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1. - Introduction

Given in $\mathbb{R}^n$, $n \geq 5$, the ball

$$\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$$

and a nonlinear analytic function

$$F: \mathbb{R}^{n^2 \times n} \ni p \mapsto F(p) \in \mathbb{R},$$

let us consider the functional

$$\Phi(u) = \int_\Omega F(\nabla u(x))dx,$$

where

$$u: \Omega \to \mathbb{R}^{n^2}$$

is a function belonging to the admissible space to be precised later on. We will prove that the above functional achieves the minimum at a point which is a Lipschitz function but not of class C^1.

In the literature, an example of "irregular" solution (not C^1) of the Euler-Lagrange system, arising from a functional of the above type, was already constructed by Nečas in the paper [5], but that counterexample was not optimal; namely, it was valid for very high Euclidean dimension.

Besides this result nothing was up to this moment known e.g. for $n = 3$, 4, 5.

So our goal was to write a counterexample which had validity in an as low as possible euclidean dimension. But all efforts for modifying it as well as for sharpening the involved estimates in order to reduce the dimension (< 5) were

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not successful; so there still remains a hope that Euler-Lagrange systems will have, at least for $n = 3$ (for $n = 2$ there is regularity), only regular solutions provided the functional is smooth enough and convex. The case $n = 4$ is still an open problem.

Indeed, this is the last possibility to get the regularity in the three-dimensional euclidean space because, as it is well-known, famous counter-examples (se [1], [4]) have shown that for elliptic systems of the type

$$\int_\Omega A_{ij}^{rs}(x) \frac{\partial u_j}{\partial x_s} \frac{\partial \varphi_r}{\partial x_i} = 0, \quad A_{ij}^{rs}(x) \in L^\infty(\Omega), \quad \varphi \in H_0^1(\Omega, \mathbb{R}^n);$$

and

$$\int_\Omega A_{ij}^{rs}(u) \frac{\partial u_j}{\partial x_s} \frac{\partial \varphi_r}{\partial x_i} = 0, \quad A_{ij}^{rs}(u) \text{ analytic, } \varphi \in H_0^1(\Omega, \mathbb{R}^n)$$

there do exist non-regular (not $C^{0,\mu}$) solutions in $\mathbb{R}^3$ (see [2]).

Moreover, it is also known, see for instance [6], that for elliptic systems of the form

$$-\frac{\partial}{\partial x_i}(A_i^r(\nabla u)) = 0$$

with A_i^r real analytic satisfying usual conditions of coercivity and monotocity, non-smooth solutions do exist from $n \geq 3$.

For further details concerning regularity and counterexamples to the regularity to non-linear elliptic systems the reader can refer to [2], [3], [7].

2. - Notations and auxiliary results

Throughout this paper we will make use of the Einstein's summation convention and the following notations:

(2.1) $$\Omega = \{x \in \mathbb{R}^n : |x| < 1\}, \ n \geq 5$$

(2.2) $$u = (u_{ij}(x)), \quad i, \ j = 1, \ldots, n,$$

(2.3) $$|\nabla u| = \left(\frac{\partial u_{ij}}{\partial x_k} \frac{\partial u_{ij}}{\partial x_k} \right)^{1/2},$$

(2.4) $$\tau_i u = \frac{\partial u_{jj}}{\partial x_i},$$

(2.5) $$\nabla_i u = \frac{1}{2} \left(\frac{\partial u_{ij}}{\partial x_j} + \frac{\partial u_{ji}}{\partial x_j} + \tau_i u \right),$$

$$\begin{aligned}\nabla_{ijk}u &= \frac{1}{6}\left(\frac{\partial u_{ij}}{\partial x_k}+\frac{\partial u_{ki}}{\partial x_j}+\frac{\partial u_{jk}}{\partial x_i}+\frac{\partial u_{ji}}{\partial x_k}+\frac{\partial u_{kj}}{\partial x_i}+\frac{\partial u_{ik}}{\partial x_j}\right)\\ &\quad -\frac{2n-1}{3(n^2-1)}(\delta_{ij}\nabla_k u+\delta_{ki}\nabla_j u+\delta_{jk}\nabla_i u).\end{aligned} \tag{2.6}$$

Moreover we will set

$$\tilde{u} = \frac{x_i x_j}{|x|} - \frac{1}{n}\delta_{ij}|x|. \tag{2.7}$$

From (2.7) we deduce easily:

$$\frac{\partial \tilde{u}_{ij}}{\partial x_k} = \frac{\delta_{ik}x_j+\delta_{jk}x_i}{|x|} - \frac{x_i x_j x_k}{|x|^3} - \frac{1}{n}\delta_{ij}\frac{x_k}{|x|}. \tag{2.8}$$

Let us note that because of $\tilde{u} \in C^{0,1}(\Omega, \mathbb{R}^{n^2})$, then its gradient exists almost everywhere and it is bounded and measurable.

Put then

$$I_1 = \int_\Omega \frac{x_i x_j}{|x|^3}\varphi_{ij}, \quad I_2 = \int_\Omega \frac{\varphi_{ii}}{|x|}, \quad \forall \varphi \in W_0^{1,4}(\Omega, \mathbb{R}^{n^2});$$

it easy to deduce, integrating by parts, that:

$$\int_\Omega \frac{\partial \tilde{u}_{ij}}{\partial x_k}\frac{\partial \varphi_{ij}}{\partial x_k} = (n+1)I_1 - \frac{n+1}{n}I_2, \tag{2.9}$$

$$\int_\Omega \nabla_{ijk}\tilde{u}\nabla_{ijk}\varphi = \frac{n^3-n^2+n}{n^2-1}I_1 - \frac{2n-1}{2(n-1)}I_2, \tag{2.10}$$

$$\int_\Omega \nabla_i\tilde{u}\nabla_i\varphi = \frac{n^2-1}{n}I_1 - \frac{(n^2-1)(n+1)}{2n}I_2, \tag{2.11}$$

$$\int_\Omega \nabla_i\tilde{u}\nabla_j\tilde{u}\nabla_k\tilde{u}\nabla_{ijk}\varphi = -\left(\frac{n^2-1}{n}\right)^3\int_\Omega \nabla_{ijk}\tilde{u}\nabla_{ijk}\varphi, \tag{2.12}$$

$$\int_\Omega \nabla_i\varphi\nabla_j\tilde{u}\nabla_k\tilde{u}\nabla_{ijk}\tilde{u} = -\frac{n^2-1}{n}\int_\Omega \nabla_i\tilde{u}\nabla_i\varphi, \tag{2.13}$$

$$\int_\Omega \nabla_i\tilde{u}\tau_i\varphi = \frac{(1-n)(n^2-1)}{n}I_2. \tag{2.14}$$

3. - The counterexample

We will prove that the matrix-valued function (2.7), which belongs to $C^{0,1}(\Omega, \mathbb{R}^{n^2})$ but not to $C^1(\Omega, \mathbb{R}^{n^2})$, is the global (unique) minimum of the functional

$$
\begin{aligned}
\Phi(u) = {} & \frac{\alpha}{4}\int_\Omega |\nabla u|^4 + \frac{\alpha_0}{2}\int_\Omega |\nabla u|^2 + \frac{\alpha_1}{2}\int_\Omega \nabla_m u \nabla_m u \nabla_{ijk} u \nabla_{ijk} u \\
& + \frac{\alpha_2}{4}\int_\Omega (\nabla_{ijk} u \nabla_{ijk} u)^2 + \frac{\alpha_3}{4}\int_\Omega (\nabla_i u \nabla_i u)^2 \\
& + \alpha_4 \int_\Omega \nabla_i u \nabla_j u \nabla_k u \nabla_{ijk} u + \alpha_5 \int_\Omega \nabla_i u \tau_i u + \frac{\alpha_6}{2}\int_\Omega \tau_i u \tau_i u
\end{aligned}
\tag{3.1}
$$

being $u \in \{u \in W^{1,4}(\Omega, \mathbb{R}^{n^2}) : u = \tilde{u} \text{ on } \partial\Omega\}$ and

$$
\begin{cases}
\alpha = \dfrac{n}{(2n^2 - n - 1)^3} \\
\alpha_0 = \left(\dfrac{1}{n^2 - 1}\right)^2 \\
\alpha_1 = \begin{cases} \dfrac{n^4}{(n^2 + 2)(n^2 - 1)} & n \le 50 \\ 0 & n > 50 \end{cases} \\
\alpha_2 = n^2 - 1 - \left(\dfrac{n^2 - 1}{n}\right)^2 \alpha_1 \\
\alpha_3 = \left(\dfrac{n}{n^2 - 1}\right)^3 \left[-\alpha_1 \dfrac{n^2 - 1}{n} + 3n + n(n^2 - n + 4)(\beta_1 + \beta_2) - \beta_3(n + 1)\right] \\
\alpha_4 = \left(\dfrac{n}{n^2 - 1}\right)^2 (1 + \beta_1 + \beta_2) n \\
\alpha_5 = \begin{cases} \dfrac{n - 4 + \beta\beta_3(n + 2)}{2\beta(n - 1)} & n \le 50 \\ \dfrac{1 + \beta\beta_3(n + 2)}{2\beta(n - 1)} & n > 50 \end{cases} \\
\alpha_6 = \dfrac{3\alpha_5^2}{\alpha_0} n^3 \\
\beta = \dfrac{1 - n}{n(n^3 - n^2 - n + 1)} \\
\beta_1 = -\beta(n + 2)\beta_3 \\
\beta_2 = 2\beta(n - 1)\alpha_5 \\
\beta_3 = \alpha_0 + \left(\dfrac{1}{2n^2 - n - 1}\right)^2 .
\end{cases}
\tag{3.2}
$$

The proof will consist of several steps. To begin, we perform the Gateaux derivative of $\Phi(u)$:

$$
\begin{aligned}
[\Phi'(u+t\varphi)]_{t=0} &= \alpha \int_\Omega |\nabla u|^2 \frac{\partial u_{ij}}{\partial x_k}\frac{\partial \varphi_{ij}}{\partial x_k} \\
&+ \alpha_0 \int_\Omega \frac{\partial u_{ij}}{\partial x_k}\frac{\partial \varphi_{ij}}{\partial x_k} + \alpha_1 \int_\Omega \nabla_{ijk}u\nabla_{ijk}u\nabla_m u\nabla_m\varphi \\
&+ \alpha_1 \int_\Omega \nabla_m u\nabla_m u\nabla_{ijk}u\nabla_{ijk}\varphi + \alpha_2 \int_\Omega \nabla_{ijk}u\nabla_{ijk}u\nabla_{abc}u\nabla_{abc}\varphi \\
&+ \alpha_3 \int_\Omega \nabla_i u\nabla_i u\nabla_i u\nabla_j u\nabla_j\varphi + \alpha_2 \int_\Omega \nabla_i u\nabla_j u\nabla_k u\nabla_{ijk}\varphi \\
&+ 3\alpha_4 \int_\Omega \nabla_i\varphi\nabla_j u\nabla_k u\nabla_{ijk}u + \alpha_5 \int_\Omega \nabla_i u\tau_i\varphi \\
&+ \alpha_5 \int_\Omega \tau_i u\nabla_i\varphi + \alpha_6 \int_\Omega \tau_i u\tau_i\varphi,
\end{aligned} \tag{3.3}
$$

$\forall\varphi \in W_0^{1,4}(\Omega, \mathbb{R}^{n^2})$.

Then, because it must be

$$
[\Phi'(\tilde{u}+t\varphi)]_{t=0} = 0, \quad \forall\varphi \in W_0^{1,4}(\Omega, \mathbb{R}^{n^2}), \tag{3.4}
$$

using (2.8)-(2.14) we must thus satisfy the linear system

$$
\begin{cases}
\beta_3(n+1) + \left[\alpha_2 + \left(\frac{n^2-1}{2}\right)^2 \alpha_1\right] \frac{n^3-n^2+n}{n^2-1} \\
\qquad + \left[\alpha_3\left(\frac{n^2-1}{n}\right)^2 + \alpha_1\right]\frac{n^2-1}{n} - \alpha_4\left(\frac{n^2-1}{n}\right)^2 (n^2-n+4) = 0 \\
\beta_3\frac{2}{n} + \left[\alpha_2 + \left(\frac{n^2-1}{n}\right)^2\alpha_1\right]\frac{2n-1}{n^2-1} + \left[\alpha_3\left(\frac{n^2-1}{n}\right)^2 + \alpha_1\right]\frac{n^2-1}{n} \\
\qquad - \alpha_4\left(\frac{n^2-1}{n}\right)^2 \frac{5n-1}{n}\alpha_5(1-n)\frac{n^2-1}{n}\frac{2}{n+1} = 0
\end{cases}
$$

where α, α_0, $\alpha_1, \dots, \alpha_6$ are unknowns. Such a system is satisfied if we chooce α, α_0, $\alpha_1, \dots, \alpha_6$ as in (3.2).

To prove that $\tilde{u}$ is the unique minimum of our functional we must show that (see later on Theorem 3) there exists a constant $c_1 > 0$ such that

$$[\Phi''(u+t\varphi)]_{t=0} \geq c_1 \int_\Omega (1+|\nabla u|^2)|\nabla\varphi|^2, \quad \forall u,\ \varphi, \tag{3.5}$$

where

$$\begin{aligned}
[\Phi''(u+t\varphi)]_{t=0} &= \alpha \int_\Omega |\nabla u|^2|\nabla\varphi|^2 \\
&+ 2\alpha \int_\Omega \left(\frac{\partial u_{ij}}{\partial x_k}\frac{\partial \varphi_{ij}}{\partial x_k}\right)^2 + \alpha_0 \int_\Omega |\nabla\varphi|^2 \\
&+ \alpha_1 \int_\Omega \nabla_{ijk}u\nabla_{ijk}u\nabla_m\varphi\nabla_m\varphi + \alpha_1 \int_\Omega \nabla_m u\nabla_m u\nabla_{ijk}\varphi\nabla_{ijk}\varphi \\
&+ 4\alpha_1 \int_\Omega \nabla_m u\nabla_m\varphi\nabla_{ijk}u\nabla_{ijk}\varphi + \alpha_2 \int_\Omega \nabla_{ijk}u\nabla_{ijk}u\nabla_{abc}\varphi\nabla_{abc}\varphi \\
&+ 2\alpha_2 \int_\Omega (\nabla_{ijk}u\nabla_{ijk}\varphi)^2 + \alpha_3 \int_\Omega \nabla_i u\nabla_i u\nabla_j\varphi\nabla_j\varphi + 2\alpha_3 \int_\Omega (\nabla_i u\nabla_i\varphi)^2 \\
&+ 6\alpha_4 \int_\Omega \nabla_i u\nabla_j\varphi\nabla_k\varphi\nabla_{ijk}u + 6\alpha_4 \int_\Omega \nabla_i\varphi\nabla_j u\nabla_k u\nabla_{ijk}\varphi \\
&+ 2\alpha_5 \int_\Omega \nabla_i\varphi\tau_i\varphi + \alpha_6 \int_\Omega \tau_i\varphi\tau_i\varphi.
\end{aligned} \tag{3.6}$$

More precisely we shall prove that for the Euler-Lagrange system the following condition of strong ellipticity is satisfied:

$$D^2_{p_{abc}p_{ijk}}F(\nabla u)\xi_{abc}\xi_{ijk} > 0, \quad \forall\xi = (\xi_{ijk}) \in \mathbb{R}^{n^2\times n}.$$

We prove the following two lemmas:

LEMMA 1. *The following two inequalities hold: for any real number K*

$$
\begin{aligned}
&\nabla_{ijk}\varphi\nabla_{ijk}\varphi\nabla_m u\nabla_m u + \nabla_m\varphi\nabla_m\varphi\nabla_{ijk}u\nabla_{ijk}u \\
&\quad + 6K^2[\nabla_i u\nabla_i u\nabla_j\varphi\nabla_j\varphi + 2(\nabla_i u\nabla_i\varphi)^2] \\
&\quad - \frac{6}{n+2}\left(\frac{n}{n^2-1} - K\right)^2 \nabla_i u\nabla_i u\nabla_j\varphi\nabla_j\varphi \\
&\quad + \left(\frac{24n}{(n+2)(n^2-1)}K - \frac{48}{n+2}K^2\right)(\nabla_i u\nabla_i\varphi)^2 \\
&\quad + 6K\nabla_i u\nabla_j\varphi\nabla_k\varphi\nabla_{ijk}u + 6K\nabla_i\varphi\nabla_j u\nabla_k u\nabla_{ijk}\varphi \geq 0;
\end{aligned}
\tag{3.7}
$$

$$
2K_1\nabla_i\varphi\tau_i\varphi + K_1^2\tau_i\varphi\tau_i\varphi + \frac{9}{4}n^3|\nabla\varphi|^2 \geq 0, \quad K_1 = \frac{3^\alpha 5n^3}{\alpha_0}. \tag{3.8}
$$

PROOF. To prove (3.7) put, at first

$$0 \neq c^2 = \nabla_l u\nabla_l u,$$

$$
\begin{aligned}
\Theta_{ijk} = c\Bigg[&\nabla_{ijk}\varphi + \frac{K}{c^2}(\nabla_i\varphi\nabla_j u\nabla_k u + \nabla_i u\nabla_j\varphi\nabla_k u + \nabla_i u\nabla_j u\nabla_k\varphi) \\
&+ \frac{\frac{n}{n^2-1} - K}{n+2}(\nabla_i\varphi\delta_{jk} + \nabla_k\varphi\delta_{ij} + \nabla_j\varphi\delta_{ik}) \\
&- \frac{2K}{c^2(n+2)}(\nabla_i u\delta_{jk} + \nabla_k u\delta_{ij} + \nabla_j u\delta_{ik})(\nabla_i u\nabla_i\varphi)\Bigg];
\end{aligned}
$$

we have of course

$$0 \leq \Theta_{ijk}\Theta_{ijk}. \tag{3.9}$$

Analogously, put

$$0 \neq \tilde{c}^2 = \nabla_l\varphi\nabla_l\varphi,$$

$$
\begin{aligned}
\tilde{\Theta}_{ijk} = \tilde{c}\Bigg[&\nabla_{ijk}u + \frac{K}{\tilde{c}^2}(\nabla_i u\nabla_j\varphi\nabla_k\varphi + \nabla_i\varphi\nabla_j u\nabla_k\varphi + \nabla_i\varphi\nabla_j\varphi\nabla_k u) \\
&+ \frac{\frac{n}{n^2-1} - K}{n+2}(\nabla_i u\delta_{jk} + \nabla_k u\delta_{ij} + \nabla_j u\delta_{ik}) \\
&- \frac{2K}{\tilde{c}^2(n+2)}(\nabla_i\varphi\delta_{jk} + \nabla_k\varphi\delta_{ij} + \nabla_j\varphi\delta_{ik})(\nabla_i u\nabla_i\varphi)\Bigg]
\end{aligned}
$$

we have

$$0 \leq \tilde{\Theta}_{ijk}\tilde{\Theta}_{ijk}. \tag{3.10}$$

The proof is achieved using (2.5), (2.6) and summing together (3.9) and (3.10).

To prove (3.8) we argue in the same way noticing that

$$\begin{aligned} 0 &\leq (\nabla_i\varphi + K_1\tau_i\varphi)^2 = \nabla_i\varphi\nabla_i\varphi + 2K_1\nabla_i\varphi\tau_i\varphi + K_1^2\tau_i\varphi\tau_i\varphi \\ &\leq 2K_1\nabla_i\varphi\tau_i\varphi + K_1^2\tau_i\varphi\tau_i\varphi + \frac{9}{4}n^3|\nabla\varphi|^2. \end{aligned}$$

∎

LEMMA 2. *Let $K_2 > 0$ and $K \in R$ be constants such that*

$$K_2\left[\frac{3}{n+2}\left(\frac{n}{n^2-1} - K\right)^2 + \frac{24}{n+2}K^2 - \frac{12n}{(n+2)(n^2-1)}K\right] \geq \frac{2}{3}; \tag{3.11}$$

then

$$\begin{aligned} &K_2[\nabla_{ijk}u\nabla_{ijk}u\nabla_{abc}\varphi\nabla_{abc}\varphi + 2(\nabla_{ijk}u\nabla_{ijk}\varphi)^2] \\ &+ 4\nabla_m u\nabla_m\varphi\nabla_{ijk}u\nabla_{ijk}\varphi + \frac{6}{n+2}\left(\frac{n}{n^2-1} - K\right)^2\nabla_i u\nabla_i u\nabla_j\varphi\nabla_j\varphi \\ &+ \left(\frac{48}{n+2}K^2 - \frac{24n}{(n+2)(n^2-1)}K\right)(\nabla_i u\nabla_i\varphi)^2 \geq 0. \end{aligned} \tag{3.12}$$

PROOF. By the Cauchy-Schwartz inequality it turns out that:

$$\begin{aligned} &K_2[\nabla_{ijk}u\nabla_{ijk}u\nabla_{abc}\varphi\nabla_{abc}\varphi + 2(\nabla_{ijk}u\nabla_{ijk}\varphi)^2] + 4\nabla_m u\nabla_m\varphi\nabla_{ijk}u\nabla_{ijk}\varphi \\ &+ \frac{6}{n+2}\left(\frac{n}{n^2-1} - K\right)^2\nabla_i u\nabla_i u\nabla_j\varphi\nabla_j\varphi \\ &+ \left(\frac{48}{n+2}K^2 - \frac{24n}{(n+2)(n^2-1)}K\right)(\nabla_i u\nabla_i\varphi)^2 \\ &\geq 3K_2(\nabla_{ijk}u\nabla_{ijk}\varphi)^2 + 4\nabla_m u\nabla_m\varphi\nabla_{ijk}u\nabla_{ijk}\varphi \\ &+ \left[\frac{6}{n+2}\left(\frac{n}{n^2-1} - K\right)^2 + \left(\frac{48}{n+2}K^2 - \frac{24n}{(n+2)(n^2-1)}K\right)\right](\nabla_i u\nabla_i\varphi)^2. \end{aligned} \tag{3.13}$$

and the quadratic form on the right-hand side of (3.13) is positive-definite by hypothesis. ∎

Lemma 1 and Lemma 2 enable us to prove the following final:

THEOREM 3. Sufficient condition for satisfying (3.5) is that $n \geq 5$.

PROOF. Suppose at first that $5 \leq n \leq 50$: put

$$K = \frac{\alpha_4}{\alpha_1},\ K_2 = \frac{2}{3}\left[\frac{3}{n+2}\left(\frac{n}{n^2-1} - K\right)^2 + \frac{24}{n+2}K^2 - \frac{12n}{(n+2)(n^2-1)}K\right]^{-1},$$

from (3.7), (3.8) and (3.12) we deduce

$$\begin{aligned} V = {} & \alpha_1[\nabla_{ijk}\varphi\nabla_{ijk}\varphi\nabla_m u\nabla_m u + \nabla_m\varphi\nabla_m\varphi\nabla_{ijk}u\nabla_{ijk}u \\ & + 4\nabla_m u\nabla_m\varphi\nabla_{ijk}u\nabla_{ijk}\varphi] \\ & + K_2\alpha_1[\nabla_{ijk}u\nabla_{ijk}u\nabla_{abc}\varphi\nabla_{abc}\varphi + 2(\nabla_{ijk}u\nabla_{ijk}\varphi)^2] \\ & + 6K^2\alpha_1[\nabla_i u\nabla_i u\nabla_j\varphi\nabla_j\varphi + 2(\nabla_i u\nabla_i\varphi)^2] + 6\alpha_4\nabla_i u\nabla_j\varphi\nabla_k\varphi\nabla_{ijk}u \\ & + 6\alpha_4\nabla_i\varphi\nabla_j u\nabla_k u\nabla_{ijk}\varphi + 2\alpha_5\nabla_i\varphi\tau_i\varphi \\ & + \frac{3\alpha_5^2 n^3}{\alpha_0}\tau_i\varphi\tau_i\varphi + \frac{3}{4}\alpha_0|\nabla\varphi|^2 \geq 0. \end{aligned} \tag{3.14}$$

And because of $n \geq 5$ we have:

$$\begin{cases} \alpha_2 \geq \alpha_1 K_2 \\ \alpha_3 \geq 6K^2\alpha_1 \end{cases} \tag{3.15}$$

and so

$$[\Phi''(u+t\varphi)]_{t=0} \geq \alpha\int_\Omega (1+|\nabla u|^2)|\nabla\varphi|^2 + \int_\Omega V. \tag{3.16}$$

When $n \geq 50$, taking into account (2.5) and (2.6); we notice that

$$\begin{aligned} & \nabla_{ijk}\varphi\nabla_{ijk}\varphi\nabla_m u\nabla_m u + \nabla_m\varphi\nabla_m\varphi\nabla_{ijk}u\nabla_{ijk}u \\ & \qquad \leq 2\,\frac{(n^2-1)^2}{n}\nabla_{ijk}u\nabla_{ijk}u\nabla_{abc}\varphi\nabla_{abc}\varphi \end{aligned} \tag{3.17}$$

and then we get:

$$
\begin{aligned}
0 \leq\ & \nabla_{ijk}\varphi\nabla_{ijk}\varphi\nabla_m u\nabla_m u + \nabla_m\varphi\nabla_m\varphi\nabla_{ijk}u\nabla_{ijk}u \\
& + 6K^2[\nabla_i u\nabla_i u\nabla_j\varphi\nabla_j\varphi + 2(\nabla_i u\nabla_i\varphi)^2] \\
& - \frac{6}{n+2}\left(\frac{n}{n^2-1} - K\right)^2 \nabla_i u\nabla_i u\nabla_j\varphi\nabla_j\varphi \\
& + \left(\frac{24n}{(n+2)(n^2-1)}K - \frac{48}{n+2}K^2\right)(\nabla_i u\nabla_i\varphi)^2 \\
& + 6K\nabla_i u\nabla_j\varphi\nabla_k\varphi\nabla_{ijk}u + 6K\nabla_i\varphi\nabla_j u\nabla_k u\nabla_{ijk}\varphi \\
\leq\ & \frac{2(n^2-1)^2}{n}\nabla_{ijk}u\nabla_{ijk}u\nabla_{abc}\varphi\nabla_{abc}\varphi + 6K^2[\nabla_i u\nabla_i u\nabla_j\varphi\nabla_j\varphi + 2(\nabla_i u\nabla_i\varphi)^2] \\
& + \frac{12n}{(n+2)(n^2-1)}K[\nabla_i u\nabla_i u\nabla_j\varphi\nabla_j\varphi + 2(\nabla_i u\nabla_i\varphi)^2] \\
& + 6K\nabla_i u\nabla_j\varphi\nabla_k\varphi\nabla_{ijk}u + 6K\nabla_i\varphi\nabla_j u\nabla_k u\nabla_{ijk}\varphi.
\end{aligned}
$$

Thus, set $K = 2\alpha_4\,\dfrac{n^2-1}{n}$, as in the previous case we obtain (3.5). ■

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