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1. – Introduction

We shall be concerned with the large time behaviour of solutions to the following convection-diffusion problems:

$$\begin{cases} u_t - \Delta u + a \cdot \nabla(|u|^{q-1}u) = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}^+, n \geq 1 \\ u(z,0) = u_0(z) & \text{in } \mathbb{R}^n \end{cases}$$

with initial data $u_0 \in L^1(\mathbb{R}^n)$, $a \in \mathbb{R}^n$ and $1 < q < 1 + 1/n$.

Without loss of generality we may consider the equation in the form

(CD) $\qquad u_t - \Delta u + \partial_y(|u|^{q-1}u) = 0 \quad \text{in } \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^+$

with $u = u(x, y, t)$, $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$. Here Δ stands for the Laplacian in all the spatial variables x, y: $\Delta = \Delta_x + \partial^2/\partial^2 y$. It is known that for any $u_0 = u_0(x, y) \in L^1(\mathbb{R}^n)$ there exists a unique solution u of (CD) taking u_0 as initial datum. Moreover,

$$u \in C([0, \infty), L^1(\mathbb{R}^n)) \cap C((0, \infty), W^{2,p}(\mathbb{R}^n)) \text{ for every } p \in (1, \infty).$$

The case $n = 1$ was dealt with in [EVZ1]. In this case (CD) takes the form:

(CD)$_1$ $\qquad u_t - u_{yy} + \partial_y(|u|^{q-1}u) = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^+$

They proved that the large time behaviour of a nonnegative solution of the convection-diffusion problem (CD)$_1$ taking any integrable initial datum $u_0 \geq 0$ is given by the fundamental entropy solution of the scalar hyperbolic conservation law

(HCL) $\qquad u_t + (|u|^{q-1}u)_y = 0$

with initial datum $(\int_{\mathbb{R}} u_0 \, dy)\delta$, where δ denotes the Dirac mass concentrated at $y = 0$. An analogous result in dimension $n \geq 2$ was established in [EVZ2] for solutions of (CD) with nonnegative integrable data u_0. It turns out that the large time behaviour of such a solution is given by the nonnegative fundamental entropy solution of the "reduced equation":

$$\text{(R)} \qquad u_t - \Delta_x u + \partial_y(|u|^{q-1}u) = 0 \quad \text{in} \quad \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^+$$

with initial datum $(\int_{\mathbb{R}^n} u_0 \, dx \, dy)\delta$ where δ denotes the Dirac mass concentrated at $(x, y) = (0, 0)$. Here Δ_x stands for the Laplacian in the variable x. We remark that in both cases the diffusion in the y-direction disappears so that an entropy condition is needed in order to ensure uniqueness.

Our goal is to extend these results to general solutions without restrictions on the sign. Let us first sketch the proof in [EVZ2] (the idea in [EVZ1] is the same) in order to see where the assumption on the sign of the solution is needed.

For $\lambda > 0$ we introduce the functions

$$u_\lambda(x, y, t) = \lambda^\alpha u(\lambda^{1/2}x, \lambda^\beta y, \lambda t), \quad \alpha = \frac{n+1}{2q}, \quad \beta = \frac{n+1+q-nq}{2q}$$

which solve:

$$\text{(CD}_\lambda) \quad u_{\lambda,t} - \Delta_x u_\lambda - \lambda^{1-2\beta}\partial_{yy}^2 u_\lambda + \partial_y(|u_\lambda|^{q-1}u_\lambda) = 0 \quad \text{in} \quad \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^+$$

with initial data:
$$u_\lambda(x, y, 0) = \lambda^\alpha u_0(\lambda^{1/2}x, \lambda^\beta y).$$

These initial data converge to $(\int_{\mathbb{R}^n} u_0 \, dx \, dy)\delta$ as $\lambda \to \infty$ in the narrow sense of measures. Since $1 - 2\beta$ is negative, one expects u_λ to converge to an entropy solution of

$$v_t - \Delta_x v + \partial_y(|v|^{q-1}v) = 0 \quad \text{in} \quad \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^+$$

with initial datum:

$$v(x, y, 0) = \left(\int_{\mathbb{R}^n} u_0 \, dx \, dy \right)\delta.$$

In order to prove this, some estimates are needed. Thanks to the "entropy" estimate satisfied by nonnegative solution of (CD$_\lambda$):

$$\text{(E)} \qquad\qquad\qquad \partial_y(u_\lambda^{q-1}) \leq \frac{1}{qt}$$

one can get uniform estimates for the L^p norms in space of u_λ and some uniform estimates for the derivatives of u_λ giving compactness. To establish the entropy estimate (E), the restriction on the sign is needed. This restriction

is also needed to ensure the uniqueness of the limiting function v. Once this is done, we know that the whole family u_λ converges to v, which turns out to be invariant under our scaling transformation. Going back to the initial variables the result on the asymptotic behaviour of u follows ([EVZ2]).

If we do not assume u to be nonnegative (or nonpositive, since the non-linearity is odd) we cannot prove the entropy estimate (E) for u_λ so that we lack estimates on the derivatives and another way to get compactness must be found. On the other hand, we must prove the uniqueness of the limit v without assuming it to have constant sign. The main contributions of this paper are that we are able to overcome these two difficulties.

We shall follow the same general scheme as in [EVZ2] to study the large time behaviour of a general solution u of (CD) taking any integrable datum. A comparison argument allows to get uniform estimates on the L^p norms of the u_λ. Once this is done, even if we lack estimates on the derivatives, we can get compactness thanks to a variant of the compactness results obtained in [LPT] by means of kinetic formulations. Concerning the uniqueness of v, it suffices to prove that if v is a fundamental entropy solution of (R) with mass $M \geq 0$, then v is nonnegative, so that the uniqueness result in [EVZ2] applies.

More precisely, we prove the following:

THEOREM 1. *Let us assume that either $n \geq 3$ and $q \in (1, 1 + 2/(n - 1))$ or $n = 1, 2$ and $q \in (1, 2]$. Then, for any real valued constant M:*
    a) *There exists at most one solution*

$$u = u(x, y, t) \in C((0, \infty); L^1(\mathbb{R}^n)) \cap L^\infty(0, \infty; L^1(\mathbb{R}^n))$$

of

(CD) $$u_t - \Delta u + \partial_y(|u|^{q-1}u) = 0 \quad in \quad \mathbb{R}^n \times \mathbb{R}^+$$

such that $u \in L^\infty_{loc}(0, \infty; L^\infty(\mathbb{R}^n))$ and

$$\lim_{t \to 0} \int_{\mathbb{R}^n} u(x, y, t)\varphi(x, y)dy\, dx = M\varphi(0, 0) \quad \forall \varphi \in BC(\mathbb{R}^n)$$

(H) $$\int_{\mathbb{R}^{n-1}} dx \int_{|y|>r} dy|u|(x, y, t) \to 0 \quad t \to 0, \quad for\ any\ fixed\ r > 0.$$

    b) *There exists at most one entropy solution*

$$u = u(x, y, t) \in C((0, \infty); L^1(\mathbb{R}^n)) \cap L^\infty(0, \infty; L^1(\mathbb{R}^n))$$

of

(R) $$u_t - \Delta_x u + \partial_y(|u|^{q-1}u) = 0 \quad in \quad \mathbb{R}^n \times \mathbb{R}^+$$

such that $u \in L^\infty_{\text{loc}}(0, \infty; L^\infty(\mathbb{R}^n))$ and

$$\lim_{t \to 0} \int_{\mathbb{R}^n} u(x, y, t)\varphi(x, y)dy\, dx = M\varphi(0, 0) \quad \forall \varphi \in BC(\mathbb{R}^n)$$

(H) $$\int_{\mathbb{R}^{n-1}} dx \int_{|y|>r} dy |u|(x, y, t) \to 0 \quad t \to 0, \quad \text{for any fixed } r > 0.$$

REMARKS. – We denote by $BC(\mathbb{R}^n)$ the space of bounded and continuous functions in $\mathbb{R}^n$.

– When $n = 1$ ([LP]) the fundamental solution with mass M of the reduced problem (R) is known to exist and to be unique for any $q > 1$ and any real number M.

– When $n \geq 1$, the nonnegative (or nonpositive) fundamental solutions with mass M of both the convection-diffusion problem (CD) and the reduced problem (R) are known to be unique for $q > 1$ ([EVZ2]).

– In case b), no fundamental solution of (R) exists when $n \geq 2$ and $q \geq 1 + 2/n - 1$. They do exist when $q \in (1, 1 + 2/n - 1)$ for $n \geq 3$ and when $q \in (1, 2]$ for $n = 2$. The case $n = 2$ and $q \in (2, 3)$ remains open. In case a), fundamental solution of (CD) are known to exist in the same ranges of q ([EVZ2]).

– In case b) diffusion is lacking in one direction. Therefore, we must deal with weak solutions and introduce an adequate entropy condition, as it is usually done for hyperbolic systems. The entropy condition reads (see [EVZ3] for more details):

$$\frac{\partial}{\partial t} |u - \psi(x)| - \Delta_x |u - \psi(x)|$$

$$\leq -\frac{\partial}{\partial y}(\|u|^{q-1}u - |\psi|^{q-1}\psi|) + \text{sign}(u - \psi(x))\Delta_x \psi(x)$$

in the sense of distributions, for any test function $\psi(x) \in C^\infty_c(\mathbb{R}^{n-1})$.

Theorem 1 allows us to extend the results on the asymptotic behaviour obtained in [EVZ2] for nonnegative data to solutions taking any integrable initial data:

THEOREM 2. *Let u be a solution of*

(CD) $$\begin{cases} u_t - \Delta u + \partial_y(|u|^{q-1}u) = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}^+ \\ u(x, y, 0) = u_0(x, y) & \text{in } \mathbb{R}^n \end{cases}$$

with initial datum $u_0 \in L^1(\mathbb{R}^n)$, $n \geq 1$ and $1 < q < 1 + 1/n$. Let M denote the mass of the initial datum u_0, i.e., $\int_{\mathbb{R}^n} u_0(x, y)dx\, dy$. Then, for any $p \in [1, \infty)$,

$$\lim_{t \to \infty} t^{\frac{n+1}{2q} \frac{p-1}{p}} \|u(t) - v(t)\|_p = 0$$

where v is the unique entropy solution of the reduced equation

(R) $$v_t - \Delta_x v + \partial_y(|v|^{q-1}v) = 0 \quad in \quad \mathbb{R}^n \times \mathbb{R}^+$$

taking the initial datum $M\delta$ in the narrow sense of measures, that is,

$$\lim_{t \to 0} \int_{\mathbb{R}^n} v(x, y, t)\varphi(x, y)dy\,dx = M\varphi(0, 0) \quad \forall \varphi \in BC(\mathbb{R}^n),$$

which is known to be selfsimilar, i.e., invariant under the scaling transformation:

$$v_\lambda(x, y, t) = \lambda^\alpha v(\lambda^{1/2}x, \lambda^\beta y, \lambda t), \quad \alpha = \frac{n+1}{2q}, \quad \beta = \frac{n+1+q-nq}{2q}.$$

The solution v has then the particular structure

$$v(x, y, t) = t^{-\alpha} f(xt^{-1/2}, yt^{-\beta})$$

with

$$\alpha = \frac{n+1}{2q}, \qquad \beta = \frac{n+1+q-nq}{2q}$$

where the profile f is the unique entropy solution of:

$$\Delta_x f + \frac{1}{2}x\nabla f + \beta y \partial_y f + \alpha f - \partial_y(|f|^{q-1}f) = 0 \quad in \quad \mathbb{R}^n$$

with mass M. Moreover, v is supported in a region of the form:

$$\{(x, y, t) \; s.t. \; 0 \le y \le Ct^\beta\}$$

for some positive constant C.

REMARK. We see that the fundamental solutions of the reduced equation have compact support in the direction where diffusion is lacking.

In Sections 2 and 3 below we give the proofs of both theorems.

2. – Uniqueness

In this section we prove Theorem 1.

PROOF OF a). Before giving the proof, we shall sketch the main steps. We can assume that $M \geq 0$. Otherwise, $-u$ will be a solution with mass $-M \geq 0$ and the same proof applies.

Step 1: We prove that the positive and negative parts of u are subsolutions of (CD).

Step 2: Given any sequence of times (t_n) such that $t_n \to 0$ as $n \to \infty$, there exist a subsequence $(t_j) \to 0$ and finite nonnegative measures μ, ν such that $M\delta = \mu - \nu$ and

$$u^+(t_j) \rightharpoonup \mu, \quad u^-(t_j) \rightharpoonup \nu$$

in the narrow sense of measures (any test function in $BC(\mathbb{R}^n)$ is allowed) when $t_j \to 0$. Both μ and ν are supported at 0, hence they must be Dirac measures.

Step 3: Let m_j be a sequence of nonnegative functions in $C_c^\infty(\mathbb{R}^n)$ converging narrowly to $M\delta$. We denote by h_j and g_j the solutions of (CD) taking initial data $u^+(t_j)$ and $u^-(t_j) + m_j$, respectively. Then,

$$0 \leq u_j^+(t) \leq h_j(t), \quad 0 \leq u_j^-(t) \leq g_j(t)$$

where $u_j^+(t) = u^+(t_j + t)$ and $u_j^-(t) = u^-(t_j + t)$.

Step 4: We prove the convergence of h_j and g_j to some positive solutions h and g (respectively) of (CD) with initial datum $\mu \geq 0$.

Step 5: The problem

$$v_t - \Delta v + \partial_y(|v|^{q-1}v) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}^+$$

$$v(x, y, 0) = \mu \quad \text{in the narrow sense of measures}$$

has a unique nonnegative solution.

Step 6: $\nu = 0$ and $u^- = 0$ therefore, the uniqueness result for nonnegative solutions in [EVZ2] applies.

Let us detail the proofs.

Step 1: u^+ and u^- are subsolutions of (CD). Let U be a convex smooth function. Multiplying the equation by $U'(u)$ we get:

$$U'(u)u_t - U'(u)\Delta u + U'(u)q|u|^{q-1}\partial_y u = 0.$$

In the view of the smoothness of u, this yields

$$(U(u))_t + \left(\int_0^u U'(s)q|s|^{q-1}ds \right)_y = U'(u)\Delta u .$$

By convexity:

$$U'(u)\Delta u = \Delta U(u) - U''(u)|\nabla u|^2 \leq \Delta U(u)$$

so that:

$$(U(u))_t + \left(\int_0^u U'(s)q|s|^{q-1}ds \right)_y \leq \Delta U(u) .$$

Applying this inequality to families of smooth convex functions $U_\varepsilon^-(s)$ and $U_\varepsilon^+(s)$ converging to s^- and s^+, respectively, when $\varepsilon \to 0$ and taking into account the oddness of the nonlinear term, it follows that:

$$u_t^+ - \Delta u^+ + \partial_y(|u^+|^{q-1}u^+) \leq 0$$

$$u_t^- - \Delta u^- + \partial_y(|u^-|^{q-1}u^-) \leq 0$$

in $\mathcal{D}'(\mathbb{R}^n \times (0, \infty))$.

Step 2: Initial data. Since $u(t)$ is compact for the narrow convergence as $t \to 0$, the positive and negative parts $u^+(t)$ and $u^-(t)$ also are (see [LP] and [V]). Therefore, we may affirm that for any sequence $(t_n) \to 0$, there exist a subsequence $(t_j) \to 0$ and two finite nonnegative measures μ, ν such that $M\delta = \mu - \nu$ and

$$u^+(t_j) \to \mu, \quad u^-(t_j) \to \nu$$

in the narrow sense of measures when $t_j \to 0$. On the other hand, the condition (H) satisfied by u

$$\int_{|x|+|y|>r} dx\, dy|u|(x, y, t) \to 0 \; t \to 0, \quad \text{for any fixed } r > 0$$

implies that μ and ν must be supported at 0. Thus, they must be Dirac measures.

Step 3: Maximum principle. We need the following:

LEMMA 2.1. *Let V be a solution of* (CD) *with initial datum* $V(0) \in (L^1 \cap L^\infty)(\mathbb{R}^n)$ *and v a subsolution for* (CD) *with initial datum* $v(0) \in (L^1 \cap L^\infty)(\mathbb{R}^n)$ *such that $v(0) \leq V(0)$. We assume that*

$$v, V \in C^1([0, T]; L^2(\mathbb{R}^n)) \cap L^\infty([0, T]; H^2(\mathbb{R}^n)) \cap L^\infty((0, T) \times \mathbb{R}^n)$$

for every $T > 0$. Then, $v \leq V$.

PROOF. The function $w = v - V$ satisfies

$$w_t - \Delta w + \partial_y(|v|^{q-1}v) - \partial_y(|V|^{q-1}V) \leq 0$$

and $w(0) \leq 0$. Multiplying the inequality by w^+ and integrating by parts, we obtain

$$\frac{d}{dt} \int \frac{|w^+(t)|^2}{2} + \int |\nabla w^+(t)|^2 \leq \int a w^+(s) \partial_y w^+(s) ds$$

where $a(x,t) = \dfrac{|v|^{q-1}v - |V|^{q-1}V}{v - V}$ is a bounded function. Integrating in t we get:

$$\frac{\|w^+(t)\|_2^2}{2} + \int_0^t \|\nabla w^+(s)\|_2^2 \, ds \leq K_1 \int_0^t \|w^+(s)\|_2^2 \, ds + \varepsilon \int_0^t \|\nabla w^+(s)\|_2^2 \, ds.$$

Thus,

$$\|w^+(t)\|_2^2 \leq 2K_2 \int_0^t \|w^+(s)\|_2^2 \, ds$$

so that, by Gronwall's inequality, $w^+(t) = 0$.   □

Let m_j be a sequence of nonnegative functions in $C_c^\infty(\mathbb{R}^n)$ converging narrowly to $M\delta$. We denote by h_j and g_j the solutions of (CD) taking initial data $u^+(t_j)$ and $u^-(t_j) + m_j$, respectively. Applying lemma 2.1 it follows that

$$0 \leq u_j^+(t) \leq h_j(t), \quad 0 \leq u_j^-(t) \leq g_j(t)$$

where $u_j^+(t) = u^+(t_j + t)$ and $u_j^-(t) = u^-(t_j + t)$.

Step 4: Estimates and passage to the limit. The functions g_j and h_j are solutions to (CD) with initial data in $L^1(\mathbb{R}^n)$ (in fact, in $L^p(\mathbb{R}^n)$ for every p). They are also nonnegative, since the initial data are nonnegative.

In this step, we prove that h_j and g_j converge to some nonnegative functions h and g, which solve the equation (CD):

$$v_t - \Delta v + \partial_y(|v|^{q-1}v) = 0 \quad \text{in} \quad D'(\mathbb{R}^n \times \mathbb{R}^+)$$

and take the measure μ as initial datum. We shall do this following the arguments in [EVZ2] (Sections 2 and 6) which do not use the diffusion in the y direction. In this way, this proof can be easily extended to equations where the term $-\partial_{yy}$ does not appear.

The following bounds are known to hold for any nonnegative solution w of (CD) ([EVZ2], Section 2) with integrable initial data, provided that $q \in (1, 2]$ (which is our case):

(E1) $\quad \int w(t) = \int w(0) = M(w)$;

(E2) $\quad 0 \le w(t) \le C(M(w))t^{-\frac{n+1}{2q}}$;

(E3) $\quad \|w(t)\|_p \le C(M(w))t^{-\frac{n+1}{2q}\frac{p-1}{p}}$;

(E4) $\quad \int |\partial_y(w^{q+r})(t)| \le C(M(w))t^{-1-\frac{r(n+1)}{2q}}$;

(E5) $\quad \int |\nabla_x(w^{q+r})(t)| \le C(\tau, M(w))(t-\tau)^{\frac{-1}{2}}t^{-(-q-1+r)\frac{(n+1)}{2q}} \quad t > \tau > 0$.

Applying these estimates to h_j and g_j, we can pass to the limit in the distributional formulation of the equation (CD) satisfied by them, exactly as in [EVZ2], Section 6 (in the scaled equation there, here no λ appears so that no term vanishes).

Let us see how this can be done for h_j, the proof for g_j being similar. Applying estimates (E1), (E2), (E3), (E4), (E5) to h_j and taking into account the fact that the masses $\int h_j(0) = M(h_j)$ remain bounded, we conclude that:

 – (h_j^{q+r}) is uniformly bounded in $L^\infty_{\text{loc}}(0, \infty; W^{1,1}(\mathbb{R}^n))$ for any $r > 0$.
 – Using the equation we get a uniform bound for $h_{j,t}$ in $L^2_{\text{loc}}(0, \infty; H^{-s}(\Omega))$ for some $s > 0$ and every bounded domain Ω of $\mathbb{R}^n$.
 – h_j is uniformly bounded $L^2_{\text{loc}}(0, \infty; L^2_{\text{loc}}(\mathbb{R}^n))$.

Taking into account that $L^2(\Omega)$ is compactly embedded in $H^{-\varepsilon}(\Omega)$ for any $\varepsilon > 0$ and that $H^{-\varepsilon}(\Omega)$ is continuously embedded in $H^{-s}(\Omega)$ for any $s > \varepsilon$, we deduce that u_λ is relatively compact in $C_{\text{loc}}((0, \infty), H^{-\varepsilon}(\Omega))$. Therefore, we may extract a subsequence $j_n \to \infty$ in such a way that:

$$h_{j_n} \to h \quad \text{in} \quad C_{\text{loc}}((0, \infty), H^{-\varepsilon}(\Omega)) \,.$$

Since $h_j(t)$ is relatively compact in $L^p_{\text{loc}}(\mathbb{R}^n)$ for any $1 \le p < \infty$ and for any $t > 0$ we get that $h_{j_n}(t)$ converges to $h(t)$ in $L^p_{\text{loc}}(\mathbb{R}^n)$ for every finite p. Now, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$ and for every $0 < \tau < t$ we get:

$$\iint h_j(x, y, t)\varphi(x, y)dx\,dy - \iint h_j(x, y, \tau)\varphi(x, y)dx\,dy$$

$$= \int_\tau^t \iint h_j(x, y, s)\Delta\varphi(x, y)dx\,dy\,ds$$

$$+ \int_\tau^t \iint |h_j|^{q-1}h_j(x, y, s)\varphi_y(x, y)dx\,dy\,ds \,.$$

Passing to the limit in each term (using weak convergence, except the last one which uses Lebesgue's dominated convergence theorem), it follows that h solves (CD) in the sense of distributions in $\mathbb{R}^n \times \mathbb{R}^+$.

In view of the convergence of the initial data for h_j and g_j to μ, one expects the limit functions h and g to take the initial datum μ in the narrow sense of measures. In order to prove this, we need $t^{-\frac{n+1}{2q}(q-1)} \in L^1(0, T)$, that is, $q < 1 + \dfrac{2}{n-1}$. This implies the upper bound $q \leq 2$ when $n \geq 3$. Indeed, taking test function $\varphi \in C_c^2(\mathbb{R}^n)$ we get:

$$\left| \iint h_j(x, y, t)\varphi(x, y)dx\, dy - \iint h_j(x, y, 0)\varphi(x, y)dx\, dy \right|$$

$$\leq \left| \int_0^t \iiint h_j(x, y, s)\Delta\varphi(x, y)dx\, dy\, ds + \int_0^t \iiint |h_j|^{q-1}h_j(x, y, s)\varphi_y(x, y)dx\, dy\, ds \right|$$

$$\leq \|\Delta\varphi\|_\infty C(M)t + \|\varphi_y\|_\infty C(M)t^{-\frac{n+1}{2q}(q-1)+1}$$

and this upper bound tends to zero as $t \to 0$, uniformly in j. Therefore,

$$\left| \iint h_j(x, y, t)\varphi(x, y)dx\, dy - \langle \mu, \varphi \rangle \right| \to 0$$

as $t \to 0$, uniformly in j. Passing to the limit when $j \to \infty$ we conclude that

$$\left| \iint h(x, y, t)\varphi(x, y)dx\, dy - \langle \mu, \varphi \rangle \right| \to 0$$

as $t \to 0$. The convergence for $\varphi \in BC(\mathbb{R}^n)$ is a consequence of the following "tail control estimate":

$$\int_{|x|+|y|>r} h_j(x, y, t)dx\, dy \to 0 \quad \text{as} \quad r \to \infty$$

uniformly in $t \in [0, t_0]$ and $j \geq 1$. We recall the proof of this kind of estimates in the Appendix (Section 4). This estimate also yields the convergence $h_{j_n}(t) \to h(t)$ in $L^p(\mathbb{R}^n)$ for every finite p. The same argument works for g_j.

Step 5: Uniqueness of positive solutions. The uniqueness of the nonnegative solution of (CD) taking as initial datum a Dirac measure

$$v_t - \Delta v + \partial_y(|v|^{q-1}v) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}^+$$

$$v(x, y, 0) = M\delta \quad \text{in the narrow sense of measures}$$

was proved in [EVZ2]. Both h and g are nonnegative solutions of a problem of this type so that $h = g$.

Step 6: Conclusion. From the previous step we know that $h = g$, so that, in the limit when $t_j \to 0$:

$$u^+(t) + u^-(t) \leq h(t) .$$

Letting $t \to 0$ we conclude that

$$\mu + v \leq \mu .$$

Therefore, $v = 0$ and u^- is a subsolution to (CD) with zero initial datum. The function $v = \int u^- dy$ is then a subsolution of the heat equation with zero datum. This implies $v = 0$ and we conclude that $u^- = 0$, $u = u^+$, $\mu = M\delta$. The uniqueness result in [EVZ2] applies.

PROOF OF b). The proof is an adaptation of the previous one with slight changes.

Step 1: We prove that the positive and negative parts of u are subsolutions of the reduced equation (R).

This follows from the entropy condition (see [LP] and [EVZ3]), but we can also use an approximation procedure which will be useful in other steps. By definition $u \in C((0, \infty); L^1(\mathbb{R}^n))$. Next, we observe that entropy solutions of (R) with data in $L^1(\mathbb{R}^n)$ are obtained as limits of entropy solutions of (R) with smooth initial data. Such solutions are obtained as limits of smooth solutions for the regularized problems

(CD$_\varepsilon$) $v_{\varepsilon,t} - \Delta_x v_\varepsilon - \varepsilon \partial_{yy} v_\varepsilon + \partial_y(|v_\varepsilon|^{q-1} v_\varepsilon) = 0$

when $\varepsilon \to 0$. Since the positive and negative parts of such v_ε are subsolutions of the regularized equations, in the limit we get the result for u^+ and u^-:

$$u_t^+ - \Delta_x u^+ + \partial_y(|u^+|^{q-1} u^+) \leq 0$$

$$u_t^- - \Delta_x u^- + \partial_y(|u^-|^{q-1} u^-) \leq 0$$

in $\mathcal{D}'(\mathbb{R}^n \times (0, \infty))$.

Step 2: Exactly as in case a). Given any $(t_n) \to 0$, there exist a subsequence $(t_j) \to 0$ and finite nonnegative measures μ, v such that $M\delta = \mu - v$ and

$$u^+(t_j) \rightharpoonup \mu, \quad u^-(t_j) \rightharpoonup v$$

in the narrow sense of measures (any test function in $BC(\mathbb{R}^n)$ is allowed) when $t_j \to 0$. Due to (H), the measures μ and v must be supported at 0, hence they must be Dirac measures.

Step 3: Let m_j be a sequence of nonnegative functions in $C_c^\infty(\mathbb{R}^n)$ converging narrowly to $M\delta$. We denote by h_j and g_j the entropy solutions of (R) taking initial data $u^+(t_j)$ and $u^-(t_j) + m_j$, respectively. Then,

$$0 \le u_j^+(t) \le h_j(t), \quad 0 \le u_j^-(t) \le g_j(t)$$

where $u_j^+(t) = u^+(t_j + t)$ and $u_j^-(t) = u^-(t_j + t)$.

To prove this, let us consider the solutions v_ε of the regularized equations (CD_ε) converging to the entropy solution u of (R). We set $v_{\varepsilon,j}^+(t) = v_\varepsilon^+(t_j + t)$ and $v_{\varepsilon,j}^-(t) = v_\varepsilon^-(t_j + t)$. Let $h_{\varepsilon,j}$ and $g_{\varepsilon,j}$ be the solutions of (CD_ε) taking initial data $v_\varepsilon^+(t_j)$ and $v_\varepsilon^-(t_j) + m_j$, respectively. By Lemma 2.1, we know that:

$$0 \le v_{\varepsilon,j}^+(t) \le h_{\varepsilon,j}(t), \quad 0 \le v_{\varepsilon,j}^-(t) \le g_{\varepsilon,j}(t).$$

Letting $\varepsilon \to 0$ we get the result.

Step 4: We prove the convergence of h_j and g_j to some positive entropy solutions h and g (respectively) of (R) with datum $\mu \ge 0$.

Estimates (E1), (E2), (E3), (E4), (E5) in Step 4 of part a) also hold for any nonnegative entropy solution w of (R) taking any integrable initial datum $w(0)$. Applying these estimates to h_j and g_j, we can pass to the limit in the distributional formulation of (R) and in the initial data exactly as in a) Step 4 (the proof there does not use the diffusion in the y-direction). Now, we must also pass to the limit in the entropy conditions satisfied by h_j and g_j. This can be done thanks to the estimates we have.

Step 5: By the results in [EVZ2] the problem

$$v_t - \Delta_x v + \partial_y(|v|^{q-1}v) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}^+$$

$$v(x, y, 0) = \mu \quad \text{in the narrow sense of measures}$$

has a unique nonnegative entropy solution. Therefore, $h = g$.

Step 6: As before, $v = 0$. Therefore u^- is a subsolution of the reduced equation (R) with zero initial datum. The function $v = \int u^- dy$ is then a subsolution of the heat equation (it is obvious for the solutions of the regularized problems, here it follows in the limit) with zero initial datum. This implies $v = 0$ and we conclude that $u^- = 0$, $u = u^+$, $\mu = M\delta$. The uniqueness result in [EVZ2] applies.

REMARK. In this proof, the condition

$$\int_{\mathbb{R}^{n-1}} dx \int_{|y|>r} dy |u|(x, y, t) \to 0 \quad t \to 0, \quad \text{for any fixed} \quad r > 0$$

in the statement of Theorem 1 allows us to reduce the proof of the uniqueness of u to the uniqueness of positive solutions taking as initial data Dirac measures. If we drop this assumption, the initial datum for u^+ may be any finite measure μ. To conclude the proof of either a) or b), in step 6 we should have to prove the uniqueness of positive solutions v of (CD) or (R) taking as initial datum an arbitrary finite measure μ. Following the proof in [EVZ2] we need

$$\int_{\mathbb{R}^{n-1}} dx \int_{|y|>r} dy \, v(x, y, t) \to 0 \quad t \to 0, \quad \text{for any fixed} \quad r > 0.$$

This is immediate when v is a positive solution with initial datum concentrated at zero, but not for an arbitrary finite measure μ. Without that condition the proof in [EVZ2] does not seem to work. The uniqueness of positive solutions of (CD) and (R) taking as initial data arbitrary finite measures is an open question.

3. – Asymptotic behaviour

In this section we prove Theorem 2.

Step 1: Decay estimates.

PROPOSITION 3.1. *Given any solution $u(x, y, t)$ with initial data $u_0 \in L^1(\mathbb{R}^n)$ of*

(CD) $\qquad u_t - \Delta u + \partial_y(|u|^{q-1}u) = 0 \quad in \quad \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^+$

with $1 < q \leq 2$, the following estimates hold for $t > 0$:

(E1) $\qquad \|u(t)\|_1 \leq \|u_0\|_1 , \int_{\mathbb{R}^n} u(x, y, t)dx \, dy = \int_{\mathbb{R}^n} u_0(x, y)dx \, dy ;$

(E2) $\qquad \|u(t)\|_\infty \leq C(\|u_0\|_1)t^{-\frac{n+1}{2q}} ;$

(E3) $\qquad \|u(t)\|_p \leq C(\|u_0\|_1)t^{-\frac{n+1}{2q}\frac{p-1}{p}}$

where C is a constant which depends continuously on $\|u_0\|_1$.

PROOF. These estimates were proved in [EVZ2], Section 2: (E1) for any solution, (E2) and (E3) for nonnegative solutions. To eliminate this restriction on the sign it suffices to remark that

$$|u(x, y, t)| \leq |\overline{u}(x, y, t)|$$

where $\overline{u}$ is the solution of (CD) with initial datum $|u_0|$.

Step 2: Scaling. For $\lambda > 0$ the functions

$$u_\lambda(x, y, t) = \lambda^\alpha u(\lambda^{1/2}x, \lambda^\beta y, \lambda t), \quad \alpha = \frac{n+1}{2q}, \quad \beta = \frac{n+1+q-nq}{2q}$$

satisfy the equations:

(CD$_\lambda$) $\quad u_{\lambda,t} - \Delta_x u_\lambda - \lambda^{1-2\beta}\partial^2_{yy}u_\lambda + \partial_y(|u_\lambda|^{q-1}u_\lambda) = 0 \quad$ in $\quad \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^+$

with initial data:

$$u_\lambda(x, y, 0) = \lambda^\alpha u_0(\lambda^{1/2}x, \lambda^\beta y).$$

These initial data converge to $(\int_{\mathbb{R}^n} u_0 dx\, dy)\delta$ as $\lambda \to \infty$ in the narrow sense of measures.

Step 3: Limit equation. For any $\varphi \in C_c^\infty(\mathbb{R}^n \times (0, \infty))$ the following identity holds:

$$
\begin{aligned}
&\iint u_\lambda(x, y, t)\varphi(x, y, t)dx\, dy - \iint u_\lambda(x, y, \tau)\varphi(x, y, \tau)dx\, dy \\
&= \int_\tau^t \!\!\! \iint u_\lambda(x, y, s)\varphi_t(x, y, s)dx\, dy\, ds \\
\text{(I_λ)} \quad &+ \int_\tau^t \!\!\! \iint |u_\lambda|^{q-1}u_\lambda(x, y, s)\varphi_y(x, y, s)dx\, dy\, ds \\
&+ \int_\tau^t \!\!\! \iint u_\lambda(x, y, s)\Delta_x\varphi(x, y, s)dx\, dy\, ds \\
&+ \lambda^{1-2\beta} \int_\tau^t \!\!\! \iint u_\lambda(x, y, s)\varphi_{y,y}(x, y, s)dx\, dy\, ds.
\end{aligned}
$$

In order to pass to the limit we shall need the following compactness result:

LEMMA 3.1. *Let* $\rho_\varepsilon \in C([0, \infty); L^1(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n \times (0, \infty))$ *be a family of solutions of:*

(CD$_\varepsilon$) $\quad \dfrac{\partial\rho_\varepsilon}{\partial t} + \dfrac{\partial}{\partial y}(A(\rho_\varepsilon)) - \Delta_x\rho_\varepsilon - \varepsilon\dfrac{\partial^2\rho_\varepsilon}{\partial y^2} = 0 \quad$ *in* $\quad \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^+$

with $A(\rho) = |\rho|^{q-1}\rho$, $1 < q < 1 + 1/n$.
   If ρ_ε *is bounded in* $L^\infty(\mathbb{R}^n \times (0, \infty)) \cap L^\infty(0, \infty; L^1(\mathbb{R}^n))$ *uniformly in* ε, *then*

$$\rho_\varepsilon \quad \text{is compact in} \quad L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^+).$$

PROOF. The result is an adaptation of a compactness lemma in [LPT]. In our setting, that lemma yields compactness when:

1) A is $C^{2,\alpha}$ and satisfies the following nondegeneracy condition:

$$\text{(ND1)} \qquad \text{meas} \left\{ |v| \le R \quad \text{such that} \quad \tau + A'(v)\xi_n = 0, \sum_{i=1}^{n-1} \xi_i^2 = 0 \right\} = 0$$

for any $R > 0$ whenever $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n$ with $\tau^2 + |\xi|^2 = 1$, or also

$$\text{(ND2)} \quad \text{meas} \left\{ |v| \le R \quad \text{such that} \quad |\tau + A'(v)\xi_n| \le \delta, \left| \sum_{i=1}^{n-1} \xi_i^2 + \varepsilon \xi_n^2 \right| \le \delta \right\} \to 0$$

as $\delta \to 0$, uniformly in ε, for $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n$ with $\tau^2 + |\xi|^2 = 1$ and for any $R > 0$. In our case this nondegeneracy conditions are satisfied but we deal with $A(\rho) = |\rho|^{q-1}\rho$, $1 < q < 1 + 1/n$ which is not smooth enough at $\rho = 0$.

2) For all convex functions S and for all $\varepsilon > 0$

$$\frac{\partial S(\rho_\varepsilon)}{\partial t} + \frac{\partial}{\partial y}(\eta(\rho_\varepsilon)) - \Delta_x S(\rho_\varepsilon) - \varepsilon \frac{\partial^2 S(\rho_\varepsilon)}{\partial y^2} \le 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^+)$$

where $\eta(t) = \int_0^t S'(s)A'(s)ds$. This condition follows immediatly if for each ε we multiply the equation (CD$_\varepsilon$) by $S'(\rho_\varepsilon)$ for S smooth.

The result in [LPT] is proved by introducing a kinetic formulation of the equation satisfied by the functions ρ_ε. Letting $f_\varepsilon(v, x, y, t) = \chi_{\rho_\varepsilon(x,y,t)}(v)$ where

$$\chi_\alpha(v) = \begin{cases} 1 & \text{if} \quad 0 < v < \alpha \\ -1 & \text{if} \quad \alpha < v < 0 \\ 0 & \text{otherwise} \end{cases}$$

we get for $f_\varepsilon(v, x, y, t)$ the equation:

$$\frac{\partial f_\varepsilon}{\partial t} + A'(v)\frac{\partial}{\partial y}f_\varepsilon - \Delta_x f_\varepsilon - \varepsilon \frac{\partial^2 f_\varepsilon}{\partial y^2} = \frac{\partial m_\varepsilon}{\partial v} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^+$$

for some bounded nonnegative measures $m_\varepsilon(v, x, y, t)$ defined in $\mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^+$.

Then, in order to prove compactness for the family ρ_ε it suffices to prove compactness for the means $\int_\mathbb{R} f_\varepsilon(v, x, y, t)\psi(v)dv$ with $\psi(v) = 1_{[-R_0, R_0]}$. Note that:

$$\int_{-R_0}^{R_0} f_\varepsilon(v, x, y, t)dv = \rho_\varepsilon(x, t).$$

For the sake of completeness, we recall the main ideas in the proof of the compactness result when A is smooth enough.

In view of the bounds we have for ρ_ε it is clear that f_ε is bounded in any $L^p_{\text{loc}}([-R_0, R_0] \times \mathbb{R}^n \times \mathbb{R}^+)$, $p \geq 1$, where R_0 is such that $\|\rho_\varepsilon\|_\infty \leq R_0$.

In order to apply the arguments in [LPT] and [DLM] to get compactness of the means by using condition (ND1) we need to include the term $-\varepsilon \frac{\partial^2 f_\varepsilon}{\partial y^2}$ in the right-hand side by writing it in the form $\frac{\partial m'_\varepsilon}{\partial v}$ for some $m'_\varepsilon(v, x, y, t)$. To this end, we set

$$\frac{\partial m'_\varepsilon}{\partial v} = \varepsilon \frac{\partial^2 f_\varepsilon}{\partial y^2} .$$

Thus, we can rewrite the equation for f_ε as

$$\frac{\partial f_\varepsilon}{\partial t} + A'(v) \frac{\partial}{\partial y}(f_\varepsilon) - \Delta_x f_\varepsilon = \frac{\partial(m_\varepsilon + m'_\varepsilon)}{\partial v} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^+ .$$

By using Sobolev embeddings we can write:

$$m_\varepsilon = (-\Delta_{x,y,t} + I)^{1/2}(-\Delta_v + I)^{r/2} g_\varepsilon$$

for some $r \geq 0$, g_ε being compact in some $L^p(\mathbb{R} \times \mathbb{R}^n \times (0, \infty))$, $1 < p \leq 2$. To see this, we argue as in [LPT]. Since m_ε is a bounded family of non-negative measures on $\mathbb{R} \times \mathbb{R}^n \times (0, \infty)$, we know that m_ε is bounded in $W^{-s,p}(\mathbb{R} \times \mathbb{R}^n \times (0, \infty))$ for $p < \frac{n+2}{n+1}$ and $\frac{n+2}{p'} < s < 1$. From this, we get the above expression for m_ε with $r > 1 + \frac{n+2}{p'}$.

In our case, we must also deal with m'_ε. From the definition of f_ε we get that:

$$\frac{\partial f_\varepsilon}{\partial y} = \delta_0(v - \rho_\varepsilon) p_\varepsilon = \frac{\partial(H_\varepsilon p_\varepsilon)}{\partial v}$$

with

$$|p_\varepsilon(x, y, t)| \leq \left| \frac{\partial \rho_\varepsilon}{\partial y}(x, y, t) \right|$$

and

$$H_\varepsilon = \begin{cases} 1 & \text{if } v > \rho_\varepsilon(x, y, t), \\ -1 & \text{if } v < \rho_\varepsilon(x, y, t). \end{cases}$$

Therefore,

$$m'_\varepsilon = \varepsilon \frac{\partial}{\partial y}(H_\varepsilon p_\varepsilon) = (-\Delta_{x,y,t} + I)^{1/2} g'_\varepsilon$$

with g'_ε compact in $L^2((-R_0, R_0) \times \mathbb{R}^n \times (0, \infty))$ since $\sqrt{\varepsilon} \frac{\partial \rho_\varepsilon}{\partial y}$ is bounded in $L^2(\mathbb{R}^n \times (0, \infty))$.

Now, we split

$$\int_{\mathbb{R}} f_\varepsilon(v, x, y, t)\psi(v)dv$$

$$=\mathcal{F}^{-1}\Bigg(\int_{\mathbb{R}} \hat{f}_\varepsilon(v, k', k_n, \tau)\psi_\delta(v)\varphi\Big(\frac{|\tau + A'(v)k_n|^2 + |k'|^4}{\delta}\Big)dv$$

$$+\int_{\mathbb{R}} \hat{f}_\varepsilon(v, k', k_n, \tau)\psi_\delta(v)(1 - \varphi)\Big(\frac{|\tau + A'(v)k_n|^2 + |k'|^4}{\delta}\Big)dv$$

$$+\int_{\mathbb{R}} \hat{f}_\varepsilon(v, k', k_n, \tau)(\psi - \psi_\delta)(v)dv\Bigg),$$

where $\hat{f}$ stands for the Fourier transform of f in the variables (x, y, t), $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, $\psi_\delta = \psi * \eta_\delta$ is a regularization of ψ and φ is a smooth cutoff function such that $\varphi = 1$ on $[-1, 1]$ and $\varphi = 0$ outside $[-2, 2]$. The Fourier variables corresponding to (x, y, t) are, respectively, (k', k_n, τ) with $k' = (k_1, \ldots, k_{n-1})$. We denote $\sum_{i=1}^{n-1} k_i^2 = |k'|^2$.

The first term is shown to tend to zero in L^p as $\delta \to 0$, uniformly in ε. This is independent of the smoothness of the A, it suffices that the measures of the sets

$$\{|v| \le R_0 + 1, |\tau + A'(v)k_n|^2 + |k'|^4 \le 2\delta\}$$

tend to zero as $\delta \to 0$.

The third term tends also to zero in L^p as $\delta \to 0$, uniformly in ε.

As far as the second term is concerned, it is compact in L^p for any fixed δ. To prove this, we take the Fourier transform of the equation satisfied by f_ε and use the compactness of g_ε and g'_ε. In this step the smoothness of the coefficients is needed.

Therefore, we may write $\int_{\mathbb{R}} f_\varepsilon(v, x, y, t)\psi(v)dv$ as the sum of two terms tending to zero as $\delta \to 0$, uniformly in ε plus a third one, which is compact for any fixed δ. This yields compactness of the means in L^p.

Since in our case A is smooth away from zero, the above arguments yield the compactness of $\int_{\mathbb{R}} f_\varepsilon(v, x, y, t)\psi(v)dv$ for $\psi \in L^\infty$ vanishing in a neigbourhood of zero. For a general $\psi \in L^\infty$ we may split the integral as follows:

$$\int_{\mathbb{R}} f_\varepsilon(v, x, y, t)\psi(v)dv = \int_{|v|\le\alpha} f_\varepsilon(v, x, y, t)\psi(v)dv + \int_{|v|>\alpha} f_\varepsilon(v, x, y, t)\psi(v)dv.$$

The first term goes to zero uniformly in ε as α tends to zero ($|f_\varepsilon| \le 1$). The second one is compact for any fixed α. Therefore, the means $\int_{\mathbb{R}} f_\varepsilon(v, x, y, t)\psi(v)dv$ are relatively compact.

This argument shows that the compactness result remains true in spite of the lack of regularity of the nonlinearity A at $\rho = 0$. $\qquad\square$

For any $\tau > 0$ the functions $u_\lambda(t + \tau), t \geq 0, \lambda \geq 1$, are solutions of the equation with data $u_\lambda(\tau) \in L^1 \cap L^\infty$. Since they are bounded uniformly in λ in $L^\infty(0, \infty, L^1 \cap L^\infty(\mathbb{R}^n))$, we may apply the above result to obtain compactness in $L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^+)$ for u_λ.

Therefore, we can extract a subsequence u_{λ_j} converging strongly in $L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^+)$ and a.e. (x, y, t) to some limit v. On the other hand,

$$\|u_{\lambda_j}(s)\|_p \leq C(p, \tau) \quad s \geq \tau, \ \forall p$$

so that (up to the extraction of subsequences):

$$|u_{\lambda_j}|^{q-1} u_{\lambda_j} \rightharpoonup \chi \quad \text{weak star in} \quad L^\infty(\tau, \infty; L^p(\mathbb{R}^n))$$

for any $p > 1$. Since $u_{\lambda_j} \to v$ a.e., we conclude that $\chi = |v|^{q-1} v$.

Passing to the limit in the distributional formulation of (CD_λ) given by the identity (I_λ) we deduce that v solves the equation (R) in $\mathcal{D}'(\mathbb{R}^n \times (0, \infty))$.

The limit v inherits the uniform estimates (Proposition 3.1) satisfied by the u_λ. It satisfies the entropy inequality (it has been obtained by "vanishing viscosity") and belongs to $L^\infty(0, \infty; L^1(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n \times (\tau, \infty))$.

Step 4: Initial data. Taking test functions $\varphi \in C^2_c(\mathbb{R}^n)$ we get:

$$\left| \iint u_\lambda(x, y, t)\varphi(x, y)dx\,dy - \iint u_\lambda(x, y, 0)\varphi(x, y,)dx\,dy \right|$$

$$\leq \left| \int_0^t \iiint u_\lambda(x, y, s)\Delta_x\varphi(x, y)dx\,dy\,ds \right.$$

$$+ \int_0^t \iiint |u_\lambda|^{q-1}u_\lambda(x, y, s)\varphi_y(x, y)dx\,dy\,ds$$

$$\left. + \lambda^{1-2\beta} \int_0^t \iiint u_\lambda(x, y, s)\varphi_{y,y}(x, y, s)dx\,dy\,ds \right|$$

$$\leq \|\Delta_x\varphi\|_\infty \|u_0\|_1 t + \lambda^{1-2\beta}\|\varphi_{y,y}\|_\infty \|u_0\|_1 t + \|\varphi_y\|_\infty C(\|u_0\|_1) t^{-\frac{n+1}{2q}(q-1)+1}$$

and this upper bound tends to zero as $t \to 0$, uniformly when $\lambda \geq 1$. Therefore,

$$\left| \iint u_\lambda(x, y, t)\varphi(x, y) - \left(\int u_0 \right)\varphi(0, 0) \right| \to 0$$

as $t \to 0$, uniformly in $\lambda \geq 1$. Passing to the limit when $\lambda \to \infty$ we conclude that

$$\left| \iint v(x, y, t)\varphi(x, y)dx\,dy - \left(\int u_0 \right)\varphi(0, 0) \right| \to 0$$

as $t \to 0$. The convergence for $\varphi \in BC(\mathbb{R}^n)$ is a consequence of the next step.

Step 5: Tail control. We want to prove that:

$$\int\limits_{|x|+|y|>R} |u_\lambda|(t) \to 0$$

when $R \to \infty$, uniformly in $\lambda \geq 1$ and $t \in [0, t_0]$, for any fixed t_0. We sketch the proof here, for more details see the appendix and [EVZ2].

We have already observed that

$$|u_\lambda| \leq \overline{u}_\lambda$$

where $\overline{u}_\lambda$ stands for the solution with datum $|u_\lambda(0)|$. Using the results obtained in [EVZ2] (Lemma 6.2) for nonnegative solutions and $1 < q < 1 + 1/n$ it follows that:

$$\int\limits_{|x|>R} dx \int dy\, |u_\lambda|(t) \leq \int\limits_{|x|>R} dx \int dy\, \overline{u}_\lambda(t) \to 0$$

when $R \to \infty$ uniformly in $\lambda \geq 1$ and $t \in [0, t_0]$ and

$$\int\limits_{-\infty}^{-k} dy \int dx\, |u_\lambda|(t) \leq \int\limits_{-\infty}^{-k} dy \int dx\, \overline{u}_\lambda(t) \to 0$$

$$\int\limits_{k+Ct^\beta}^{\infty} dy \int dx\, |u_\lambda|(t) \leq \int\limits_{k+Ct^\beta}^{\infty} dy \int dx\, \overline{u}_\lambda(t) \to 0$$

when $\lambda \to \infty$, for any fixed $k > 0, t > 0$ and for some positive constant C.

The two last estimates also imply that

$$\int\limits_{-\infty}^{-k} dy \int dx\, |v|(t) = \int\limits_{k+Ct^\beta}^{\infty} dy \int dx\, |v|(t) = 0.$$

Thus, any limit function v must be supported in a region of the form $\{(x, y, t)$ s.t. $0 \leq y \leq Ct^\beta\}$.

Step 6: Strong convergence. The function $v \in C((0, \infty) : L^1(\mathbb{R}^n))$ is a solution of the equation (R) in the sense of distributions taking the initial datum $(\int u_0\, dx\, dy)\delta$ in the narrow sense of measures. Moreover, v satisfies the entropy condition and verifies also:

$$\int\limits_{|x|+|y|>R} |v|(t)\, dx\, dy \to 0$$

when $t \to 0$ for any fixed $R > 0$ (see Section 4). There exists a unique function v having these properties, therefore the whole family u_λ must converge to v, which must be a positive function of selfsimilar form supported in a region of the form $\{(x, y, t) \ s.t. \ 0 \le y \le Ct^\beta\}$.

From the known convergence of u_λ to v in $L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^n)$ and the step 5 we obtain strong convergence in $L^1(\mathbb{R}^n \times (\tau_1, \tau_2))$ for any $\tau_2 > \tau_1 > 0$.

The equations (CD_λ) and estimate (E3) in Proposition 3.1 yield for $u_{\lambda,t}$ a bound in $L^2_{\text{loc}}((0, \infty), H^{-s}(\Omega))$ for any bounded domain $\Omega \subset \mathbb{R}^n$ and for some $s > 0$. As a consequence of (E3) we also have that the family u_λ is bounded in $L^2_{\text{loc}}((0, \infty), L^2_{\text{loc}}(\mathbb{R}^2))$. Taking into account that $L^2(\Omega)$ is compactly embedded in $H^{-\varepsilon}(\Omega)$ for any $\varepsilon > 0$ and that $H^{-\varepsilon}(\Omega)$ is continuously embedded in $H^{-s}(\Omega)$ for any $s > \varepsilon$, we deduce that u_λ is relatively compact in $C_{\text{loc}}((0, \infty), H^{-\varepsilon}(\Omega))$. Therefore, we may extract a subsequence $\lambda_n \to \infty$ in such a way that:

$$u_{\lambda_n} \to \overline{u} \quad \text{in} \quad C_{\text{loc}}((0, \infty), H^{-\varepsilon}(\Omega)) \,.$$

In view of the strong convergence of u_{λ_n} to v in $L^1(\mathbb{R}^n \times (\tau_1, \tau_2))$ for any $\tau_2 > \tau_1 > 0$ we conclude that $\overline{u} = v$ and that for a fixed $t > 0$:

$$\|u_{\lambda_n}(t) - v(t)\|_1 \to 0 \quad \text{as} \quad n \to \infty \,.$$

The limit v is independent of the subsequence, it must be the selfsimilar entropy solution of (R) with mass $\int u_0$ satisfying the condition (H). Therefore, the whole sequence converges:

$$\|u_\lambda(t) - v(t)\|_1 \to 0 \quad \text{as} \quad \lambda \to \infty \,.$$

Interpolating and using the L^∞ bound (Proposition 3.1) we get convergence in $L^p(\mathbb{R}^n)$ for any $p \in [1, \infty)$.

Changing to the original variables we obtain the result we searched for:

$$\lambda^{\frac{n+1}{2q} \frac{p-1}{p}} \|u(\lambda t) - v(\lambda t)\|_p = \|u_\lambda(t) - v(t)\|_p \to 0$$

when $\lambda \to \infty$.

4. – Appendix: Tail control

In Step 4 of the proof of parts a) and b) of Theorem 1 (Section 2) and in Step 5 and 6 of the proof of Theorem 2 (Section 3), we used some tail control estimates without detailing the proofs. In this appendix we briefly recall how the estimates are obtained.

For a solution u of (CD) or (R) with initial datum $u_0 \in L^1$ we have $|u| \le \overline{u}$, where $\overline{u}$ is the solution corresponding to the datum $|u_0|$. Therefore, it suffices to prove this kind of estimates for nonnegative solutions.

In the sequel we assume $(u_j)_{j \geq 0}$ to be nonnegative solutions of (CD) corresponding to nonnegative data $u_j(0) \in L^1(\mathbb{R}^n)$. We are interested in data $u_j(0)$ with the properties:

(P1) $u_j(0) \to M\delta$ in the narrow sense of measures as $t \to \infty$,

(P2) $\|u_j(0)\|_{L^1(|x|+|y|>r)} \to 0$ as $j \to \infty, r > 0$ fixed.

From (P2) it follows that:

$$\|u_j(0)\|_{L^1(|x|+|y|>r)} \to 0 \quad \text{as} \quad r \to \infty, \quad \text{uniformly in} \quad j \geq 1$$

$$\|u_j(0)\|_{L^1(\mathbb{R}^n)} \leq C, \qquad\qquad\qquad \text{uniformly in} \quad j.$$

For instance, we may consider the initial data in Section 3 ($\lambda = j$):

$$u_j(x, y, 0) = j^\alpha u_0(j^{1/2}x, j^\beta y)$$

or the data $h_j(0)$ and $g_j(0)$ in Step 4 in the Proof of Theorem 1.

We want to prove two kinds of tail controls:

(1) $\displaystyle\int_{|x|+|y|>r} u_j(x, y, t) \to 0 \quad \text{as} \quad r \to \infty$

uniformly in $t \in [0, t_0]$ and $j \geq 1$

(2) $\displaystyle\int_{|x|+|y|>r} u_j(x, y, t) \to 0 \quad \text{letting} \quad j \to \infty, \quad \text{and then} \quad t \to 0$

for any fixed $r > 0$.

The first one is needed in Step 4 in part a) (resp. b)) of the uniqueness proof, when proving that h_j, g_j converge to fundamental solutions of (CD) (resp. (R)) with Dirac measures as initial data. It is also needed in Step 5 of the Proof of Theorem 2 (Section 3) when proving that u_{λ_n} converge to an entropy fundamental solution of (R) which takes as initial datum $(\int u_0)\delta$. The second one is used in Step 6 in the Proof of Theorem 2. It ensures that the fundamental solution v obtained in that way satisfies:

$$\int_{|x|+|y|>r} v(x, y, t) \to 0 \quad \text{as} \quad t \to 0$$

for any fixed $r > 0$ and, therefore, it is unique.

Let us sketch the proof of those estimates. As in [EVZ2] we have

$$\int_{|x|>r} dx \int dy\, u_j(x, y, t) \leq \int_{|x|>r} dx\, v_j(x, t)$$

where $w_j(x, t)$ is the solution of the heat equation (in $\mathbb{R}^{n-1}$) with data

$$w_j(x, 0) = \int dy \, u_j(x, y, 0).$$

We consider the equation satisfied by $w_{j,r} = v_j \varphi_r$ with $\varphi_r(x) = \varphi(\frac{x}{r})$ and φ such that:

$$\varphi \in BC^2(\mathbb{R}^{n-1}), \quad 0 \leq \varphi \leq 1, \quad \varphi = 0 \quad \text{for} \quad |x| < 1$$

where BC^2 denotes the space of bounded functions of class C^2 in $\mathbb{R}^n$. From the corresponding integral inequality we get

$$\int\limits_{|x|>r} dx \, v_j(x, t) \leq C\|G(t) * v_j(0)\varphi_r\|_1 + C\|v_j(0)\|_1 \left(\frac{t^{1/2}}{R} + \frac{t}{R^2} \right).$$

We conclude that:

$$\int\limits_{|x|>r} dx \int dy \, u_j(x, y, t)$$

tends to zero:
   – uniformly in $t \in [0, t_0]$ and in $j \geq 1$ when $r \to \infty$
   – as $t \to 0$ and $j \to \infty$ for any fixed $r > 0$.

As before, following [EVZ2] we have for $k > 0$

$$\int dx \int\limits_{-\infty}^{-k} dy \, u_j(x, y, t) \leq v_j(-k, t)$$

$$\int dx \int\limits_{k}^{\infty} dy \, u_j(x, y, t) \leq w_j\left(k - \frac{C}{\beta} t^\beta, t\right) \qquad \beta = \beta(q) > 0$$

where
   – v_j is the solution of the one dimensional heat equation with data

$$v_j(y, 0) = \int dx \int\limits_{-\infty}^{y} ds \, u_j(x, s, 0)$$

   – w_j is the solution of the one dimensional heat equation with data

$$w_j(y, 0) = \int dx \int\limits_{y}^{\infty} ds \, u_j(x, s, 0).$$

Taking $u_j(x, y, 0) = j^{-\alpha} u_0(j^{-1/2}x, j^{-\beta}y)$ we have:

$$v_j(-k, t) = \int\limits_{\mathbb{R}} dz \frac{e^{\frac{|z|^2}{4t}}}{(4\pi t)^{n/2}} \int dx \int\limits_{-\infty}^{-k-z} ds \, j^{-\alpha} u(j^{-1/2}x, j^{-\beta}s, 0)$$

$$= \int\limits_{\mathbb{R}} dz \frac{e^{\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} \int dx \int\limits_{-\infty}^{-kj^{-\beta}-z} ds \, u(x, s, 0) j^{\beta}$$

In the same way

$$w_j\left(k - \frac{C}{\beta} t^{\beta}, t\right) = \int\limits_{\mathbb{R}} dz \frac{e^{\frac{|z|^2}{4t}}}{(4\pi t)^{n/2}} \int dx \int\limits_{-z+j^{-\beta}k-j^{-\beta}\frac{C}{\beta}t^{\beta}}^{\infty} ds \, u(x, s, 0) j^{\beta} \, .$$

In both cases we get convergence to zero:
— when $k \to \infty$, uniformly in $t \in [0, t_0]$ and in $j \geq 1$
— when $j \to \infty$ and $t \to 0$ for a fixed $k > 0$.

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