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1. – Introduction and definitions

Lord Rayleigh conjectured last century that among all clamped plates with a given area, the disk has the minimal fundamental tone. N. Nadirashvili [25] recently proved this conjecture by improving a technique of G. Talenti [34], and M. S. Ashbaugh and R. D. Benguria [3] have established the analogous result in $\mathbf{R}^3$, where now the "plate" has a given volume.

We extend these methods to $\mathbf{R}^n, n \geq 4$, and prove Rayleigh's conjecture up to a constant factor d_n, with $d_4 \approx 0.95$, for example. Since $d_n \to 1$ as $n \to \infty$, one can say that Rayleigh's conjecture holds asymptotically for high dimensions.

Our results provide a computable lower bound for the fundamental tone of an arbitrary clamped plate in terms just of its volume. The bound is the best known of this type, improving on the earlier work of Talenti by a factor of almost 2, in high dimensions.

Inspired by Lord Rayleigh's conjecture, G. Pólya and G. Szegö conjectured in 1951 that among all clamped plates of the same area subjected to uniform lateral compression, the disk has minimal buckling load. The conjecture remains open, but by straightforwardly generalizing an observation of J. H. Bramble and L. E. Payne [5] to higher dimensions, we establish the best partial result to date, proving the conjecture in $\mathbf{R}^n$ up to a constant factor c_n with $c_n \to 1$ as $n \to \infty$. Thus the Pólya–Szegö conjecture also holds asymptotically in high dimensions. Note that these results provide a computable lower bound for the buckling load of an arbitrary clamped plate in terms just of its volume.

In this first section we define the eigenvalues to be used and estimated, namely the critical buckling load Λ of the plate and the fundamental tones Γ of the plate and λ_1 of the membrane. Then in Sections 2 and 3 we present our lower bounds on Λ and Γ respectively, with Sections 4 and 5 containing the proofs. In Sections 6 and 7 we examine a few particular plane domains for which good estimates on the buckling load and fundamental tone are known

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already. While estimates of the type in this paper do not improve on these known bounds, they do give results of very much the right order of magnitude. This seems a satisfactory outcome: since our results apply to *all* shapes of plate, we would be surprised if they were the best known for any intensively–studied particular shape. In Section 8 we mention related results and open problems.

Now we commence the definitions. Given a bounded, connected open set Ω in $\mathbf{R}^n, n \geq 2$, define the *buckling load* of Ω to be

$$\Lambda(\Omega) := \inf \left\{ \frac{\int_\Omega (\Delta u)^2 \, dx}{\int_\Omega |\nabla u|^2 \, dx} : u \in W_0^{2,2}(\Omega), \, \nabla u \not\equiv 0 \right\},$$

where dx denotes Lebesgue measure on $\mathbf{R}^n$. A smooth function u_1 exists that attains the infimum for $\Lambda(\Omega)$ and satisfies

$$\Delta\Delta u_1 + \Lambda(\Omega)\Delta u_1 = 0.$$

By virtue of its definition, $\Lambda(\Omega)$ is the principal eigenvalue of this equation.

Physically, when $n = 2$ the eigenfunction u_1 models the transverse deflection of a homogeneous thin plate Ω with clamped boundary that is subjected to a uniform compressive stress or load on that boundary. We have expressed the clamping implicitly: roughly speaking, each function u in the Sobolev space $W_0^{2,2}(\Omega)$ vanishes on the boundary of Ω, as does its normal derivative $\partial u/\partial n$. The critical buckling load of the plate is proportional (see [36]) to the eigenvalue $\Lambda(\Omega)$, and so to establish a buckle-free stress level for the plate we must estimate $\Lambda(\Omega)$ from below. This we do in Theorem 1 and Corollary 2

Notice that since we defined Λ as an infimum, to obtain an upper bound for it we have only to choose a test function $u \in W_0^{2,2}(\Omega)$ and compute the integrals of $(\Delta u)^2$ and $|\nabla u|^2$. There is no obvious method for computing a lower bound for Λ, and this is also the case for the fundamental tone, defined below. Nevertheless, several authors have developed and applied methods for obtaining lower bounds. We mention A. Weinstein's method of intermediate problems [41] (see also D. W. Fox and W. C. Rheinboldt [11] and H. F. Weinberger [40]), G. Fichera's method of orthogonal invariants [10], and the methods of B. Knauer and W. Velte [15,38] and J. R. Kuttler [17]. There are also methods that can yield a good bound for one of the eigenvalues of an equation, without telling which eigenvalue it is, *e.g.* [4]. For further references to the various methods, see [38].

Next, define the *fundamental tone* of the plate Ω to be

$$\Gamma(\Omega) := \inf \left\{ \frac{\int_\Omega (\Delta v)^2 \, dx}{\int_\Omega v^2 \, dx} : v \in W_0^{2,2}(\Omega), \, v \not\equiv 0 \right\}.$$

Again, a smooth function v_1 exists that attains the infimum for $\Gamma(\Omega)$ and satisfies

$$\Delta\Delta v_1 = \Gamma(\Omega) v_1,$$

and $\Gamma(\Omega)$ is the lowest eigenvalue of this equation.

When $n = 2$, we interpret the eigenfunction v_1 physically as describing a transverse mode of vibration of the homogeneous thin plate Ω with clamped boundary. The frequency of vibration of the plate is proportional (see [20, (1.5)]) to $\Gamma(\Omega)^{1/2}$. In order to establish a lower bound on that fundamental frequency, we estimate $\Gamma(\Omega)$ from below in Theorem 4.

Thirdly, we have the familiar Rayleigh quotient for the fundamental tone of Ω regarded as a fixed membrane:

$$\lambda_1(\Omega) := \inf \left\{ \frac{\int_\Omega |\nabla w|^2 \, dx}{\int_\Omega w^2 \, dx} : w \in W_0^{1,2}(\Omega), w \not\equiv 0 \right\},$$

and a smooth eigenfunction w_1 exists that attains the infimum for $\lambda_1(\Omega)$ and satisfies

$$-\Delta w_1 = \lambda_1(\Omega) w_1.$$

Of course, $\lambda_1(\Omega)$ is the principal eigenvalue of $-\Delta$.

Write $|\Omega|$ for the volume (*i.e.*, Lebesgue measure) of Ω, and employ $\Omega^\#$ to denote the open ball centered at the origin of $\mathbf{R}^n$ having the same volume as Ω. Let

$v_n :=$ volume of the unit ball in R^n, *e.g.* $v_2 = \pi$;

$\nu := (n-2)/2$;

$J_\nu :=$ the ν-th Bessel function of the first kind;

$I_\nu :=$ the ν-th modified Bessel function of the first kind;

$j_\nu :=$ the first zero of $J_\nu(t), t > 0$, *e.g.* $j_0 \approx 2.40482556$;

$k_\nu :=$ the first zero of $J_\nu(t) I_{\nu+1}(t) + J_{\nu+1}(t) I_\nu(t), t > 0$, *e.g.* $k_0 \approx 3.19622062$;

$c_n := 2^{2/n}(j_\nu / j_{\nu+1})^2$;

$d_n := 2^{4/n}(j_\nu / k_\nu)^4$.

We will often use the fact that the ball B_r of radius r has buckling load $\Lambda(B_r) = j_{\nu+1}^2 / r^2$ and fundamental tone $\Gamma(B_r) = k_\nu^4 / r^4$; these formulas can be proved by separation of variables, using the J_μ, I_μ, and spherical harmonics. (In Section 4 we provide also an alternative method for evaluating $\Lambda(B_r)$.) Notice that consequently $\Lambda(\Omega^\#) = j_{\nu+1}^2 (v_n / |\Omega|)^{2/n}$ and $\Gamma(\Omega^\#) = k_\nu^4 (v_n / |\Omega|)^{4/n}$.

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2. – Estimates on the buckling load

We would like to be able to prove the following conjecture, estimating $\Lambda(\Omega)$ from below just in terms of the volume of Ω.

CONJECTURE (Pólya–Szegö [29]).

$$\Lambda(\Omega) \geq \frac{j_{\nu+1}^2 v_n^{2/n}}{|\Omega|^{2/n}} = \Lambda(\Omega^\#),$$

with equality if and only if Ω *is a ball.*

This conjecture remains open, but we prove a partial result.

THEOREM 1.

$$\Lambda(\Omega) > \frac{2^{2/n} j_\nu^2 v_n^{2/n}}{|\Omega|^{2/n}} = c_n \Lambda(\Omega^\#).$$

COROLLARY 2 (BRAMBLE–PAYNE [5, (3.14)]). *In particular, when* $n = 2$ *and* Ω *lies in the plane,*

$$\Lambda(\Omega) > \frac{2\pi j_0^2}{Area(\Omega)}.$$

We prove Theorem 1 in Section 4 by generalizing J. H. Bramble and L. E. Payne's proof of Corollary 2 to higher dimensions in a straightforward way.

Theorem 1 falls not too far short of Pólya and Szegö's conjecture, since

$c_2 \geq 0.7877 \qquad c_3 \geq 0.7759 \qquad c_4 \geq 0.7872 \qquad c_5 \geq 0.8020 \qquad c_6 \geq 0.8163$

and since

$$\begin{aligned} c_n &= 1 - (4 - \log 4)/n + O(n^{-5/3}) \\ &\to 1 \end{aligned} \tag{2.1}$$

as $n \to \infty$, by [1, p. 371, eq. 9.5.14]. In particular, (2.1) and Theorem 1 show that the Pólya-Szegö conjecture holds asymptotically as the dimension n approaches infinity.

Note also that almost fifty years ago, G. Szegö [33] did prove the Pólya–Szegö conjecture under the additional hypothesis that an eigenfunction u_1 corresponding to $\Lambda(\Omega)$ never changes sign. Szegö's hypothesis fails to hold for many domains, however, and we discuss this further in Section 8.

In Section 6, we examine several simple plates in the plane and compare the estimate on Λ provided by the Bramble–Payne bound $\Lambda(\Omega) > 2\pi j_0^2/\text{Area}(\Omega)$ with previously–known bounds.

3. – Estimates on the fundamental tone

For the vibrating plate, too, an outstanding conjecture motivates our work.

CONJECTURE (Lord Rayleigh [30, p. 382]).

$$\Gamma(\Omega) \geq \frac{k_\nu^4 v_n^{4/n}}{|\Omega|^{4/n}} = \Gamma(\Omega^\#),$$

with equality if and only if Ω *is a ball.*

In the plane, the conjecture has been proved.

THEOREM 3 (Nadirashvili [25]). *When $n = 2$ and Ω lies in the plane,*

$$\Gamma(\Omega) \geq \frac{\pi^2 k_0^4}{Area(\Omega)^2} = \Gamma(\Omega^\#),$$

with equality if and only if Ω is a disk.

Furthermore, M.S. Ashbaugh and R.D. Benguria [3] have proved Rayleigh's conjecture in $\mathbf{R}^3$ (and $\mathbf{R}^2$) by means of the same general method. This method cannot prove Rayleigh's conjecture in dimensions 4 and higher, but we exploit it nonetheless to obtain the best known partial result, estimating $\Gamma(\Omega)$ from below in terms simply of the volume of Ω.

THEOREM 4. *For $n \geq 4$,*

$$\Gamma(\Omega) > \frac{2^{4/n} j_\nu^4 v_n^{4/n}}{|\Omega|^{4/n}} = d_n \Gamma(\Omega^\#).$$

Under the additional hypothesis that an eigenfunction v_1 corresponding to $\Gamma(\Omega)$ never changes sign, G. Szegö [33] did prove Rayleigh's conjecture by techniques that apply in all dimensions. It soon became clear, though, that just as for the buckling load, Szegö's hypothesis fails to hold for many simple domains; we provide references for this in Section 8.

Theorem 4 provides the best known partial result for Rayleigh's conjecture in $\mathbf{R}^n, n \geq 4$, and the constant d_n can easily be evaluated:

$d_4 \geq 0.9537 \quad d_5 \geq 0.9218 \quad d_6 \geq 0.9077 \quad d_7 \geq 0.9018 \quad d_8 \geq 0.8998 \quad d_9 \geq 0.9001$

and

$$\begin{aligned} d_n > c_n^2 &\geq 1 - 2(4 - \log 4)/n + O(n^{-5/3}) \\ &\to 1 \end{aligned}$$

as $n \to \infty$, by (2.1) and the fact that $k_\nu < j_{\nu+1}$. (See [21] for this and other facts about k_ν.) Thus Rayleigh's conjecture holds asymptotically as the dimension n approaches infinity.

G. Talenti [34] proved that $\Gamma(\Omega) \geq d'_n \Gamma(\Omega^\#)$ in $\mathbf{R}^n, n \geq 2$, with

$d'_2 \approx 0.9777 \quad d'_3 \approx 0.7391 \quad d'_4 \approx 0.6524 \quad d'_5 \approx 0.6093 \quad d'_6 \approx 0.5839$

and $d'_n \geq 1/2$, for all n. Note that Nadirashvili's result in $\mathbf{R}^2$ and Ashbaugh–Benguria's in $\mathbf{R}^3$ improve both the constants d'_2 and d'_3 to 1, and that our inequality $\Gamma(\Omega) > d_n \Gamma(\Omega^\#)$ is stronger than Talenti's for $n = 4, 5, 6$, since $d_n > d'_n$ for those values of n.

In fact, our inequality is stronger than Talenti's for all $n \geq 4$, as we show in Section 4. We further show that $\limsup_{n\to\infty} d'_n \leq 1/2$: thus our Theorem 4 improves Talenti's result by a factor of almost 2 for large n.

In Section 7, we consider several simple plates in the plane and compare the estimate on Γ provided by the Nadirashvili bound $\Gamma(\Omega) \geq \Gamma(\Omega^{\#})$ with bounds from the literature.

Before we commence the proofs, it is interesting to note that the estimate $\Gamma(\Omega) > 2^{-2/n} d_n \Gamma(\Omega^{\#})$, which is weaker than Theorem 4, follows easily for all $n \geq 2$ from Theorem 1 and the facts that $\Gamma \geq \Lambda\lambda_1$ (almost trivially) and $\lambda_1(\Omega) \geq j_\nu^2 v_n^{2/n}/|\Omega|^{2/n}$ (by the Faber–Krahn Theorem [13, p. 89]). Bramble and Payne [5, (3.15)] used this weaker estimate in the $n = 2$ case. An even weaker estimate, which still improves on Talenti's result for high dimensions, is that $\Gamma(\Omega) > 2^{-4/n} d_n \Gamma(\Omega^{\#})$; this follows from Faber–Krahn together with A. Weinstein's observation [41, p. 191] that $\Gamma > \lambda_1^2$.

4. – Proof of Theorem 1

Write $\lambda_2(\Omega)$ for the second eigenvalue of the Laplacian on Ω with Dirichlet boundary conditions. Payne [27, p. 523] proved that

$$\Lambda(\Omega) \geq \lambda_2(\Omega).$$

Theorem 1 follows now from the next lemma, proved by E. Krahn [16]. (P. Szego later rediscovered this lemma; see [28, p. 336].)

LEMMA 5.
$$\lambda_2(\Omega) > \frac{2^{2/n} j_\nu^2 v_n^{2/n}}{|\Omega|^{2/n}}.$$

For convenience, we include a modern proof of Lemma 5. Let $w_2 \in W_0^{1,2}(\Omega)$ be a (smooth) eigenfunction of the Laplacian on Ω with eigenvalue $\lambda_2(\Omega)$, so that $-\Delta w_2 = \lambda_2(\Omega) w_2$. Put $\Omega_+ := \{x \in \Omega : w_2(x) > 0\}$ and $\Omega_- := \{x \in \Omega : w_2(x) < 0\}$. Neither Ω_+ nor Ω_- can be empty, because the first eigenfunction w_1 of the Laplacian on Ω, corresponding to $\lambda_1(\Omega)$, is positive and satisfies $\int_\Omega w_1 w_2 \, dx = 0$. Since the restriction of w_2 to $\Omega_\pm$ belongs to $W_0^{1,2}(\Omega_\pm)$ and satisfies $-\Delta w_2 = \lambda_2(\Omega) w_2$, we have that $\lambda_2(\Omega)$ is an eigenvalue of $-\Delta$ on $\Omega_\pm$. Hence

$$\lambda_2(\Omega) \geq \lambda_1(\Omega_\pm) \geq \lambda_1(\Omega_\pm^{\#}) = \frac{j_\nu^2 v_n^{2/n}}{|\Omega_\pm|^{2/n}},$$

where the second inequality rests upon the Faber–Krahn theorem [13, p. 89]. Summing this last inequality over Ω_+ and Ω_- gives that

$$2\lambda_2(\Omega) \geq j_\nu^2 v_n^{2/n} \left(\frac{1}{|\Omega_+|^{2/n}} + \frac{1}{|\Omega_-|^{2/n}} \right) \geq 2 j_\nu^2 v_n^{2/n} \left(\frac{2}{|\Omega|} \right)^{2/n}, \tag{4.1}$$

since $|\Omega_+| + |\Omega_-| \leq |\Omega|$ and since $t \mapsto 1/t^{2/n}$ is convex. If equality held throughout (4.1) then Ω_+ and Ω_- would have to be (disjoint) balls, by the equality statement of the Faber–Krahn theorem [13, p. 92]. Since they would also need to have volume $|\Omega|/2$ each, Ω would not be connected. This proves Lemma 5.

Next we prove that $\Lambda(\Omega^{\#}) = j_{\nu+1}^2 v_n^{2/n}/|\Omega|^{2/n}$, a fact needed for the equality in Theorem 1. By a dilation we may assume $|\Omega| = v_n$ and $\Omega^{\#}$ equals the unit ball B. We show $\Lambda(B) = j_{\nu+1}^2$. Define a radial function

$$u(x) := \frac{J_\nu(j_{\nu+1}|x|)}{|x|^\nu} - J_\nu(j_{\nu+1}).$$

Then u is smooth on $\mathbf{R}^n$ and both u and its normal derivative vanish on the boundary of B since $(d/dt)(J_\nu(t)/t^\nu) = -J_{\nu+1}(t)/t^\nu$ [1, p. 361, eq. 9.1.30]. Thus $u \in W_0^{2,2}(B)$. From Bessel's equation we get that u satisfies a Helmholtz-type equation $\Delta u + j_{\nu+1}^2(u + J_\nu(j_{\nu+1})) = 0$, so that $\Delta\Delta u + j_{\nu+1}^2 \Delta u = 0$, and hence $j_{\nu+1}^2 \geq \Lambda(B)$. For the reverse inequality, notice that $\Lambda(B) \geq \lambda_2(B)$ by Payne's inequality above, and $\lambda_2(B) = j_{\nu+1}^2$. Therefore $\Lambda(B) = j_{\nu+1}^2$.

As an aside, we elaborate a little on Payne's proof that $\Lambda(\Omega) \geq \lambda_2(\Omega)$, using his notation. First, although Payne states the theorem in the plane, his proof extends directly to $\mathbf{R}^n$ for all $n \geq 2$. Second, in the language of Sobolev spaces, the functions ψ belong to $W_0^{1,2}(\mathcal{D})$ and the function W_1 belongs to $W_0^{2,2}(\mathcal{D})$. Third, the constants a_1 and a_2 in the proof cannot be chosen in the desired manner if $\iint_{\mathcal{D}} u_1 W_1 \, dA = 0$, but in that case W_1 itself is admissible for λ_2 and so the inequality $\Lambda_1 \geq \lambda_2$ follows from (43).

5. – Proof of Theorem 4; comparison with Talenti's results

We begin by roughly outlining the approach developed by Talenti [34], Nadirashvili [25] and Ashbaugh–Benguria [3]. Every step works in each $\mathbf{R}^n$, $n \geq 2$, until the end.

Divide the region Ω into two pieces, Ω_+ and Ω_-, in which the fundamental mode v_1 is positive and negative respectively. Then split both the numerator and denominator of the Rayleigh quotient $\Gamma(\Omega) = \int_\Omega (\Delta v_1)^2 \, dx / \int_\Omega v_1^2 \, dx$ into integrals over Ω_+ and Ω_-. The value of this Rayleigh quotient does not change when we replace in it the functions $-\Delta v_1$ on Ω_+ and Δv_1 on Ω_- with their symmetric decreasing rearrangements f_+ and f_-. The Rayleigh quotient decreases when we replace v_1 on Ω_+ and Ω_- with the functions $w_+ = -\Delta^{-1} f_+$ on $\Omega_+^{\#}$ and $w_- = \Delta^{-1} f_-$ on $\Omega_-^{\#}$ respectively. One has still to estimate from below this new type of Rayleigh quotient, a quotient involving integrals over the two balls $B_a := \Omega_+^{\#}$ and $B_b := \Omega_-^{\#}$. Call this quotient $Q(a, b)$.

At this point, Nadirashvili takes $n = 2$ and applies a transplantation method to shrink one of the balls to a point and to expand the other ball up to $\Omega^{\#}$, showing that the least possible value of $Q(a, b)$ decreases under the transplantation and that its value approaches $\Gamma(\Omega^{\#})$. In contrast, Ashbaugh and Benguria proceed more explicitly, determining the least possible value of $Q(a, b)$ for each choice of a and b subject to the constraint that $v_n a^n + v_n b^n = |\Omega_+| + |\Omega_-| = |\Omega|$. After rescaling, they are able to assume that $a^n + b^n = 1, 0 \leq a \leq b \leq 1$. We simply quote their result (see (32) and (39) of [3]), which holds for all $n \geq 2$:

$$\Gamma(\Omega) \geq k_\nu(a)^4 (v_n/|\Omega|)^{4/n} \qquad \text{for some } 0 \leq a \leq 2^{-1/n}, \tag{5.1}$$

where $k_\nu(2^{-1/n}) := 2^{1/n} j_\nu$ and $k_\nu(a)$ is defined for $0 \leq a < 2^{-1/n}$ to be the first positive zero k of

$$h_\nu(k) := f_\nu(ka) + f_\nu(kb),$$

with f_ν defined by

$$f_\nu(x) := x^{2\nu+1} \left[\frac{J_{\nu+1}}{J_\nu}(x) + \frac{I_{\nu+1}}{I_\nu}(x) \right], \qquad 0 \leq x < \infty.$$

Regard the parameter a as fixed, in the definition of h_ν. See [1, 39] for information on J_ν and I_ν. Notice that $f_\nu(0) = 0$ and so $k_\nu(0) = k_\nu$.

Ashbaugh and Benguria [3, p. 9] show that when $n = 2, 3$, the inequality $k_\nu(a) \geq k_\nu(0) = k_\nu$ holds for all a, which proves Rayleigh's conjecture in view of (5.1) and the remark near the end of Section 1 that $\Gamma(\Omega^{\#}) = k_\nu^4 (v_n/|\Omega|)^{4/n}$. Ashbaugh and Benguria emphasize that one can only hope to prove Rayleigh's conjecture via (5.1) in dimensions n for which $2^{1/n} j_\nu = k_\nu(2^{-1/n}) \geq k_\nu$; this inequality fails, however, in dimension 4. (Actually, it fails for all $n \geq 4$, but that is a consequence of Theorem 4, which we are in the process of proving.) The problem with the above approach to proving Rayleigh's conjecture seems to be that when we split Ω into two pieces and symmetrize each piece separately, we lose too much information about the function v_1. In dimensions 2 and 3 the method scrapes home regardless, but not in dimensions higher than that.

Nevertheless, the method suffices to establish the partial result in Theorem 4: we prove below that for any $n \geq 4$,

$$k_\nu(a) > k_\nu(2^{-1/n}) = 2^{1/n} j_\nu, \qquad 0 \leq a < 2^{-1/n}. \tag{5.2}$$

Clearly Theorem 4 follows from this and (5.1), together with the fact that the inequality in (5.1) is strict if $a = 2^{-1/n}$ (since Ω_+ and Ω_- cannot both be balls and so either (21) or (22) of [3] is strict).

Geometrically, we can summarize by saying that for dimensions 2 and 3 the worst case in (5.1) is when $a = 0, b = 1$, and the ball B_a has degenerated to a point, but for dimensions $n \geq 4$ the worst case occurs when $a = b = 2^{-1/n}$ and the balls B_a and B_b have the same size.

We start the proof of (5.2) by collecting in this paragraph some useful facts from [3, p. 10]. In what follows, we denote by $j_{\nu,m}$ the m-th positive zero of J_ν, so that $j_\nu = j_{\nu,1}$. Also, we assume always that $0 \le a < 2^{-1/n}$. Then f_ν is defined except at the zeros $j_{\nu,m}$ of J_ν, and f_ν increases strictly on its intervals of definition. Hence f_ν is positive on $(0, j_{\nu,1})$ and increases from $-\infty$ to ∞ on $(j_{\nu,1}, j_{\nu,2})$. Since $j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2}$ and $f_\nu(j_{\nu+1,1}) > 0$, it follows that $j_{\nu,1} < k_\nu < j_{\nu+1,1}$. By similar reasoning, $h_\nu(k)$ increases strictly wherever it is defined, and it is defined on the interval $j_{\nu,1}/b < k < \min(j_{\nu,1}/a, j_{\nu,2}/b)$. Plainly $k_\nu(a)$ lies in this interval for $0 \le a < 2^{-1/n}$. Incidentally, this shows that $\lim_{a \to 2^{-1/n}-} k_\nu(a) = 2^{1/n} j_\nu$, since $k_\nu(a)$ lies between $j_{\nu,1}/b$ and $j_{\nu,1}/a$.

Notice that

$$2^{1/n} j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} \le j_{\nu,2}/b,$$

with the first inequality justified, for example, by applying Theorem 1 to a ball. Hence $j_{\nu,1}/b < 2^{1/n} j_{\nu,1} < \min(j_{\nu,1}/a, j_{\nu,2}/b)$. Thus we aim to show that $h_\nu(2^{1/n} j_{\nu,1}) < 0$, for then the strictly increasing nature of h_ν implies that $2^{1/n} j_{\nu,1} < k_\nu(a)$, which is (5.2). Writing

$$F_\nu(a) := f_\nu(2^{1/n} j_{\nu,1} a) + f_\nu(2^{1/n} j_{\nu,1} b), \qquad 0 \le a < 2^{-1/n}, \quad a^n + b^n = 1,$$

we see that our goal is to prove

$$F_\nu(a) < 0, \qquad 0 \le a < 2^{-1/n}. \tag{5.3}$$

From now on, we regard a as the variable, not k. Notice that F_ν is continuous for $0 \le a < 2^{-1/n}$.

We begin proving (5.3) by showing that

$$\lim_{a \to 2^{-1/n}-} F_\nu(a) < 0. \tag{5.4}$$

From evaluating Bessel's equation $x^2 J_\nu''(x) + x J_\nu'(x) + (x^2 - \nu^2) J_\nu(x) = 0$ at $x = j_\nu$, and from the following recursion relations [1, p. 361]

$$J_{\nu+1}(x) = -J_\nu'(x) + \nu J_\nu(x)/x, \qquad J_{\nu+1}'(x) = -J_\nu''(x) + \nu J_\nu'(x)/x - \nu J_\nu(x)/x^2,$$

we get that

$$J_\nu''(j_\nu) = -J_\nu'(j_\nu)/j_\nu, \quad 0 \ne J_{\nu+1}(j_\nu) = -J_\nu'(j_\nu), \quad J_{\nu+1}'(j_\nu) = (\nu+1) J_\nu'(j_\nu)/j_\nu.$$

Hence for x near $j_\nu = j_{\nu,1}$,

$$
\begin{aligned}
f_\nu(x) &= x^{2\nu+1}\left[\frac{J_{\nu+1}(x)}{J_\nu(x)} + \frac{I_{\nu+1}(x)}{I_\nu(x)}\right] \\
&= x^{2\nu+1}\left[\frac{J_{\nu+1}(j_\nu) + J'_{\nu+1}(j_\nu)(x-j_\nu) + o(x-j_\nu)}{J'_\nu(j_\nu)(x-j_\nu) + J''_\nu(j_\nu)(x-j_\nu)^2/2 + o(x-j_\nu)^2}\right. \\
&\qquad\qquad \left. + \frac{I_{\nu+1}(j_\nu)}{I_\nu(j_\nu)} + o(1)\right] \\
&= x^{2\nu+1}\left[\frac{-J'_\nu(j_\nu) + (\nu+1)J'_\nu(j_\nu)(x-j_\nu)/j_\nu + o(x-j_\nu)}{J'_\nu(j_\nu)(x-j_\nu) - J'_\nu(j_\nu)(x-j_\nu)^2/(2j_\nu) + o(x-j_\nu)^2}\right. \\
&\qquad\qquad \left. + \frac{I_{\nu+1}(j_\nu)}{I_\nu(j_\nu)}\right] + o(1) \\
&= (j_\nu^{2\nu+1} + (2\nu+1)j_\nu^{2\nu}(x-j_\nu) + o(x-j_\nu)) \\
&\qquad \cdot \left[\frac{1 + (x-j_\nu)/(2j_\nu) + o(x-j_\nu)}{J'_\nu(j_\nu)(x-j_\nu)}\right. \\
&\qquad\qquad \cdot \{-J'_\nu(j_\nu) + (\nu+1)J'_\nu(j_\nu)(x-j_\nu)/j_\nu + o(x-j_\nu)\} \\
&\qquad\qquad \left. + \frac{I_{\nu+1}(j_\nu)}{I_\nu(j_\nu)}\right] + o(1) \\
&= j_\nu^{2\nu+1}\left[\frac{-1}{x-j_\nu} - \frac{2\nu+1}{2j_\nu} + \frac{I_{\nu+1}(j_\nu)}{I_\nu(j_\nu)}\right] + o(1).
\end{aligned}
$$

For a near $2^{-1/n}$, both $2^{1/n}a$ and $2^{1/n}b$ are near 1, and so by writing $a = 2^{-1/n}(1-t)^{1/n}$ and $b = 2^{-1/n}(1+t)^{1/n}$, we get from the preceding formula that

$$
\begin{aligned}
&\lim_{a\to 2^{-1/n}-} F_\nu(a)/j_\nu^{2\nu+1} \\
&\quad = \lim_{a\to 2^{-1/n}-} [f_\nu(2^{1/n}j_\nu a) + f_\nu(2^{1/n}j_\nu b)]/j_\nu^{2\nu+1} \\
&= \lim_{t\to 0+}\left\{\frac{-1}{j_\nu}\left[\frac{1}{(1-t)^{1/n}-1} + \frac{1}{(1+t)^{1/n}-1}\right] - \frac{2\nu+1}{j_\nu} + 2\frac{I_{\nu+1}(j_\nu)}{I_\nu(j_\nu)} + o(1)\right\} \\
&\quad = -\frac{4\nu+2}{j_\nu} + 2\frac{I_{\nu+1}(j_\nu)}{I_\nu(j_\nu)},
\end{aligned}
$$

since $n = 2\nu + 2$. We have yet to show that this last quantity is negative.

From the recursion relation for $J_{\nu+1}$ and the infinite product representation of J_ν [1, p. 370] we deduce that

$$
\frac{J_{\nu+1}(x)}{J_\nu(x)} = -(\log|J_\nu|)'(x) + \frac{\nu}{x} = 2x\sum_{m=1}^{\infty}\frac{1}{j_{\nu,m}^2 - x^2}, \tag{5.5}
$$

$$
\frac{I_{\nu+1}(x)}{I_\nu(x)} = 2x\sum_{m=1}^{\infty}\frac{1}{j_{\nu,m}^2 + x^2}. \tag{5.6}
$$

In particular,

$$\sum_{m=1}^{\infty} \frac{1}{j_{\nu,1}^2} = \lim_{x\to 0+} \frac{J_{\nu+1}(x)}{2xJ_\nu(x)} = \frac{1}{4(\nu+1)} \tag{5.7}$$

by the series expansions for J_ν and $J_{\nu+1}$. By (5.6) and (5.7),

$$\begin{aligned}
2\frac{I_{\nu+1}(j_\nu)}{I_\nu(j_\nu)} &= 4j_{\nu,1}\sum_{m=1}^{\infty}\frac{1}{j_{\nu,m}^2+j_{\nu,1}^2} = \frac{2}{j_{\nu,1}} + 4j_{\nu,1}\sum_{m=2}^{\infty}\frac{1}{j_{\nu,m}^2+j_{\nu,1}^2}\\
&< \frac{2}{j_{\nu,1}} + 4j_{\nu,1}\sum_{m=2}^{\infty}\frac{1}{j_{\nu,m}^2}\\
&= \frac{-2}{j_{\nu,1}} + \frac{j_{\nu,1}}{(\nu+1)}\\
&< \frac{2(\nu+2)}{j_{\nu,1}},
\end{aligned}$$

with the final inequality following from the bound $j_{\nu,1}^2 < 2(\nu+1)(\nu+3)$ [39, p. 486]. We conclude that

$$\lim_{a\to 2^{-1/n}-} F_\nu(a)/j_\nu^{2\nu+1} < -\frac{4\nu+2}{j_\nu} + \frac{2\nu+4}{j_\nu} = \frac{2}{j_\nu}(1-\nu) \le 0,$$

since $2\nu+2 = n \ge 4$, which proves (5.4).

Given (5.4), our desired goal (5.3) will follow (in the case that $a > 0$) once we establish the following:

$$\text{if } 0 < a < 2^{-1/n} \text{ and } F_\nu(a) = 0 \text{ then } F_\nu'(a) > 0. \tag{5.8}$$

Towards that end, recall from Appendix 1 of [3] that

$$\begin{aligned}
f_\nu'(x) &= x^{2\nu+1}\left(2 + \frac{J_{\nu+1}(x)^2}{J_\nu(x)^2} - \frac{I_{\nu+1}(x)^2}{I_\nu(x)^2}\right)\\
&= 2x^{n-1} + f_\nu(x)\left[\frac{J_{\nu+1}(x)}{J_\nu(x)} - \frac{I_{\nu+1}(x)}{I_\nu(x)}\right].
\end{aligned} \tag{5.9}$$

Suppose that $0 < a < 2^{-1/n}$ and $F_\nu(a) = 0$. Since $db/da = -a^{n-1}/b^{n-1}$,

$$F_\nu'(a) = 2^{1/n}j_\nu a^{n-1} f_\nu(2^{1/n}j_\nu a)G_\nu(a)$$

where

$$\begin{aligned}
G_\nu(a) :=& \frac{1}{a^{n-1}}\left[\frac{J_{\nu+1}(2^{1/n}j_\nu a)}{J_\nu(2^{1/n}j_\nu a)} - \frac{I_{\nu+1}(2^{1/n}j_\nu a)}{I_\nu(2^{1/n}j_\nu a)}\right]\\
&+ \frac{1}{b^{n-1}}\left[\frac{J_{\nu+1}(2^{1/n}j_\nu b)}{J_\nu(2^{1/n}j_\nu b)} - \frac{I_{\nu+1}(2^{1/n}j_\nu b)}{I_\nu(2^{1/n}j_\nu b)}\right].
\end{aligned}$$

Next, from (5.5) and (5.6) we see that

$$\begin{aligned}
G_\nu(a) &= \frac{4(2^{1/n}j_\nu)^3}{a^{n-4}}\sum_{m=1}^{\infty}\frac{1}{j_{\nu,m}^4-(2^{1/n}j_\nu a)^4}+\frac{4(2^{1/n}j_\nu)^3}{b^{n-4}}\sum_{m=1}^{\infty}\frac{1}{j_{\nu,m}^4-(2^{1/n}j_\nu b)^4}\\
&> \frac{4(2^{1/n}j_\nu)^3}{a^{n-4}}\frac{1}{j_{\nu,1}^4-(2^{1/n}j_\nu a)^4}+\frac{4(2^{1/n}j_\nu)^3}{b^{n-4}}\frac{1}{j_{\nu,1}^4-(2^{1/n}j_\nu b)^4}\\
&= \frac{4(2^{3/n})}{j_\nu a^{n-4}b^{n-4}}\frac{2^{4/n}-a^{n-4}-b^{n-4}}{(1-(2^{1/n}a)^4)((2^{1/n}b)^4-1)}\\
&\geq 0
\end{aligned}$$

since each term in the numerator is nonnegative and each term in the denominator is positive; here we use that $n \geq 4$. The positivity of G_ν establishes (5.8) and thus proves (5.3) when $a > 0$. By continuity, then, $F_\nu(0) \leq 0$.

We must still prove the $a = 0$ case of (5.3), *i.e.*, that $F_\nu(0) < 0$. When $n = 4$ and $\nu = 1$, we simply compute that $F_\nu(0) = f_1(2^{1/4}j_1) < 0$; equivalently, $2^{1/4}j_1 \approx 4.56 < 4.61 \approx k_1$. For the remainder of this part of the proof, we assume $n > 4$. Assume in contradiction to what we desire that $F_\nu(0) = 0$, so that $f_\nu(2^{1/n}j_\nu) = 0$ and thus $f'_\nu(2^{1/n}j_\nu) = 2(2^{1/n}j_\nu)^{n-1}$ by (5.9). The Taylor expansion of f_ν around $2^{1/n}j_\nu$ is

$$f_\nu(x) = 2(2^{1/n}j_\nu)^{n-1}(x-2^{1/n}j_\nu)+O(x-2^{1/n}j_\nu)^2,$$

and $b-1 = -a^n/n+O(a^{2n})$ for a near 0, so that for a near 0 (and b near 1),

$$f_\nu(2^{1/n}j_\nu b) = -2(2^{1/n}j_\nu)^n a^n/n+O(a^{2n}) = -(2^{1/n}j_\nu)^n a^n/(\nu+1)+O(a^{2n}).$$

On the other hand, (5.5), (5.6) and (5.7) yield that for x near 0,

$$\begin{aligned}
f_\nu(x) &= 4x^n\sum_{m=1}^{\infty}\frac{j_{\nu,m}^2}{j_{\nu,m}^4-x^4} = x^n\left(4\sum_{m=1}^{\infty}\frac{1}{j_{\nu,m}^2}+Cx^4+O(x^8)\right)\\
&= x^n/(\nu+1)+Cx^{n+4}+O(x^{n+8})
\end{aligned}$$

for some positive constant C. Thus for a near 0,

$$f_\nu(2^{1/n}j_\nu a) = (2^{1/n}j_\nu)^n a^n/(\nu+1)+C(2^{1/n}j_\nu)^{n+4}a^{n+4}+O(a^{n+8}).$$

Since $n > 4$ by assumption in this part of the proof, we deduce that

$$F_\nu(a) = f_\nu(2^{1/n}j_\nu a)+f_\nu(2^{1/n}j_\nu b) = C(2^{1/n}j_\nu)^{n+4}a^{n+4}+o(a^{n+4}) > 0$$

for all a near 0. Because this contradicts the fact proved above that $F_\nu(a) < 0$ for $0 < a < 2^{-1/n}$, we must conclude that $F_\nu(0) < 0$, as desired. We have proved (5.3) and hence also Theorem 4.

Finally, we show that Theorem 4 improves on the results of Talenti [34] mentioned in Section 3.

By the values given in Section 3, we know that $d_n' < d_n$ for $n = 4, 5, 6$. For any n, putting $t = 1/2$ in [34, (2.18)] yields that

$$d_n' \leq p(1)/(2p(1/2)), \tag{5.10}$$

using Talenti's function "$p(t)$". By [34, Th.2] we get that $p(1) = k_\nu^{-4}$ and that $\tilde{z} := (1/2)^{1/n} p(1/2)^{-1/4}$ is the smallest positive root of the equation

$$P(z) := 1 - \frac{\nu + 1}{z} \left\{ \frac{I_{\nu+1}(z)}{I_\nu(z)} + \frac{J_{\nu+1}(z)}{J_\nu(z)} \right\} = 2.$$

Observe that $P(z) < 1$ for all $0 < z < j_\nu$, that $P(z)$ is undefined at $z = j_\nu$, that $P(z) > 2$ when z is just larger than j_ν, and that $P(k_\nu) = 1$. Hence $j_\nu < \tilde{z} < k_\nu$. From this and (5.10) we deduce that for any n,

$$d_n' < 2^{-1+4/n}. \tag{5.11}$$

For $5 \leq n \leq 9$, direct calculation shows that $d_n' < 2^{-1+4/n} \leq 2^{-1/5} < d_n$. For $n \geq 10$,

$$d_n' < 2^{-1+4/n} < 2^{4/n}(1 + 4/n)^{-2} < 2^{4/n}(j_\nu/j_{\nu+1})^4 < 2^{4/n}(j_\nu/k_\nu)^4 = d_n,$$

where the third inequality results from applying the Payne–Pólya–Weinberger/Thompson bound $\lambda_2/\lambda_1 \leq 1 + 4/n$ to a ball —see [2 p. 1631] for this and related inequalities.

We have established that $d_n' < d_n$ for all n. Furthermore, $\limsup_{n\to\infty} d_n' \leq 1/2$ by (5.11), and so our Theorem 4 improves Talenti's result by a factor of almost 2 for large n.

6. – Buckling plates in the plane

We examine here various planar regions Ω and the lower estimates for the buckling load $\Lambda(\Omega)$ provided by Bramble and Payne's Corollary 2. We do not claim that these lower estimates are the best known; our goal here is rather to show that they are at least of the right order of magnitude.

In the following table, the three columns list the domain Ω, the lower bound on $\Lambda(\Omega)$ provided by Corollary 2, and for some domains, a better lower bound on the buckling load taken from the literature (truncated to three significant figures).

Ω	Cor. 2	Other [ref.]
Square with side a.	$36.3/a^2$	$52.3/a^2$ [35]
Rectangle with sides $a, 2a$.	$18.1/a^2$	
Equilateral triangle with side a.	$83.9/a^2$	
Right–triangle with sides $a, a, a\sqrt{2}$.	$72.6/a^2$	$140/a^2$ [42, p. 143]
Ellipse with semi–axes a, b.	$11.5/ab$	see below

For ellipses with a/b equal to $1.25, 2$ and 3 respectively, Y. Shibaoka [31, p. 532] showed that Λ most probably has values approximately equal to $15.3/ab, 20.9/ab$ and $29.8/ab$. For ellipses with a/b equal to $1.2, 1.4, 1.6, 2$ and 4 respectively, J.W. McLaurin [23, p. 40] found that Λ is greater than or equal to $15.1/ab, 16.1/ab, 17.5/ab, 20.8/ab$ and $39.0/ab$.

For each domain considered above, Corollary 2 provides a respectable, if conservative, lower bound for $\Lambda(\Omega)$. Furthermore, Corollary 2 is extremely easy to apply in practice, as we need only compute the area of Ω.

Note that most of the numerical estimates quoted above have been rescaled from the original references, by the relation $\Lambda(t\Omega) = t^{-2}\Lambda(\Omega)$.

Keep in mind that for a geometrically "nice" domain, we might be able to use Payne's inequality $\Lambda \geq \lambda_2$ directly (without resorting to Krahn's Lemma 5) by somehow finding the exact value of λ_2, or at least a good lower bound. For a square of side a this gives $\Lambda \geq \lambda_2 = 5\pi^2/a^2 > 49.3/a^2$, which is remarkably close to the actual value $52.3/a^2$ of the buckling load. For a rectangle of sides a, b, we get $\Lambda \geq \lambda_2 = \pi^2(a^{-2} + 4b^{-2})$. For ellipses, B.A. Troesch [37, p. 769] has shown that $\lambda_2 \geq 12.3/ab$, which improves somewhat on the table entry above. We can also obtain results for the buckling loads of domains with re–entrant corners. For example, the L–shape made up of three squares of side a has $\lambda_2 > 15.1/a^2$ (see [12]).

We do not claim to have exhausted in this section the list of particular domains treated by other authors; we simply hope to have provided a few enlightening examples.

To finish the section, we point out three other methods for estimating the critical buckling load. When Ω is very nearly a disk, the perturbation methods of Pólya and Szegö [29] apply. For more general domains, P.-Y. Shih and H. L. Schreyer [32] have exploited a variational method for bounding $\Lambda(\Omega)$ from below, using certain Rayleigh quotients of the vibration of Ω under prescribed loads. They get $\Lambda \geq 28/a^2$ for the square. Their paper also surveys work in recent decades on buckling problems. Lastly, the method of inclusion sometimes yields useful bounds: first find a larger domain $\Omega' \supset \Omega$ for which a lower estimate on $\Lambda(\Omega')$ exists, then use the fact that $\Lambda(\Omega) \geq \Lambda(\Omega')$ since $W_0^{2,2}(\Omega) \subset W_0^{2,2}(\Omega')$.

7. – Vibrating plates in the plane

As in the previous section, we examine various planar regions Ω, but this time we focus on the lower estimates for the fundamental tone $\Gamma(\Omega)$ provided by Nadirashvili's result Theorem 3. These lower estimates are not the best known for any of the domains considered. One should regard that as the price to be paid for the unusual breadth and ease of application of Nadirashvili's result: it applies to *all* domains, and to apply it one need only compute the area of the domain. Even so, the price paid need not be too great; the table below indicates that for domains that are not too elongated in any direction, Theorem 3 gives lower bounds of the right order of magnitude.

The three columns of the following table list the domain Ω, the lower bound on $\Gamma(\Omega)$ provided by Theorem 3, and the best lower bound on the fundamental tone known to us (truncated to four significant figures).

Ω	Th. 3	Other [ref.]
Square with side a.	$1030/a^4$	$1294/a^4$ [20, p. 58] [4, p. 197]
Rectangle with sides $a, 2a$.	$257.5/a^4$	$603.8/a^4$ [4, p. 197] [19, p. 588]
Equilateral triangle with side a.	$5493/a^4$	$9798/a^4$ [19, p. 588]
Right–triangle with sides $a, a, a\sqrt{2}$.	$4120/a^4$	$8724/a^4$ [19, p. 588]
Ellipse with semi–axes a, b.	$104.3/a^2b^2$	see below

Considering ellipses with a/b equal to 1.25, 2 and 3 respectively, Y. Shibaoka [20, p. 37] found that Γ most probably has values approximately equalling $109.8/a^2b^2$, $189.0/a^2b^2$ and $359.7/a^2b^2$. For ellipses with a/b equal to 1.1, 1.2, 2 and 4 respectively, J. McLaurin [22, p. 681] found that Γ is greater than or equal to $105.7/a^2b^2$, $109.4/a^2b^2$, $187.3/a^2b^2$ and $587.2/a^2b^2$.

Note that most of the numerical estimates quoted above have been rescaled from the original references, by the relation $\Gamma(t\Omega) = t^{-4}\Gamma(\Omega)$.

For a rectangular plate Ω of sides a and b, Theorem 3 gives that $\Gamma(\Omega) > 1030/a^2b^2$. For long rectangles, say with $b/a > 2.2$, it turns out that a better bound is

$$\Gamma \geq \Lambda\lambda_1 \geq \lambda_2\lambda_1 = \pi^4(a^{-2} + b^{-2})(a^{-2} + 4b^{-2}).$$

When Ω is very nearly a disk, Pólya and Szegö's [29] perturbation methods apply, and for arbitrary domains Ω, the inclusion method of Section 6 again sometimes yields useful bounds.

As in the previous section, we have aimed simply to provide a small, representative sample of work on particular domains. For further reading we suggest in particular the rigorous works of A. Weinstein and W. Stenger [41], N.W. Bazley, D.W. Fox and J.T. Stadter [4], and J.R. Kuttler and V.G. Sigillito [18,19]. See also the books of A. W. Leissa [20] and G. Fichera [10] for expositions of much numerical and theoretical work on the first few eigenvalues and modes of vibration.

8. – Related results and open problems

A number of questions about buckling and vibrating plates remain open. To begin with, does the Pólya-Szegö conjecture $\Lambda(\Omega) \geq \Lambda(\Omega^{\#})$ hold true? Next, can one characterize geometrically the domains for which a first eigenfunction u_1 does not change sign? For these domains, Szegö has proved the Pólya-Szegö conjecture. We must regard such domains as exceptional, however, since u_1 does change sign for every rectangle and also for certain smoothly–bounded convex regions, by recent work of V. A. Kozlov, V. A. Kondrat'ev and V. G. Maz'ya [14].

For vibrating plates, does the Rayleigh conjecture $\Gamma(\Omega) \geq \Gamma(\Omega^{\#})$ hold in $\mathbf{R}^n, n \geq 4$? Also, can one characterize geometrically the domains for which a first eigenfunction v_1 does not change sign? Szegö has proved Rayleigh's conjecture for such domains, but we must again regard these domains as exceptional. Indeed, over forty years ago R. J. Duffin and D. H. Shaffer found that a sufficiently thick annulus will have a nodal line (and a principal eigenvalue of multiplicity two); see [7] for references and recent work[1]. C.V. Coffman [6] showed that for a domain whose boundary contains a right angle, a principal eigenfunction must oscillate infinitely often along every ray approaching the right angle. Even worse, perhaps, is that principal eigenfunctions change sign for certain smooth simply connected regions, again by Kozlov, Kondrat'ev and Maz'ya's work [14]. Incidentally, a number of authors have considered the similar question of the positivity of the Green function for the biharmonic operator, a question first raised by Hadamard. This Green function is positive for the unit disk and for ellipses of small eccentricity, but changes sign for very eccentric ellipses and oscillates badly if the domain has a right–angle corner in its boundary, as S. Osher [26] showed. Positivity of the Green function can result in surprising properties: P.L. Duren et al. [9] found contractive zero–divisors in each Bergman space $A^p, 1 \leq p < \infty$, over the unit disk by exploiting the positivity of the Green function.

E. Mohr [24] contributed in a different manner to the work on Rayleigh's conjecture: he showed that if among all bounded domains of a given area there exists one that minimizes the principal frequency, then the domain must be circular. The question of the existence of a minimizing domain remains open.

Returning to buckling plates, we might hope that by adapting Nadirashvili's proof (or Ashbaugh–Benguria's) from the vibrating plate in $\mathbf{R}^2$ to the buckling plate, we could get that $\Lambda(\Omega) \geq \Lambda(\Omega^{\#})$. However, this approach merely provides another proof of Theorem 1. We cannot explain intuitively why this line of proof succeeds completely in showing the minimality of the ball's principal eigenvalue for the *vibrating* plate yet succeeds only partially for the *buckling* plate.

Since we are unable to solve the original problem, we raise an easier one.

[1] For descriptions of recent work by Behnke, Owen and Wieners, see Section 2 of E.B. Davies, "L^p spectral theory of higher order elliptic differential operators", preprint.

CONJECTURE.

$$\Lambda(\Omega)\lambda_1(\Omega) \geq \frac{j_{\nu+1}^2 j_\nu^2 v_n^{4/n}}{|\Omega|^{4/n}} = \Lambda(\Omega^\#)\lambda_1(\Omega^\#),$$

with equality if and only if Ω is a ball.

In view of the Faber–Krahn inequality, this last conjecture would follow from the one of Pólya and Szegö in Section 2, and hence is presumably easier to prove.

Another way to improve our results might be to restrict attention to plates Ω that are convex, which is reasonable for applications. For convex Ω, Krahn's estimate on λ_2 in Lemma 5 is no longer optimal, since Ω can no longer approach the case of two disjoint balls. Thus even a partial result for the following problem might improve our bound on Λ.

PROBLEM. *Minimize $\lambda_2(\Omega)$, assuming Ω is convex and has fixed volume. Describe the shape of the extremal domain Ω, if possible.*

B.A. Troesch [37] raised this problem and found the minimum for elliptical regions in the plane. We remark also that an extremal domain is known to exist for the general version of the problem, at least in $\mathbf{R}^2$, by work of S.J. Cox and M. Ross [8, Th.2.3]. Still, for general convex membranes Ω, we do not know of any explicit lower estimate on $\lambda_2(\Omega)$ in terms simply of the area/volume of Ω, except for Krahn's Lemma 5.

One final lower bound for $\Lambda(\Omega)$ deserves mention, in light of Theorem 1. By Green's formula, Cauchy–Schwarz and the definition of $\Gamma(\Omega)$, we obtain that

$$\Lambda(\Omega) \geq \Gamma(\Omega)^{1/2}. \tag{8.1}$$

Thus for dimensions $n \geq 4$, our estimate on the buckling load in Theorem 1 follows from our estimate on the fundamental tone in Theorem 4. Any improvement in Theorem 4, such as a proof of Rayleigh's conjecture, would hence improve Theorem 1, when $n \geq 4$. Notice, however, that $2^{1/n} j_\nu > k_\nu$ for $n = 2, 3$, and so combining (8.1) with Nadirashvili's and Ashbaugh–Benguria's results does *not* improve Theorem 1 in the case that $n = 2, 3$. Interestingly, that $2^{1/n} j_\nu > k_\nu$ for $n = 2, 3$, is precisely what allows the proofs of Nadirashvili and of Ashbaugh and Benguria to succeed; for $n \geq 4$, however, taking Ω to be a ball in Theorem 4 shows that $k_\nu > 2^{1/n} j_\nu$.

To conclude the paper, we summarize for the reader's convenience several inequalities used in the above discussions:

$$\Lambda^2 \geq \Gamma \geq \Lambda\lambda_1 \geq \lambda_2\lambda_1 > \lambda_1^2.$$

REFERENCES

[1] M. ABRAMOWITZ - I.A. STEGUN, Eds., Handbook of Mathematical Functions. National Bureau of Standards Applied Mathematics Series, Vol. 55. U.S. Government Printing Office, Washington, D.C., 1964

[2] M.S. ASHBAUGH - R.D. BENGURIA, *More bounds on eigenvalue ratios for Dirichlet Laplacians in N dimensions*, SIAM J. Math. Anal., **24** (1993), 1622-1651.

[3] M.S. ASHBAUGH - R.D. BENGURIA, *On Rayleigh's conjecture for the clamped plate and its generalization to three dimensions*, Duke Math. J., **78** (1995), 1-17.

[4] N.W. BAZLEY - D.W. FOX - J.T. STADTER, *Upper and lower bounds for the frequencies of rectangular clamped plates*, Z. Angew. Math. Mech., **47** (1967), 191-198.

[5] J.H. BRAMBLE - L.E. PAYNE, *Pointwise bounds in the first biharmonic boundary value problem*, J. Math. & Phys., **42** (1963), 278-286.

[6] C.V. COFFMAN, *On the structure of solutions to $\Delta\Delta u = \lambda u$ which satisfy the clamped plate conditions on a right angle*, SIAM J. Math. Anal., **13** (1982), 746-757.

[7] C.V. COFFMAN - R.J. DUFFIN, *On the fundamental eigenfunctions of a clamped punctured disk*, Adv. in Appl. Math., **13** (1992), 142-151.

[8] S.J. COX - M. ROSS, *Extremal eigenvalue problems for starlike planar domains*, J. Differential Equations., **120** (1995), 174-197.

[9] P.L. DUREN - D.KHAVINSON - H.S. SHAPIRO - C. SUNDBERG, *Contractive zero-divisors in Bergman spaces*, Pacific J. Math., **157** (1993), 37-56.

[10] G. FICHERA, Numerical and Quantitative Analysis, Trans. by S. Graffi. Pitman, London, 1978.

[11] D.W. FOX - W.C. RHEINBOLDT, *Computational methods for determining lower bounds for eigenvalues of operators in Hilbert space*, SIAM Rev., **8** (1996), 427-462.

[12] L. FOX - P. HENRICI - C. MOLER, *Approximations and bounds for eigenvalues of elliptic operators*, SIAM J. Numer. Anal., **4** (1967), 89-103.

[13] B. KAWOHL, Rearrangements and Convexity of Level Sets in PDE. Springer-Verlag, Berlin, 1985.

[14] V.A. KOZLOV - V.A. KONDRAT'EV - V.G. MAZ'YA, *On sign variation and the absence of "strong" zeros of solutions of elliptic equations*, Math. USSR-Izv., **34** (1990), 337-353.

[15] B. KNAUER, *Untere Schranken für die Eigenwerte selbstadjungierter positiv-definiter Operatoren*, Numer. Math., **17** (1971), 166-171.

[16] E. KRAHN, *Über Minimaleigenschaften der Kugel in drei und mehr Dimensionen*, Acta Comm. Univ. Tartu (Dorpat), **A9** (1926), 1-44. English translation in: Edgar Krahn 1894-1961: A Centenary Volume. Ed: Ü. Lumiste and J. Peetre, pp.139-174. IOS Press, Amsterdam, 1994.

[17] J.R. KUTTLER, *Upper and lower bounds for eigenvalues by finite differences*, Pacific J. Math., **35** (1970), 429-440.

[18] J.R. KUTTLER - V.G. SIGILLITO, *Upper and lower bounds for frequencies of clamped rhombical plates*, J. Sound Vibration, **68** (1980), 597-607.

[19] J.R. KUTTLER - V.G. SIGILLITO, *Upper and lower bounds for frequencies of trapezoidal and triangular plates*, J. Sound Vibration, **78** (1981), 585-590.

[20] A.W. LEISSA, Vibration of Plates. NASA SP-160. Office of Technology Utilization, NASA, Washington, D.C., 1969.

[21] L. LORCH, *Monotonicity of the zeros of a cross product of Bessel functions*, Methods Applicat. Anal., **1** (1994), 75-80.

[22] J. MCLAURIN, *Bounding eigenvalues of clamped plates*, Z. Angew. Math. Phys., **19** (1968), 676-681.

[23] J.W. MCLAURIN, Bounds for Vibration Frequencies and Buckling Loads of Clamped Plates. Dissertation No. 4415, ETH, Zürich. Juris-Druck Verlag, Zürich, 1969.

[24] E. MOHR, *Über die Rayleighsche Vermutung: Unter allen Platten von gegebener Fläche und konstanter Dichte und Elastizität hat die kreisförmige den tiefsten Grundton*, Ann. Mat. Pura Appl., **104** (1975), 85-122.

[25] N. NADIRASHVILI, *Rayleigh's conjecture on the principal frequency of the clamped plate*, Arch. Ration. Mech. Anal., **129** (1995), 1-10. See also *New isoperimetric inequalities in mathematical physics*. In: Partial Differential Equations of Elliptic Type. Ed: A. Alvino, E. Fabes and G. Talenti. Cambridge University Press, Cambridge, 1994.

[26] S. OSHER, *On Green's function for the biharmonic equation in a right-angle wedge*, J. Math. Anal. Appl., **43** (1973), 705-716.

[27] L.E. PAYNE, *Inequalities for eigenvalues of membranes and plates*, J. Rat. Mech. Anal., **4** (1955), 517-529.

[28] G. PÓLYA, *On the characteristic frequencies of a symmetric membrane*, Math. Z., **63** (1955), 331-337.

[29] G. PÓLYA - G. SZEGÖ, Isoperimetric Inequalities in Mathematical Physics. Princeton University Press, Princeton, N.J., 1951.

[30] LORD RAYLEIGH (J.W. STRUTT), The Theory of Sound, Vol. 1. Dover Publications, New York, 1945.

[31] Y. SHIBAOKA, *On the buckling of an elliptic plate with clamped edge, II*, J. Phys. Soc. Japan, **12** (1957), 529-532. See also *On the buckling of an elliptic plate with clamped edge, I.* J. Phys. Soc. Japan, **11** (1957), 1088-1091.

[32] P.-Y. SHIH - H.L. SCHREYER, *Lower bounds to fundamental frequencies and buckling loads of columns and plates*, Internat. J. Solids and Structures, **14** (1978), 1013-1026.

[33] G. SZEGÖ, *On membranes and plates*, Proc. Nat. Acad. Sci. U.S.A., **36** (1950), 210-216. See also *Note to my paper "On membranes and plates,"* Proc. Nat. Acad. Sci. U.S.A., **44** (1958), 314-316.

[34] G. TALENTI, *On the first eigenvalue of the clamped plate*, Ann. Mat. Pura Appl., **129** (1981), 265-280.

[35] G.I. TAYLOR, *The buckling load for a rectangular plate with four clamped edges*, Z. Angew. Math. Mech., **13** (1933), 147-152.

[36] S. TIMOSHENKO - J.M. GERE, Theory of Elastic Stability. 2nd Ed. McGraw-Hill, New York, 1961.

[37] B.A. TROESCH, *Elliptical membranes with smallest second eigenvalue*, Math. Comp., **27** (1973), 767-772.

[38] W. VELTE, *Bounds for critical values and eigenfrequencies of mechanical systems*, in: Proceedings of the German Italian Symposium "Applications of Mathematics in Technology", held in Rome, 1984, pp. 469-484. B.G. Teubner, Stuttgart, 1984.

[39] G.N. WATSON, Theory of Bessel Functions, 2nd Ed. Cambridge University Press, Cambridge, 1944.

[40] H.F. WEINBERGER, Variational Methods for Eigenvalue Approximation. Regional conference series in applied mathematics, **15**. SIAM, Philadelphia, 1974.

[41] A. WEINSTEIN - W. STENGER, Methods of Intermediate Problems for Eigenvalues: Theory and Ramifications. Academic Press, New York, 1972.

[42] W.H. WITTRICK, *Symmetrical buckling of right-angled isosceles triangular plates*, Aeronaut. Quart., **5** (1954), 131-143.

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