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Everywhere regularity for a class of elliptic systems without growth conditions

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Everywhere Regularity for a Class of Elliptic Systems Without Growth Conditions

PAOLO MARCELLINI

Contents

1. - Introduction and statement of the main results

In this paper we study the local regularity (the local Lipschitz-continuity and then, as a consequence, the $C^{1,\alpha}_{\text{loc}}$ and C^{∞} regularity) of weak solutions of elliptic systems of the type

$$\sum_{i} \frac{\partial}{\partial x_i} a_i^{\alpha}(Du) = 0, \quad \forall\ \alpha = 1, 2, \ldots, N, \tag{1.1}$$

in an open set Ω of $\mathbb{R}^n$ ($n \geq 2$), where Du is the gradient of a vector-valued function $u: \Omega \to \mathbb{R}^N$ ($N \geq 1$). We assume that the vector field $a_i^{\alpha}: \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ is the gradient of a function $f: \mathbb{R}^{N\times n} \to \mathbb{R}$.

Under our conditions, every weak solution of the elliptic system (1.1) is a minimizer of the integral functional of the Calculus of Variations

$$F(v) = \int_{\Omega} f(Dv)dx \tag{1.2}$$

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and vice-versa. In this paper, by a *weak solution* to the elliptic system (1.1) we mean a function $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ such that $f(Du) \in L^1_{\mathrm{loc}}(\Omega)$, which satisfies (1.1) in the sense of distributions. Similarly, here by a *minimizer* of the integral (1.2) we mean a function $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ such that $f(Du) \in L^1_{\mathrm{loc}}(\Omega)$, with the property that $F(u) \leq F(u+\varphi)$ for every $\varphi \in C^1_0(\Omega, \mathbb{R}^N)$.

In this context of vector-valued problems *in general* it is natural to look for *almost everywhere regularity* of minimizers, introduced in the late 60's (precisely in 1968-1969) by Morrey [24], Giusti-Miranda [14] and Giusti [12]. In fact, counterexamples to the *everywhere regularity* of weak solutions of elliptic systems have been given, first by De Giorgi [5] in 1968, then in the same year by Giusti-Miranda [13] and, more specifically for minimizers of integrals of the type of (1.2) with f of quadratic growth, by Nečas ([25]; see also Example 3.4 of Chapter II in [10]) in 1975-1977.

Nevertheless Uhlenbeck [29], in a well-known paper in 1977, proved that, for every fixed $p \geq 2$, then minimizers of the integral

$$\int_\Omega |Dv|^p dx, \tag{1.3}$$

are *everywhere* smooth, precisely they are of class $C^{1,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$.

Therefore, the study of everywhere regularity of vector-valued problems explains our restriction to integral functionals of the type

$$\int_\Omega f(Dv)dx, \quad \text{with} \quad f(Dv) = g(|Dv|) \tag{1.4}$$

(here we pose $f(\xi) = g(|\xi|)$ for every $\xi \in \mathbb{R}^{N\times n}$; with $\xi = (\xi_i^\alpha)$, $i = 1,2,\dots,n$, $\alpha = 1,2,\dots,N$), where $g\colon [0,+\infty) \to \mathbb{R}$ is a convex function.

Similarly, in terms of systems, we restrict ourselves to elliptic systems of the form

$$\sum_i \frac{\partial}{\partial x_i} a_i^\alpha(Du) = 0, \quad \forall\ \alpha = 1,2,\dots,N, \ \text{with}\ a_i^\alpha(Du) = a(|Du|)u^\alpha_{x_i}, \tag{1.5}$$

with $a\colon [0,+\infty) \to [0,+\infty)$ related to g in (1.4) by the condition $a(t) = g'(t)/t$.

In recent years interest has increased in the study of regularity of solutions of elliptic equations and systems under *general growth conditions*; an extensive list of references can be found in [22]. In particular, since it is more related to the context considered in this paper, we mention the recent regularity results for the specific integral functional (1.4) with $f(\xi) = \exp(|\xi|^2)$, given by Lieberman [17] in the scalar case $N = 1$, and by Duc-Eells [7] in the vector-valued case $N \geq 1$. In [18, Example 3] and [22, Section 6] it has been proposed an approach to the regularity of a class of elliptic problems under general growth conditions, including the *slow exponential growth*, i.e. the case

$$f(\xi) \sim \exp(|\xi|^{\alpha}) \text{ as } |\xi| \to +\infty, \quad \text{with } \alpha \in \mathbb{R}^{+} \text{ small depending on the dimension } n. \tag{1.6}$$

For reference we mention also the partial regularity for systems, proved under anisotropic growth conditions in [1]; the Lipschitz and $C^{1,\alpha}$ regularity obtained for systems in [3] (quadratic growth), [11], [26], [28] (p-growth, with $p \geq 2$), and [4] (like in (1.4), with functions g of class Δ_2); the local boundedness of minimizers of integrals of the Calculus of Variations of the type (1.4), with g of class Δ_2 studied in the scalar case in [9], [23].

A main reason to classify elliptic problems with respect to growth conditions relies on the fact that growth conditions involving powers, like in (1.3), give rise to *uniformly* elliptic problems (as $|\xi| \to +\infty$); on the contrary, more general growth conditions, as for example the exponential growth conditions, give rise to *non-uniformly* elliptic problems.

In this paper we do not assume growth conditions. Instead, we assume the following *nonoscillatory conditions*:

$$\begin{cases} \text{(i) } \textit{the function } a\colon [0,+\infty) \to [0,+\infty) \textit{ is increasing;} \\ \text{(ii) } \textit{for every } \alpha > 1 \textit{ the limit } \displaystyle\lim_{t\to+\infty} \frac{a'(t)\cdot t}{[a(t)]^{\alpha}} \textit{ exists.} \end{cases} \tag{1.7}$$

To avoid trivial situations, we consider functions a not identically equal to zero; however, we do not assume that $a(0) > 0$ and, more generally, we allow $a(t)$ to be equal to zero in $[0, t_0]$, with $t_0 > 0$. Up to a rescaling, we will assume that $t_0 < 1$, so that $a(1) > 0$.

Let us explicitly note (see Remark 2.10) that, if the limit in (1.7ii) exists, then necessarily it is equal to zero. However, the spirit of the *non-oscillatory assumption* (1.7ii) is to avoid functions $a(t)$ whose derivative $a'(t)$ oscillates too much (see Example 2.9).

Of course (1.7) implies a type of "growth" conditions: first (i) implies that $a(t) \geq a(1)$ for all $t \geq 1$ (so that f in (1.4) has at least quadratic growth, like in the context (1.3), with $p \geq 2$, considered by Uhlenbeck [29]; secondly, the derivative $a'(t)$ can be bounded in terms of the α-power of $a(t)$ (see Lemma 2.4). However these growth restrictions are weak enough to be satisfied, for example, not only by the family of functions of exponential growth in (1.6) with $\alpha > 0$ small, but also by

$$f(\xi) = \exp(|\xi|^{p}), \quad \text{with} \quad p \geq 2, \tag{1.8}$$

or even by any finite composition of functions of the type

$$f(\xi) = (\exp(\ldots(\exp(\exp|\xi|^{2})^{p_1})^{p_2})\ldots)^{p_k}, \quad \text{with} \quad p_i \geq 1, \quad \forall\, i = 1, 2, \ldots, k. \tag{1.9}$$

The *nonoscillatory conditions* (1.7) are flexible enough to be compatible also with functions of non-power growth, considered in [9], [27], and defined by

$$f(\xi)=\begin{cases}|\xi|^{p+1-\sin\log\log|\xi|}, & \text{if } |\xi|\in(e,+\infty)\\ e|\xi|^{p}, & \text{if } |\xi|\in[0,e]\end{cases}, \quad \text{with } p\geq 3 \tag{1.10}$$

($p\geq 2$ is a sufficient condition for the convexity of f in (1.10), while $p\geq 3$ is a sufficient condition for the monotonicity of $a(t)=g'(t)/t$); note that, to apply the following regularity results to (1.10) and to the other previous examples, it is not necessary to impose restrictions on p in dependence on n.

In the statement of the next Theorem we will denote by B_ρ, B_R balls of radii ρ and R ($\rho<R$) contained in Ω and with the same center.

THEOREM 1.1. *Under the previous nonoscillatory conditions* (1.7)*, let u be a weak solution to the elliptic system* (1.5)*. Then $u\in W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and, for every $\epsilon>0$ and $R>\rho>0$, there exists a constant $c=c(\epsilon,n,\rho,R)$ such that*

$$\|Du\|^2_{L^\infty(B_\rho;\mathbb{R}^{N\times n})}\leq c\left\{\int_{B_R}[1+f(Du)]dx\right\}^{1+\epsilon}. \tag{1.11}$$

REMARK 1.2. If a satisfies, instead of (1.7), the growth condition

$$0\leq a'(t)\cdot t\leq ca(t),\quad \forall\ t\geq 0 \tag{1.12}$$

(equivalent, in terms of the function g, to

$$0\leq g'(t)\leq g''(t)\cdot t\leq cg'(t),\quad \forall\ t\geq 0), \tag{1.13}$$

then the conclusion (1.11) of Theorem 1.1 holds with $\epsilon=0$ too. However in this case the system (1.5) is *uniformly elliptic* and variational problems with, for example, f satisfying either (1.8) or (1.9), are ruled out.

Once we have the estimate (1.11) for the L^∞-norm of the gradient, then the behaviour as $t\to+\infty$ of $a(t)$ becomes irrelevant to obtain further regularity of solutions. Therefore we can apply the results already known, with assumptions on the behavior of $a(t)$ as $t\to 0^+$. We refer in particular to the papers [4], [11], [28], [29] for the vector-valued case $N\geq 1$ and to [6], [8], [16], [19] for the scalar case $N=1$. We obtain the following consequence (that can be applied to the examples (1.8), (1.9), (1.10)).

COROLLARY 1.3. *Assume that $a\in C^1((0,+\infty))$ satisfies the nonoscillatory conditions* (1.7)*, and that there exist an exponent $p\geq 2$ and two positive constants m, M such that*

$$mt^{p-2}\leq a(t)\leq a(t)+t\cdot a'(t)\leq Mt^{p-2},\quad \forall\ t\in(0,1] \tag{1.14}$$

(or, equivalently, in terms of the function $g \in C^2([0,+\infty))$*),*

$$(1.15) \qquad m t^{p-2} \leq \frac{g'(t)}{t} \leq g''(t) \leq M t^{p-2}, \quad \forall\ t \in (0,1]);$$

then every weak solution to the elliptic system (1.5) *is of class* $C^{1,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$.

If $a(0) > 0$ we are in the case (1.14) with $p = 2$ and the problem is uniformly elliptic as $t = |\xi| \to 0$. Thus, since $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, each component of the gradient Du is a weak solution to a system (see (3.5)) with Hölder-continuous coefficients. Then the regularity theory for linear elliptic systems with smooth coefficients applies (see for example Section 3 of Chapter 3 of [10]) and we obtain:

COROLLARY 1.4. *Assume that a is a function of class $C^{k-1,\alpha}([0,+\infty))$ for some $k \geq 2$, satisfying the nonoscillatory conditions* (1.7) *and* $a(0) > 0$ *(in terms of g, equivalently, $g \in C^{k,\alpha}([0,+\infty))$ with $g''(0) > 0$). Then every weak solution to the elliptic system* (1.5) *is of class* $C^{k,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$.

2. - Ellipticity estimates

With the aim to study integrals of the Calculus of Variations of the type

$$(2.1) \qquad \int_\Omega f(Dv)dx, \quad \text{with} \quad f(Dv) = g(|Dv|),$$

we consider $f(\xi) = g(|\xi|)$, for $\xi \in \mathbb{R}^{N \times n}$ ($\xi = (\xi_i^\alpha), i = 1,2,\dots,n, \alpha = 1, 2,\dots,N$), where

$$(2.2) \qquad \begin{cases} g\colon [0,+\infty) \to [0,+\infty) \text{ is a convex function of class } C^2([0,+\infty)), \\ \text{with } g'(t)/t \text{ increasing in } (0,+\infty). \end{cases}$$

Since $g'(t)/t$ is increasing, then necessarily $g'(0) = 0$. Moreover, without loss of generality, by adding a constant to g, we can reduce to the case $g(0) = 0$. Finally, not to consider a trivial situation, we assume that g is not identically equal to zero and, up to a rescaling, we can reduce to the case $g(1) > 0$.

LEMMA 2.1. *Under the previous notations* (2.1) *and assumptions* (2.2) *on f and g, the following conditions hold:*

(i) $\quad g'(t) \leq g''(t) \cdot t, \quad \forall\ t \geq 0;$

(ii) $$\frac{g'(|\xi|)}{|\xi|} |\lambda|^2 \leq \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(\xi) \lambda_i^\alpha \lambda_j^\beta \leq g''(|\xi|)|\lambda|^2, \quad \forall\ \lambda,\ \xi \in \mathbb{R}^{N \times n}.$$

PROOF. Since $g'(t)/t$ is increasing then

$$\frac{d}{dt}\frac{g'(t)}{t} = \frac{g''(t)\cdot t - g'(t)}{t^2} \geq 0, \quad \forall\, t > 0. \tag{2.3}$$

and thus (i) holds. To prove (ii), like in Section 6 of [22], we have

$$\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(\xi)\lambda_i^\alpha \lambda_j^\beta \begin{cases} \leq \max\{g''(|\xi|), g'(|\xi|)/|\xi|\} \cdot |\lambda|^2 \\ \geq \min\{g''(|\xi|), g'(|\xi|)/|\xi|\} \cdot |\lambda|^2 \end{cases}, \quad \forall\, \lambda,\ \xi \in \mathbb{R}^{N\times n};$$

therefore (ii) is consequence of (i). □

REMARK 2.2. Note that the assumption that $g'(t)/t$ is an increasing function is an intermediate condition between the convexity of g and the convexity of g'. In fact, if $a(t) = g'(t)/t \geq 0$ is increasing, then $g'(t) = a(t)\cdot t$ is increasing too; while, if $g'(t)$ is convex, since $g'(0) = 0$, then we have $0 = g'(0) \geq g'(t) + g''(t)\cdot(-t)$; thus, by (2.3), $g'(t)/t$ is increasing.

LEMMA 2.3. *Let $a(t) = g'(t)/t$, with g satisfying* (2.2). *Then the following conditions are equivalent to each other:*

(i) *for every $\alpha > 1$ the limit* $\displaystyle\lim_{t\to+\infty} \frac{a'(t)\cdot t}{[a(t)]^\alpha}$ *exists;*

(ii) *for every $\alpha > 1$ the limit* $\displaystyle\lim_{t\to+\infty} \frac{g''(t)\cdot t^\alpha}{[g'(t)]^\alpha}$ *exists;*

(iii) $$\lim_{t\to+\infty} \frac{g'(t)\cdot t^{2\alpha-1}}{[g(t)]^\alpha} < +\infty, \quad \lim_{t\to+\infty} \frac{g''(t)\cdot t^\alpha}{[g'(t)]^\alpha} < +\infty, \quad \forall\, \alpha > 1.$$

PROOF. (i) $\Leftrightarrow$ (ii): with the notation $g'(t) = a(t)\cdot t$ and $g''(t) = a'(t)\cdot t + a(t)$, we obtain the identity

$$\frac{g''(t)\cdot t^\alpha}{[g'(t)]^\alpha} = \frac{a'(t)\cdot t}{[a(t)]^\alpha} + [a(t)]^{1-\alpha};$$

therefore (i) is equivalent to (ii), since $[a(t)]^{1-\alpha}$ is a monotone function, bounded for $t \geq 1$.

(ii) $\Rightarrow$ (iii) (the opposite implication is obvious): since $g'(t)/t$ is increasing and $\alpha > 1$, then we have

$$\lim_{t\to+\infty} \left[\frac{g'(t)}{t}\right]^{1-\alpha} \leq [g'(1)]^{1-\alpha}.$$

By l'Hôpital's rule then we obtain

$$+\infty > \lim_{t\to+\infty} \frac{[g'(t)]^{1-\alpha}}{t^{1-\alpha}} = \lim_{t\to+\infty} \frac{[g'(t)]^{-\alpha} g''(t)}{t^{-\alpha}} = \lim_{t\to+\infty} \frac{g''(t)\cdot t^\alpha}{[g'(t)]^\alpha};$$

here we have used the assumption that the limit in the right-hand side exists, and the fact that we have a form $0/0$, because $g'(t) \geq g'(1) \cdot t \to +\infty$ as $t \to +\infty$.

Therefore the second limit in (iii) is obtained. To prove the first limit relation we compute equivalently:

$$\lim_{t \to +\infty} \frac{[g'(t)]^{1/\alpha} \cdot t^{2-1/\alpha}}{g(t)} =$$

$$= \lim_{t \to +\infty} \left\{ \frac{1}{\alpha} \left[\frac{t}{g'(t)} \right]^{2-1/\alpha} \cdot g''(t) + \left(2 - \frac{1}{\alpha} \right) \left[\frac{t}{g'(t)} \right]^{1-1/\alpha} \right\} < +\infty;$$

in the last step it has been possible to utilize the second limit relation of (iii), since the exponent $2 - 1/\alpha$ is greater than 1. □

LEMMA 2.4. *The following conditions are equivalent each other:*

(iv) *for every $\alpha > 1$ there exists a constant $c = c(\alpha)$ such that*

$$g'(t) \cdot t^{2\alpha - 1} \leq c[g(t)]^{\alpha}, \quad g''(t) \cdot t^{\alpha} \leq c[g'(t)]^{\alpha}, \quad \forall\, t \geq 1;$$

(v) *for every $\alpha > 1$ there exists a constant $c = c(\alpha)$ such that*

$$g''(t) \cdot t^{2\alpha} \leq c[g(t)]^{\alpha}, \quad \forall\, t \geq 1;$$

and they are consequence of anyone of conditions (i), (ii), (iii) *of Lemma* 2.3. *The constant c in* (iv) *(respectively* (v)*) depends only on the exponent α and on the constant c in* (v) *(respectively* (iv)*).*

PROOF. (iii) $\Rightarrow$ (iv): routine (since $g(1) > 0$, then also $g'(t)$, being increasing, is non equal to zero at $t = 1$).

(iv) $\Rightarrow$ (v): by iterating the two inequalities in (iv), for every $t \geq 1$ we have

$$g''(t) \cdot t^{2\alpha^2} = [g''(t) \cdot t^{\alpha}] \cdot t^{\alpha(2\alpha-1)} \leq c[g'(t)]^{\alpha} \cdot t^{\alpha(2\alpha-1)} =$$

$$= c[g'(t) \cdot t^{2\alpha-1}]^{\alpha} \leq c^{1+\alpha}[g(t)]^{\alpha^2};$$

since α^2 is a generic real number greater than 1, (v) is proved.

(v) $\Rightarrow$ (iv): by the inequality $g'(t) \leq g''(t) \cdot t$ (see Lemma 2.1), from (v) we deduce the first of (iv)

$$g'(t) \cdot t^{2\alpha-1} \leq g''(t) \cdot t^{2\alpha} \leq c[g(t)]^{\alpha}, \quad \forall\, t \geq 1.$$

By the convexity inequality $g(t) \leq g'(t) \cdot t$ (in fact $0 = g(0) \geq g(t) + g'(t) \cdot (-t)$), from (v) we obtain the second of (iv)

$$g''(t) \cdot t^{\alpha} = g''(t) \cdot t^{2\alpha} \cdot t^{-\alpha} \leq c[g(t)]^{\alpha} \cdot t^{-\alpha} \leq c[g'(t)]^{\alpha}, \quad \forall\, t \geq 1. \qquad \square$$

REMARK 2.5. If $g'(t)/t \to +\infty$ (and this is the most interesting case considered in this paper), then either (iv) or (v) are equivalent to anyone of the conditions (i), (ii), (iii). To show this fact, it is sufficient to prove that (iv) implies (iii) (we prove only the first limit relation of (iii), the proof of the other being similar): by the first inequality in (iv), with α replaced by $(\alpha+1)/2$, we have

$$\frac{g'(t)\cdot t^{2\alpha-1}}{[g(t)]^{\alpha}}=\frac{g'(t)\cdot t^{\alpha}}{[g(t)]^{(\alpha+1)/2}}\cdot\frac{t^{\alpha-1}}{[g(t)]^{(\alpha-1)/2}}\leq c\cdot\left[\frac{t^2}{g(t)}\right]^{(\alpha-1)/2}$$

and the right-hand side converges to zero as $t\to+\infty$, as a consequence of the assumption $g'(t)/t\to+\infty$.

On the contrary, if $g'(t)/t$ has a finite limit as $t\to+\infty$, then the condition $\alpha>1$ becomes not relevant and it can be more convenient to consider (iv) with $\alpha=1$ as an assumption, instead of (1.7ii). In this case we can see that the conclusion (1.11) of Theorem 1.1 holds with $\epsilon=0$ too. For completeness, we mention that it is easy to obtain from the second inequality in (iv) with $\alpha=1$:

$$g''(t)\cdot t\leq cg'(t),\quad\forall\ t\geq 0,\tag{2.4}$$

the first one (with $\alpha=1$ and c replaced by $2c$); in fact it is sufficient to integrate both sides over $[0,t]$ (and then to integrate by parts the left-hand side). Finally, under the notation $a(t)=g'(t)/t$, (2.4) is equivalent to

$$a'(t)\cdot t\leq(c-1)\cdot a(t),\quad\forall\ t\geq 0\tag{2.5}$$

(note that the constant c in (2.4) is greater than or equal to 1, by (i) of Lemma 2.1). Therefore the statement in Remark 1.2 is justified.

LEMMA 2.6. *If g satisfies* (2.2) *and the conditions stated in Lemma* 2.4 *then, for every $\beta>2$, there exists a constant c such that*

$$g''(t)\leq c\left\{\frac{1}{t}\int_0^t\sqrt{\frac{g'(s)}{s}}\,ds\right\}^{\beta},\quad\forall\ t\geq 1.\tag{2.6}$$

The constant c depends on the constants $c=c(\alpha)$ appearing in (iv) *and* (v) *of Lemma* 2.4, *on β, on $g(1)$, $g'(1)$, and on a lower bound for $\int_0^1\sqrt{g'(s)/s}\,ds$.*

PROOF. By condition (v) of Lemma 2.4, being $g''(t)\leq c[g(t)]^{\alpha}/t^{2\alpha}$, to obtain (2.6) it is sufficient to prove that, for some constant c_1,

$$[g(t)]^{\frac{\alpha}{\beta}}\cdot t^{1-\frac{2\alpha}{\beta}}\leq c_1\int_0^t\sqrt{\frac{g'(s)}{s}}\,ds,\quad\forall\ t\geq 1.\tag{2.7}$$

Let us fix $\alpha\in(1,\beta/2)$ and let us limit ourselves to consider c_1 greater than or equal to $[g(1)]^{\alpha/\beta}\left\{\int_0^1\sqrt{g'(s)/s}\,ds\right\}^{-1}$. Then (2.7) is satisfied for $t=1$; thus we will obtain the conclusion by proving that it is possible to choose c_1 so that the derivative with respect to t of the left-hand side of (2.7) is less than or equal to the derivative of the right-hand side, i.e.:

$$\frac{\alpha}{\beta}\,[g(t)]^{\frac{\alpha}{\beta}-1}\cdot g'(t)\cdot t^{1-\frac{2\alpha}{\beta}}+\left(1-\frac{2\alpha}{\beta}\right)[g(t)]^{\frac{\alpha}{\beta}}\cdot t^{-\frac{2\alpha}{\beta}}\le c_1\sqrt{\frac{g'(t)}{t}},\quad\forall\ t\ge 1.$$

Since the two addenda in the left-hand side are non-negative, up to some computations we obtain the equivalent system of inequalitites for $t\ge 1$

$$g'(t)\cdot t^{3-\frac{4\alpha}{\beta}}\le c_2[g(t)]^{2-\frac{2\alpha}{\beta}},\qquad [g(t)]^{\frac{2\alpha}{\beta}}\cdot t^{1-\frac{4\alpha}{\beta}}\le c_3 g'(t). \tag{2.8}$$

The first of (2.8) holds by Lemma 2.4(iv), with α replaced by $2-2\alpha/\beta$. By the convexity inequality $g(t)\le g'(t)\cdot t$, the second one can be reduced to

$$1\le c_3\left[\frac{g'(t)}{t}\right]^{1-\frac{2\alpha}{\beta}}$$

and it holds with $c_3=[g(1)]^{-1+2\alpha/\beta}$. □

With the following Lemma we give two more inequalities, to be used in the proves of the *a priori* estimates of Section 3.

LEMMA 2.7. *If g satisfies* (2.2) *and the conditions stated in Lemma* 2.4 *then:*

(i) *for every $\alpha>1$ there exists a constant $c=c(\alpha)$ such that*

$$1+g''(t)\cdot t^2\le c[1+g(t)]^{\alpha},\quad\forall\ t\ge 0;$$

(ii) *for every $\beta>2$ there exists a constant $c=c(\beta)$ such that*

$$1+\left(\frac{t^{\gamma+1}}{\gamma+1}\right)^{\beta}g''(t)\le c\left[1+\int_0^t s^{\gamma}\sqrt{\frac{g'(s)}{s}}\,ds\right]^{\beta},\quad\forall\ t\ge 0,\quad\forall\ \gamma\ge 0.$$

The constant c depends respectively on α and β, on the constants of Lemmata 2.4, 2.6, *and on $g''(1)$.*

PROOF. By Lemma 2.4(v), for every $t\ge 1$, we have

$$1+g''(t)\cdot t^2\le 1+g''(t)\cdot t^{2\alpha}\le 1+c[g(t)]^{\alpha}\le 2c[1+g(t)]^{\alpha}; \tag{2.9}$$

moreover, if $t\in[0,1]$, then the left-hand side is bounded by the constant $1+g''(1)$, while the right-hand side is greater than or equal to $2c$. Thus (2.9) holds for every $t\ge 0$, with constant equal to $\max\{2c,1+g''(1)\}$.

To prove (ii) we begin to observe that, since the functions $t \to t^\gamma$ and $t \to g'(t)/t$ are increasing, then we have (see for example Lemma 3.4(v) of [22])

$$\int_0^t s^\gamma \sqrt{\frac{g'(s)}{s}}\,ds \geq \frac{1}{t}\int_0^t s^\gamma ds \cdot \int_0^t \sqrt{\frac{g'(s)}{s}}\,ds = \frac{t^\gamma}{\gamma+1}\cdot \int_0^t \sqrt{\frac{g'(s)}{s}}\,ds$$

which, combined with Lemma 2.6, gives, for every $t \geq 1$

$$2c\cdot\left\{1+\int_0^t s^\gamma\sqrt{\frac{g'(s)}{s}}\,ds\right\}^\beta \geq 1+\left\{\frac{t^{\gamma+1}}{\gamma+1}\right\}^\beta\cdot g''(t); \tag{2.10}$$

moreover, if $t \in [0,1]$, then the right-hand side is bounded by the constant $1+g''(1)$ (independent of γ), while the left-hand side is greater than or equal to $2c$. Thus (2.10) holds for every $t \geq 0$, again with constant equal to $\max\{2c, 1+g''(1)\}$. □

REMARK 2.8. By the previous analysis we can see that the *nonoscillatory conditions* (1.7i) and (1.7ii) on the function a correspond, in terms of the convex function g, respectively to the monotonicity assumption (2.2) and to one of the (equivalent each other) conditions of Lemma 2.4. For this reason, the conclusion (1.11) of Theorem 1.1 (and its consequences in Corollaries 1.3 and 1.4) holds under the assumption that $g = g(t)$ satisfies (2.2) and, for example, the condition:

$$\begin{cases} \textit{for every } \alpha > 1 \textit{ there exists a constant } c = c(\alpha) \textit{ such that} \\ g''(t)\cdot t^{2\alpha} \leq c[g(t)]^\alpha, \quad \forall\, t \geq 1. \end{cases} \tag{2.11}$$

We may ask if either the *nonoscillatory condition* (1.7ii), or (2.11), are consequences of the other conditions; by the following example we show that they may be not satisfied ($g(t) = t^2/(T-t)$, with $t \in [0,T)$, is a simple example of function that does not satisfy (2.11) for $t \to T^-$, but it is not definite (or finite) all over $[0,+\infty)$).

EXAMPLE 2.9. Let φ be a locally bounded function with the properties

$$\varphi \in L^1([0,+\infty)); \quad \varphi(t) \geq 0, \text{ a.e. } t \in [0,+\infty); \quad \limsup_{t\to+\infty} \varphi(t) > 0.$$

Let us define

$$a(t) = 1 + \int_0^t \varphi(s)ds.$$

Then it is clear that, for every $\alpha \geq 1$,

$$\limsup_{t\to+\infty} \frac{a'(t)\cdot t}{[a(t)]^\alpha} = +\infty, \qquad \liminf_{t\to+\infty} \frac{a'(t)\cdot t}{[a(t)]^\alpha} = 0;$$

in particular the second limit relation holds, since the condition $a'(t)\cdot t/[a(t)]^\alpha > \epsilon > 0$ for t large implies $a'(t) > \epsilon \cdot [a(0)]^\alpha/t = \epsilon/t$, that is in contradiction with the assumption that $a' = \varphi$ is summable in $[0,+\infty)$.

REMARK 2.10. Finally we prove what already stated in the introduction about the *nonoscillatory condition* (1.7ii):

$$\text{(2.12)} \qquad \textit{for every } \alpha > 1 \textit{ the limit } \lim_{t\to+\infty} \frac{a'(t)\cdot t}{[a(t)]^\alpha} \textit{ exists,}$$

that is equivalent to the condition

$$\text{(2.13)} \qquad \textit{for every } \alpha > 1 \ \lim_{t\to+\infty} \frac{a'(t)\cdot t}{[a(t)]^\alpha} = 0.$$

In fact, if the limit (2.12) exists, then by L'Hôpital's rule, for every $\alpha > 1$ we have

$$0 = \lim_{t\to+\infty} \frac{[a(t)]^{1-\alpha}}{\log t} = (1-\alpha) \lim_{t\to+\infty} \frac{a'(t)\cdot t}{[a(t)]^\alpha}.$$

We emphasize, however, that the spirit of the *nonoscillatory assumption* (1.7ii) is to avoid *oscillatory conditions* of the type presented in the previous Example 2.9.

3. - A priori estimates

In this Section we consider the integral of the Calculus of Variations

$$\text{(3.1)} \qquad \int_\Omega f(Dv)dx, \quad \text{with} \quad f(Dv) = g(|Dv|),$$

and we make the following supplementary assumption: there exist two positive constants m, M such that

$$\text{(3.2)} \qquad m|\lambda|^2 \le \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(\xi)\lambda_i^\alpha \lambda_j^\beta \le M|\lambda|^2, \quad \forall\ \lambda,\ \xi \in \mathbb{R}^{N\times n}.$$

(or, equivalently, $m \le g'(t)/t \le g''(t) \le M, \quad \forall\ t > 0$).

For convenience of the reader, we observe that in Sections 4 and 5 we will remove this assumption. The reason that will make it possible this removal relies on the fact that the constants m and M do not enter in the *a priori estimate* (3.3) for the L^∞-norm of the gradient, neither enter in (3.16); on the contrary, here we will make use of the estimates obtained in the previous Section for the function g and its derivatives g' and g''.

We will denote by B_ρ, B_R balls of radii respectively ρ and R ($\rho < R$) contained in Ω and with the same center. Recall that g is related to the function a in Section 1 by the condition $g'(t) = a(t) \cdot t$.

LEMMA 3.1. *Under the previous assumptions* (1.7), (3.2), *let* $u \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ *be a minimizer of the integral* (3.1). *Then* $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ *and there exists a constant* c *such that*

$$\|Du\|^2_{L^\infty(B_\rho; \mathbb{R}^{N\times n})} \leq \frac{c}{(R-\rho)^n} \int_{B_R} (1 + |Du|^2 g''(|Du|))dx, \tag{3.3}$$

for every ρ, R ($\rho < R$), *(if* $n = 2$ *then the exponent* n *in the right-hand side must be replaced by* $2+\epsilon$, *with* $\epsilon > 0$*). The constant* c *depends on the the dimension* n *and on the constants of Lemmata of Section* 2, *but it is independent of the constants* m, M *in* (3.2).

PROOF. Let u be a minimizer of (3.1). By the left-hand side of (3.2), $u \in W^{1,2}(\Omega; \mathbb{R}^N)$ and, by the right-hand side of (3.2), it satisfies the Euler's first variation:

$$\int_\Omega \sum_{i,\alpha} f_{\xi_i^\alpha}(Du) \cdot \phi^\alpha_{x_i} dx = 0, \quad \forall\ \phi \equiv (\phi^\alpha) \in W^{1,2}_0(\Omega; \mathbb{R}^N). \tag{3.4}$$

Again, by using the right-hand side of (3.2), with the technique of the difference quotient, we obtain (see, for example Theorem 1.1 of Chapter II of [10]; in the specific context of non standard growth conditions see also [2], [20], [21], [22]) that u admits second derivatives, precisely that $u \in W^{2,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$; moreover u satisfies the second variation:

$$\begin{gathered} \int_\Omega \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\beta_{x_j x_k} \phi^\alpha_{x_i} dx = 0, \\ \forall\ k = 1, 2, \ldots, n, \quad \forall\ \phi \equiv (\phi^\alpha) \in W^{1,2}_0(\Omega; \mathbb{R}^N). \end{gathered} \tag{3.5}$$

Fixed $k \in \{1, 2, \ldots, n\}$, let $\eta \in C^1_0(\Omega)$ and $\phi^\alpha = \eta^2 u^\alpha_{x_k} \Phi(|Du|)$ for every $\alpha = 1, 2, \ldots, N$, where Φ is a positive, increasing, bounded, Lipschitz continuous function in $[0, +\infty)$, (in particular Φ and Φ' are bounded in $[0, +\infty)$, so that $\phi \equiv (\phi^\alpha) \in W^{1,2}_0(\Omega; \mathbb{R}^N)$); then

$$\phi^\alpha_{x_i} = 2\eta\eta_{x_i} u^\alpha_{x_k} \Phi(|Du|) + \eta^2 u^\alpha_{x_i x_k} \Phi(|Du|) + \eta^2 u^\alpha_{x_k} \Phi'(|Du|) \cdot (|Du|)_{x_i}$$

and from the equation (3.5) we deduce

$$
\begin{aligned}
&\int_\Omega 2\eta\Phi(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\beta_{x_j x_k} \eta_{x_i} u^\alpha_{x_k}\, dx \\
&+ \int_\Omega \eta^2\Phi(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\beta_{x_j x_k} u^\alpha_{x_i x_k}\, dx \\
&+ \int_\Omega \eta^2\Phi'(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\beta_{x_j x_k} u^\alpha_{x_k} (|Du|)_{x_i}\, dx = 0.
\end{aligned} \tag{3.6}
$$

We can estimate the first integral in (3.6) by using the Cauchy-Schwarz inequality and the inequality $2ab \leq \frac{1}{2} a^2 + 2b^2$

$$
\begin{aligned}
&\left| \int_\Omega 2\eta\Phi(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\beta_{x_j x_k} \eta_{x_i} u^\alpha_{x_k}\, dx \right| \\
&\leq \int_\Omega 2\phi(|Du|) \left(\eta^2 \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\alpha_{x_i x_k} u^\beta_{x_j x_k} \right)^{\frac{1}{2}} \\
&\qquad\qquad \left(\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) \eta_{x_i} u^\alpha_{x_k} \eta_{x_j} u^\beta_{x_k} \right)^{\frac{1}{2}} dx \\
&\leq \int_\Omega \eta^2\Phi(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\alpha_{x_i x_k} u^\beta_{x_j x_k}\, dx \\
&\qquad + 2 \int_\Omega \Phi(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) \eta_{x_i} u^\alpha_{x_k} \eta_{x_j} u^\beta_{x_k}\, dx.
\end{aligned}
$$

From (3.6) we obtain

$$
\begin{aligned}
&\frac{1}{2} \int_\Omega \eta^2\Phi(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\beta_{x_j x_k} u^\alpha_{x_i x_k}\, dx \\
&\qquad + \int_\Omega \eta^2\Phi'(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) u^\beta_{x_j x_k} u^\alpha_{x_k} (|Du|)_{x_i}\, dx \\
&\leq 2 \int_\Omega \Phi(|Du|) \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(Du) \eta_{x_i} u^\alpha_{x_k} \eta_{x_j} u^\beta_{x_k}\, dx.
\end{aligned} \tag{3.7}
$$

To estimate the second integral in (3.7), we make use of the assumption that f depends on the modulus of the gradient of u: with the representation $f(\xi) = g(|\xi|)$ for every $\xi \in \mathbb{R}^{N\times n}$ ($\xi = (\xi_i^\alpha), i = 1,2,\dots,n, \alpha = 1,2,\dots,N$) we have

$$f_{\xi_i^\alpha} = g'(|\xi|)\cdot\frac{\xi_i^\alpha}{|\xi|}, \quad f_{\xi_i^\alpha\xi_j^\beta} = \left(\frac{g''(|\xi|)}{|\xi|^2} - \frac{g'(|\xi|)}{|\xi|^3}\right)\cdot \xi_i^\alpha \xi_j^\beta + \frac{g'(|\xi|)}{|\xi|}\cdot \delta_{\xi_i^\alpha\xi_j^\beta};$$

$$\begin{aligned}
&\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(Du)u^\beta_{x_jx_k}u^\alpha_{x_k}(|Du|)_{x_i}\\
&= \left(\frac{g''(|Du|)}{|Du|^2} - \frac{g'(|Du|)}{|Du|^3}\right)\cdot \sum_{i,j,\alpha,\beta} u^\alpha_{x_i}u^\beta_{x_j}u^\beta_{x_jx_k}u^\alpha_{x_k}(|Du|)_{x_i}\\
&\qquad + \frac{g'(|Du|)}{|Du|}\cdot\sum_{i,\alpha} u^\alpha_{x_ix_k}u^\alpha_{x_k}(|Du|)_{x_i}.
\end{aligned}$$

Since

$$(|Du|)_{x_i} = \frac{1}{|Du|}\cdot\sum_{k,\alpha} u^\alpha_{x_ix_k}u^\alpha_{x_k}, \tag{3.8}$$

then it is natural to sum up with respect to $k = 1,2,\dots,n$; we obtain

$$\begin{aligned}
&\sum_k \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(Du)u^\beta_{x_jx_k}u^\alpha_{x_k}(|Du|)_{x_i}\\
&= \left(\frac{g''(|Du|)}{|Du|} - \frac{g'(|Du|)}{|Du|^2}\right)\cdot \sum_{i,k,\alpha} u^\alpha_{x_i}(|Du|)_{x_k}u^\alpha_{x_k}(|Du|)_{x_i}\\
&\qquad + g'(|Du|)\cdot\sum_i ((|Du|)_{x_i})^2\\
&= \left(\frac{g''(|Du|)}{|Du|} - \frac{g'(|Du|)}{|Du|^2}\right)\cdot \left[\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\right]^2\\
&\qquad + g'(|Du|)\cdot\sum_i ((|Du|)_{x_i})^2 \geq 0,
\end{aligned}$$

since, by Lemma 2.1, $g'(|Du|) \leq |Du|\cdot g''(|Du|)$. Therefore, from (3.7) we deduce the estimate

$$\begin{aligned}
&\int_\Omega \eta^2\Phi(|Du|)\sum_{k,i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(Du)u^\beta_{x_jx_k}u^\alpha_{x_ix_k}\,dx\\
&\leq 4\int_\Omega \Phi(|Du|)\sum_{k,i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(Du)\eta_{x_i}u^\alpha_{x_k}\eta_{x_j}u^\beta_{x_k}\,dx.
\end{aligned}$$

By applying the Cauchy-Schwarz inequality to (3.8) we have

$$|(|Du|)_{x_i}|^2 \leq \sum_{k,\alpha} (u^\alpha_{x_i x_k})^2, \quad \forall\, i = 1, 2, \ldots, n;$$

thus $|D(|Du|)|^2 \leq |D^2u|^2$ and, by the ellipticity conditions of Lemma 2.1(ii), we finally obtain

$$\begin{aligned} &\int_\Omega \eta^2 \Phi(|Du|) \frac{g'(|Du|)}{|Du|} \cdot |D(|Du|)|^2 dx \\ &\leq 4 \int_\Omega |D\eta|^2 \Phi(|Du|) g''(|Du|) \cdot |Du|^2 dx. \end{aligned} \tag{3.9}$$

We recall that, till this point, Φ is a positive, increasing, local Lipschitz continuous function in $[0, +\infty)$, with Φ and Φ' bounded in $[0, +\infty)$. If we consider a more general Φ not bounded, with derivative Φ' not bounded too, then we can approximate it by a sequence of functions Φ_r, each of them being equal to Φ in the interval $[0, r]$, and then extended to $(r, +\infty)$ with the constant value $\Phi(r)$. We insert Φ_r in (3.9) and we go to the limit as $r \to +\infty$ by the monotone convergence theorem. We obtain the validity of (3.9) for every Φ positive, increasing, local Lipschitz continuous function in $[0, +\infty)$.

Let us define

$$G(t) = 1 + \int_0^t \sqrt{\Phi(s) \frac{g'(s)}{s}}\, ds, \quad \forall\, t \geq 0;$$

since g' and Φ are increasing and $g'(t) \leq tg''(t)$, then we have

$$[G(t)]^2 \leq \left[1 + t \cdot \sqrt{\Phi(t) \frac{g'(t)}{t}}\right]^2 \leq 2(1 + \Phi(t) \cdot g'(t) \cdot t) \leq 2(1 + \Phi(t) \cdot g''(t) \cdot t^2),$$

and

$$\begin{aligned} &|D(\eta G(|Du|)|^2 \leq 2|D\eta|^2 [G(|Du|)]^2 + 2\eta^2 [G'(|Du|)]^2 |D(|Du|)|^2 \\ &\leq 4|D\eta|^2 (1 + \Phi(|Du|) \cdot g''(|Du|) \cdot |Du|^2) \\ &\qquad + 2\eta^2 \Phi(|Du|) \frac{g'(|Du|)}{|Du|} |D(|Du|)|^2. \end{aligned} \tag{3.10}$$

From (3.9), (3.10) we deduce that

$$\int_\Omega |D(\eta G(|Du|)|^2 dx \leq 4 \int_\Omega |D\eta|^2 (1 + 3\Phi(|Du|) \cdot g''(|Du|) \cdot |Du|^2) dx. \tag{3.11}$$

Let us denote by 2^* the Sobolev's exponent, i.e. $2^* = 2n/(n-2)$ if $n \geq 3$, while 2^* is any fixed real number greater than 2, if $n = 2$. By Sobolev's inequality, there exist a constant c_1 such that

$$\left\{ \int_\Omega [\eta G(|Du|)]^{2^*} dx \right\}^{2/2^*} \leq c_1 \int_\Omega |D(\eta G(|Du|)|^2 dx. \tag{3.12}$$

Let us define $\Phi(t) = t^{2\gamma}$, with $\gamma \geq 0$ (so that Φ is increasing). By Lemma 2.7(ii) with $\beta = 2^*$, there exists a constant $c_2 > 0$ such that

$$[G(t)]^{2^*} = \left[1 + \int_0^t \sqrt{s^{2\gamma} \frac{g'(s)}{s}}\, ds \right]^{2^*} \geq c_2 \left[1 + \left(\frac{t^{\gamma+1}}{\gamma+1} \right)^{2^*} g''(t) \right], \tag{3.13}$$

for every $t \geq 0$ and for every γ. From (3.11), (3.12), (3.13), we deduce that

$$\left\{ \int_\Omega \eta^{2^*} (1 + |Du|^{(\gamma+1)2^*} g''(|Du|)) dx \right\}^{2/2^*}$$
$$\leq c_3 (\gamma+1)^2 \int_\Omega |D\eta|^2 (1 + |Du|^{2(\gamma+1)} g''(|Du|)) dx.$$

Let us denote by B_R and B_ρ balls compactly contained in Ω, of radii respectively R, ρ, with the same center. Let η be a test function equal to 1 in B_ρ, whose support is contained in B_R, such that $|D\eta| \leq 2/(R-\rho)$. Let us denote by $\delta = 2(\gamma+1)$ (note that, since $\gamma \geq 0$, then $\delta \geq 2$). We have

$$\left\{ \int_{B_\rho} (1 + |Du|^{\delta \frac{2^*}{2}} g''(|Du|)) dx \right\}^{2/2^*}$$
$$\leq c_3 \left(\frac{4\delta}{R-\rho} \right)^2 \int_{B_R} (1 + |Du|^\delta g''(|Du|)) dx. \tag{3.14}$$

Fixed R_0 and ρ_0, for all $i \in \mathbb{N}$ we rewrite (3.14) with $R = \rho_{i-1}$ and $\rho = \rho_i$, where $\rho_i = \rho_0 + (R_0 - \rho_0)/2^i$; moreover, for $i = 1, 2, 3, \ldots$ we put δ equal to 2, $2(2^*/2)$, $2(2^*/2)^2$, ... By iterating (3.14), since $R - \rho = (R_0 - \rho_0) \cdot 2^{-i}$, for every $i \in \mathbb{N}$ we obtain

$$\left\{ \int_{B_{\rho_i}} (1 + |Du|^{2\left(\frac{2^*}{2}\right)^i} g''(|Du|)) dx \right\}^{\left(\frac{2}{2^*}\right)^i} \leq c_4 \int_{B_{R_0}} (1 + |Du|^2 g''(|Du|)) dx,$$

where

$$
\begin{aligned}
c_4 &= \prod_{i=1}^{\infty}\left[\frac{c_3\cdot 2^8\cdot(2^*)^{i-1}}{(R_0-\rho_0)^2}\right]^{\left(\frac{2}{2^*}\right)^{i-1}} \\
&= (2^*)^{\sum_{i=1}^{\infty}(i-1)\cdot\left(\frac{2}{2^*}\right)^{i-1}}\cdot\left[\frac{c_3\cdot 2^8}{(R_0-\rho_0)^2}\right]^{\sum_{i=1}^{\infty}\left(\frac{2}{2^*}\right)^{i-1}} \\
&= c_5(R_0-\rho_0)^{-\frac{2}{1-2/2^*}} = c_5(R_0-\rho_0)^{-n}
\end{aligned}
\tag{3.15}
$$

if $n \geq 3$; otherwise, if $n = 2$, then for every $\epsilon > 0$ we can choose 2^* so that $c_4 = c_5(R_0-\rho_0)^{-2-\epsilon}$ for some constant c_5.

Since $g'(t)/t$ is increasing, then by Lemma 2.1(i) we have $g''(t) \geq g'(t)/t \geq g'(1)$ for every $t \geq 1$. Therefore for every $t \geq 0$ and $\alpha > 0$ we obtain $g''\cdot t^\alpha + 1 \geq c_6\cdot t^\alpha$, with $c_6 = \min\{g'(1), 1\}$. Finally we go to the limit as $i \to +\infty$:

$$
\begin{aligned}
\sup\{|Du(x)|^2 : x\in B_{\rho_0}\} &= \lim_{i\to+\infty}\left\{\int_{B_{\rho_0}} |Du|^{2\left(\frac{2^*}{2}\right)^i}dx\right\}^{\left(\frac{2}{2^*}\right)^i} \\
&\leq \limsup_{i\to+\infty}\left\{\frac{1}{c_6}\int_{B_{\rho_i}}(1+|Du|^{2\left(\frac{2^*}{2}\right)^i}g''(|Du|))dx\right\}^{\left(\frac{2}{2^*}\right)^i} \\
&\leq c_4\int_{B_{R_0}}(1+|Du|^2 g''(|Du|))dx
\end{aligned}
$$

and, by the representation of c_4 in (3.15), we have the conclusion (3.3). □

LEMMA 3.2. *Under the assumptions* (1.7), (3.2), *let* $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ *be a minimizer of the integral* (3.1). *Then, for every* $\epsilon > 0$ *and for every* ρ, R $(0 < \rho < R)$, *there exists a constant* $c = c(\epsilon, \rho, R)$ *such that*

$$
\int_{B_\rho}(1+|Du|^2 g''(|Du|))dx \leq c\left\{\int_{B_R}[1+g(|Du|)]dx\right\}^{1+\epsilon}; \tag{3.16}
$$

c depends also on the constants of Lemmata of Section 2 *and on the dimension* n, *but does not depend on the constants* m, M *in* (3.2).

PROOF. Under the notations of the previous Lemma 3.1, let us consider again the estimates (3.11), (3.12), with Φ identically equal to 1 (or, equivalently, with $\gamma = 0$):

$$\left\{ \int_{\Omega} [\eta G(|Du|)]^{2^*} dx \right\}^{2/2^*} \leq 4c_1 \int_{\Omega} |D\eta|^2 (1 + 3g''(|Du|) \cdot |Du|^2) dx. \tag{3.17}$$

We apply Lemma 2.7(ii) with $\gamma = 0$ and $\beta \in (2, 2^*)$, and we represent β under the form $\beta = 2^*/\delta$ (thus $1 < \delta < 2^*/2$). Then there exists a constant c_2 such that, for all $t \geq 0$,

$$[G(t)]^{2^*} = \left[1 + \int_0^t \sqrt{\frac{g'(s)}{s}}\, ds \right]^{\frac{2^*}{\delta} \cdot \delta} \geq c_2 \left[1 + t^{\frac{2^*}{\delta}} g''(t) \right]^{\delta}. \tag{3.18}$$

From (3.17), (3.18), since $2^*/\delta > 2$, we deduce that

$$\left\{ \int_{\Omega} \eta^{2^*} (1 + |Du|^2 \cdot g''(|Du|))^{\delta} dx \right\}^{2/2^*} \leq c_3 \int_{\Omega} |D\eta|^2 (1 + |Du|^2 \cdot g''(|Du|)) dx.$$

Like in the previous Lemma, we consider a test function η equal to 1 in B_ρ, with support contained in B_R and such that $|D\eta| \leq 2/(R - \rho)$; we obtain

$$\left\{ \int_{B_\rho} V^{\delta} dx \right\}^{2/2^*} \leq \frac{4c_3}{(R-\rho)^2} \int_{B_R} V dx,$$

where $V = V(x) = 1 + |Du|^2 \cdot g''(|Du|)$.

Let $\gamma > 2^*/2$. By Hölder's inequality then we have

$$\begin{aligned} \left\{ \int_{B_\rho} V^{\delta} dx \right\}^{2/2^*} &\leq \frac{4c_3}{(R-\rho)^2} \int_{B_R} V^{\frac{\delta}{\gamma}} \cdot V^{1 - \frac{\delta}{\gamma}} dx \\ &\leq \frac{4c_3}{(R-\rho)^2} \left\{ \int_{B_R} V^{\delta} dx \right\}^{\frac{1}{\gamma}} \cdot \left\{ \int_{B_R} V^{\frac{\gamma - \delta}{\gamma - 1}} dx \right\}^{\frac{\gamma - 1}{\gamma}} \end{aligned} \tag{3.19}$$

Fixed R_0 and ρ_0, for all $i \in \mathbb{N}$ we consider (3.19) with $R = \rho_i$ and $\rho = \rho_{i-1}$, with $\rho_i = R_0 - (R_0 - \rho_0)/2^i$. By iterating (3.19), since $R - \rho = (R_0 - \rho_0) \cdot 2^{-i}$, similarly to the computation in (3.15) we obtain

$$\int_{B_{\rho_0}} V^\delta dx$$

$$\leq \left\{ \int_{B_{\rho_i}} V^\delta dx \right\}^{\left(\frac{2^*}{2\gamma}\right)^i} \cdot \prod_{i=1}^{\infty} \left\{ \frac{c_3 4^{i+1}}{(R_0-\rho_0)^2} \right\}^{\gamma\left(\frac{2^*}{2\gamma}\right)^i} \left\{ \int_{B_{R_0}} V^{\frac{\gamma-\delta}{\gamma-1}} dx \right\}^{(\gamma-1)\left(\frac{2^*}{2\gamma}\right)^i}$$

$$\leq \left\{ \int_{B_{\rho_i}} V^\delta dx \right\}^{\left(\frac{2^*}{2\gamma}\right)^i} \cdot c_4 \left\{ \frac{1}{(R_0-\rho_0)^2} \right\}^{\frac{2^*\gamma}{2\gamma-2^*}} \left\{ \int_{B_{R_0}} V^{\frac{\gamma-\delta}{\gamma-1}} dx \right\}^{\frac{2^*(\gamma-1)}{2\gamma-2^*}}$$

We use Lemma 2.7(i) with $\alpha = (\gamma-1)/(\gamma-\delta) > 1$ and we go to the limit as $i \to +\infty$; we obtain

$$\int_{B_{\rho_0}} V^\delta dx \leq c_5 \left\{ \frac{1}{(R_0-\rho_0)^2} \right\}^{\frac{2^*\gamma}{2\gamma-2^*}} \left\{ \int_{B_{R_0}} [1+g(|Du|)]dx \right\}^{\frac{2^*(\gamma-1)}{2\gamma-2^*}}$$

Finally

$$\begin{aligned} \int_{B_{\rho_0}} V dx &\leq \operatorname{meas}\{B_{\rho_0}\}^{1-\frac{1}{\delta}} \left\{ \int_{B_{\rho_0}} V^\delta dx \right\}^{\frac{1}{\delta}} \\ &\leq c_6 \left\{ \frac{1}{(R_0-\rho_0)^2} \right\}^{\frac{2^*\gamma}{(2\gamma-2^*)\delta}} \left\{ \int_{B_{R_0}} [1+g(|Du|)]dx \right\}^{\frac{2^*(\gamma-1)}{(2\gamma-2^*)\delta}} \end{aligned} \tag{3.20}$$

Since the exponents in the right-hand side of (3.20) converge to 1 as $\delta \to 2^*/2$ and $\gamma \to +\infty$, then we have the conclusion (3.16). □

4. - Approximation of the original problem with regular variational problems

Like in Section 1 we consider again the function $a: [0,+\infty) \to [0,+\infty)$, related to the convex function g by the condition $a(t) = g'(t)/t$. Let us recall that, by assumption, a is an increasing function of class $C^1([0,+\infty))$, with $a(1) = g'(1) > 0$.

Let $t_0 = \inf\{t \geq 0 \colon a(t) > 0\}$; then $t_0 \in [0,1)$. For every $k \in \mathbb{N}$ let us denote by $t_k = t_0 + (1 - t_0)/k$ and let us define the function a_k by

$$a_k(t) = \begin{cases} a(t_k) & \text{if } t \in [0, t_k) \\ a(t) & \text{if } t \in [t_k, k] \\ a(k) & \text{if } t \in (k, +\infty) \end{cases} . \tag{4.1}$$

Then, for every $k \in \mathbb{N}$, a_k is an increasing function on $[0,+\infty)$.

Let us denote by g_k the function of class $C^1([0,+\infty))$, with $g(0) = 0$ and whose first derivative is given by $g_k'(t) = a_k(t) \cdot t$. Since $g_k'(t)$ is increasing with respect to t, then g_k is convex in $[0,+\infty)$.

REMARK 4.1. For every $k \in \mathbb{N}$, g_k is a function of class $C^1([0,+\infty))$ and in general not of class $C^2([0,+\infty))$; but it is clear that we could modify g_k in neighbourhoods of $t = t_k$ and $t = k$ to make it of class $C^2([0,+\infty))$. However, the second derivative $g_k''(t)$ exists for every $t \neq t_k$, $t \neq k$, and, by Lemma 2.4 (see also Lemma 4.3), $g_k'(t)$ is Lipschitz continuous on bounded subsets of $[0,+\infty)$. Since the chain rule holds for compositions of functions of $W^{1,2}$ by Lipschitz continuous functions, then it is not difficult to see that, in fact, our analysis works for convex functions g of class $W^{2,\infty}_{\text{loc}}([0,+\infty))$. In terms of a, our results hold for increasing functions a of class $W^{1,\infty}_{\text{loc}}([0,+\infty))$. These can be considered, from the very beginning, as classes of regularity for a and g.

LEMMA 4.2. (i) *for every $k \in \mathbb{N}$, the following conditions of uniform ellipticity for g_k (see* (3.2)*) hold*

$$m_k \leq \frac{g_k'(t)}{t} \leq g_k''(t) \leq M_k;$$

for some positive constants m_k, M_k;

(ii) *there exists a constant c, independent of k, such that*

$$g_k(t) \leq c(1 + g(t)), \quad \forall\ t \geq 0, \quad \forall\ k \in \mathbb{N}.$$

PROOF. Since $a_k(t)$ is increasing for $t \in [0,+\infty)$ then, by Lemma 2.1(i), we have $g_k'(t) \leq g_k''(t) \cdot t$; moreover, $g_k''(t)$ is equal either to $g''(t)$ (if $t \in [t_k, k]$, in fact $g_k'(t) = g'(t)$), or to a constant (respectively $g_k''(t) = a(k)$ if $t > k$, $g_k''(t) = a(t_k)$ if $t \in [0, t_k)$). Thus we have

$$0 < a(t_k) \leq a_k(t) = \frac{g_k'(t)}{t} \leq g_k''(t) \leq \max\{a(k), \max\{g''(t) \colon t \in [t_k, k]\}\}$$

and (i) is proved. To prove (ii), let us observe that $a_k(t) \leq a(t)$ for every $t \geq 1$, while $a_k(t) \leq a(1)$ if $t < 1$. Since $g_k'(t) = a_k(t) \cdot t$ and $g'(t) = a(t) \cdot t$, then we obtain

$$g_k(t) \leq g(1) + \int_1^t g'(s)ds = g(t), \quad \forall\ t \geq 1,$$

and, if $t < 1$, then $g_k(t) < g(1)$. Finally, for every $t \geq 0$,

$$g_k(t) \leq \max\{g(1), g(t)\} \leq g(1) + g(t) \leq (1 + g(1)) \cdot (1 + g(t)). \qquad \square$$

To apply to g_k the *a priori estimates* of Section 3, it remains to show that g_k satisfies the ellipticity conditions of Section 2, as stated in the following lemma.

LEMMA 4.3. *If g satisfies the conditions of the Lemmata of Section 2 then g_k satisfies the same conditions too, with constants independent of k.*

PROOF. Let us first observe that, since $t_k < 1 < k$ for every $k > 1$, then $g_k'(1) = g'(1)$, $g_k''(1) = g''(1)$; moreover $g(1) \leq g_k(1) \leq g_1(1)$ and $\int_0^1 \sqrt{g_k'(s)/s}\, ds \geq \int_0^1 \sqrt{g'(s)/s}\, ds$. Thus, by the equivalences proved in Section 2,

it is sufficient to show that, for example, the condition (v) of Lemma 2.4 holds with constant independent of k, since the other conditions follow as consequence.

If $t \in [1, k)$, then $g_k'' = g''$ and $g_k \geq g$; thus

$$g_k''(t) \cdot t^{2\alpha} = g''(t) \cdot t^{2\alpha} \leq c[g(t)]^\alpha \leq c[g_k(t)]^\alpha, \quad \forall\ k \in \mathbb{N}. \tag{4.2}$$

If $t \geq k$ then (by $g_k''(k)$ we denote the right second derivative, equal to $g'(k)/k$)

$$\begin{aligned} g_k(t) &= g_k(k) + g_k'(k)(t-k) + \frac{1}{2} g_k''(k)(t-k)^2 \\ &\geq g(k) + g'(k)(t-k) + \frac{g'(k)}{2k}(t-k)^2 = g(k) + \frac{g'(k)}{2k}(t^2 - k^2) \end{aligned}$$

and, by the convexity inequality $g'(k) \cdot k \geq g(k)$,

$$g_k(t) \geq g(k) \cdot \frac{t^2 + k^2}{2k^2} = g(k) \cdot \frac{t^2 + k^2}{t^2} \cdot \frac{t^2}{2k^2} \geq g(k) \cdot \frac{t^2}{2k^2}.$$

By the first of (iv) of Lemma 2.4, then we obtain

$$g_k''(t) \cdot t^{2\alpha} = \frac{g'(k)}{k} \cdot t^{2\alpha} = g'(k) \cdot k^{2\alpha-1} \left(\frac{t}{k}\right)^{2\alpha} \leq c[g(k)]^\alpha \cdot \left(\frac{t}{k}\right)^{2\alpha} \leq 2^\alpha c[g_k(t)]^\alpha,$$

for every $k \in \mathbb{N}$ and for every $t \geq k$, which, together with (4.2), gives the conclusion.

5. - Passage to the limit

Let us consider, for every $k \in \mathbb{N}$, the integral functional

$$F_k(v) = \int_\Omega f_k(Dv)dx, \quad \text{with} \quad f_k(Dv) = g_k(|Dv|), \tag{5.1}$$

where g_k is defined trough its derivative $g_k'(t) = a_k(t) \cdot t$, $g_k(0) = 0$, and a_k is given by (4.1). Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, such that $f(Dv) \in L^1_{\mathrm{loc}}(\Omega)$, be a minimizer of the integral $F(v)$ in (1.4).

Let B_R be a ball of radius R contained in Ω and let $u_k \in W^{1,2}(B_R, \mathbb{R}^N)$ be a minimizer of the integral in (5.1), with the Dirichlet condition $u_k = u$ on the boundary ∂B_R of B_R, i.e.

$$\int_{B_R} g_k(|Du_k|)dx \leq \int_{B_R} g_k(|Dv|)dx, \quad \forall\ v \in W_0^{1,2}(B_R, \mathbb{R}^N) + u \tag{5.2}$$

(since the ellipticity conditions of Lemma 4.2(i), it is equivalent to consider in (5.2) test functions $v - u$ either in $W_0^{1,2}(B_R, \mathbb{R}^N)$, or in $C_0^1(B_R, \mathbb{R}^N)$). Then, in particular,

$$\int_{B_R} g_k(|Du_k|)dx \leq \int_{B_R} g_k(|Du|)dx. \tag{5.3}$$

By the analysis of Section 4, for every $k \in \mathbb{N}$ g_k satisfies the ellipticity conditions of Lemma 4.2(i), and, uniformly with respect to $k \in \mathbb{N}$, the ellipticity estimates of Section 2. Therefore we can apply to g_k the *a priori estimates* of Section 3: by Lemmata 3.1 and 3.2, for every $\epsilon > 0$ and for every ball B_ρ, of radius $\rho < R$ and with the same center of B_R, there exist constants c_1, c_2 such that (we use the notation $\rho' = \rho + (R - \rho)/2$)

$$\begin{aligned} ||Du_k||^2_{L^\infty(B_\rho;\mathbb{R}^{N\times n})} &\leq c_1 \int_{B_{\rho'}} (1 + |Du_k|^2 g_k''(|Du_k|))dx \\ &\leq c_2 \left\{ \int_{B_R} [1 + g_k(|Du_k|)]dx \right\}^{1+\epsilon}, \quad \forall\ k \in \mathbb{N}. \end{aligned} \tag{5.4}$$

We emphasize that c_1 and c_2 do not depend on k. By (5.3) and by Lemma

4.2(ii) we obtain

$$
\begin{aligned}
\|Du_k\|^2_{L^\infty(B_\rho;\mathbb{R}^{N\times n})} &\leq c_2 \left\{ \int_{B_R} [1 + g_k(|Du|)]dx \right\}^{1+\epsilon} \\
&\leq c_3 \left\{ \int_{B_R} [1 + g(|Du|)]dx \right\}^{1+\epsilon}, \quad \forall\ k \in \mathbb{N}.
\end{aligned} \tag{5.5}
$$

By (5.5), up to a subsequence, u_k converges in the weak* topology of $W^{1,\infty}(B_\rho, \mathbb{R}^N)$ to a function w. Fixed $k_0 \in \mathbb{N}$, we consider $k \geq k_0$ and, by using (5.2), we pass to the limit as $k \to +\infty$:

$$
\int_{B_\rho} g_{k_0}(|Dw|)dx \leq \liminf_{k \to +\infty} \int_{B_\rho} g_{k_0}(|Du_k|)dx \leq \int_{B_\rho} g_{k_0}(|Dv|)dx,
$$

for all $v \in C^1_0(B_\rho, \mathbb{R}^N) + u$. As $k_0 \to +\infty$ (we can go to the limit by the dominated convergence theorem) we obtain that w is a minimizer of the integral $F(v)$ in (1.4). Moreover, by (5.5),

$$
\|Dw\|^2_{L^\infty(B_\rho;\mathbb{R}^{N\times n})} \leq c_3 \left\{ \int_{B_R} [1 + g(|Du|)]dx \right\}^{1+\epsilon}. \tag{5.6}
$$

Note that our assumptions on f do not guarantee uniqueness of the minimizer for the Dirichlet problem. However $g''(t)$ and $g'(t) \cdot t$ are positive for $t \geq 1$; thus $f(\xi) = g(|\xi|)$ is locally strictly convex for $|\xi| > 1$. This implies that $|Dw(x)| = |Du(x)|$ for almost every $x \in B_R$ such that $|Du(x)| > 1$, and thus Du satisfies (5.6) too. This completes the proof of Theorem 1.1.

REMARK 5.1. As stated in Remarks 1.2 and 2.5, under the assumption (1.12), the conclusion (1.11) of Theorem 1.1 holds with $\epsilon = 0$ too; in fact in this case, since $1 + g''_k(t) \cdot t^2 \leq c(1 + g_k(t))$, then the use of Lemma 3.2 in (5.4) becomes unnecessary.

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