^n\,,$$

which completes the proof. □

The next result is essentially the *Chern-Levine-Nirenberg inequality.*

COROLLARY 3.3. *Let $0 < r < R$. If we denote $B^R = B(0,R)$ the n for $u \in \mathrm{PSH} \cap C^\infty(B^R)$, $u \le 0$,*

$$\int_{B^r} Mu \le \frac{n!\lambda(B^R)}{(R^2-r^2)^n}\|u\|^n_{B^R}\,. \tag{3.1}$$

PROOF. Let $g(z) := |z|^2 - R^2$ and $h_\varepsilon \in \mathrm{PSH}\cap C^\infty(B_R)$ be such that $\max\{g, -\varepsilon\} \le h_\varepsilon \le 0$ and $h_\varepsilon = g$ on a neighborhood of ∂B_R. If we now let ε tend to 0, Theorem 3.2 with $v \equiv 0$ gives

$$\int_{B^R} (-g)^n Mu \le n!\lambda(B^R)\|u\|^n_{B^R}\,,$$

which implies (3.1). □

THEOREM 3.4. ([BT1]). *If u is continuous and psh then Mu can be uniquely defined as a nonnegative Borel measure in such a manner that*

$$Mu = \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar z_k}\right) \tag{3.2}$$

if u is smooth, and uniform convergence $u_j \longrightarrow u$ implies weak convergence $Mu_j \longrightarrow Mu$.

PROOF. Take $u \in \mathrm{PSH} \cap C(\Omega)$ and suppose r and R are such that $B^r \Subset B^R \Subset \Omega$. We may assume that $-K \leq u \leq 0$ on B^R. Take a nonnegative test function φ such that $\varphi^{1/n} \in C_0^\infty(B^r)$. One can find $g, h \in \mathrm{PSH} \cap C^\infty(\Omega)$ such that $\varphi^{1/n} = h - g$.

By Theorem 3.1 for any $v_j \in \mathrm{PSH} \cap C^\infty(B^R)$ with $-K \leq v_j \leq 0$, $j = 1, 2$, the following estimate holds:

$$\left| \int_{B^r} \varphi(Mv_1 - Mv_2) \right| \leq 2^n n! n K^{n-1} \|v_1 - v_2\|_{B^R} \int_{B^R} Mg \tag{3.3}$$

(because $v_1 - \|v_1 - v_2\| \leq v_2$).

Choose a sequence $\{u_j\} \subset \mathrm{PSH} \cap C^\infty(B^R)$ uniformly convergent to u and such that $-K \leq u_j \leq 0$. By (3.3) the sequence $\int_{B^r} \varphi M u_j$ is convergent and, again by (3.3), the limit is independent of the choice of $\{u_j\}$. We may therefore define $Mu(\varphi)$ for $\varphi \geq 0$ with $\varphi^{1/n} \in C_0^\infty(B^r)$ and by Corollary 3.3

$$Mu(\varphi) = \lim_{j \to \infty} \int_{B^r} \varphi M u_j \leq C \|\varphi\|_{B^r} ,$$

where C does not depend on φ. This implies that we can well define $Mu(\varphi)$ for every $\varphi \in C_0(\Omega)$, thus Mu is a nonnegative Borel measure on B^r. Since the problem is purely local, the definition of Mu is valid in whole Ω. Moreover, approximation arguments show that (3.3) remains valid if v_1 and v_2 are only continuous.

Finally, let $\{u_j\}$ be an arbitrary sequence converging uniformly to u. After adding appropriate constants we may assume that $-K \leq u_j \leq 0$ and the sequence is decreasing. Then by the extended version of (3.3) $Mu_j(\varphi) \longrightarrow Mu(\varphi)$ for $\varphi \geq 0$ with $\varphi^{1/n} \in C_0^\infty(\Omega)$. It follows that Mu_j converges weakly to Mu which completes the proof. □

The right hand-side of (3.2) is also well defined for C^2, and even $C^{1,1}$ functions, that is C^1 functions with Lipschitz first partial derivatives. It follows from Rademarcher theorem that a $C^{1,1}$ function is twice differentiable almost everywhere with respect to the Lebesgue measure. Moreover, second partial derivatives of such a function, defined pointwise as locally bounded functions, coincide with distributional derivatives (this follows from the fact that Lipschitz functions of one variable can be integrated by parts). The Sobolev theorem implies that every distribution with locally bounded second partial derivatives is a $C^{1,1}$ function.

PROPOSITION 3.5. *The equality* (3.2) *remains valid for $C^{1,1}$ psh functions. Moreover, if $u_\delta := u * \rho_\delta$ is a regularization of such a function u then $Mu_\delta \longrightarrow Mu$ locally in the L^p-norm for every $p < \infty$.*

PROOF. It is enough to prove the second claim of the proposition. Since the second partial derivatives of u are locally bounded, we have for $p < \infty$ the convergence in $\mathrm{L}^p_{\mathrm{loc}}$

$$\frac{\partial^2 u_\delta}{\partial z_j \partial \bar{z}_k} = \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} * \rho_\delta \longrightarrow \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} .$$

To complete the proof it suffices to observe that if two sequences of functions $\{f_j\}$ and $\{g_j\}$ are locally uniformly bounded and convergent to f and g, respectively, then $f_j g_j \longrightarrow fg$ in L^p_{loc}. Indeed, write

$$f_j g_j - fg = f_j(g_j - g) + g(f_j - f)$$

and the proposition follows. □

An important property of the complex Monge-Ampère operator is its good behavior under a holomorphic change of coordinates:

PROPOSITION 3.6. *Let Ω_1 and Ω_2 be open in $\mathbb{C}^n$ and suppose that $H : \Omega_1 \longrightarrow \Omega_2$ is a holomorphic mapping with non-vanishing jacobian. Then for $u \in \mathrm{PSH} \cap C(\Omega_2)$ we have*

$$M(u \circ H) = |\mathrm{Jac} H|^2 H^* Mu .$$

PROOF. $H^* : \mathcal{D}'(\Omega_2) \longrightarrow \mathcal{D}'(\Omega_1)$ is a continuous linear mapping and, since the problem is purely local, we may assume that u is smooth. Then we have an equality of matrices

$$\left(\frac{\partial^2 (u \circ H)}{\partial z_j \partial \bar{z}_k} \right) = A^T \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_m} \circ H \right) \bar{A} ,$$

where $A = (\partial H_p / \partial z_q)$, and the proposition follows. □

Next theorem is called the *comparison principle*.

THEOREM 3.7. ([BT1]). *If $\Omega \Subset \mathbb{C}^n$, u, $v \in \mathrm{PSH}(\Omega) \cap C(\bar{\Omega})$ are such that $u \le v$ on $\partial\Omega$ and $Mu \ge Mv$, then $u \le v$ on Ω.*

PROOF. Suppose the set $\{u < v\}$ is non-empty. Let C be such that $\psi(z) := |z|^2 - C \le 0$ on Ω; then for some $\varepsilon > 0$ the set $S := \{u + \varepsilon\psi > v\}$ is non-empty too. For $\delta > 0$ set $v_\delta := \max\{u + \varepsilon\psi, v + \delta\}$. Then $v_\delta \downarrow u + \varepsilon\psi$ on S, if $\delta \downarrow 0$, and $v_\delta = v + \delta$ on a neighborhood of ∂S. From Stokes theorem and the weak convergence $Mv_\delta \longrightarrow M(u + \varepsilon\psi)$ we get

$$Mv(S) = \liminf_{\delta \longrightarrow 0} Mv_\delta(S) \ge M(u + \varepsilon\psi)(S) \ge Mu(S) + \varepsilon^n > Mu(S)$$

which is a contradiction. (The operator M is subadditive - this follows for example from Corollary 2.5). □

THEOREM 3.8. ([BT1]). *If u and v are psh and continuous then*

$$M \max\{u, v\} \ge 1_{\{u > v\}} Mu + 1_{\{u \le v\}} Mv ,$$

where 1_A stands for the characteristic function of a set A.

PROOF. It is enough to show that for compact $K \subset \{u = v\}$ one has $M \max\{u, v\}(K) \geq Mu(K)$. For $\delta > 0$ let $u_\delta := \max\{u + \delta, v\}$; then $u_\delta = u + \delta$ on a neighborhood of K and $u_\delta \downarrow \max\{u, v\}$. The weak convergence $Mu_\delta \longrightarrow M \max\{u, v\}$ gives

$$Mu(K) = \limsup_{\delta \longrightarrow 0} Mu_\delta(K) \leq M \max\{u, v\}(K) . \qquad \square$$

We will need the following version of the comparison principle:

THEOREM 3.9. *Let Ω and u be as in Theorem 3.7. Suppose that $v \in C^2(\Omega) \cap C(\bar{\Omega})$, $u \leq v$ on $\partial\Omega$ and $Mu \geq Mv$ on Γ_v, the open set where $(\partial^2 v/\partial z_j \partial \bar{z}_k)$ is positive. Then $u \leq v$ on Ω.*

PROOF. It is similar to the proof of Lemma 5.2 in [RT]. Let C be such that $\psi(z) := |z|^2 - C \leq 0$ on Ω. Suppose that the set $\{u > v\}$ is non-empty; then $S := \{u + \varepsilon\psi > v\} \neq \emptyset$ for some $\varepsilon > 0$. By a denote the maximum of the function $u + \varepsilon\psi - v$ and by W the set where it is attained. W is a compact subset of S.

Suppose that $W \subset \Gamma_v$. Then for some $a' < a$ we would have $u + \varepsilon\psi - v \leq a'$ on $\partial\Gamma_v$ and by the classical comparison principle we would get a contradiction. We may therefore assume that there exists $z_0 \in W \backslash \Gamma_v$. Then the matrix $(\partial^2 v/\partial z_j \partial \bar{z}_k(z_0))$ is not positive, hence for some $\alpha \in \mathbb{C}^n$

$$\frac{\partial^2 v(z_0 + \zeta\alpha)}{\partial\zeta\partial\bar{\zeta}}(0) \leq 0 .$$

Therefore we have

$$\frac{\partial^2 (u + \varepsilon\psi - v)(z_0 + \zeta\alpha)}{\partial\zeta\partial\bar{\zeta}}(0) > 0$$

which contradicts the fact that $u + \varepsilon\psi - v$ has a local maximum at z_0. $\square$

REMARK. Theorem 3.9 is not true for arbitrary $v \in C(\bar{\Omega})$. A counterexample can be easily constructed even in the case $n = 1$.

For $A = (a_{jk}) \in \mathcal{A}$, where $\mathcal{A}$ is defined in Lemma 2.4, put

$$\Delta_A := \frac{1}{n} \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2}{\partial z_j \partial \bar{z}_k} .$$

By Lemma 2.4 and Proposition 3.5 we have

$$(Mu)^{1/n} = \inf_{A \in \mathcal{A}} \Delta_A u \quad \text{if} \quad u \in \mathrm{PSH} \cap C^{1,1} . \tag{3.4}$$

THEOREM 3.10. *Let u be a continuous psh function and let F be continuous and nonnegative. Then the following are equivalent:*

i) $Mu \geq F$;

ii) $\Delta_A u \geq F^{1/n}$, $A \in \mathcal{A}$;

iii) $(Mu_\delta)^{1/n} \geq F^{1/n} * \rho_\delta$, $\delta > 0$, *where* $u_\delta = u * \rho_\delta$.

PROOF. The implication iii) $\Rightarrow$ i) follows from the weak convergence $Mu_\delta \longrightarrow Mu$ and the uniform convergence $F^{1/n} * \rho_\delta \longrightarrow F^{1/n}$. From ii) it follows that $\Delta_A u_\delta = (\Delta_A u) * \rho_\delta \geq F^{1/n} * \rho_\delta$ and (3.4) implies iii). It remains therefore to show the implication i) $\Rightarrow$ ii). Fix $A \in \mathcal{A}$. After simple approximation arguments we may assume that $F^{1/n}$ is smooth. Take any ball B and let $v \in C(\bar{B})$ be such that $\Delta_A v = F^{1/n}$ and $v = u$ on ∂B. Then v is smooth in B and by (3.4) $Mv \leq F$ on Γ_v. Now, Theorem 3.9 implies that $u \leq v$ and it follows that $\int_B \Delta_A u \geq \int_B F^{1/n} d\lambda$ which completes the proof of the theorem. □

In Section 4 we will use the following extension of the subadditivity of the Monge-Ampère operator:

THEOREM 3.11. *Let $u, v \in$ PSH $\cap C$ be such that $Mu \geq F$, $Mv \geq G$, where F and G are continuous and nonnegative. Then*

$$M\left(\frac{u+v}{2}\right) \geq \left(\frac{F^{1/n}+G^{1/n}}{2}\right)^n . \tag{3.5}$$

PROOF. If u and v are smooth then (3.5) follows directly from Lemma 2.5. In order to prove it for arbitrary u and v take a nonnegative test function φ and use Theorem 3.10:

$$\begin{aligned}\int \varphi M\left(\frac{u+v}{2}\right) &= \lim_{\delta\to 0} \int \varphi M\left(\frac{u_\delta+v_\delta}{2}\right)\\ &\geq \liminf_{\delta\to 0} \int \varphi \left(\frac{(Mu_\delta)^{1/n}+(Mv_\delta)^{1/n}}{2}\right)^n\\ &\geq \liminf_{\delta\to 0} \int \varphi \left(\frac{F^{1/n}*\rho_\delta+G^{1/n}*\rho_\delta}{2}\right)^n\\ &= \int \varphi \left(\frac{F^{1/n}+G^{1/n}}{2}\right)^n .\end{aligned}$$ □

4. – The Dirichlet problem in B-regular domains

The aim of this section is to prove the following theorem:

THEOREM 4.1. ([BT1]). *Assume $\Omega \Subset \mathbb{C}^n$ is a B-regular domain. Let $f \in C(\partial\Omega)$, $F \in C(\bar{\Omega})$, $F \geq 0$. Then there exists a unique solution $u = u_\Omega(f, F)$ of the following Dirichlet problem:*

$$\begin{cases} u \in \mathrm{PSH}(\Omega) \cap C(\bar{\Omega}) \\ Mu = F \\ u|_{\partial\Omega} = f . \end{cases} \tag{4.1}$$

In Section 6 we will generalize the above theorem to the class of hyperconvex domains.

PROOF OF THEOREM 4.1. The uniqueness follows from the comparison principle. Moreover, the solution, if exists, must be of the form $u = \sup \mathcal{B}$, where

$$\mathcal{B} = \{v \in PSH(\Omega) \cap C(\bar{\Omega}) : v|_{\partial\Omega} = f, \quad Mv \geq F\}.$$

(Elements of $\mathcal{B}$ are called *subsolutions* of the problem (4.1).) Let ψ be as in Theorem 1.7. The family $\mathcal{B}$ is nonempty, for if $v \in PSH(\Omega) \cap C(\bar{\Omega})$, $v|_{\partial\Omega} = f$ then $M(v + A\psi) \geq Mv + A^n M\psi \geq A^n \geq F$ for sufficiently big A. By Theorem 3.8 the family $\mathcal{B}$ has a lattice property:

$$v_1, v_2 \in \mathcal{B} \Rightarrow \max\{v_1, v_2\} \in \mathcal{B}. \tag{4.2}$$

Theorem 4.1 will be proved in several steps.

STEP I. $u := \sup \mathcal{B} \in \mathcal{B}$.

Choose any $v_0 \in \mathcal{B}$ and let h be harmonic on Ω, continuous on $\bar{\Omega}$ and such that $h|_{\partial\Omega} = f$. We have $v_0 \leq u \leq h$ which implies $u_* = u^* = f$ on $\partial\Omega$.

By the definition u is lower semicontinuous. We want to show that it is also upper semicontinuous. The arguments will be similar to those from the proof of Theorem 1.5. Fix $z_0 \in \Omega$ and $\varepsilon > 0$. We can find $0 < \delta < \text{dist}(z_0, \partial\Omega)$ such that

$$z \in \bar{\Omega},\ w \in \partial\Omega,\ |z - w| \leq 2\delta \Rightarrow |v_0(z) - f(w)| < \varepsilon,\ |h(z) - f(w)| < \varepsilon \tag{4.3}$$

(by the compactness of $\partial\Omega$) and

$$|\tau| \leq \delta,\ z \in \bar{\Omega},\ z + \tau \in \bar{\Omega} \Rightarrow |F(z + \tau) - F(z)| \leq (\varepsilon/\|\psi\|_\Omega)^m \tag{4.4}$$

(by the uniform continuity of F on $\bar{\Omega}$). Fix $\tilde{z} \in B(z_0, \delta)$. We can find $v \in \mathcal{B}$ such that $u(\tilde{z}) \leq v(\tilde{z}) + \varepsilon$. From (4.2) it follows that we can take $v \geq v_0$, thus by (4.3) we have

$$z \in \bar{\Omega},\ w \in \partial\Omega,\ |z - w| \leq 2\delta \Rightarrow |v(z) - v(w)| < \varepsilon. \tag{4.5}$$

Define

$$\hat{v}(z) := \begin{cases} \max\{v(z), v(z + \tilde{z} - z_0) - 2\varepsilon\} & \text{if } z + \tilde{z} - z_0 \in \bar{\Omega}, \\ v(z) & \text{if } z + \tilde{z} - z_0 \notin \bar{\Omega}. \end{cases}$$

If $z+\tilde{z}-z_0 \in \partial\Omega$ then by (4.5) $v(z+\tilde{z}-z_0) \leq v(z)+\varepsilon$, hence $\hat{v} \in PSH(\Omega)\cap C(\bar{\Omega})$. From (4.5) it follows also that $\hat{v} = v$ on a neighborhood of $\partial\Omega$. By (4.4) and Theorem 3.8 $M\hat{v} \geq F - (\varepsilon/\|\psi\|_\Omega)^n$, and it follows that $\hat{v}(z) + \varepsilon\psi/\|\psi\|_\Omega \in \mathcal{B}$. Therefore we have $u(z_0) \geq \hat{v}(z_0) - \varepsilon \geq v(\tilde{z}) - 3\varepsilon \geq u(\tilde{z}) - 4\varepsilon$, so u is lower semicontinuous. We have thus shown that $u \in PSH(\Omega) \cap C(\bar{\Omega})$, $u|_{\partial\Omega} = f$.

Choquet lemma (cf. [Doo]) implies that there exists a sequence $\{v_j\} \subset \mathcal{B}$ such that $u = \sup v_j$. Then, for $u_j := \max\{v_1, \dots, v_j\} \in \mathcal{B}$ we have $u_j \uparrow u$. The convergence is uniform, hence $Mu \geq F$ and $u \in \mathcal{B}$.

STEP II. We may reduce the proof to the case when Ω is the unit ball B, $F^{1/n} \in C^\infty(\bar{B})$ and $f \in C^\infty(\partial B)$.

By Step I, to prove Theorem 4.1 it remains to show that $Mu = F$. First assume we have done it for a ball and let Ω be arbitrary. Fix $\tilde{B} = B(z_0, r) \Subset \Omega$ and put $\tilde{u} := u_{\tilde{B}}(f|_{\partial\tilde{B}}, F|_{\tilde{B}})$. By the comparison principle $\tilde{u} \geq u$ on $\tilde{B}$ and $\tilde{u} = u$ on $\partial\tilde{B}$. From Theorem 3.8 it follows that the function

$$v := \begin{cases} \tilde{u} & \text{on } \tilde{B} \\ u & \text{on } \bar{\Omega}\setminus\tilde{B} \end{cases}$$

belongs to $\mathcal{B}$ (because $v = \max\{u, v\}$). This implies that $\tilde{u} = u$ on B, hence $Mu = F$ on $\tilde{B}$. Therefore we may assume that $\Omega = B$.

Assume that Theorem 4.1 is true for smooth data and let f, F be continuous. We can find sequences $\{f_j\} \subset C^\infty(\partial B)$, decreasing to f, $\{F_j^{1/n}\} \subset C^\infty(\bar{B})$, increasing to $F^{1/n}$, and solutions $u_j := u_B(f_j, F_j)$. By the comparison principle

$$u_j(z) + \|F_j - F_k\|_{\bar{B}}^{1/n}(|z|^2 - 1) \leq \|f_j - f_k\|_{\partial B} + u_k(z), \quad z \in B\,,$$

hence

$$\|u_j - u_k\|_{\bar{B}} \leq \|f_j - f_k\|_{\partial B} + \|F_j - F_k\|_{\bar{B}}^{1/n}\,.$$

Thus the sequence $\{u_j\}$ is uniformly convergent on $\bar{B}$ and $u := \lim u_j$ is the required solution.

STEP III. If $f \in C^{1,1}(\partial B)$ and $F^{1/n} \in C^{1,1}(\bar{B})$ then $u \in C^{1,1}(B)$.

For $a \in B$ set

$$T_a(z) := \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}\,,$$

where $P_a(z) = \frac{\langle z,a\rangle}{|a|^2}a$ is the projection of z on the complex line $\mathbb{C}a$, $Q_a(z) = z - P_a(z)$, $s_a = \sqrt{(1-|a|^2)}$ ($T_0(z) = -z$). Then T_a is a holomorphic automorphism of B, $T_a^{-1} = T_a$ and $T_a(0) = a$ (cf. [Rud]).

Fix $\varepsilon > 0$ and define

$$L(a, h, z) := T_{a+h}^{-1}(T_a(z)) = T_{a+h}(T_a(z))\,.$$

Then $L \in C^\infty\left(\overline{B^{1-\varepsilon}} \times B^\varepsilon \times \bar{B}\right)$ ($B^R = B(0, R)$), because we can compute that

$$T_a(z) = \frac{a - s_a z + \dfrac{\langle z, a\rangle}{s_a + 1}a}{1 - \langle z, a\rangle}\,.$$

If we put

$$V(a,h,z) := \frac{1}{2}[u(a,h,z)) + u(L(a,-h,z))],$$

then $V \in C^{1,1}\left(\overline{B^{1-\varepsilon}} \times B^{\varepsilon} \times \partial B\right)$ and

$$V(\cdot,0,z) = u(z), \tag{4.6}$$

$$V(z,h,z) = \frac{1}{2}[u(z+h) + u(z-h)]. \tag{4.7}$$

First we want to show that there are positive constants K_1 and K_2, depending only on u and ε, such that for $(a,h) \in \overline{B^{1-\varepsilon}} \times \overline{B^{\varepsilon/2}}$ the function

$$v(z) := V(a,h,z) - K_1|h|^2 + K_2(|z|^2 - 1)|h|^2$$

belongs to $\mathcal{B}$. For $a \in \overline{B^{1-\varepsilon}}$ and $z \in \partial B$ we have $\partial V/\partial h(a,0,z) = 0$, thus at the points where V is twice differentiable the Taylor expansion of V gives

$$V(a,h,z) = V(a,0,z) + \frac{1}{2}\frac{\partial^2 V}{\partial h^2}(a,\tilde{h},z)\cdot h^2$$

for some $|\tilde{h}| \le |h|$. By (4.6) it is enough to take

$$K_1 := \frac{1}{2}\left\|\frac{\partial^2 V}{\partial h^2}\right\|_{\overline{B^{1-\varepsilon}}\times\overline{B^{\varepsilon/2}}\times\partial B},$$

to get $v|_{\partial\mathcal{B}} \le f$. By proposition 3.6, Theorem 3.11 and since $Mu \ge F$,

$$Mv \ge \left(\frac{F(L(a,h,z))^{1/n}|\mathrm{Jac}_z L(a,h,z)|^{2/n} + F(L(a,-h,z))^{1/n}|\mathrm{Jac}_z L(a,-h,z)|^{2/n}}{2}\right)^n + K_2|h|^2.$$

Now, using the Taylor expansion again, we get that with the constant

$$\tilde{K} := \frac{1}{2}\left\|\frac{\partial^2(F(L(a,h,z))|\mathrm{Jac}_z L(a,h,z)|^2)}{\partial h^2}\right\|_{\overline{B^{1-\varepsilon}}\times\overline{B^{\varepsilon/2}}\times\overline{B}}$$

we have

$$Mv \ge (F^{1/n} - \tilde{K}|h|^2)^n + K_2|h|^2$$

and thus $v \in \mathcal{B}$ for K_2 big enough.

If $a = z$ then from (4.7) we obtain the estimate

$$u(z+h) + u(z-h) - 2u(z) \le K|h|^2, \quad |z| \le 1-\varepsilon,\ |h| \le \varepsilon/2, \tag{4.8}$$

where the constant $K := K_1 + K_2$ depends only on ε, f and F.

Now we shall show that if a psh function u fulfills the estimate (4.8) then $u \in C^{1,1}$. First observe that the estimate holds also for the regularizations $u_\delta := u * \rho_\delta$. Indeed, by (4.8) for $\delta < \varepsilon/2$ we have

$$u_\delta(z+h) + u_\delta(z-h) - 2u_\delta(z) \leq K|h|^2, \quad |z| \leq 1 - 2\varepsilon, \ |h| \leq \varepsilon/2 .$$

This implies that $u_\delta''(z).h^2 \leq K|h|^2$. The functions u_δ are psh, thus

$$u_\delta''(z).h^2 + u_\delta''(z).(ih)^2 = 4\sum_{j,k=1}^{4} \frac{\partial^2 u_\delta}{\partial z_j \partial \bar{z}_k}(z) h_j \bar{h}_k \geq 0 .$$

Therefore we have

$$u_\delta''(z).h^2 \geq -u_\delta''(z).(ih)^2 \geq -K|h|^2 ,$$

hence $|u_\delta''(z)| \leq K$. Since $\{u_\delta''\}$ is weakly convergent to u'', we get that the second partial derivatives of u are locally bounded and therefore $u \in C^{1,1}$.

STEP IV. $Mu = F$, if $u \in C^{1,1}$.

By Step I we have $Mu \geq F$. It follows from Proposition 3.5 that it is enough to show that

$$\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) = F \tag{4.9}$$

at point where u is twice differentiable. Suppose for some z_0 we have a strict inequality in (4.9). We may assume that the matrix $(\partial^2 u/\partial z_j \partial \bar{z}_k(z_0))$ has a diagonal form $(\lambda_j \delta_{jk})$, $\lambda_j > 0$. Let $0 < \tilde{\lambda}_j < \lambda_j$ be such that $\tilde{\lambda}_1 \ldots \tilde{\lambda}_n > F(z_0)$. The Taylor expansion gives

$$u(z_0 + h) = Re\ P(h) + \lambda_1|h_1|^2 + \cdots + \lambda_n|h_n|^2 + o(|h|^2) ,$$

where

$$P(h) = u(z_0) + 2\sum_{j=1}^{n} \frac{\partial u}{\partial z_j}(z_0)h_j + \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial z_k}(z_0) h_j h_k .$$

Therefore we can find $r > 0$ and $\varepsilon > 0$ such that $F(z_0 + h) < \tilde{\lambda}_1 \ldots \tilde{\lambda}_n$, if $|h| \leq r$, and

$$Q(h) := Re\ P(h) + \tilde{\lambda}_1|h_1|^2 + \cdots + \tilde{\lambda}_n|h_n|^2 + \varepsilon < u(z_0 + h) ,$$

if $|h| = r$. Then the function

$$v(z) := \begin{cases} \max\{u(z), Q(z - z_0)\} & \text{if } z \in B(z_0, r) , \\ u(z) & \text{if } z \notin B(z_0, r) \end{cases}$$

belongs to $\mathcal{B}$, hence $v \leq u$. But $v(z_0) \geq Re\ P(0) + \varepsilon = u(z_0) + \varepsilon$ which leads to a contradiction.

The proof of Theorem 4.1 is complete. □

Incidently, in Step III of the above proof we obtained the following regularity theorem:

THEOREM 4.2. *If B is the unit ball in $\mathbb{C}^n$, $f \in C^{1,1}(\partial B)$ and $F^{1/n} \in C^{1,1}(\bar{B})$ then $u_B(f, F) \in C^{1,1}(B)$.* □

5. – Stability of the complex Monge-Ampère operator

DEFINITION. *Let p, $q \in (0, \infty]$. We say that the complex Monge-Ampère operator is (p, q)-*stable *if there is a constant C, depending only on p, q and n, such that*

$$\|u_B(0, F)\|_{L^p(B)} \leq C\|F\|_{L^q(B)}^{1/n}, \quad F \in C(\bar{B}), \quad F \geq 0. \tag{5.1}$$

It follows from Hölder's inequality that (p_0, q_0)-stability implies (p, q)-stability for $p \leq p_0$ and $q \geq q_0$. By the comparison principle for $F \in C(\bar{B})$, $F \geq 0$, we have

$$\|F\|_{\bar{B}}^{1/n}(|z|^2 - 1) \leq u_B(0, F)(z), \quad z \in B,$$

and it follows that the Monge-Ampère operator is (∞, ∞)-stable. This means that it is also (p, ∞)-stable for every $p > 0$.

We shall now see when the Monge-Ampère operator is not stable:

EXAMPLE. Let u be psh, smooth on $\mathbb{C}^n$ such that $u(z) = \log|z|$ if $|z| \geq 1$. Define $u_j(z) := u(jz) - \log j$. Then $u_j(z) = \log|z|$, if $|z| \geq 1/j$, $u_j \downarrow \log|z|$, if $j \uparrow \infty$, and $Mu_j(z) = j^{2n}Mu(jz)$. Therefore we have $\|u_j\|_{L^p(B)} \longrightarrow \|\log|z|\|_{L^p(B)}$ and, since $Mu_j(z) = M(\log|z|) = 0$ if $|z| \geq 1/j$, we get $\|Mu_j\|_{L^q(B)} = j^{2n(1-1/q)}\|Mu\|_{L^q(B)}$. This shows that the Monge-Ampère operator is not (p, q)-stable if either $q < 1$ or $p = \infty$ and $q = 1$.

(p, q)-stability means that the L^p-norm of a sufficiently regular psh function u on the unit ball, vanishing on the boundary, can be estimated by the L^q-norm of the density of Mu. It turns out that considering only a ball and psh functions is not essential:

THEOREM 5.1. *Let Ω be a domain in $\mathbb{C}^n$ contained in a ball of radius R. Assume that the Monge-Ampère operator is (p, q)-stable. Then*

i) *For u, $v \in \mathrm{PSH}(\Omega) \cap C(\bar{\Omega})$ such that $u = v$ on $\partial\Omega$ and Mu, $Mv \in C(\bar{\Omega})$ we have*

$$\|u - v\|_{L^p(\Omega)} \leq C_R\|Mu - Mv\|_{L^q(\bar{\Omega})}^{1/n},$$

where

$$C_R = CR^{2\left(\frac{n}{p} + \frac{q-1}{q}\right)}.$$

ii) *For $\varphi \in C^2(\bar{\Omega})$ such that $\varphi|_{\partial\Omega} = 0$ we have*

$$\|\varphi\|_{L^p(\Omega)} \leq 2C_R\|M\varphi\|_{L^q(\bar{\Omega})}^{1/n}.$$

PROOF. We may assume that $R = 1$ and $\Omega \subset B$ (the exponent at R one can get using the linear transformation $z \mapsto Rz$).

To prove i) choose $F_j \in C(\bar{B})$ such that $F_{j\bar{\Omega}_j} = |Mu - Mv|$ and $F_j \downarrow 0$ on $\bar{B}\backslash\bar{\Omega}$. If we set $u_j := u_B(0, F_j)$ then by the comparison principle $|u-v| \le -u_j$ on Ω. Thus (p,q)-stability gives

$$\|u - v\|_{L^p(\Omega)} \le \|u_j\|_{L^p(B)} \le C\|F_j\|^{1/n}_{L^q(B)} \longrightarrow C_R\|Mu - Mv\|^{1/n}_{L^q(\bar{\Omega})}.$$

To show ii) let Γ_φ be as in Theorem 3.9. As before we can find G_j^+, $G_j^- \in C(\bar{B})$ such that $G_j^\pm = M(\pm\varphi)$ in $\bar{\Gamma}_{\pm\varphi}$ and $G_j^\pm \downarrow 0$ on $\bar{B}\backslash\bar{\Gamma}_{\pm\varphi}$. Now, if $u_j^\pm := u_B(0, G_j^\pm)$, then Theorem 3.9 gives $u_j^+ \le \varphi \le -u_j^-$, Eventually

$$\|\varphi\|_{L^p(\Omega)} \le \|u_j^+\|_{L^p(B)} + \|u_j^-\|_{L^p(B)} \le C\left(\|G_j^+\|^{1/n}_{L^q(B)} + \|G_j^-\|^{1/n}_{L^q(B)}\right)$$
$$\longrightarrow C\left(\|M_\varphi\|^{1/n}_{L^q(\bar{\Gamma}_\varphi)} + \|M\varphi\|^{1/n}_{L^q(\bar{\Gamma}_{-\varphi})}\right) \le 2C\|M\varphi\|^{1/n}_{L^q(\bar{\Omega})}. \qquad \square$$

On the other hand the next theorem shows that it is enough to verify the stability only for smooth functions.

THEOREM 5.2. *Assume that for some p, q, C and $\tilde{C}$ we the following estimate holds:*

(5.2) $\|u\|_{L^p(B)} \le \tilde{C}\|u\|_{\partial B} + C\|Mu\|^{1/n}_{L^q(B)}, \quad u \in \mathrm{PSH}(B) \cap C^\infty(\bar{B}), \quad u \le 0.$

Then the Monge-Ampère operator is (p,q)-stable.

PROOF. Since the estimate (5.2) does not hold for $q < 1$ and the Monge-Ampère operator is (p,∞)-stable for every $p > 0$, we may assume that $1 \le q < \infty$.

To show (5.1) take first $F \ge 0$ with $F^{1/n}$ smooth on $\bar{B}$. By Theorem 4.2 the function $u := u_B(0,F)$ is $C^{1,1}$ in B. Fix $\varepsilon > 0$ and let $u_\delta := u * \rho_\delta \in \mathrm{PSH} \cap C^\infty(B_\delta)$ ($B_\delta = B(0, 1-\delta)$) be a regularization of u. Let $\tilde{\delta} > 0$ be such that $\|u\|_{B\backslash B_{\tilde{\delta}}} \le \varepsilon$. Then

(5.3) $$\|u\|^p_{L^p(B)} \le \|u\|^p_{L^p(B_{\tilde{\delta}})} + \varepsilon^p\lambda(B) = \lim_{\delta\to 0}\|u_\delta\|^p_{L^p(B_{\tilde{\delta}})} + \varepsilon^p\lambda(B).$$

By (5.2) and Proposition 3.5

(5.4) $$\lim_{\delta\to 0}\|u_\delta\|_{L^p(B_{\tilde{\delta}})} \le \lim_{\delta\to 0}\left(\tilde{C}\varepsilon + C\|Mu_\delta\|^{1/n}_{L^q(B_{\tilde{\delta}})}\right) = \tilde{C}\varepsilon + C\|F\|^{1/n}_{L^q(B_{\tilde{\delta}})}.$$

Combining (5.3) and (5.4) one obtains

$$\|u\|^p_{L^p(B)} \le \left(\tilde{C}\varepsilon + C\|Mu\|^{1/n}_{L^q(B)}\right)^p + \varepsilon^p\lambda(B),$$

which implies (5.1) for $F^{1/n}$ smooth.

Now let F be arbitrary and take F_j decreasing to F with $F_j^{1/n}$ smooth on $\bar{B}$. By the comparison principle we have

$$u + \|F - F_j\|^{1/n}_{\bar{B}}(|z|^2 - 1) \le u_j \le u_{j+1} \le u, \quad z \in B,$$

where $u = u_B(0,F)$, $u_j = u_B(0,F_j)$; hence $u_j \uparrow u$. In particular $\|u_j\|_{L^p(B)} \longrightarrow \|u\|_{L^p(B)}$ and $\|F_j\|_{L^q(B)} \longrightarrow \|F\|_{L^q(B)}$ which completes the proof. $\square$

Now we prove a result of Cegrell and Persson which will be used in the next section:

THEOREM 5.3. ([CP]). *The complex Monge-Ampère operator is* $(\infty, 2)$*-stable.*

The proof will rely on two lemmas:

LEMMA 5.4. (([GT]), Lemma 9.2). *Let* B *be the unit ball in* $\mathbb{R}^m$. *Then for* $u \in C^2(B) \cap C(\bar{B})$ *one has*

$$-\inf_{\bar{B}} u \leq -\inf_{\partial B} u + \frac{2}{\lambda(B)^{1/m}} \left(\int_{\{D^2 u \geq 0\}} \det D^2 u \, d\lambda \right)^{1/m},$$

where $D^2 u = (\partial^2 u / \partial x_j \partial x_k)_{j,k=1,\dots,m}$. □

LEMMA 5.5. *Suppose that* u *is twice differentiable at* $z_0 \in \mathbb{C}^n$ *and the matrix*

$$\left(\frac{\partial^2 u}{\partial x_j \partial x_k}(z_0) \right)_{j,k=1,\dots,2n}$$

is nonnegative (*we use the notation* $z_j = x_j + i x_{n+j}$, $j = 1, \dots, n$). *Then*

$$\det \left(\frac{\partial^2 u}{\partial x_j \partial x_k}(z_0) \right) \leq 4^n \left(\det \left(\frac{\partial^2 u}{\partial z_p \partial \bar{z}_q}(z_0) \right) \right)^2.$$

PROOF. Denote

$$X := \left(\frac{\partial^2 u}{\partial x_j \partial x_k}(z_0) \right)_{j,k=1,\dots,2n}, \quad B := \left(\frac{\partial^2 u}{\partial z_p \partial \bar{z}_q}(z_0) \right)_{p,q=1,\dots,n}.$$

One can easily compute that, with notations from Lemmas 2.3 and 2.4, for $A \in \mathcal{A}$ one has $4\,\mathrm{tr}(AB) = \mathrm{tr}(\tilde{A}^T X)$. Then from Lemmas 2.3 and 2.4 we get

$$(\det B)^{1/n} = \frac{1}{4n} \inf\{\mathrm{tr}(\tilde{A}^T X) : A \in \mathcal{A}\} \geq \frac{1}{2} (\det X)^{1/2n}. \quad \square$$

PROOF OF THEOREM 5.3. Take $u \in PSH(B) \cap C^\infty(\bar{B})$, $u \leq 0$. By Lemmas 5.4 and 5.5 we have

$$\|u\|_B \leq \|u\|_{\partial B} + C \left(\int_{\{D^2 u \geq 0\}} (Mu)^2 d\lambda \right)^{1/2n}.$$

$(\infty, 2)$-stability follows from Theorem 5.2. □

REMARK. Kołodziej [Koł] has recently shown that the complex Monge-Ampère is (p, q)-stable for all pairs (p, q) except the ones from the example at the beginning of this section. His methods however are more complicated than the ones used above.

6. – The Monge-Ampère operator in hyperconvex domains

Using the $(\infty, 2)$-stability we can generalize Theorem 4.1 to the class of hyperconvex domains:

THEOREM 6.1. ([Bło2]). *Let $\Omega \Subset \mathbb{C}^n$ be a hyperconvex domain. Suppose that $f \in C(\partial\Omega)$ can be continuously extended to a psh function on Ω, that is there exists $v_0 \in \mathrm{PSH}(\Omega) \cap C(\bar{\Omega})$ such that $v_0|_{\partial\Omega} = f$. Let F be as in Theorem 4.1. Then there exists a solution $u = u_\Omega(f, F)$ of* (4.1).

PROOF. First suppose that $F \equiv 0$ and let u be as in Theorem 1.5. We can find h, harmonic on Ω, continuous on $\bar{\Omega}$ and such that $h|_{\partial\Omega} = f$. Then $v_0 \le u \le h$ and, by Theorem 1.5, $u \in \mathrm{PSH}(\Omega) \cap C(\bar{\Omega})$. If $\tilde{B} = \tilde{B}(z_0, r) \Subset \Omega$ then we can easily show that $u = u_{\tilde{B}}(u|_{\partial\tilde{B}}, 0)$ on $\tilde{B}$, hence $Mu = 0$. We may therefore assume in addition that $Mv_0 = 0$.

Next suppose that F has compact support in Ω. Let ψ be as in Theorem 1.6. Then $M(v_0 + A\psi) \ge F$ for A big enough (because F has compact support). Let $\Omega_j \uparrow \Omega$ be B-regular domains. Theorem 4.1 provides solutions $u_j := u_{\Omega_j}(v_0|_{\partial\Omega_j}, F|_{\Omega_j})$. By the comparison principle

$$v_0 + A\psi \le u_{j+1} \le u_j \le v_0 \quad \text{on} \quad \Omega_j \,. \tag{6.1}$$

We want to show that the sequence $\{u_j\}$ is locally uniformly convergent on Ω. Take $K \Subset \Omega$, $\varepsilon > 0$ and let k_0 be such that $K \subset \Omega_{k_0}$ and $|A\psi| \le \varepsilon$ on $\partial\Omega_{k_0}$. Then by (6.1) and the comparison principle for $j, k \ge k_0$ one has

$$\|u_j - u_k\|_K \le \|u_j - u_k\|_{\Omega_{k_0}} = \|u_j - u_k\|_{\partial\Omega_{k_0}} \le \|A\psi\|_{\partial\Omega_{k_0}} \le \varepsilon \,,$$

thus the sequence $\{u_j\}$ is locally uniformly convergent on Ω. We can now easily show that $u := \lim u_j$ is the desired solution.

Let now F be arbitrary. We can find $F_j \in C_0^\infty(\Omega)$, $F_j \ge 0$, such the $F_j \longrightarrow F$ in $\mathrm{L}^2(\Omega)$. It is enough to show that the solutions $u_j := u_\Omega(f, F_j)$ are uniformly convergent on $\bar{\Omega}$. We may assume that $\Omega \subset B$. Then by the comparison principle we have on Ω

$$|u_j - u_k| \le -u_\Omega(0, |F_j - F_k|) \le -u_B(0, |F_j - F_k|) \,,$$

hence by Theorem 5.3

$$\|u_j - u_k\|_{\bar{\Omega}} \le \|u_B(0, |F_j - F_k|)\|_B \le C\|F_j - F_k\|_{\mathrm{L}^2(\Omega)}^{1/n} \,.$$

Therefore the sequence $\{u_j\}$ is uniformly convergent on $\bar{\Omega}$ and this completes the proof. □

REMARK. In the proof of Theorem 6.1 it is not necessary to use the full strength of Theorem 1.6. For if $K \Subset \Omega$ then without appealing to Theorem 1.1 we can easily construct an exhaustion function which is smooth and strictly psh only on a neighborhood of K, and this is sufficient in the proof of Theorem 6.1.

Moreover, we do not need to use stability of the Monge-Ampère operator if we know that Ω admits an exhausting ψ with $M\psi \ge F$. This is so in the case of a bidisc:

EXAMPLE. The bidisc Δ^2 in $\mathbb{C}^2$ is hyperconvex but not B-regular. For $\varepsilon \in (0, 1]$ set

$$\psi(z) := -(1-|z_1|^2)^\varepsilon(1-|z_2|^2)^\varepsilon, \qquad z = (z_1, z_2) \in \bar{\Delta}^2.$$

Then $\psi|_{\partial\Delta^2} = 0$, ψ is smooth, separately subharmonic on Δ^2 and

$$M\psi(z) = \varepsilon^2(1-|z_1|^2)^{2\varepsilon-2}(1-|z_2|^2)^{2\varepsilon-2}(1-\varepsilon(|z_1|^2+|z_2|^2)),$$

thus ψ is psh on Δ^2 if $\varepsilon \le 1/2$.

This shows (without appealing to the results of Sections 1 and 5) that for $\Omega = \Delta^2$ the problem (4.1) has a solution if f is as in Theorem 6.1 and $F \in C(\Delta^2)$ is such that

$$0 \le F(z) \le \frac{C}{(1-|z_1|^2)^\beta(1-|z_2|^2)^\beta}, \quad z \in \Delta^2,$$

for some $C \ge 0$ and $\beta < 2$. This is precisely a result of Levenberg and Okada ([LO], Theorem 3.1) who use however much more complicated probabilistic methods in their proof.

We also want to prove that a function $f \in C(\partial\Delta^2)$ satisfies the hypothesis of Theorem 6.1 if and only if it is separately subharmonic on $\partial\Delta^2$, that is

(6.2) for every $\zeta \in \partial\Delta$ $f(\zeta, \cdot)$ and $f(\cdot, \zeta)$ are subharmonic in Δ.

The part "only if" is obvious. To show "if" put

$$u := \sup\{v \in \mathrm{PSH}(\Delta^2) : v^*|_{\partial\Delta^2} \le f\}.$$

From the regularity of Δ^2 it follows that $u^* \le f$ on $\partial\Delta^2$. Thus, by Theorem 1.5, it is enough to show that $u_* \ge f$ on $\partial\Delta^2$. We may assume that $f \in C(\bar{\Delta}^2)$ and f satisfies (6.2). Fix $\zeta \in \partial\Delta$ and $\varepsilon > 0$. We can find a barrier $\varphi \in C(\bar{\Delta})$, subharmonic in Δ and such that $\varphi(\zeta) = 0$, $\varphi|_{\bar{\Delta}\setminus\{\zeta\}} < 0$. We claim that for A big enough the function

$$v(z_1, z_2) := f(\zeta, z_2) + A\varphi(z_1) - \varepsilon, \quad (z_1, z_2) \in \bar{\Delta}^2,$$

satisfies $v \le f$ on $\bar{\Delta}^2$. From the uniform continuity of f one can get $r > 0$ such that $|f(z_1, z_2) - f(\zeta, z_2)| \le \varepsilon$ if $|z_1 - \zeta| < r$. Now, taking A so large that

$$A \sup_{|z_1-\zeta|\ge r} \varphi(z_1) \le \inf_{|z_1-\zeta|\ge r, z_2\in\bar{\Delta}} (f(\zeta, z_2) - f(z_1, z_2) - \varepsilon),$$

we have $v \le f$. Therefore $u_*(\zeta, z_2) \ge f(\zeta, z_2) - \varepsilon$ and $u_* \ge f$ on $\partial\Delta \times \bar{\Delta}$. Of course, in the same way one shows the desired inequality also on $\bar{\Delta} \times \partial\Delta$ which completes the proof of the contention.

Finally, we want to show that in hyperconvex domains the problem (4.1) has smooth subsolutions:

THEOREM 6.2. *Let Ω, F and f be as in Theorem 6.1. Then one can find $u \in \mathrm{PSH} \cap C^\infty(\Omega) \cap C(\bar{\Omega})$ such that $u|_{\partial\Omega} = f$ and $Mu \ge F$.*

If $F \equiv 1$ and $f \equiv 0$, we get a smooth exhaustion function u with $Mu \ge 1$. Theorem 6.1 strengthens therefore Theorem 1.6.

To prove Theorem 6.1 we shall define a suitable Richberg sheaf:

DEFINITION. *If Ω is open in $\mathbb{C}^n$ then by $\mathcal{F}(\Omega)$ denote the set of all continuous strictly psh functions u on Ω with $Mu > 1$ (that is for $\Omega' \Subset \Omega$ there exists $a > 1$ such that $Mu \geq a$ in Ω').*

Observe that if u is psh and $C^{1,1}$ then $Mu > 0$ implies that u is strictly psh. However, it is not so for arbitrary psh u as the following example shows:

EXAMPLE. For $z = (z_1, z_2) \in \mathbb{C}^2$ set $u(z) := (1 + |z_1|^2)|z_2|$. Then u is separately subharmonic in $\mathbb{C}^2$, smooth on $\{z_2 \neq 0\}$ and $Mu = 1/4$ there. This implies that $u \in PSH \cap C(\mathbb{C}^2)$ and $Mu \geq 1/4$ in $\mathbb{C}^2$. (In fact one can show that $\int_{\{z_2=0\}} Mu = 0$ and thus $Mu = 1/4$ everywhere). To see that u is not strictly psh it is enough to observe that $\partial^2 u/\partial z_1 \partial \bar{z}_1 = |z_2|$.

PROPOSITION 6.3. *$\mathcal{F}$ is a Richberg sheaf.*

PROOF. The definition is local, thus $\mathcal{F}$ is a sheaf. To show that $\mathcal{F}$ satisfies (1.1) take $u \in \mathcal{F}(\Omega)$, $\varphi \in C_0^\infty(\Omega)$ and $\Omega' \Subset \Omega$. We may assume that $\operatorname{supp}\varphi \subset \Omega'$ and $Mu \geq a > 1$ in Ω'. If $a^{-1/n} < \lambda < 1$, then for some $\varepsilon_0 > 0$ we have $(1-\lambda)u + \varepsilon\varphi \in PSH(\Omega)$ for $\varepsilon \in [0, \varepsilon_0]$ and $M(u + \varepsilon\varphi) \geq \lambda^n Mu \geq \lambda^n a > 1$, which implies (1.1). (1.2) follows directly from Theorem 3.8.

Finally, let Ω', Ω, θ and u be as in (1.3). If D is as in the proof of Proposition 1.2, in the same way as there we can get uniform convergence of the partial derivatives $\partial^2 u_{\delta\theta}/\partial z_j \partial \bar{z}_k \longrightarrow \partial^2 u/\partial z_j \partial \bar{z}_k$ as $\delta \downarrow 0$ on D, whereas on a neighborhood of $\Omega' \backslash D$ we have $u_{\delta\theta} = u_\delta$ for δ small enough. Now it is enough to apply Theorem 3.10 with $F \equiv 1$ to see that $\mathcal{F}$ satisfies (1.3). □

PROOF OF THEOREM 6.2. First assume that $F \equiv 1$. Then, in view of Proposition 6.3 and Theorem 1.3, it suffices to find $u \in \mathcal{F}(\Omega) \cap C(\bar{\Omega})$ with $u|_{\partial\Omega} = f$. Let $v := u_\Omega(f, 1)$ be given by Theorem 6.1 and ψ by Theorem 1.6. Put $u := v + \psi$. Then u is strictly psh and $Mu \geq Mv + M\psi > 1$. Thus, Theorem 6.2 is proved for $F \equiv 1$.

Let now F be arbitrary. By the previous part we can find $u_1, u_2 \in \mathcal{F} \cap C^\infty(\Omega) \cap C(\bar{\Omega})$ with $u_1|_{\partial\Omega} = 0$ and $u_2|_{\partial\Omega} = f$. Then $M(cu_1 + u_2) \geq c^n \geq F$ for c big enough, which completes the proof. □

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