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1. - Introduction

For $n > 0$, A^{-n} is defined as the Banach space of all analytic functions f in the unit disc U such that

$$\|f\|_{A^{-n}} = \sup_{z\in U} |f(z)|(1-|z|^2)^n < \infty.$$

If M is a subset of A^{-n} and if $a \in U$ we consider the extremal problem

$$\sup_{f\in M, \|f\|\leq 1} |f(a)|.$$

Any function which attains the supremum will be called an extremal function for M at a. We note that if M is closed under locally uniform convergence then M has extremal functions at every $a \in U$, for the functions of norm ≤ 1 in A^{-n} form a normal family. However, it should be emphasized that not every norm-closed subspace of A^{-n} is closed in this sense. A simple example of such a subspace is

$$A_0^{-n} = \{f \in A^{-n}: \lim_{|z|\to 1} |f(z)|(1-|z|^2)^n = 0\}.$$

A function $f \in M$ is called a contractive divisor for M if $\|f\| = 1$ and for every $g \in M$, $g/f \in A^{-n}$ with

$$\left\|\frac{g}{f}\right\| \leq \|g\|.$$

In [2] and [3] the importance and interdependence of the above concepts were amply demonstrated in the case of the Bergman spaces which are closely related to A^{-n}.

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In Section 2 of the present work, we shall give a geometric characterization of extremal functions in A^{-n} and some conditions relating them to contractive divisors. Section 3 is devoted to the explicit calculation of extremal functions in some simple cases. The authors wish to thank Dr. E. Beller for helpful discussions pertaining to this work.

2. - The main results

For $K \subset \mathbb{C}$ compact define the polynomial hull of K by

$$\hat{K} = \{z \in \mathbb{C}: |p(z)| \leq \sup_{w \in K} |p(w)| \text{ for all analytic polynomials } p\}.$$

The following elementary characterization of $\hat{K}$ is proved in the first chapter of [4].

PROPOSITION 2.1. *$\hat{K} = K$ together with all of the bounded components of $\mathbb{C}\backslash K$.*

DEFINITION 2.2. If $f \in A^{-n}$, if $\|f\| = 1$, and if $\varepsilon > 0$ define

$$K_\varepsilon (= K_\varepsilon(f)) = \{z \in U: |f(z)|(1 - |z|^2)^n > 1 - \varepsilon\}$$
$$K_0 (= K_0(f)) = \{z \in U: |f(z)|(1 - |z|^2)^n = 1\}.$$

DEFINITION 2.3. Let $a \in U$. We say that $f \in A^{-n}$ has property P_a if $\|f\| = 1$ and for every $\varepsilon > 0$, $a \in \hat{\overline{K}}_\varepsilon$. (Roughly, this means that every K_ε "surrounds" a.) We say that f has property P'_a if $a \in \hat{\overline{K}}_0$.

THEOREM 2.4. *Let M be a subset of A^{-n} invariant under multiplication by nonvanishing entire functions (e.g., a closed subspace invariant under multiplication by z or the set of all functions in A^{-n} vanishing precisely on a given set in U.) Then if f is an extremal function for M at some point $a \in U$, f has property P_a.*

PROOF. Clearly, $\|f\| = 1$. Assume that f does not have property P_a. Then for some $\varepsilon > 0$, $a \notin \hat{\overline{K}}_\varepsilon$. However, $\mathbb{C}\backslash\hat{\overline{K}}_\varepsilon$ is just the unbounded component of $\mathbb{C}/\overline{K}_\varepsilon$, which is open and connected. It follows that also $\mathbb{C}\backslash\hat{\overline{K}}_\varepsilon \cup \{a\}$ is open and connected. Therefore, by Runge's theorem there exists a polynomial p such that $p(a) > 0$ and $\operatorname{Re} p(z) < 0$ on $\overline{K}_\varepsilon$. Define

$$h(z) = \exp\left(\frac{1}{m} p(z)\right),$$

where m is chosen so large that $|h(z)| \leq \dfrac{1}{1-\varepsilon}$ everywhere in U. By our assumption, $hf \in M$. Furthermore, on K_ε, $|h(z)| < 1$ so $|h(z)|\,|f(z)|(1 - |z|^2)^n < 1$. On $U \backslash K_\varepsilon$, $|f(z)|(1 - |z|^2)^n \le 1 - \varepsilon$, so $|h(z)|\,|f(z)|(1 - |z|^2)^n \le 1$. We conclude that $\|hf\| \le 1$. But $|h(a)| > 1$, so $|(hf)(a)| > |f(a)|$, contradicting the extremality of f at a. □

DEFINITION 2.5. For $f \in A^{-n}$, I_f represents the smallest z-invariant closed subspace containing f, i.e., the norm closure of all functions pf such that p is a polynomial. Also, define $J_f = \{g \in A^{-n} : g/f \text{ is analytic in } U\}$.

With this notation we can prove a converse to Theorem (2.4).

THEOREM 2.6. *If f has property P_a for some $a \in U$ then f is extremal at a for the subspace I_f.*

PROOF. Clearly, it suffices to show that if p is a polynomial such that $\|pf\| = 1$ then $|p(a)| \le 1$. To that end let $\varepsilon > 0$ be given. By property P_a applied to f, $a \in \hat{\overline{K}}_\varepsilon$. Thus

$$|p(a)| \le \sup_{z \in \overline{K}_\varepsilon} |p(z)| = \sup_{z \in K_\varepsilon} |p(z)| = \sup_{z \in K_\varepsilon} \frac{|p(z)|\,|f(z)|(1 - |z|^2)^n}{|f(z)|(1 - |z|^2)^n} \le \frac{1}{1 - \varepsilon}.$$

Letting $\varepsilon \to 0$ we obtain the result. □

Next we prove strengthened versions of Theorems (2.4) and (2.6) for extremal functions with "tame" boundary behavior; i.e., for functions f of norm 1 which satisfy

$$\varlimsup_{|z| \to 1} |f(z)|(1 - |z|^2)^n = 1 - \delta, \text{ with } \delta > 0. \tag{2.7}$$

THEOREM 2.8. *Let M and f be as in Theorem* (2.4) *with the additional assumption that f satisfies* (2.7). *Then f has property P_a' (as in Definition* (2.3)*).*

PROOF. By (2.7) $K_0(f)$ is a compact subset of U. If f does not have property P_a' then by the geometric characterization of $\hat{K}_0$ there must be a curve γ connecting a to the boundary of U which does not intersect K_0. This together with (2.7) implies that for some ε, with $0 < \varepsilon < \delta$,

$$|f(z)|(1 - |z|^2)^n < 1 - \varepsilon \text{ on } \gamma.$$

It follows that γ is disjoint from $\overline{K}_\varepsilon$. But this means that f does not have property P_a. In view of Theorem (2.4) we have arrived at a contradiction to the assumption that f is extremal. □

THEOREM 2.9. *Assume that $\|f\| = 1$ and f satisfies* (2.7). *Then f is extremal at every point of $\hat{K}_0$ both for I_f and for the generally larger subspace J_f.*

PROOF. The condition (2.7) implies that K_0 is a compact subset of U. Now if $a \in \hat{K}_0$ and if $g \in J_f$ while $\|g\| = 1$ then

$$\left|\frac{g(a)}{f(a)}\right| \leq \max_{z \in K_0} \left|\frac{g(z)}{f(z)}\right| = \max_{z \in K_0} \frac{|g(z)|(1-|z|^2)^n}{|f(z)|(1-|z|^2)^n} \leq 1.$$

Therefore, f is extremal at a. □

Our next two results concern the contractive property.

THEOREM 2.10. *Let f be a contractive divisor for some set $M \subset A^{-n}$. Then f is extremal for M at zero.*

PROOF. If $g \in M$ and if $\|g\| \leq 1$, then

$$\left|\frac{g(0)}{f(0)}\right| \leq \left\|\frac{g}{f}\right\| \leq \|g\| \leq 1,$$

so $|g(0)| \leq |f(0)|$. □

THEOREM 2.11. *Assume that $f \in A^{-n}$ is continuous on $\overline{U}$ and is a contractive divisor for the subspace J_f. Then f has property P_0' and $|f(z)| \geq 1$ on the circle $T = \{z: |z| = 1\}$. Conversely, if f is continuous on $\overline{U}$, $|f(z)| \geq 1$ on T, and if all of the zeros of f are contained in $\hat{K}_0$, then f is a contractive divisor on J_f.*

PROOF. If f is a contractive divisor on J_f, Theorem (2.10) gives that f is extremal for J_f at zero. If, moreover, f is continuous on $\overline{U}$, then it certainly satisfies (2.7). Thus we can conclude from Theorem (2.8) that f has property P_0'. For the next assertion we argue by contradiction. If at some $z_0 \in T$ $|f(z_0)| < 1$, then there is a neighborhood S of z_0 and a number $\varepsilon > 0$ such that

$$|f(z)| \leq 1 - \varepsilon \text{ in } S \cap \overline{U}.$$

Now for any positive integer m the function

$$g(z) = \frac{1}{(1-\overline{z}_0 z)^n}\left(\frac{z+z_0}{2}\right)^m$$

has norm 1, and if we choose m sufficiently large we can arrange that $|g(z)| \leq 1 - \varepsilon$ outside of S. It follows immediately that $\|fg\| \leq 1 - \varepsilon$ while $\|g\| = 1$. Since $fg \in J_f$ we have contradicted the contractive property of f. For the converse assertion, if $g \in J_f$ and $\|g\| = 1$ then Theorem 2.9 implies that

$$|g(z)| \leq |f(z)| \text{ for all } z \in \hat{K}_0,$$

or

$$\left|\frac{g(z)}{f(z)}\right| \leq 1 \text{ on } \hat{K}_0. \tag{2.12}$$

On the other hand, $U\backslash\hat{K}_0$ is an open connected set whose boundary is contained in $T \cup K_0$. By hypothesis f is nonvanishing in $U\backslash\hat{K}_0$ and $|f(z)| \geq 1$ on T. Furthermore, on K_0

$$|f(z)| = \frac{1}{(1-|z|^2)^n} \geq 1,$$

and so we conclude by the minimum principle that

$$|f(z)| \geq 1 \text{ for all } z \in U\backslash\hat{K}_0.$$

Thus if $g \in J_f$ and $||g|| = 1$

$$\left|\frac{g}{f}\right| \leq |g| \text{ in } U\backslash\hat{K}_0.$$

This together with (2.12) proves that f is a contractive divisor on J_f. □

We conclude this section with the observation that although we have presented our results in the context of A^{-n} they apply essentially verbatim to spaces of analytic functions in U which are bounded by arbitrary weight functions.

3. - Construction of extremal functions for finite zero sets

An important special case of the theory presented in Section 2 is obtained by choosing a finite set $P = \{z_1 \dots z_m\} \subset U$ and positive integers $\{k_1 \dots k_m\}$ and letting

$$M = \{f \in A^{-n} : f \text{ vanishes at each } z_i \in P \text{ with multiplicity at least } k_i\}$$

$$M' = \{f \in A^{-n} : f \text{ vanishes at each } z_i \in P \text{ with multiplicity } k_i \text{ and nowhere else in } U\}.$$

By a normal families argument, M and M' both contain extremal functions at each point of U. Now if h is an extremal function for M' at some point $a \in U$, and if h satisfies (2.7) then by Theorem (2.8) h has property P'_a. Theorem (2.9) then implies that h is extremal at a for J_h, which in this case is just M. In particular, we obtain an extremal function for M which has no extraneous zeros. These considerations together with the results of Section 2 accentuate the importance of the following question which we have been unable to resolve.

QUESTION. *With M and M' as above, what can be said about the boundary behavior of their extremal functions?*

We turn to the problem of explicit construction of extremal functions for M and M' at a point $a \in U$ in some simple cases. If we presume that these functions satisfy (2.7) then they must have property P'_a. It is convenient to hypothesize the slightly stronger property

P^*_a: $\|f\| = 1$, f satisfies (2.7), and there exists an analytic Jordan curve γ surrounding a on which $|f(z)|(1-|z|^2)^n \equiv 1$.

Clearly, any function f satisfying P^*_a is extremal for J_f at a. Moreover, P^*_a implies that

$$\nabla(|f(z)|(1-|z|)^2)^n) = 0 \text{ on } \gamma$$

or

$$\frac{\partial}{\partial z}(|f(z)|^2(1-|z|^2)^{2n}) = 0 \text{ on } \gamma.$$

which gives the differential equation

$$\frac{f'(z)}{f(z)} = \frac{2n\bar{z}}{1-|z|^2} \text{ on } \gamma. \tag{3.1}$$

Thus is $S(z)$ is the "Schwarz function" (see [1] and [6]) analytic near γ and satisfying $S(z) = \bar{z}$ on γ, we have

$$\frac{f'(z)}{f(z)} = \frac{2nS(z)}{1-zS(z)} \text{ on } \gamma, \tag{3.2}$$

and by analytic continuation this must persist in the whole unit disc.

Conversely, if γ is an analytic Jordan curve in U which surrounds a and if the Schwarz function S of γ is such that

$$\frac{2nS(z)}{1-zS(z)}$$

is analytic in U except for simple poles at z_i $(i = 1,2\ldots m)$ with positive integral residues k_i then we can integrate (3.2) to produce $f \in M'$ (as defined above) such that $|f(z)|^2(1-|z|^2)^{2n}$ has zero gradient on γ. Now if this f also satisfies (2.7) and if $S(z) = \bar{z}$ *only* on γ (so that $|f(z)|^2(1-|z|^2)^{2n}$ has no other critical points in U) we can conclude that $|f(z)|(1-|z|^2)^n$ takes its maximum identically on γ, so that if we normalize f to have norm 1 it has property P^*_a, and in particular f is extremal for M' and for M.

Let us apply these ideas in the case of a single zero of order $k \geq 1$ at the origin. Here

$$M_k = \{f \in A^{-n}: f(z)/z^k \text{ is analytic in } U\}$$

$$M'_k = \{f \in A^{-n}: f(z)/z^k \text{ is analytic and nonvanishing in } U\}.$$

If $f \in M'_k$, the function $zf'(z)/f(z)$ is analytic in U. If this f satisfies (3.1) we have

$$\frac{zf'(z)}{f(z)} = \frac{2n|z|^2}{1-|z|^2} \text{ on } \gamma. \tag{3.3}$$

In particular, the analytic function $zf'(z)/f(z)$ is positive on γ which is possible only if this function is identically constant. Since $f \in M_k'$ we must have

$$\frac{zf'(z)}{f(z)} = k; \quad f(z) = cz^k.$$

By (3.3)

$$\frac{2n|z|^2}{1 - |z|^2} = k \text{ on } \gamma,$$

from which we deduce that γ must be the circle $|z| = \sqrt{\dfrac{k}{2n + k}}$. Letting

$$\alpha_{k,n} = \left(\sqrt{\frac{k}{2n + k}}\right)^k \left(1 - \frac{k}{2n + k}\right)^n,$$

we find that $f(z) = \dfrac{z^k}{\alpha_{k,n}}$ is a function of norm 1 which attains its norm identically on the circle γ. By the reasoning outlined at the beginning of this section we conclude that f is extremal both for M' and for M at every point inside or on γ. Clearly, $|f(z)| > 1$ on the boundary so by Theorem (2.11) f is a contractive divisor in M. All of this can easily be checked by a direct argument using the maximum principle.

At points outside of γ we have yet to determine an extremal function for M_k'. Since (3.3) cannot be fulfilled on any curve surrounding such points, one might expect that for each a outside of γ there are extremal functions which attain the value $\dfrac{1}{(1 - |a|^2)^n}$ at a, as is indeed the case on γ itself where f attains this value identically. In fact, we can construct such functions explicitly, as follows: Consider the two parameter family of functions

$$h(z) = h_{k,\varepsilon}(z) = \frac{z^k}{(1 - z)^{n-\varepsilon}} \quad (0 < \varepsilon < n).$$

If we substitute this h into formula (3.1) for critical points of $|h(z)|^2(1 - |z|^2)^{2n}$ we find that critical points occur when

$$k + \frac{(n - \varepsilon)z}{1 - z} = \frac{2n|z|^2}{1 - |z|^2}. \tag{3.4}$$

By inspection of the function $|h(z)|^2(1 - |z|^2)^{2n}$ it is clear that its maximum must occur at some point $z = r > 0$. Conversely, if we choose $r > 0$ such that $\dfrac{2nr^2}{1 - r^2} > k$ we find that (3.4) is satisfied uniquely at r if we choose

$$n - \varepsilon = \frac{2nr}{1 + r} - \frac{k(1 - r)}{r}. \tag{3.5}$$

This choice is always possible since the right side of (3.5) increases precisely from 0 to n as r proceeds from the circle γ to 1. Finding ε as indicated by (3.5) and normalizing, we find that the function $\dfrac{h_{k,\varepsilon}(z)}{h_{k,\varepsilon}(r)(1-r^2)^n}$ is extremal at r, where it attains the value $\dfrac{1}{(1-r^2)^n}$. By a rotation we can extend the above construction to arbitrary points outside of γ. For the case $k=1$ we can summarize our results in the following "Schwarz Lemma for A^{-n}".

LEMMA 3.6. *If* $f \in A^{-n}$, $\|f\| = 1$, *and if* $f(0) = 0$ *then the following estimates are sharp:*

$$|f(z)| \le \begin{cases} \dfrac{(2n+1)^{n+1/2}}{(2n)^n}|z| & ; \quad |z| < \left(\dfrac{1}{2n+1}\right)^{1/2} \\[2ex] \dfrac{1}{(1-|z|^2)^n} & ; \quad |z| \ge \left(\dfrac{1}{2n+1}\right)^{1/2} \end{cases}.$$

Equality is attained in the first estimate only by the functions

$$f(z) = e^{i\theta}\frac{(2n+1)^{n+1/2}}{(2n)^n}z.$$

Next we consider extremal functions vanishing at an arbitrary single point $\alpha \in U$. We use the important fact that for every such α the transformation

$$(Tf)(z) = \frac{(1-|\alpha|^2)^n}{(1-\overline{\alpha}z)^{2n}} f\left(\frac{\alpha - z}{1-\overline{\alpha}z}\right) \quad \text{is an isometry on } A^{-n}.$$

(T was used extensively in [5].) In particular, if f has norm 1 and vanishes at α, Tf has norm 1 and vanishes at zero. Thus we can generalize the Schwarz Lemma as follows:

LEMMA 3.7. *If* $f \in A^{-n}$, $\|f\| = 1$, *and* $f(\alpha) = 0$ *then the following estimates are sharp:*

$$|f(z)| \le \begin{cases} \dfrac{(2n+1)^{n+1/2}}{(2n)^n}\dfrac{(1-|\alpha|^2)^n}{|1-\overline{\alpha}z|^{2n}}\left|\dfrac{\alpha-z}{1-\overline{\alpha}z}\right|; & \left|\dfrac{\alpha-z}{1-\overline{\alpha}z}\right| \le \dfrac{1}{\sqrt{2n+1}} \\[2ex] \dfrac{1}{(1-|z|^2)^n} & \textit{otherwise.} \end{cases}$$

Equality is obtained in the first estimate only by the extremal functions

$$G_\alpha(z) = e^{i\theta}\frac{(2n+1)^{n+1/2}}{(2n)^n}\frac{(1-|\alpha|^2)^n}{(1-\overline{\alpha}z)^{2n}}\frac{\alpha-z}{1-\overline{a}z}.$$

It is interesting to note that the extremal function G_α is not always a contractive divisor. By Theorem (2.11) it is contractive if and only if $|G_\alpha(z)| \ge 1$

whenever $|z| = 1$. One easily sees that this occurs if and only if

$$\frac{1-|\alpha|}{1+|\alpha|} \geq \frac{2n}{(2n+1)^{1+1/2n}} \text{ on } |\alpha| \leq \frac{(2n+1)^{1+1/2n}-2n}{(2n+1)^{1+1/2n}+2n}.$$

We can show further that if

$$\frac{(2n+1)^{1+1/2n}-2n}{(2n+1)^{1+1/2n}+2n} < |\alpha| < \frac{1}{\sqrt{2n+1}}$$

there are no contractive divisors for the space $M_\alpha = \{f \in A^{-n} : f(\alpha) = 0\}$.

Indeed, suppose that α lies in the indicated region, $g \in M_\alpha$, and g is contractive. Then by (3.7) $|g(0)| \leq |G_a(0)|$, and by the contractive property

$$\left|\frac{G_a(0)}{g(0)}\right| \leq \left\|\frac{G_\alpha}{g}\right\| \leq 1.$$

Hence $|G_\alpha(0)| = |g(0)|$ which by Lemma 3.7 implies that $G_\alpha(z) = g(z)$, which contradicts the assumption that g is contractive.

This situation is in sharp contrast with the case of the Bergman spaces, see [2].

As a final example we consider the problem of extremal functions for the set of functions in A^{-n} which have simple zeros at two symmetric points $\pm z_0$ and are nonvanishing elsewhere. By the remarks following equation (3.1) the main problem is to produce an appropriate curve γ and an appropriate Schwarz function S. To that end we use some ideas from Shapiro's notes [6].

Let A and R be positive numbers such that

$$R < A < \frac{R^2-1}{2}. \tag{3.8}$$

Then the function

$$z = \varphi(w) = \frac{2Aw}{w^2+R^2} \tag{3.9}$$

maps $\overline{U}$ univalently into U. Specifically, the inverse is given by

$$w = \frac{A - \sqrt{A^2 - R^2 z^2}}{z}, \tag{3.10}$$

where we choose that branch of the square root which makes $w = 0$ correspond to $z = 0$. Now when $|w| = 1$, z traces a Jordan curve γ on which

$$\overline{z} = \frac{2A\overline{w}}{\overline{w}^2+R^2} = \frac{2Aw}{1+R^2w^2}, \tag{3.11}$$

so by inserting (3.10) into (3.11) we obtain a Schwarz function $S(z)$ for γ. Equation (3.2) now becomes

$$\frac{f'(z)}{f(z)} = \frac{4nAw}{1 + R^2w^2 - 2Awz} = \frac{4An}{\frac{1}{w} + R^2w - 2Az}.$$

By (3.10) $\frac{1}{w} = \frac{A + \sqrt{A^2 + R^2z^2}}{R^2z}$ so

$$\frac{f'(z)}{f(z)} = \frac{4nAR^2z}{A(1+R)^4 + [(1-R^4)\sqrt{A^2 - R^2z^2}] - 2AR^2z^2}$$
$$= \frac{4nz}{\left(\frac{1}{R^2} + R^2\right) - 2z^2 + \left(\frac{1}{R^2} - R^2\right)\sqrt{1 - \frac{R^2}{A^2}z^2}}.$$

At this point it is convenient to define new parameters:

(3.12) $$a = R^2 + \frac{1}{R^2}; \quad b = R^2 - \frac{1}{R^2}; \quad c = \frac{R^2}{A^2}.$$

It follows from (3.8) that a, b and c are positive and $c < 1$. Clearly, $a^2 - b^2 = 4$. In these parameters we have the equation

$$\frac{f'(z)}{f(z)} = \frac{4nz}{a - 2z^2 - b\sqrt{1 - cz^2}}.$$

Thus if g satisfies

(3.13) $$\frac{g'(z)}{g(z)} = \frac{2n}{a - 2z - b\sqrt{1 - cz}}$$

and has a single zero in U, we can take $f(z) = g(z^2)$ to obtain a solution of (3.2) having two symmetric zeros. Using $a^2 - b^2 = 4$ we observe that

$$\frac{2n}{a - 2z - b\sqrt{1 - cz}} + \frac{2n}{a - 2z + b\sqrt{1 - cz}} = \frac{n(a - 2z)}{z^2 - \left(a - \frac{b^2c}{4}\right)z + 1}.$$

The expression on the right has two reciprocal poles, say at r with $|r| \leq 1$ and at $1/r$. However, by a calculation one deduces from (3.8) and (3.12) that $a - \frac{b^2c}{4} < -2$ which implies that $-1 < r < 0$. Since a, b, and c are positive, $a - 2z + b\sqrt{1 - cz}$ cannot vanish when z is negative, so we conclude that the expression on the right side of (3.13) has exactly one simple pole in U, namely at r, and one additional pole at $1/r$. By partial fractions

$$\frac{n(a-2z)}{z^2-\left(a-\frac{b^2c}{4}\right)z+1}=\frac{\alpha}{z-r}+\frac{\beta}{z-1/r}, \tag{3.14}$$

where

$$\alpha=\frac{n(a-2r)}{r-1/r}, \tag{3.15}$$

which we make equal to one by an appropriate choice of a. Equating coefficients of z in (3.14) we then find that

$$\alpha+\beta=-2n, \text{ or } \beta=-2n-1.$$

Solving (3.13) and inserting f in place of g we conclude that

$$f(z)=c\left(\frac{z^2-r}{z^2-1/r}\right)\frac{1}{(z^2-1/r)^{2n}}h(z^2), \tag{3.16}$$

where

$$h(w)=\exp\left(\int_0^w\frac{-2ndz}{1-2z+b\sqrt{1-cz}}\right).$$

The integration for h can be carried out explicitly to obtain a closed expression for f. We prefer a different approach. Namely, going back to (3.13) we note that

$$\int\frac{2ndz}{a-2z-b\sqrt{1-cz}}=(w=\sqrt{1-cz})\int\frac{-2nwdw}{w^2-\frac{bc}{2}w+\left(\frac{ac}{2}-1\right)}$$
$$=\int\frac{\alpha}{w-w_1}+\frac{\beta}{w-w_2}\,dw.$$

From (3.13) we obtain

$$f(z)=c_1(\sqrt{1-cz^2}-w_1)^{\alpha}(\sqrt{1-cz^2}-w_2)^{\beta}.$$

Comparing with (3.16) we conclude that

$$f(z)=c_1\frac{\sqrt{1-cz^2}-\sqrt{1-cr}}{(\sqrt{1-cz^2}-\sqrt{1-c/r})^{2n+1}}, \tag{3.17}$$

where c_1 is chosen to give $\|f\|=1$.

It remains only to verify that the function we have constructed really is extremal. Since this function is analytic in a neighborhood of $\overline{U}$ our remarks at the beginning of this section imply that f will be proved extremal if we can verify that in the above construction

$$S(z) = \overline{z} \textit{ only } \text{on } \gamma.$$

But this can easily be checked via the parametric equation

$$\frac{2Aw}{1 + R^2 w^2} = S(z) = \overline{z} = \frac{2A\overline{w}}{\overline{w}^2 + R^2}$$

which one readily sees is satisfied only if $|w| = 1$; i.e., on γ, or if $w = 0$, which corresponds to $z = 0$. However, the point $z = 0$ is in general an extraneous critical point of the function $|f(z)|^2(1 - |z|)^2)^{2n}$, introduced by the fact that this is a smooth function of z^2. One sees this clearly in the case where $|r| < \frac{1}{2n+1}$, for then the Schwarz Lemma (3.7) prevents any function of f of norm 1 which vanishes at the point $\sqrt{r}$ from taking the value 1 at the origin. Thus zero cannot be a maximum point of $|f(z)|^2(1 - |z|^2)^{2n}$ in this case, and all the more so if f also vanishes at $-\sqrt{r}$. So in general the function constructed in (3.17) really attains its norm on γ, and we can conclude that it is extremal at all points inside or on γ for the subspace of functions vanishing at the points $z = \pm\sqrt{r}$.

Finally, we compute the range of r and n for which our last example is applicable. Now if $n > 0$ and $r \in (-1, 0)$ are given, formula (3.15) shows that we must choose the parameter a so that

$$a = 2r + \frac{1}{n}\left(r - \frac{1}{r}\right).$$

By (3.12) $b = \sqrt{4 - a^2}$, and by (3.4) $a - \frac{b^2 c}{4} = r + \frac{1}{r} \Rightarrow c = \left(r + \frac{1}{r} - a\right)/1 - \frac{a^2}{4}$.

The restrictions (3.8) together with (3.12) imply that $a > 1$, $c < 1$ and $c > \frac{4}{a-2}$. However, the last inequality is an automatic consequence of our explicit formula for c, together with the fact that $-1 < r < 0$. So really the only restrictions are

$$a > 6, \quad c < 1,$$

from which one can find the exact range of applicability of the example. Qualitatively one sees that as $n \to \infty$ we can accept r's only from a progressively smaller neighborhood of zero, and as $n \to 0$ the range of r expands to the whole interval $(-1, 0)$.

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