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Orbifold Riemann Surfaces and the Yang-Mills-Higgs Equations

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0. - Introduction

In this paper we study the $U(2)$ Yang-Mills-Higgs equations on orbifold Riemann surfaces. Among other aspects, we discuss existence theorems for solutions of the Yang-Mills-Higgs equations, the analytic construction of the moduli space of such solutions, the connectivity and topology of this space, its holomorphic symplectic structure and its reinterpretations as a space of orbifold Higgs bundles or $SL_2(\mathbb{C})$-representations of (a central extension of) the orbifold fundamental group. We follow Hitchin's original paper for (ordinary) Riemann surfaces [14] quite closely but there are many novelties in the orbifold situation. (There is some overlap with a recent preprint of Boden and Yokogawa [4].)

It may help to mention here a few of our motivations.

1. In studying the orbifold moduli space, we are also studying the parabolic moduli space (see § 5A, and [27]).
2. The moduli space provides interesting examples of non-compact hyper-Kähler manifolds in all dimensions divisible by 4.
3. As a special case of the existence theorem for solutions of the Yang-Mills-Higgs equations we get the existence of metrics with conical singularities and constant sectional curvature on 'marked' Riemann surfaces (see Corollary 3.4, Theorem 6.17 and compare [15]).
4. The orbifold fundamental groups we study are Fuchsian groups and their central extensions: these include the fundamental groups of elliptic surfaces and of Seifert manifolds. We obtain results on varieties of $SL_2(\mathbb{C})$- and $SL_2(\mathbb{R})$-representations of such groups (see § 6 and compare e.g. [17]). In particular, we prove that Teichmüller space for a Fuchsian group or, equivalently, for a 'marked' Riemann surface is homeomorphic to a ball (Theorem 6.16).
5. Moduli of parabolic Higgs bundles and of marked Riemann surfaces have potential applications in Witten's work on Chern-Simons gauge theory.

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Let E be a Hermitian rank 2 V-bundle (i.e. orbifold bundle) over an orbifold Riemann surface of negative Euler characteristic, equipped with a normalised volume form, Ω. Let A be a unitary connexion on E and ϕ an End(E)-valued $(1,0)$-form. Then the Yang-Mills-Higgs equations are

$$F_A + [\phi, \phi^*] = -\pi i c_1(E)\Omega I_E \quad \text{and} \quad \overline{\partial}_A \phi = 0.$$

See § 3A for details. These equations arise by dimensional-reduction of the 4-dimensional Yang-Mills equations. Another interpretation is that they arise if we split projectively flat $SL_2(\mathbb{C})$-connexions into compact and non-compact parts (see § 6A).

Just as for ordinary Riemann surfaces, the moduli space, $\mathcal{M}$, of solutions to the Yang-Mills-Higgs equations has an extremely rich geometric structure which we study in the later sections of this paper. Let us indicate the main results and outline the contents of each section. The first is devoted to preliminaries on orbifold Riemann surfaces and V-bundles (i.e. orbifold bundles): § 1A covers the very basics, for the sake of revision and in order to fix notation, while § 1B deals with the correspondence between divisors and holomorphic line V-bundles on an orbifold Riemann surface (some of this may have been anticipated in unpublished work of B. Calpini). We particularly draw attention to the notational conventions concerning rank 2 V-bundles and their rank 1 sub-V-bundles established in § 1A which are used throughout this paper.

The second section introduces Higgs V-bundles and the appropriate stability condition (§ 2A) and studies the basic algebraic-geometric properties of stable Higgs V-bundles (§ 2B) - the principal result here is Theorem 2.8. This material roughly parallels [14, § 3], an important difference being that [14, Proposition 3.4] does not generalise to the orbifold case.

The third section introduces the Yang-Mills-Higgs equations (§ 3A), discusses the existence of solutions on stable Higgs V-bundles (§ 3B) and gives the analytic construction of $\mathcal{M}$ (§ 3C). These first three subsections parallel [14, §§ 4-5] and only in § 3B would any significant alteration to Hitchin's work be necessary to allow for the orbifold structure. The main results are Theorem 3.3 and Theorem 3.5. The Riemannian structure of the moduli space (including the fact that the moduli space is hyper-Kähler) is also discussed briefly in § 3C, following [14, § 6]. There is one other subsection: § 3D sketches alternative, equivariant, arguments that can be used for the existence theorem and the construction of $\mathcal{M}$. This last subsection also discusses the pull-back map between moduli spaces which arises when an orbifold Riemann surface is the base of a branched covering by a Riemann surface - see Theorem 3.13. We stress that equivariant arguments *cannot* easily be applied throughout the paper - difficulties arise e.g. in § 2B, § 5 and § 6.

The fourth section discusses the topology of $\mathcal{M}$, following [14, § 7]. The results are Theorem 4.1 and Corollary 4.2. General formulæ for the Betti numbers are not given but it is clear how to calculate the Poincaré polynomial in any given instance (however, see [4]).

The fifth section is devoted to the holomorphic symplectic structure on $\mathcal{M}$: following [14, § 8], $\mathcal{M}$ is described as a completely integrable Hamiltonian system via the determinant map $\det\colon \mathcal{M} \to H^0(K^2)$, defined by taking the determinant of the Higgs field. This result is given as Theorem 5.1 (we believe that a similar result was obtained by Peter Scheinost). There are a number of stages to the proof: first, it is simpler to use parabolic Higgs bundles and these are discussed in § 5A; § 5C contains the major part of the proof, with two special cases which arise in the orbifold case being dealt with separately in § 5B and § 5D. Moreover, it is shown that with respect to the determinant map $\mathcal{M}$ is a fibrewise compactification of the cotangent bundle of the moduli space of stable V-bundles (§ 5E).

The final section deals with the interpretation of the moduli space as a space of projectively flat connexions (§ 6A) or $SL_2(\mathbb{C})$-representations of (a central extension of) the orbifold fundamental group (§ 6B), the identification of the submanifold of $SL_2(\mathbb{R})$-representations (§ 6C) and the interpretation of one of the components as Teichmüller space (§ 6D), which leads to a proof that Teichmüller space is homeomorphic to ball. The proofs are much like those of [14, §§ 10-12] and [6] and accordingly we concentrate on those aspects of the orbifold case which are less familiar.

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1. - Orbifold Riemann Surfaces

This section compiles some basic facts about orbifold Riemann surfaces and fixes some notations which we will need in the sequel.

1A. - *Introduction to Orbifold Surfaces*

We start with the definition and basic properties of orbifold surfaces (or V-surfaces). The notion of a V-manifold was introduced by Satake [25] and re-invented as 'orbifold' by Thurston. By an ORBIFOLD SURFACE (respectively ORBIFOLD RIEMANN SURFACE) M we mean a closed, connected, smooth, real 2-manifold (respectively complex 1-manifold) together with a finite number (assumed non-zero) of 'marked' points with, at each marked point, an associated order of isotropy α (an integer greater than one). (See [25] or [26] for full details of the definition.) Notice that M has an 'underlying' surface where we forget about the marked points and orders of isotropy.

Although every point of a surface has a neighbourhood modeled on D^2 (the open unit disc), we think of a neighbourhood of a *marked* point as having the form $D^2/\mathbb{Z}_\alpha$, where $\mathbb{Z}_\alpha$ acts on $\mathbb{R}^2 \cong \mathbb{C}$ in the standard wayas the α^{th} roots of unity. We make this distinction because M is to be thought of as an orbifold. Orbifold ideas do not seem to have been widely used in the study of 'surfaces with marked points'. For instance the tangent V-bundle to $D^2/\mathbb{Z}_\alpha$ is $(D^2 \times \mathbb{R}^2)/\mathbb{Z}_\alpha$ - this leads to an idea of an orbifold Riemannian metric on M which corresponds to that of a metric on the underlying surface with conical singularities at the marked points (see § 6D).

We introduce the following notations, which will remain fixed throughout this paper. Let M be an orbifold (Riemann) surface with topological genus g; denote by $\widetilde{M}$ the 'underlying' (Riemann) surface obtained by forgetting the marked points and isotropy. Denote the number of marked points of M by n, the points themselves by $p_1, \dots, p_n$ and the associated orders of isotropy by $\alpha_1, \dots, \alpha_n$. Let σ_i denote the standard representation of $\mathbb{Z}_{\alpha_i}$, with generator $\varsigma_i = e^{2\pi i/\alpha_i}$. At a point where M is locally D^2 or D^2/σ_i use z for the standard (holomorphic) coordinate on D^2; call this a local UNIFORMISING coordinate and at a marked point let $w = z^{\alpha_i}$ denote the associated local coordinate. When giving local arguments centred at a marked point, drop the subscript i's; i.e. use p for p_i and so on.

Given a surface which is the base of a branched covering we naturally consider it to be an orbifold surface by marking a branch point with isotropy given by the ramification index. In this way we arrive at a definition of the ORBIFOLD FUNDAMENTAL GROUP $\pi_1^V(M)$ (see [26]): it has the following presentation

$$\begin{aligned}\pi_1^V(M) = \langle a_1, b_1, \dots, a_g, b_g, q_1, \dots, q_n \quad | \\ q_i^{\alpha_i} = 1, q_1 \dots q_n[a_1, b_1] \dots [a_g, b_g] = 1 \rangle.\end{aligned}$$

In this presentation a_1, $b_1, \dots, a_g$, b_g generate the fundamental group of the underlying surface while $q_1, \dots, q_n$ are represented by small loops around the marked points. Similarly, in this situation, the Riemann-Hurwitz formula suggests the following definition of the EULER CHARACTERISTIC of an orbifold surface:

$$\chi(M) = 2 - 2g - n + \sum_{i=1}^{n} \frac{1}{\alpha_i}.$$

We always work with orbifold surfaces with $\chi(M) < 0$ - note that this includes cases with $g = 0$ or $g = 1$ in contrast to the situation for ordinary surfaces.

A V-BUNDLE, E, with fibre $\mathbb{C}^r$, is as follows. We ask for a local trivialisation around each point of M with smooth (or holomorphic) transition functions; at a marked point p this should be of the form $E|_{D^2/\sigma} \xrightarrow{\cong} (D^2 \times \mathbb{C}^r)/(\sigma \times \tau)$, where τ is an ISOTROPY REPRESENTATION $\tau: \mathbb{Z}_\alpha \to GL_r(\mathbb{C})$. We can always choose coordinates in a V-bundle which RESPECT THE V-STRUCTURE: that is, if the isotropy representation is $\tau: \mathbb{Z}_\alpha \to GL_r(\mathbb{C})$ then we can choose coordinates so that τ decomposes as $\tau = \sigma^{x_1} \oplus \sigma^{x_2} \oplus \cdots \oplus \sigma^{x_r}$, where, for $j = 1, \dots, r$, x_j is an integer with $0 \leq x_j < \alpha$ and the x_j are increasing.

We will mostly be interested in rank 2 and rank 1 V-bundles and for these we introduce particular notations for the isotropy, which will be fixed throughout: for a rank 2, respectively rank 1, V-bundle, denote the isotropy at a marked point by x and x', respectively by y, with $0 \leq x,\ x',\ y < \alpha$. In the rank 2 case order x and x' so that $x \leq x'$. If a rank 1 V-bundle is a sub-V-bundle of a rank 2 V-bundle then of course $y \in \{x, x'\}$: in this case, let $\epsilon \in \{-1, 0, 1\}$ describe the isotropy of the sub-V-bundle, with $\epsilon = 0$ if $x = x'$, $\epsilon = -1$ if $y = x$ and $\epsilon = 1$ if $y = x'$. Add subscript i's, when necessary, to indicate the marked point in question. Call a vector (ϵ_i) with $\epsilon_i = 0$ if $x_i = x'_i$ and $\epsilon_i \in \{\pm 1\}$ if not an ISOTROPY VECTOR. For a rank 2 V-bundle let $n_0 = \#\{i: x_i = x'_i\}$ and for a rank 1 sub-V-bundle let $n_\pm = \#\{i: \epsilon_i = \pm 1\}$.

If a V-bundle is, at a marked point, locally like $(D^2 \times \mathbb{C}^r)/(\sigma \times \tau)$ then by a Hermitian metric we mean, locally, a Hermitian metric on $D^2 \times \mathbb{C}^r$ which is equivariant with respect to the action of $\mathbb{Z}_\alpha$ via $\sigma \times \tau$. Considering the tangent V-bundle, we can also define the concepts of Riemannian metric and orientation for an orbifold surface (an orientation of an orbifold surface is just an orientation of the underlying surface).

We introduce the notion of a connexion in a V-bundle in the obvious way. The first Chern class or degree of a V-bundle can be defined using Chern-Weil theory. Notice that the degree of a V-bundle is a *rational* number, congruent modulo the integers to the sum $\sum_{i=1}^{n} (y_i/\alpha_i)$, where (y_i) is the isotropy of the determinant line V-bundle.

When E is a rank 2 V-bundle with isotropy (x_i, x'_i), as above then we write

$$c_1(\Lambda^2 E) = l + \sum_{i=1}^{n} \frac{x'_i + x_i}{\alpha_i},$$

for $l \in \mathbb{Z}$. Similarly, if L is a sub-V-bundle with isotropy given by an isotropy vector (ϵ_i) in the manner explained above then we write

$$c_1(L) = m + \sum_{i=1}^{n} \frac{\epsilon_i(x_i' - x_i) + (x_i' + x_i)}{2\alpha_i}$$

for $m \in \mathbb{Z}$. These meanings of l and m will be fixed throughout.

Topologically, $U(1)$ and $U(2)$ V-bundles are classified by their isotropy representations and first Chern class: we quote the following classification result from [10].

PROPOSITION 1.1 (Furuta-Steer). *Let M be an orbifold surface. Then, over M:*

1. *any complex line V-bundle is topologically determined by its isotropy representations and degree,*
2. *any $SU(2)$ V-bundle is topologically determined by its isotropy representations (necessarily of the form $\sigma^x \oplus \sigma^{-x}$, where $0 \leq x \leq [\alpha/2]$) and*
3. *any $U(2)$ V-bundle is topologically determined by its isotropy representations and its determinant line V-bundle.*

REMARK 1.2. Let E be a $U(2)$ V-bundle with isotropy (x_i, x_i') and let (ϵ_i) be any isotropy vector. Then there exists a $U(1)$ V-bundle L with isotropy specified by (ϵ_i) (unique up to twisting by a $U(1)$-bundle i.e. up to specifying the integer m, above) and, topologically, $E = L \oplus L^*\Lambda^2 E$, by Proposition 1.1.

1B. - *Divisors and Line V-bundles*

The theory of divisors developed here has also been dealt with in the Geneva dissertation of B. Calpini written some time ago.

Suppose M is an orbifold Riemann surface. It is convenient to associate an order of isotropy α_p to every point p; it is 1 if the point is not one of the marked points (and α_i if $p = p_i$ for some i). A DIVISOR is then a linear combination

$$D = \sum_{p \in M} \frac{n_p}{\alpha_p} \cdot p$$

with $n_p \in \mathbb{Z}$ and zero for all but a finite number of p.

If f is a non-zero meromorphic function on M we define the DIVISOR OF f by $Df = \sum_p \nu_p(f).p$. Here $\nu_p(f)$ is defined in the usual way when $\alpha_p = 1$. When $\alpha_p = \alpha > 1$ and z is a local uniformising coordinate with $\rho: D^2 \twoheadrightarrow D^2/\sigma$ the associated projection, then on D^2 we find that $\rho^* f$ has a Laurent expansion of the form

$$\sum_{j \geq -N} a_j z^{\alpha j} \qquad \text{with } a_{-N} \neq 0$$

and we set $\nu_p(f) = -N$. (The divisor of a meromorphic function is thus an *integral* divisor.) Two divisors D and D' are LINEARLY EQUIVALENT if

$$D - D' = Df$$

for some meromorphic f. The DEGREE of a divisor $D = \sum_p (n_p/\alpha_p).p$ is defined to be $d(D) = \sum_p n_p/\alpha_p$.

The correspondence between divisors and holomorphic line V-bundles goes through in exactly the same way as for Riemann surfaces without marked points. To a point p with $\alpha_p = 1$ we associate the point line bundle L_p as in [12]. If $\alpha_p = \alpha > 1$ then to the divisor p/α we associate the following V-bundle. Let z be a local uniformising coordinate; then, making the appropriate identification locally with D^2/σ, we define

$$L_{p/\alpha} = ((D^2 \times \mathbb{C})/(\sigma \times \sigma)) \cup_\Phi ((M \setminus \{p\}) \times \mathbb{C}),$$

where $\Phi\colon (D^2 \setminus \{0\} \times \mathbb{C})/(\sigma \times \sigma) \to ((M \setminus \{p\}) \times \mathbb{C})$ is given by its $\mathbb{Z}_\alpha$-equivariant lifting

$$\hat{\Phi}\colon (D^2 \setminus \{0\}) \times \mathbb{C} \to ((D^2/\sigma) \setminus \{0\}) \times \mathbb{C}$$
$$(z, z') \mapsto (z^\alpha, z^{-1}z').$$

This V-bundle has an obvious section 'z'; this is given on $D^2 \times \mathbb{C}$ by $z \mapsto (z, z)$ and extends by the constant map to the whole of M. So $L_{p/\alpha}$ is positive. We denote by L_i the line V-bundle L_{p_i/α_i}, associated to the divisor p_i/α_i, and by s_i the canonical section 'z'.

Finally for a general divisor

$$D = \sum_{p \in M} \frac{n_p}{\alpha_p} \cdot p$$

we set

$$L_D = \otimes_p (L_{p/\alpha_p})^{n_p}.$$

As for a meromorphic function, we can define the divisor of a meromorphic section of a line V-bundle L. If p has ramification index $\alpha_p = \alpha$ and we have a local uniformising coordinate z and a corresponding local trivialisation $L|_{D^2/\sigma} \cong (D^2 \times \mathbb{C})/(\sigma \times \sigma^y)$, for some isotropy y (with, by convention, $0 \le y < \alpha$), then locally we have $s(z) = \sum_{j \ge -N'} a'_j z^j$ with $a'_{-N'} \neq 0$. However, we have $\mathbb{Z}_\alpha$-equivariance which means that $s(\varsigma \cdot z) = \varsigma^y s(z)$ (where $\varsigma = e^{2\pi i/\alpha}$ generates $\mathbb{Z}_\alpha$). It follows that $a'_j = 0$ unless $j \equiv y \pmod{\alpha}$ and hence

$$s(z) = z^y \sum_{j \ge -N} a_j z^{\alpha j} \qquad \text{with } a_{-N} \neq 0, \tag{1a}$$

where $-N\alpha + y = -N'$. We define $\nu_p(s) = -N'/\alpha = -N + y/\alpha$: so for the canonical section s_i of the line V-bundle L_i we have $\nu_{p_i}(s_i) = 1/\alpha_i$.

PROPOSITION 1.3. *The above describes a bijective correspondence between equivalence classes of divisors and of holomorphic line V-bundles. The degree $d(D)$ of a divisor D is just $c_1(L_D)$, the first Chern class of the corresponding line V-bundle.*

PROOF. Much of the proof is contained in [10]. The correspondence has been defined above and it is clear that if we start from a divisor D and pass to L_D then taking the divisor associated to the tensor product of the canonical sections we get back D. We have to show that the correspondence behaves well with respect to equivalence classes. If $D_1 \equiv D_2$, where $D_j = \sum (n_p^{(j)}/\alpha_p) \cdot p$ for $j = 1, 2$, then from what we know about divisors of meromorphic functions we see that $n_p^{(1)} \equiv n_p^{(2)} \pmod{\alpha_p}$. Now $L_{D_j} = \otimes_p (L_{p/\alpha_p})^{n_p^{(j)}}$. Since $n_p^{(1)} \equiv n_p^{(2)}$ $(\operatorname{mod} \alpha_p)$, we find that $L_{D_j} \otimes \bigotimes_{i=1}^n (L_{p_i/\alpha_i})^{-n_{p_i}^{(1)}}$ is a genuine line bundle for $j = 1$, 2. Moreover the two are equivalent because the corresponding divisors are. Hence $L_{D_1} \equiv L_{D_2}$. Similarly we show that two meromorphic sections of the same line V-bundle define equivalent divisors. □

COROLLARY 1.4. *If L is a holomorphic line V-bundle with $c_1(L) \leq 0$ then $H^0(L) = 0$, unless L is trivial.*

Let L be a holomorphic line V-bundle over M, with isotropy y_i at p_i, and let $\mathcal{O}(L)$ be the associated sheaf of germs of holomorphic sections; we take the cohomology of L over M to be the sheaf cohomology of $\mathcal{O}(L)$ over $\tilde{M}$. From (1a), $\mathcal{O}(L)$ is locally free over $\mathcal{O}_M = \mathcal{O}_{\tilde{M}}$ and hence there is a natural line bundle $\tilde{L}$ over $\tilde{M}$ with $\mathcal{O}(\tilde{L}) \cong \mathcal{O}(L)$. If we define $\tilde{L} = L \otimes L_1^{-y_1} \otimes \cdots \otimes L_n^{-y_n}$ then this gives the required isomorphism of sheaves.

PROPOSITION 1.5. *If L is a holomorphic line V-bundle then, with $\tilde{L}$ defined as above, there is a natural isomorphism of sheaves $\mathcal{O}(L) \cong \mathcal{O}(\tilde{L})$ given by tensoring with the canonical sections of the L_i.*

PROOF. Recall that $s_1, \ldots, s_n$ are the canonical sections of $L_1, \ldots, L_n$. If s is a holomorphic section of L then $\tilde{s} = s_1^{-y_1} \ldots s_n^{-y_n} s$ will be a meromorphic section of $\tilde{L}$, holomorphic save perhaps at p_i. In fact (by choosing a local coordinate) we see that $\tilde{s}$ has removable singularities at p_i and that $D(\tilde{s}) = Ds - \sum_{i=1}^{n} (y_i/\alpha_i) p_i$. Conversely, given a section $\tilde{s}$ of $\tilde{L}$, then $s_1^{y_1} \ldots s_n^{y_n} \tilde{s}$ is a section of L and the correspondence is bijective. □

As corollaries we get the orbifold Riemann-Roch theorem, originally due to Kawasaki [18] and an orbifold version of Serre duality.

THEOREM 1.6 (Kawasaki-Riemann-Roch). *Let L be a holomorphic line*

V-bundle with the isotropy at p_i given by y_i, with $0 \leq y_i < \alpha_i$. Then

$$h^0(L) - h^1(L) = 1 - g + c_1(L) - \sum_{i=1}^{n} \frac{y_i}{\alpha_i},$$

where h^i denotes the dimension of H^i.

THEOREM 1.7. *If L is a holomorphic line V-bundle and K_M is the canonical V-bundle of the orbifold Riemann surface then*

$$H^1(L) \cong H^0(L^* K_M)^*.$$

PROOF. By definition, $H^1(L) = H^1(\mathcal{O}(L)) = H^1(\tilde{L})$. So $H^1(L) \cong H^0((\tilde{L})^* K_{\tilde{M}})^*$ by the standard duality. But $(\tilde{L})^* K_{\tilde{M}} = \widetilde{L^* K_M}$ by a straightforward computation. □

2. - Higgs V-Bundles

Throughout this section $E \to M$ is a holomorphic rank 2 V-bundle over an orbifold Riemann surface with $\chi(M) < 0$ and we write $K = K_M$, the canonical V-bundle, and $\Lambda = \Lambda^2 E$, the determinant line V-bundle.

2A. - *Higgs V-Bundles*

In this subsection we introduce Higgs V-bundles - this is a straightforward extension of the basic material in Hitchin's paper [14] to orbifold Riemann surfaces.

Define a HIGGS FIELD, ϕ, to be a holomorphic section of $\mathrm{End}_0(E) \otimes K$ where $\mathrm{End}_0(E)$ denotes the trace-free endomorphisms of E. A HIGGS V-BUNDLE or HIGGS PAIR is just a pair (E, ϕ).

Let (E_1, ϕ_1) and (E_2, ϕ_2) be two Higgs V-bundles. A *homomorphism of Higgs V-bundles* is just a homomorphism of V-bundles $h: E_1 \to E_2$ such that h is holomorphic and intertwines ϕ_1 and ϕ_2. The corresponding notion of an ISOMORPHISM OF HIGGS V-BUNDLES is then clear.

A holomorphic line sub-V-bundle L of E is called a HIGGS SUB-V-BUNDLE (or 'ϕ-invariant sub-V-bundle') if $\phi(L) \subseteq KL$. A Higgs V-bundle (E, ϕ) is said to be STABLE if

(2a) $$c_1(L) < \frac{1}{2} c_1(E), \quad \text{for every Higgs sub-}V\text{-bundle, } L.$$

If we allow possible equality in (2a) then the Higgs V-bundle is called SEMI-STABLE. If a Higgs V-bundle is stable or a direct sum of two line V-bundles of equal degree with ϕ also decomposable then (it is certainly semi-stable and) it is called POLYSTABLE. If E is stable then certainly (E, ϕ) is stable for any Higgs field ϕ. The following result, due to Hitchin in the smooth case [14, Proposition 3.15], goes over immediately.

PROPOSITION 2.1. *Let (E_1, ϕ_1) and (E_2, ϕ_2) be stable Higgs V-bundles with isomorphic holomorphic determinant line V-bundles, $\Lambda^2 E_1 \cong \Lambda^2 E_2$. Suppose that $\psi: E_1 \to E_2$ is a non-zero homomorphism of Higgs V-bundles. Then ψ is an isomorphism of Higgs V-bundles. If $(E_1, \phi_1) = (E_2, \phi_2)$ then ψ is scalar multiplication.*

2B. - *Algebraic Geometry of Stable Higgs V-Bundles*

For applications in later sections we now develop some results on the possibilities for stable Higgs V-bundles. Higgs V-bundles are holomorphic V-bundles with an associated 'Higgs field'; a holomorphic $(1,0)$-form-valued endomorphism of the V-bundle. We assume familiarity with [14, § 3].

Given $E \to M$, we investigate whether there are any Higgs fields ϕ such that the Higgs pair (E, ϕ) is stable. Recall that the isotropy of E at p_i is denoted by (x_i, x_i') and that $n_0 = \#\{i: x_i = x_i'\}$. We will suppose throughout that $n_0 < n$ - this is because the case $n = n_0$ is just that of a genuine bundle twisted by a line V-bundle and so essentially uninteresting (see also § 5B).

The following lemma is a simple computation using the Kawasaki-Riemann-Roch theorem and Serre duality.

LEMMA 2.2. *We have*

$$h^0(K^2) = \chi(K^2) = 3g - 3 + n \text{ and } \chi(\mathrm{End}_0(E) \otimes K) = 3g - 3 + n - n_0.$$

If E is stable we know that the only endomorphisms of E are scalars and so $h^0(\mathrm{End}_0(E)) = 0$; consequently if $3 - 3g - n + n_0 > 0$ (this only happens if $g = 0$ and $n - n_0 \leq 2$) there are no stable V-bundles.

Suppose that L is a holomorphic sub-V-bundle of E. Then we have the short exact sequences

$$\text{(2b)} \qquad 0 \to L \xrightarrow{i} E^i \xrightarrow{j} L^*\Lambda \to 0 \quad \text{and} \quad 0 \to L\Lambda^* \xrightarrow{j^*} E^* \xrightarrow{i^*} L^* \to 0$$

from which follows

$$\text{(2c)} \qquad 0 \to E^* \otimes KL \to \mathrm{End}_0(E) \otimes K \to KL^{-2}\Lambda \to 0.$$

Associated to (2b) tensored by KL is the long exact sequence in cohomology

$$\text{(2d)} \qquad \begin{aligned} &0 \to H^0(KL^2\Lambda^*) \to H^0(E^* \otimes KL) \to H^0(K) \xrightarrow{\delta} \\ &\xrightarrow{\delta} H^1(KL^2\Lambda^*) \to H^1(E^* \otimes KL) \to H^1(K) \to 0 \end{aligned}$$

and associated to (2c) we have

$$\begin{aligned} &0 \to H^0(E^* \otimes KL) \to H^0(\mathrm{End}_0(E) \otimes K) \to H^0(KL^{-2}\Lambda) \xrightarrow{\delta} \\ \text{(2e)} \qquad & \xrightarrow{\delta} H^1(E^* \otimes KL) \to H^1(\mathrm{End}_0(E) \otimes K) \to H^1(KL^{-2}\Lambda) \to 0. \end{aligned}$$

Now let us review the strategy of the proof of [14, Proposition 3.3]: if E is stable then all pairs (E, ϕ) are certainly stable and we know something about stable V-bundles from [10]. If E is not stable then there is a destabilising sub-V-bundle L_E. Recall that L_E is unique if E is not semi-stable. Moreover, in the semi-stable case the assumption $n \neq n_0$ implies that $L_E \not\cong L_E^* \Lambda$ and so L_E is unique if E is not decomposable and if it is then L_E and $L_E^*\Lambda$ are the only destabilising sub-V-bundles. Thus there will be some ϕ such that the pair (E, ϕ) is stable unless every Higgs field fixes L_E (or $L_E^*\Lambda$, in the semi-stable, decomposable case). Moreover, the subspace of sections leaving L invariant is $H^0(E^* \otimes KL) \subset H^0(\mathrm{End}_0(E) \otimes K)$. It follows that a necessary and sufficient condition for E to occur in a stable pair is $H^0(E^* \otimes KL_E) \neq H^0(\mathrm{End}_0(E) \otimes K)$ (and similarly for $L_E^*\Lambda$, in the semi-stable, decomposable case). Considering (2e) this amounts to non-injectivity of the Bockstein operator δ, which we consider in the next lemma - proved as in the proof of [14, Proposition 3.3]. From the lemma we obtain a version of [14, Proposition 3.3].

LEMMA 2.3. *If L is a sub-V-bundle of E with* $\deg(L) \geq \deg(\Lambda)/2$ *then*

1. $H^1(E^* \otimes KL) \cong \mathbb{C}$;
2. $H^0(KL^{-2}\Lambda) \xrightarrow{\delta} H^1(E^* \otimes KL)$ *is surjective if and only if* $e_E \neq 0$, *where* $e_E \in H^1(L^2\Lambda^*)$ *is the extension class.*

PROOF. 1. Consider the long exact sequence in cohomology (2d) for L, which includes the segment

$$\text{(2f)} \qquad \cdots \to H^1(KL^2\Lambda^*) \xrightarrow{j^*} H^1(E^* \otimes KL) \xrightarrow{i^*} \mathbb{C} \to 0.$$

Then the result follows from the fact that $h^1(KL^2\Lambda^*) = 0$, using Serre duality and the vanishing theorem.

2. Consider (2e) and let i^* be the map on cohomology indicated in (2f); then the result follows from the fact that $i^* \cdot \delta$ is multiplication by the extension class e_E. $\square$

PROPOSITION 2.4. *Let E be a non-stable V-bundle. Then E appears in a stable pair if and only if one of the following holds:*

1. *E is indecomposable with* $h^0(KL_E^{-2}\Lambda) > 1$;
2. *E is decomposable, not semi-stable with* $h^0(KL_E^{-2}\Lambda) \geq 1$;
3. *E is decomposable, semi-stable with* $h^0(KL_E^{-2}\Lambda) \geq 1$ *and* $h^0(KL_E^2\Lambda^*) \geq 1$.

To find more precise results in the case that E is semi-stable we estimate $h^0(KL_E^{-2}\Lambda)$ and $h^0(KL_E^2\Lambda^*)$ using the following lemmas. For these recall the definitions of the integers n_0, $n_\pm$, l and m from § 1A.

LEMMA 2.5. *Suppose that L is any sub-V-bundle of E. Then, with the notations established above,*

$$\chi(KL^{-2}\Lambda) = l - 2m + g - 1 + n_- \text{ and}$$

$$\chi(KL^2\Lambda^*) = 2m - l + g - 1 + n_+.$$

Moreover:

1. *if $2c_1(L) - c_1(\Lambda) \geq 0$ then $h^0(KL^2\Lambda^*) = \chi(KL^2\Lambda^*) \geq g$ and $\chi(KL^{-2}\Lambda) \leq g - 2 + n - n_0$;*
2. *if $2c_1(L) - c_1(\Lambda) \leq 0$ then $h^0(KL^{-2}\Lambda) = \chi(KL^{-2}\Lambda) \geq g$ and $\chi(KL^2\Lambda^*) \leq g - 2 + n - n_0$.*

PROOF. The first part is just the Kawasaki-Riemann-Roch theorem. Now consider part 1 (part 2 is entirely similar): we have $H^1(KL^2\Lambda^*) \cong H^0(L^{-2}\Lambda)^*$ and this is zero (because the degree is non-positive and the isotropy is non-trivial as $n > n_0$). Let $\theta = \sum_{i=1}^{n} \epsilon_i(x_i' - x_i)/\alpha_i$ so that $2c_1(L) - c_1(\Lambda) \equiv \theta \pmod{\mathbb{Z}}$. Then $-n_- < \theta < n_+$ and the estimates on $\chi(KL^2\Lambda^*)$ and $\chi(KL^{-2}\Lambda)$ follow. $\square$

LEMMA 2.6. *For a given M and $n - n_0$, an E (with the given $n - n_0$) such that the bounds on $\chi(KL^2\Lambda^*)$ and $\chi(KL^{-2}\Lambda)$ in Lemma 2.5, parts 1 and 2 are attained exists if and only if*

$$\min_{\{i_1,\dots,i_{n-n_0}\}\subseteq\{1,\dots,n\}} \left\{ \sum_{j=1}^{n-n_0} \frac{1}{\alpha_{i_j}} \right\} \leq 1.$$

For a given topological E the bounds are attained for some holomorphic structure on E if and only if

$$\min_{\{(\epsilon_i):n_+ + l\equiv 1(2)\}} \left\{ n_+ - \sum_{i=1}^{n} \frac{\epsilon_i(x_i' - x_i)}{\alpha_i} \right\} \leq 1,$$

where (ϵ_i) varies over all isotropy vectors with $n_+ + l \equiv 1(2)$.

PROOF. To see this we construct examples as follows. It is sufficient to consider only topological examples and therefore, given any M and topological E, to choose (ϵ_i) and $m \in \mathbb{Z}$ to specify L topologically. (Examples where L is a topological sub-V-bundle of E exist by Remark 1.2.)

Now, given a choice of (ϵ_i) and m, we have $\chi(KL^{-2}\Lambda) = l - 2m + g - 1 + n_-$ and $\chi(KL^2\Lambda^*) = 2m - l + g - 1 + n_+$ from Lemma 2.5. So, for $2c_1(L) - c_1(\Lambda) \geq 0$

(the case $2c_1(L)-c_1(\Lambda)\leq 0$ is entirely similar), the bounds are attained provided $2m-l+n_+=1$ and $2m-l+\sum_{i=1}^{n}\epsilon_i(x_i'-x_i)/\alpha_i\geq 0$. Since we can vary m, the first equation just fixes the parity of n_+. Hence the problem reduces to finding (ϵ_i) such that

$$\sum_{i=1}^{n}\frac{\epsilon_i(x_i'-x_i)}{\alpha_i}-n_+\geq 1 \text{ and} \tag{2g}$$

$$n_+ + l\equiv 1 \pmod 2. \tag{2h}$$

This gives the desired result, for a given topological E. To see whether examples exist for a given M and $n-n_0$ as we allow E to vary over topological types with fixed $n-n_0$, we simply note that the maximum value of the left-hand side of (2g) (subject to (2h)) is

$$\max_{\{i_1,\dots,i_{n-n_0}\}\subseteq\{1,\dots,n\}}\left\{\sum_{j=1}^{n-n_0}\left(-\frac{1}{\alpha_{i_j}}\right)\right\}.$$

Thus the bounds are certainly attained if the α_i are such that this is not less than -1. □

COROLLARY 2.7. *If L is a sub-V-bundle of E with $c_1(L)=c_1(\Lambda)/2$ and ϵ_i, n_+ and n_- are defined by the isotropy of L, as before, then*

$$h^0(\mathrm{End}_0(E)\otimes K)=\begin{cases}3g-3+n-n_0 & \text{if } 0\to L\to E\to L^*\Lambda\to 0\\ & \text{is non-trivial;}\\ 3g-2+n-n_0 & \text{if it is trivial;}\end{cases}$$

$$h^0(E^*\otimes KL)=2g-1-\sum_{i=1}^{n}\frac{\epsilon_i(x_i'-x_i)}{\alpha_i}+n_+;$$

$$h^0(KL^{-2}\Lambda)=g-1+\sum_{i=1}^{n}\frac{\epsilon_i(x_i'-x_i)}{\alpha_i}+n_-;$$

$$h^0(KL^2\Lambda^*)=g-1-\sum_{i=1}^{n}\frac{\epsilon_i(x_i'-x_i)}{\alpha_i}+n_+;$$

Moreover,

$$\begin{aligned}2g\leq\ & h^0(E^*\otimes KL) && \leq n-n_0+2g-2,\\ g\leq\ & h^0(KL^{-2}\Lambda) && \leq n-n_0+g-2 \text{ and}\\ g\leq\ & h^0(KL^2\Lambda^*) && \leq n-n_0+g-2.\end{aligned}$$

These estimates are attained for all values of g and $n-n_0$ (but not necessarily for all M or E).

PROOF. The results on $h^0(KL^{-2}\Lambda)$ and $h^0(KL^2\Lambda^*)$ follow from Lemma 2.5. Moreover we know that $h^1(E^* \otimes KL) = 1$ from Lemma 2.3, part 1 and so $h^0(E^* \otimes KL)$ follows from the Kawasaki-Riemann-Roch theorem. To calculate $h^0(\mathrm{End}_0(E) \otimes K)$ we use (2e) and Lemma 2.3, part 2. The estimates on $h^0(KL^{-2}\Lambda)$ and $h^0(KL^2\Lambda^*)$ are contained in Lemma 2.5 and the estimate on $h^0(E^* \otimes KL)$ follows (as $h^0(E^* \otimes KL) = -h^0(KL^{-2}\Lambda) + 3g - 2 + n - n_0$). □

When $c_1(L) = c_1(\Lambda)/2$ it is not possible to have $n - n_0 = 1$ (because $c_1(L^2\Lambda^*)$ cannot be an integer if $n - n_0 = 1$ but, on the other hand, it is supposed zero).

Applying these results to L_E (and $L_E^*\Lambda$ in the semi-stable, decomposable case) we can strengthen Proposition 2.4 as far as it refers to semi-stable V-bundles. Adding in some necessary conditions on g and $n - n_0$ derived from our estimates above we obtain the following theorem.

THEOREM 2.8. *A holomorphic rank 2 V-bundle E occurs in a stable pair if and only if one of the following holds:*

1. *E is stable (if $g = 0$ then necessarily $n - n_0 \geq 3$);*
2. *E is semi-stable, not stable (necessarily $n - n_0 \geq 2$) with one of the following holding:*
   (a) *E is indecomposable and $g > 1$;*
   (b) *E is indecomposable, $g = 0$ or 1 and $h^0(K\widetilde{L_E^{-2}}\Lambda) > 1$ (necessarily $g + n - n_0 \geq 4$);*
   (c) *E is decomposable and $g > 0$;*
   (d) *E is decomposable, $g = 0$ and $1 \leq h^0(K\widetilde{L_E^{-2}}\Lambda) \leq n - n_0 - 3$ (necessarily $n - n_0 \geq 4$);*
3. *E is not semi-stable with one of the following holding:*
   (a) *E is indecomposable and $h^0(K\widetilde{L_E^{-2}}\Lambda) > 1$ (necessarily $g \geq 2$ or $g + n - n_0 \geq 4$; if $g = 2$ and $n - n_0 = 1$ then $\widetilde{KL_E^{-2}\Lambda}$ is necessarily canonical);*
   (b) *E is decomposable and $h^0(K\widetilde{L_E^{-2}}\Lambda) \geq 1$ (necessarily $g \geq 1$ or $n - n_0 \geq 3$; if $2g + n - n_0 = 3$ then $\widetilde{KL_E^{-2}\Lambda}$ is necessarily trivial).*

In all cases the necessary conditions are the best possible ones depending only on g and $n - n_0$.

PROOF. In part 2 the first three items follow from Corollary 2.7 together with Proposition 2.4, parts 1 and 3, while for the last item we note that when $g = 0$, $h^0(K\widetilde{L_E^2}\Lambda^*) \geq 1$ if and only if $h^0(K\widetilde{L_E^{-2}}\Lambda) < n - n_0 - 2$ (from Corollary 2.7) and apply Proposition 2.4, part 3.

Only the necessary conditions in part 3 need any additional comment. Using Lemma 2.5, part 1 we have that $\chi(K\widetilde{L_E^{-2}}\Lambda) \leq g - 2 + n - n_0$ and the bound is attained for some M and E by Lemma 2.6. Thus if $g > 2$ there are cases with $\chi(K\widetilde{L_E^{-2}}\Lambda) \geq 2$ and hence $h^0(K\widetilde{L_E^{-2}}\Lambda) \geq 2$. If $g = 2$ then there are cases with $\chi(KL_E^{-2}\Lambda) = n - n_0$, similarly. The only problem then occurs if $n - n_0 = 1$ when $c_1(\widetilde{KL_E^{-2}\Lambda}) = 2$: in order to have $h^0(KL_E^{-2}\Lambda) > 1$ we must have $\widetilde{KL_E^{-2}\Lambda} = K_{\tilde{M}}$. Similarly, if $g = 1$ we can suppose that $\chi(KL_E^{-2}\Lambda) = n - n_0 - 1$. Then for $h^0(KL_E^{-2}\Lambda) > 1$ we need $n - n_0 \geq 3$ and for $h^0(KL_E^{-2}\Lambda) \geq 1$ we need $n - n_0 \geq 1$ with $\widetilde{KL_E^{-2}\Lambda}$ trivial if $n - n_0 = 1$. Finally, if $g = 0$ we need $n - n_0 \geq 4$ for $h^0(KL_E^{-2}\Lambda) > 1$ and $n - n_0 \geq 3$ (with $\widetilde{KL_E^{-2}\Lambda}$ trivial if $n - n_0 = 3$) for $h^0(KL_E^{-2}\Lambda) \geq 1$. □

For each of the items of Theorem 2.8 examples of such V-bundles do actually exist (see also § 4 and § 5). Only items 2b, 2d, 3a and 3b pose any problem but it is fairly easy to construct the required examples using the ideas of § 1B and Lemma 2.6. Of particular interest is part 3b when $g = 0$ and $n - n_0 = 3$: we have the following result (compare § 4).

PROPOSITION 2.9. *There exist orbifold Riemann surfaces with $g = 0$ with V-bundles with $n - n_0 = 3$ over them which are decomposable but not semi-stable and exist in stable pairs. Such a stable pair contributes an isolated point to the moduli space (which is nevertheless connected - see Corollary 4.3).*

PROOF. We set $E = L_E \oplus L_E^*\Lambda$ with $2c_1(L_E) > c_1(\Lambda)$. Now, according to Theorem 2.8, part 3b, we get a stable pair if and only if $\widetilde{KL_E^{-2}\Lambda}$ is trivial. Moreover, applying § 1B or Lemma 2.6, we see that examples certainly exist.

We write the Higgs field according to the decomposition $\phi = \begin{pmatrix} t & u \\ v & -t \end{pmatrix}$. Now $h^0(KL_E^{-2}\Lambda) = 1$ implies that $h^0(KL_E^2\Lambda^*) = 0$ and hence $u = 0$. More simply, $g = 0$ implies $t = 0$ and so ϕ is given by v, with $v \in H^0(KL_E^{-2}\Lambda) \cong \mathbb{C}$ non-zero for a stable pair. Now we need to consider the action of V-bundle automorphisms: $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ acts on $H^0(KL_E^{-2}\Lambda) \cong \mathbb{C}$ by $z \mapsto \lambda^2 z$ and hence there is a single orbit. □

Notice that [14, Proposition 3.4] does *not* extend to orbifold Riemann surfaces with $\chi(M) < 0$. To prove that result Hitchin uses Bertini's theorem to show that, for a given rank 2 holomorphic bundle over a Riemann surface with negative Euler characteristic, either the generic Higgs field leaves no subbundle invariant or there is a subbundle invariant under all Higgs fields; he then shows that the latter cannot happen when the bundle exists in a stable pair. Although we have not been able to enumerate all the cases in which this result is false in the orbifold case there are three things which can go wrong:

1. Bertini's theorem may not apply and the conclusion may be false: E may be such that it exists in a stable pair, the generic Higgs field has an invariant sub-V-bundle and no sub-V-bundle is invariant by all Higgs fields;

2. E may be stable and have a sub-V-bundle invariant by all Higgs fields;
3. E may be non-stable, exist in a stable pair and have a sub-V-bundle invariant by all Higgs fields.

We give counterexamples of the first and third types. Although we suspect that counterexamples of the second type also exist we have not been able to show this. For a counterexample where Bertini's theorem doesn't apply consider the following: if $g = 1$ and $n - n_0 = 1$ then, anticipating Lemma 5.7, *every Higgs field has an invariant sub-V-bundle* and yet if E is a non-stable V-bundle which exists in a stable pair (these exist by Theorem 2.8, part 3b) then *no sub-V-bundle is invariant by all Higgs fields.* All counterexamples of the third type are given in the following proposition, which also has interesting applications in § 4.

PROPOSITION 2.10. *A non-stable V-bundle E exists in a stable pair and has a sub-V-bundle invariant by all Higgs fields if and only if $g = 0$, $E = L_E \oplus L_E^*\Lambda$ with $2c_1(L_E) > c_1(\Lambda)$ and L_E is such that the bounds in Lemma 2.5, part 1 are attained. Moreover, there exist orbifold Riemann surfaces with such E over them, with E having any given $n - n_0 \geq 3$.*

PROOF. Suppose E is non-stable, exists in a stable pair and has a sub-V-bundle invariant by all Higgs fields. Since E is non-stable and exists in a stable pair the destabilising sub-V-bundle(s) cannot be invariant by all Higgs fields. Moreover, if $h^0(KL_E^2\Lambda^*) > 0$ then, via the inclusions $H^0(KL_E^2\Lambda^*) \hookrightarrow H^0(E^* \otimes KL_E) \hookrightarrow H^0(\mathrm{End}_0(E) \otimes K)$, we get a family of Higgs fields which leave no sub-V-bundle except L_E invariant hence we must have $h^0(KL_E^2\Lambda^*) = 0$. By Lemma 2.5, part 1 this can only happen if the bounds there are attained and $g = 0$. Now consideration of the long exact sequence (2d) shows that $h^0(E^* \otimes KL_E) = g$ and hence Lemma 2.3 and (2e) together show that E is decomposable. Considering the Higgs field according to the decomposition, in the manner of Proposition 2.9, we see that $L_E^*\Lambda$ is invariant under all Higgs fields: it follows that $2c_1(L_E)$ must be strictly greater than $c_1(\Lambda)$ for E to form a stable pair.

The converse is straightforward: we suppose that $g = 0$, $2c_1(L_E) > c_1(\Lambda)$ and L_E is such that the bounds in Lemma 2.5, part 1 are attained and, exactly as in Proposition 2.9, we set $E = L_E \oplus L_E^*\Lambda$. We write the Higgs field according to the decomposition as $\phi = \begin{pmatrix} 0 & u \\ v & -0 \end{pmatrix}$. Since $g = 0$, the fact that L_E is such that the bounds in Lemma 2.5, part 1 are attained means that $h^0(KL_E^{-2}\Lambda) = n - n_0 - 2 \geq 1$ and $h^0(KL_E^2\Lambda^*) = 0$. Hence v can be chosen non-zero so that E exists in a stable pair and $u = 0$ so that $L_E^*\Lambda$ is invariant by all ϕ, as required. Finally, examples where the bounds in Lemma 2.5, part 1 are attained exist by Lemma 2.6. □

3. - The Yang-Mills-Higgs Equations and Moduli

We now prove an equivalence between stable Higgs V-bundles and the appropriate analytic objects - irreducible Yang-Mills-Higgs pairs - and use this to give an analytic construction of the moduli space. Throughout this section M is an orbifold Riemann surface of negative Euler characteristic, equipped with a normalised volume form, Ω, and E is a smooth rank 2 V-bundle over M with a fixed Hermitian metric.

3A. - *The Yang-Mills-Higgs Equations*

Given the fixed Hermitian metric on E, holomorphic structures correspond to unitary connexions. Let ϕ be a Higgs field with respect to A, i.e. a Higgs field on E_A or satisfying $\overline{\partial}_A\phi = 0$. We call the pair (A, ϕ) a HIGGS PAIR. (With the unitary structure understood Higgs pairs are entirely equivalent to the corresponding Higgs V-bundles and so we can talk about stable Higgs pairs, isomorphisms of Higgs pairs and so on.) (From some points of view it is more natural to consider the holomorphic structure as fixed and the unitary structure as varying. Of course the two approaches are equivalent.)

We impose determinant-fixing conditions in what follows; they are not essential but they remove some redundancies associated with scalar automorphisms (see Proposition 2.1), tensoring by line V-bundles and so on. We have already made the assumption that the Higgs field ϕ fixes determinants in the sense that it is trace-free; the other determinant-fixing conditions are defined as follows. A unitary structure on E induces one on the determinant line V-bundle Λ. With this fixed and a choice of isomorphism class of holomorphic structure on Λ, there is a unique (up to unitary gauge) unitary connexion on Λ which is compatible with the class of holomorphic structure and is Yang-Mills, i.e. has constant central curvature $-2\pi i c_1(\Lambda)\Omega$. Fix one such connexion and denote it A_Λ. We say that a unitary connexion or holomorphic structure on E has FIXED DETERMINANT if it induces this fixed connexion or holomorphic structure in the determinant line V-bundle. (On the other hand if we fix the holomorphic structure then we can choose a Hermitian-Yang-Mills metric on the determinant line V-bundle and fix the determinant of our metrics by insisting that they induce this metric.)

Given a unitary connexion A the trace-free part of the curvature is $F_A^0 =_{\text{def}} F_A + \pi i c_1(\Lambda)\Omega I_E$, by the Chern-Weil theory. We say that a Higgs pair (A, ϕ) (with fixed determinants understood) is YANG-MILLS-HIGGS if

$$F_A^0 + [\phi, \phi^*] = 0 \quad \text{and} \qquad \text{(3a)}$$
$$\overline{\partial}_A\phi = 0.$$

(For a Hermitian metric varying on a fixed Higgs V-bundle this is the condition for the metric to be Hermitian-Yang-Mills-Higgs.) The involution

$\phi \mapsto \phi^*$ is a combination of the conjugation $dz \mapsto d\bar{z}$ and taking the adjoint of an endomorphism with respect to the metric. The second part of the condition merely reiterates the fact that ϕ is holomorphic with respect to the holomorphic structure induced by A. Of course if $\phi = 0$ then (3a) is just the Yang-Mills equation (see [1, 10]) and we say that A is YANG-MILLS. An existence theorem for Yang-Mills connexions in stable V-bundles, generalising the Narasimhan-Seshadri theorem from the smooth case [5], is given in [10]. The first half of our correspondence between stable Higgs V-bundles and Yang-Mills-Higgs pairs is not difficult; again a result of Hitchin [14, Theorem 2.1] generalises easily.

PROPOSITION 3.1. *Let M be an orbifold Riemann surface with negative Euler characteristic. If (A, ϕ) is a Yang-Mills-Higgs pair (with respect to the fixed unitary structure on E and with fixed determinants) then the pair (A, ϕ) is [illegible]e unless it has a $U(1)$-reduction, in which case it is polystable.*

We call a pair with a $U(1)$-reduction, *as a pair*, REDUCIBLE; otherwise the pair is IRREDUCIBLE. Notice that a reducible pair is Yang-Mills-Higgs if and only if the connexions in the two line V-bundles are Yang-Mills.

Define the GAUGE GROUP $\mathcal{G}(E)$ to be the group of unitary automorphisms of E (fixing the base). This acts on Higgs fields by conjugation and has a natural action on $\bar{\partial}$-operators such that the corresponding Chern connexions transform in the standard way. Thus this action fixes the determinant line V-bundle, acts on the set of Higgs V-bundles by isomorphisms and takes one Yang-Mills-Higgs pair to another. We also consider the COMPLEXIFIED GAUGE GROUP $\mathcal{G}^c(E)$ of complex-linear automorphisms of E (fixing the base). Again this acts on Higgs V-bundles by isomorphisms. Isomorphic Higgs V-bundle structures are precisely those that lie in the same $\mathcal{G}^c(E)$-orbit. Notice that Proposition 2.1 implies that $\mathcal{G}^c(E)$ acts freely (modulo scalars) on the set of stable Higgs V-bundles. (If we think of the Higgs V-bundle (E, ϕ) as fixed and the Hermitian metric as variable then $\mathcal{G}^c(E)$ acts transitively on the space of Hermitian metrics.) Once again we easily obtain a uniqueness result due to Hitchin [14, Theorem 2.7] in the smooth case.

PROPOSITION 3.2. *Let (E_1, ϕ_1) and (E_2, ϕ_2) be isomorphic Higgs V-bundles with fixed determinants, with Chern connexions A_1 and A_2 and the same underlying rank 2 Hermitian V-bundle. Suppose that the Higgs pairs (A_1, ϕ_1) and (A_2, ϕ_2) are both Yang-Mills-Higgs. Then (E_1, ϕ_1) and (E_2, ϕ_2) are gauge-equivalent (i.e. there is an element of $\mathcal{G}(E)$ taking one to the other).*

3B. - *An Existence Theorem for Yang-Mills-Higgs Pairs*

A version of the Narasimhan-Seshadri theorem for stable Higgs V-bundles (essentially a converse to Proposition 3.1) can be proved directly for orbifolds, extending the arguments of [5, 14].

THEOREM 3.3. *Let $E \to M$ be a fixed $U(2)$ V-bundle over an orbifold Riemann surface of negative Euler characteristic. If (A, ϕ) is a polystable Higgs pair with fixed determinant on E then there exists an element $g \in \mathcal{G}^c$ of determinant 1, unique modulo elements of $\mathcal{G}$ of determinant 1, such that $g(A, \phi)$ is Yang-Mills-Higgs.*

We shall deduce the theorem from the ordinary case by equivariant arguments in § 3D, though there is some advantage to a direct proof, as an appeal to Fox's theorem is avoided and uniformisation results from the following corollary, proved as in [14, Corollary 4.23].

COROLLARY 3.4. *If M is an orbifold Riemann surface of negative Euler characteristic then M admits a unique compatible metric of constant sectional curvature -4.*

PROOF. We define a stable Higgs V-bundle by equipping $E = K \oplus 1$ with the Higgs field

$$\phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We fix a Hermitian-Yang-Mills metric on $\Lambda^2 E$. From Theorem 3.3 we have a Hermitian-Yang-Mills-Higgs metric h on E. Exactly as in [14, Corollary 4.23], this must split and we obtain a metric on K such that the dual metric in the tangent bundle has constant sectional curvature -4. □

3C. - *The Yang-Mills-Higgs Moduli Space*

We now construct the moduli space of irreducible Yang-Mills-Higgs pairs, beginning with a brief discussion of reducible Yang-Mills-Higgs pairs. Let (A, ϕ) be a reducible Yang-Mills-Higgs pair on E. The reduction means that there is a splitting of E into a direct sum $E = L \oplus L^*\Lambda$, where L and $L^*\Lambda$ have the same degree, with respect to which A and ϕ are diagonal - the resulting Higgs V-bundle is polystable but not stable. The isotropy group of the pair (A, ϕ) is S^1 or $SU(2)$ according to whether the two summands are distinct or identical; since ϕ is trace-free the latter is only possible if $\phi = 0$.

Let us now consider the question of the existence of reductions. Obviously the essential prerequisite is that L exists such that L and $L^*\Lambda$ have the same degree. If a denotes the least common multiple of the α_i's then the degrees of line V-bundles have the form s/a for $s \in \mathbb{Z}$ and all s occur. Thus a necessary condition for a reduction is that $c_1(\Lambda) = s/a$ with s even. However, even when s is even, there is a further constraint: the isotropy of E is fixed and, as before, the isotropy of L must be described by an isotropy vector (ϵ_i) with $c_1(L) \equiv \sum_{i=1}^{n} \{\epsilon_i(x_i' - x_i) + (x_i' + x_i)\}/2\alpha_i \pmod{\mathbb{Z}}$ and so the isotropy may imply a constraint to finding L with appropriate $c_1(L)$. For general M and E it is impossible in 'most' cases (see [10] for details).

From now on we make the assumption that the isotropy of M and the degree and isotropy of E are such that there are no reducible Yang-Mills-Higgs pairs on E.

We outline the deformation theory to show that the moduli space is a finite-dimensional manifold. (For the purposes of this outline we suppress the use of Sobolev spaces - this is standard; see e.g. [24].) Fix an irreducible Yang-Mills-Higgs pair (A, ϕ). The 'deformation complex' at (A, ϕ) is then the following elliptic complex:

$$
\begin{aligned}
0 \to \Gamma(su(E)) &\xrightarrow{d_1} \Gamma(su^1(E)) \oplus \Omega^{1,0}(sl(E)) \\
&\xrightarrow{d_2} \Gamma(su^2(E)) \oplus \Omega^{1,1}(sl(E)) \to 0,
\end{aligned} \tag{3b}
$$

where $su^k(E)$ denotes the bundle of skew-adjoint k-forms with values in the trace-free endomorphisms of E and $sl(E)$ denotes the bundle of trace-free endomorphisms of E. Here d_1, giving the linearisation of the action, is given by

$$d_1 : \psi \mapsto (d_A\psi, [\phi, \psi]))$$

and d_2, giving the linearisation of the Yang-Mills-Higgs equations, by

$$d_2 : (A', \phi') \mapsto (d_A A' + [\phi', \phi^*] + [\phi, \phi'^*], \overline{\partial}_A \phi' + [(A')^{0,1}, \phi]).$$

We use the orbifold Atiyah-Singer index theorem [19] to calculate the index of (3b) as $6(g-1)+2(n-n_0)$. We note that the zeroeth and second cohomology groups, H^0 and H^2, of the complex vanish - for H^0 this follows from the irreducibility of (A, ϕ) and for H^2 the duality argument given by Hitchin will suffice. Hence the first cohomology group has dimension $6(g-1)+2(n-n_0)$. Moreover the Kuranishi method shows that a neighbourhood of zero in H^1 is a local model for the moduli space and hence the moduli space is a smooth complex manifold of dimension $6(g-1)+2(n-n_0)$.

THEOREM 3.5. *Let M be an orbifold Riemann surface of negative Euler characteristic and $E \to M$ a fixed complex rank 2 V-bundle.*

1. *Suppose that E is equipped with a Hermitian metric and admits no reducible Yang-Mills-Higgs pairs. Then the moduli space of Yang-Mills-Higgs pairs on E with fixed determinants, $\mathcal{M}(E, A_\Lambda)$, is a complex manifold of dimension $6(g-1)+2(n-n_0)$.*
2. *Suppose that E admits no Higgs V-bundle structures which are polystable but not stable. Then the moduli space of stable Higgs V-bundle structures on E with fixed determinants is a complex manifold of dimension $6(g-1)+2(n-n_0)$.*

REMARK 3.6. In the smooth case there are essentially only two moduli spaces (of which only one is smooth), according to the parity of the degree. In the orbifold case, how many moduli spaces are there? Clearly it is sufficient to consider only one topological Λ in each class under the equivalence $\Lambda \sim \Lambda L^2$, for any topological line V-bundle L - 'square-free' representatives for each class will be discussed in § 6B. A further subtlety in the orbifold case is the possibility of non-trivial topological square roots of the trivial line V-bundle, or simply TOPOLOGICAL ROOTS: if L is a topological root then there is a map on moduli $\mathcal{M}(E, A_\Lambda) \leftrightarrow \mathcal{M}(E \otimes L, A_\Lambda)$ by tensoring by L, which fixes Λ but alters the topology of E. For L to be a topological root necessarily $c_1(L) = 0$ and L has 'half-trivial' isotropy, i.e. the isotropy is 0 or $\alpha/2$ at each marked point. If we consider topological line V-bundles of the form $L = \otimes_{\alpha_i \,\mathrm{even}} L_i^{\delta_i \alpha_i/2}$, for $\delta_i \in \mathbb{Z}$ where the L_i are the point V-bundles of § 1B, then it is clear that L is a topological root provided $c_1(L) = \sum \delta_i/2 = 0$. If we let n_2 denote the number of marked points where the isotropy is even, then, provided $n_2 \geq 1$, there are 2^{n_2-1} topological roots. It follows that for each topological Λ, if $n_2 \geq 1$, there will be 2^{n_2-1} different topological E's giving essentially the same moduli space. We will see another manifestation of this in § 6B.

Recall that the tangent space to the moduli space is given by the first cohomology of the deformation complex (3b), i.e. by $\ker(d_1^*) \cap \ker(d_2)$. This space admits a natural L^2 metric and, just as in [14, Theorems 6.1 & 6.7], we have the following result.

PROPOSITION 3.7. *Let E be a fixed rank 2 Hermitian V-bundle over an orbifold Riemann surface of negative Euler characteristic and suppose that E admits no reducible Yang-Mills-Higgs pairs. Then the natural L^2 metric on the moduli space $\mathcal{M}(E, A_\Lambda)$ is complete and hyper-Kähler.*

3D. - *The Yang-Mills-Higgs equations and Equivariance*

Here we sketch how many *but not all* of the previous results of this section can be treated by equivariant arguments. Further details for this subsection can be found in [22].

An orbifold Riemann surface with negative Euler characteristic, M, has a topological orbifold covering by a surface [26] and so its universal covering is necessarily a surface with negative Euler characteristic. Pulling-back the complex structure we find that the universal covering is necessarily D^2, the unit disk, with $\pi_1^V(M)$ a group of automorphisms acting properly discontinuously. In other words $\pi_1^V(M)$ is a co-compact Fuchsian group or, in the terminology of [8], an F-GROUP.

Thinking of D^2 as the hyperbolic upper half-plane or Poincaré disk, the elements of $\pi_1^V(M)$ act by orientation-preserving isometries and so we get a compatible Riemannian metric of constant sectional curvature on M. This is just Corollary 3.4. In this context we need the following result of [8].

PROPOSITION 3.8 (FOX). *If Γ is an F-group then Γ has a normal subgroup of finite index, containing no elements of finite order.*

COROLLARY 3.9. *Let M be an orbifold Riemann surface with negative Euler characteristic. Then there exists a smooth Riemann surface, $\hat{M}$, with negative Euler characteristic, together with a finite group, F, of automorphisms of $\hat{M}$, such that $M = F\backslash\hat{M}$.*

The important point here is that the covering is *finite* and hence $\hat{M}$ is compact.

The existence result of Theorem 3.3 follows from the corresponding result on $\hat{M}$, [14, Theorem 4.3], using an averaging argument (compare [11]). We will always use the notation that objects on $\hat{M}$ pulled-back from M under the covering map $\hat{M} \to M$ will be denoted by a 'hat'; $\hat{}$. In this notation the pull-back of a V-bundle $E \to M$ becomes $\hat{E} \to \hat{M}$, and so on. For the equivariant argument it is easiest to fix the Higgs V-bundle structure on E and vary the metric; therefore, rather than suppose that a *Hermitian* structure on E is given, we temporarily suppose that a *holomorphic* structure on E (and hence on $\hat{E}$) is given. We will show that if (E, ϕ) is stable then $(\hat{E}, \hat{\phi})$ is polystable and admits a Hermitian-Yang-Mills-Higgs metric which is F-invariant and so descends to the required metric on E.

PROPOSITION 3.10. *Let (E, ϕ) be a stable Higgs V-bundle and let $(\hat{E}, \hat{\phi})$ be the pull-back to $\hat{M}$. Then $(\hat{E}, \hat{\phi})$ is polystable.*

PROOF. Suppose first that $(\hat{E}, \hat{\phi})$ is *not semi-stable*. Then there is a unique destabilising Higgs sub-V-bundle $L = L_{\hat{E}}$ and the action of F cannot fix L. Therefore for some $f \in F$ we have that $f(L) \neq L$. However $f(L)$ is a Higgs sub-V-bundle of $(\hat{E}, \hat{\phi})$ (because $\hat{\phi}$ commutes with the action of $f \in F$) and has the same degree as L. This contradicts the uniqueness of L. So $(\hat{E}, \hat{\phi})$ is semi-stable. Suppose it is *not stable*. Then again there is a destabilising Higgs sub-V-bundle $L = L_{\hat{E}}$ (not necessarily unique). As before L cannot be fixed by F and so we obtain, for some $f \in F$, a Higgs sub-V-bundle $f(L) \neq L$ of the same degree as L. Let $g: f(L) \to \hat{E}/L$ be the composition of the inclusion of $f(L)$ into$\hat{E}$ with the projection onto $\hat{E}/L$: g is a homomorphism between two line bundles of the same degree and hence either zero or constant. Since $f(L) \neq L$ the map g cannot be zero and hence $f(L) = \hat{E}/L$. Since $f(L)$ is actually a Higgs sub-V-bundle, $(\hat{E}, \hat{\phi})$ is a direct sum $(L \oplus f(L), \hat{\phi}_L \oplus \hat{\phi}_{f(L)})$ and so is polystable as claimed. □

PROPOSITION 3.11. *Let (E, ϕ) be a stable Higgs V-bundle and let $(\hat{E}, \hat{\phi})$ be the pull-back to $\hat{M}$. Then the polystable Higgs V-bundle $(\hat{E}, \hat{\phi})$ admits a Hermitian-Yang-Mills-Higgs metric which is F-invariant (and unique up to scale).*

PROOF. Certainly $(\hat{E}, \hat{\phi})$ admits a Hermitian-Yang-Mills-Higgs metric (by Proposition 3.10 and [14, Theorem 4.3]). By averaging, the Hermitian-Yang-Mills-Higgs metric can be supposed F-invariant. □

An F-invariant Hermitian-Yang-Mills-Higgs metric descends to (E, ϕ), where it trivially still satisfies the Hermitian-Yang-Mills-Higgs condition. We can satisfy the determinant-fixing condition by a choice of scalar multiple and so we obtain the desired existence result - Theorem 3.3.

Suppose again that a Hermitian, rather than holomorphic, structure on E is given. We recall that Hitchin proves that if $\hat{E}$ has odd degree then there is a smooth moduli space $\mathcal{M}(\hat{E}, \hat{A}_\Lambda)$ of complex dimension $6(\hat{g}-1)$. The pull-back map $(A, \phi) \mapsto (\hat{A}, \hat{\phi})$ defines a map from Higgs pairs on E to F-invariant Higgs pairs on $\hat{E}$ - what can be said about the corresponding map on moduli? Suppose that (A, ϕ) is an irreducible Yang-Mills-Higgs pair on E. The first point to note is that $(\hat{A}, \hat{\phi})$ may be reducible, by the analogue of Proposition 3.10 for pairs. For simplicity, we will ignore this possibility in our discussion - we suppose that there are topological obstructions to the existence of reducible Yang-Mills-Higgs pairs on $\hat{E}$.

LEMMA 3.12. *Suppose that (A, ϕ) is an irreducible Yang-Mills-Higgs pair on E with an irreducible lift. Suppose further that for some $g \in \hat{\mathcal{G}}$, of determinant 1, $g(\hat{A}, \hat{\phi})$ is F-invariant. Then $f^{-1}gf = \pm g$ for all $f \in F$. Conversely, given $g \in \hat{\mathcal{G}}$ of determinant 1 such that $f^{-1}gf = \pm g$ for all $f \in F$, $g(\hat{A}, \hat{\phi})$ is irreducible and F-invariant.*

PROOF. Since $(\hat{A}, \hat{\phi})$ is F-invariant we know that $f d_{\hat{A}} = d_{\hat{A}} f$ and $f\hat{\phi} = \hat{\phi} f$ for any $f \in F$. Since the same is also true of $g(\hat{A}, \hat{\phi})$ it follows that $d_{\hat{A}} = (g^{-1}f^{-1}gf)(d_{\hat{A}})(f^{-1}g^{-1}fg)$ and similarly for the Higgs field. Since (A, ϕ) is a stable pair it follows (Proposition 2.1) that $\pm g = f^{-1}gf$. The converse is clear. □

Let $\hat{\mathcal{G}}^F$ be the subgroup of $\hat{\mathcal{G}}$ consisting of F-invariant elements of determinant 1 and let $\hat{\mathcal{G}}^\pm$ denote that of elements $g \in \hat{\mathcal{G}}$ of determinant 1 such that, for all $f \in F$, $f^{-1}gf = \pm g$. Clearly either $\hat{\mathcal{G}}^\pm = \hat{\mathcal{G}}^F$ or $\hat{\mathcal{G}}^F < \hat{\mathcal{G}}^\pm$ with even index. (In fact these groups will be equal under quite mild hypotheses, which amount to the vanishing of a certain equivariant $\mathbb{Z}_2$-characteristic class - see [22] and compare [10, Proposition 1.8, part iii)].) If these groups are unequal then $f^{-1}gf = -g$ for some $f \in F$ and $g \in \hat{\mathcal{G}}$ of determinant 1 - but such a g cannot be close to ± 1 and so does not enter the local description of the moduli space (compare [24, Theorem 4.1]). At an irreducible F-invariant pair $(\hat{A}, \hat{\phi})$ the group F acts on the deformation complex. The pull-back map induces a commutative diagram of deformation complexes and it follows immediately that $\mathcal{M}(E, A_\Lambda)$ covers a submanifold of $\mathcal{M}(\hat{E}, \hat{A}_\Lambda)$ with covering group $\hat{\mathcal{G}}^\pm/\hat{\mathcal{G}}^F$.

THEOREM 3.13. *Let M be an orbifold Riemann surface of negative Euler characteristic and $E \to M$ a fixed complex rank 2 V-bundle. Let $\hat{E}$ be the pull-back of E under the identification $M = F\backslash\hat{M}$ of Corollary 3.9.*

1. *Suppose that E is equipped with a Hermitian metric and $\hat{E}$ with the pulled-back metric and that E admits no reducible Yang-Mills-Higgs pairs. If $\hat{E}$ has odd degree then, under pull-back, the moduli space of Yang-Mills-Higgs pairs with fixed determinants on E, $\mathcal{M}(E, A_\Lambda)$, covers*

a submanifold of the corresponding moduli space on $\hat{E}$ with covering group $\hat{\mathcal{G}}^{\pm}/\hat{\mathcal{G}}^{F}$ (with $\hat{\mathcal{G}}^{\pm}$ and $\hat{\mathcal{G}}^{F}$ as above). If $\hat{E}$ has even degree then this remains true for those classes of Higgs pairs which are irreducible on $\hat{E}$.

2. *Suppose that E admits no Higgs V-bundle structures which are polystable but not stable. If $\hat{E}$ has odd degree then, under pull-back, the moduli space of stable Higgs V-bundle structures with fixed determinants on E covers a submanifold of the corresponding moduli space on $\hat{E}$ with covering group $\hat{\mathcal{G}}^{\pm}/\hat{\mathcal{G}}^{F}$ (with $\hat{\mathcal{G}}^{\pm}$ and $\hat{\mathcal{G}}^{F}$ as above). If $\hat{E}$ has even degree then this remains true for those classes of Higgs V-bundle structure which are stable on $\hat{E}$.*

Notice that in the case when $\hat{M}$ is a hyperelliptic surface of genus 2 branched over 6 points of the Riemann sphere then the dimensions of the two moduli spaces are equal (a simple arithmetic check shows that this is the only case where this happens).

4. - The topology of the moduli space

We now give some results on the topology of the moduli space using the Morse function $(A,\phi) \xrightarrow{\mu} \|\phi\|^2_{L^2}$, following [14, § 7]. Notation and assumptions remain as before; in particular, we suppose that E admits no reducible Yang-Mills-Higgs pairs, so that the moduli space $\mathcal{M} = \mathcal{M}(E, A_\Lambda)$ is smooth and recall the definitions of the integers $n_\pm$ and l from § 1A.

The function $(A,\phi) \xrightarrow{\mu} \|\phi\|^2_{L^2} = 2\mathrm{i}\int(\phi\phi^*)$ is invariant with respect to the circle action $e^{\mathrm{i}\theta}(A,\phi) = (A, e^{\mathrm{i}\theta}\phi)$ and $d\mu(Y) = -2\mathrm{i}\omega_1(X,Y)$ where X generates the S^1-action and ω_1 is as in [14, § 6]. The map μ is proper and there's an extension of [14, Proposition 7.1]. To describe it we need to consider pairs $(m,(\epsilon_i))$ where m is an integer and (ϵ_i) is an isotropy vector - such pairs describe topological sub-V-bundles of E, with isotropy described by (ϵ_i) and degree $m + \sum_{i=1}^{n} \{\epsilon_i(x_i' - x_i) + (x_i' + x_i)\}/(2\alpha_i)$ (see e.g. Remark 1.2).

THEOREM 4.1. *Let E be a fixed rank 2 Hermitian V-bundle over an orbifold Riemann surface of negative Euler characteristic and suppose that E admits no reducible Yang-Mills-Higgs pairs. If $g = 0$ then suppose that $n - n_0 \geq 3$. Let μ be as above: then, with the notations established above,*

1. *μ has critical values 0 and $2\pi\left\{2m - l + \sum_{i=1}^{n}\{\epsilon_i(x_i' - x_i)/\alpha_i\}\right\}$ for an integer m and isotropy vector (ϵ_i) with*

$$l < 2m + \sum_{i=1}^{n} \frac{\epsilon_i(x_i' - x_i)}{\alpha_i} \leq l + 2g - 2 + \sum_{i=1}^{n} \frac{\epsilon_i(x_i' - x_i)}{\alpha_i} + n_-;$$

2. *the minimum* $\mu^{-1}(0)$ *is a non-degenerate critical manifold of index 0 and is diffeomorphic to the space of stable V-bundles with fixed determinants and*
3. *the other critical manifolds are also non-degenerate and are* 2^{2g}*-fold coverings of* $S^r\tilde{M}$*, where* $r = l - 2m + 2g - 2 + n_-$*. Moreover, they are of index* $2\{2m - l + g - 1 + n_+\}$.

PROOF. The critical points are the fixed points of the induced circle action on $\mathcal{M}$. Because we are taking quotients by the gauge group, these correspond to pairs $((A,\phi),\lambda)$ where $\lambda: S^1 \to \mathcal{G}$ such that, for all θ, $\lambda(e^{i\theta})d_A\lambda(e^{-i\theta}) = d_A$ and $\lambda(e^{i\theta})\phi\lambda(e^{-i\theta}) = e^{i\theta}\phi$. If $\phi = 0$ then, holomorphically, we simply get stable V-bundles. If $\phi \neq 0$ then certainly $\lambda(e^{i\theta}) \neq 1$ for $\theta \not\equiv 0 \pmod{2\pi}$. The first equation now implies that the stabiliser $\mathcal{G}_A$ is non-trivial and A is reducible to a $U(1)$-connexion. Consequently, as a holomorphic V-bundle, E is decomposable (so, in particular, not stable) and can be written $L \oplus L^*\Lambda$. If we write $\phi = \begin{pmatrix} t & u \\ v & -t \end{pmatrix}$ and $\lambda(e^{i\theta}) = \begin{pmatrix} \mu_\theta & 0 \\ 0 & \mu_\theta^{-1} \end{pmatrix}$ with respect to this splitting then the second equation implies $t = 0$ and either $u = 0$ or $v = 0$. Replacing L by $L^*\Lambda$ if necessary, we can suppose that $u = 0$ and that $v \in H^0(KL^{-2}\Lambda)$ - v is holomorphic from the self-duality equations.

The remaining term of the Yang-Mills-Higgs equations is $*(F_A + [\phi,\phi^*]) = -\pi i dI_E$. Writing $*F_{A_L} = *F - \pi i d$, in terms of the above decomposition, so that $*F_{A_{L^*\Lambda}} = -*F - \pi i d$, we find that $F = v \wedge \bar{v}$ and

$$\deg L = \frac{i}{2\pi}\int (F - *\pi i c_1(\Lambda)) = \frac{i}{2\pi}\int (v \wedge \bar{v}) + \frac{c_1(\Lambda)}{2} = \frac{\mu}{4\pi} + \frac{c_1(\Lambda)}{2}.$$

Since $\mu > 0$ for $\phi \neq 0$, we have $2\deg L > c_1(\Lambda)$ and $L = L_E$, the destabilising sub-V-bundle of E. Moreover, because $v \neq 0$ we must have $h^0(KL^{-2}\Lambda) \geq 1$ (compare Theorem 2.8).

Now, for any $(m,(\epsilon_i))$ let $L_{(m,(\epsilon_i))}$ be the corresponding topological sub-V-bundle of E. Consider pairs $(m,(\epsilon_i))$ with $2c_1(L_{(m,(\epsilon_i))}) > c_1(\Lambda)$ and set $L = L_{(m,(\epsilon_i))}$ and $E = L \oplus L^*\Lambda$. This occurs as a stable pair (E,ϕ) provided L admits a holomorphic structure with $h^0(KL^{-2}\Lambda) \geq 1$, and the Higgs field ϕ is then given by $v \in H^0(KL^{-2}\Lambda) \setminus \{0\}$ (compare Theorem 2.8, part 3b and Proposition 2.10).

To see whether a given topological $L = L_{(m,(\epsilon_i))}$ admits an appropriate holomorphic structure we use our results from § 2B: by Lemma 2.5 we have $\chi(KL^{-2}\Lambda) = l - 2m + g - 1 + n_-$. It follows that $r = c_1(\widetilde{KL^{-2}\Lambda}) = l - 2m + 2g - 2 + n_-$. Hence, supposing that $r \geq 0$, for each effective (integral) divisor of divisor order r (if $r = 0$ then for the empty divisor) we obtain a holomorphic structure on $\widetilde{KL^{-2}\Lambda}$ with a holomorphic section determining the divisor (determined up to multiplication by elements of $\mathbb{C}^*$). Hence we get a holomorphic structure on $KL^{-2}\Lambda$ with holomorphic section v and all holomorphic sections arise in this way. Placing a corresponding holomorphic structure on L requires a choice of holomorphic square root and there are 2^{2g} such choices. For each root L the pair $(E,v) = (L \oplus L^*\Lambda, v)$ is clearly stable by construction. The section v is determined by the divisor up to a multiplicative constant $\lambda \neq 0$ but $(L \oplus L^*\Lambda, v)$ and $(L \oplus L^*\Lambda, \lambda v)$ are in the same orbit under the action of the complexified gauge group and hence equivalent. Two distinct divisors determine distinct stable pairs so that we have the critical set is a 2^{2g}-fold covering of the set of effective divisors of degree $r = l - 2m + 2g - 2 + n_-$; that is, a 2^{2g}-fold covering of $S^r\tilde{M}$ (a point if $r = 0$).

Let $E = L \oplus L^*\Lambda$ for $L = L_{(m,(\epsilon_i))}$, as above. The subset $U = \{\phi \in H^0(\mathrm{End}_0(E) \otimes K) : (E, \phi)$ is stable$\}$ is acted upon freely by $\mathrm{Aut}_0(E)/\{\pm 1\}$, where $\mathrm{Aut}_0(E)$ are the holomorphic automorphisms of determinant 1 (see Proposition 2.1). The quotient $U/(\mathrm{Aut}_0(E)/\{\pm 1\})$ is a complex manifold of dimension $3g - 3 + n - n_0$. So through each point $P \in \mathcal{M}$ there passes a $(3g-3+n-n_0)$-dimensional isotropic complex submanifold $U/(\mathrm{Aut}_0(E)/\{\pm 1\})$, invariant under S^1: it is thus *Lagrangian*. Suppose $P \in \mathcal{M}$ is fixed under the S^1-action and $P = (E, \phi)$, where $E = L \oplus L^*\Lambda$, $\phi = \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}$, as above. The homomorphism λ is given by $\lambda(\theta) = \begin{pmatrix} e^{-\mathrm{i}\theta/2} & 0 \\ 0 & e^{\mathrm{i}\theta/2} \end{pmatrix}$ with respect to this decomposition. Now $\mathrm{End}_0(E) = L^{-2}\Lambda \oplus L^2\Lambda^* \oplus \mathbb{C}$ and $\lambda(\theta)$ acts as $(e^{\mathrm{i}\theta}, e^{-\mathrm{i}\theta}, 1)$. Hence $\lambda(\theta)$ acts with negative weight solely on $H^0(KL^2\Lambda^*) \subset H^0(\mathrm{End}_0(E) \otimes K)$. As $\lambda(\theta)$ acts on ϕ by multiplication by $e^{\mathrm{i}\theta}$ there are no negative weights on $H^0(\mathrm{End}_0(E)) \cdot \phi$ and hence we find, as in [14], that the index is $2h^0(KL^2\Lambda^*) = 2\{2m - l + g - 1 + n_+\}$, by Lemma 2.5. $\square$

From this, the work of [9] and general Morse-Bott theory [2] we can, in principle, calculate the Betti numbers - see [4]. We content ourselves with Corollary 4.3, below, for which we need the following preliminary lemma.

LEMMA 4.2. *There is exactly one critical manifold of index 0 and this is connected and simply-connected.*

PROOF. Theorem 4.1 shows that if $g > 0$ then the space of stable V-bundles is the only index 0 critical manifold and this is connected and simply-connected (even when $g = 0$) by [10, Theorem 7.11]. When $g = 0$, critical manifolds of index 0 other than the moduli of stable V-bundles may occur: these have the form $S^r\tilde{M} \cong \mathbb{CP}^r$ and so are also connected and simply connected.

It remains to show that exactly one of the possibilities is non-empty in each case. Making allowances for differences in notation, the following is implicit in [10, Theorem 4.7]: the space of stable V-bundles is empty if and only if there exists a vector (ϵ_i) with $n_+ + l \equiv 1(2)$ and

$$n_+ - \sum_{i=1}^{n} \frac{\epsilon_i(x_i' - x_i)}{\alpha_i} < 1 - g. \tag{4a}$$

Since the left-hand side of (4a) is clearly not less than zero we see that the space of stable V-bundles is non-empty whenever $g > 0$.

When $g = 0$, Theorem 4.1 shows that the critical manifolds of index 0 other than the moduli of stable V-bundles consist precisely of the V-bundles considered in Proposition 2.10. The number of such critical manifolds is the number of topological types $L_{(m,(\epsilon_i))}$ satisfying the criteria of Proposition 2.10, which, using the ideas of Lemma 2.6, is

$$\#\left\{(\epsilon_i)\colon n_+ + l \equiv 1(2) \quad \text{and} \quad n_+ - \sum_{i=1}^{n} \frac{\epsilon_i(x_i' - x_i)}{\alpha_i} < 1\right\}, \tag{4b}$$

where (ϵ_i) varies over all isotropy vectors. Comparing (4b) to (4a) we see that exactly one of the two types of critical manifold must occur. Moreover, we claim that the number in (4b) is at most 1 - this is sufficient to establish the lemma.

To prove the claim suppose, without loss of generality, that $n_0 = 0$. Observe that it is an easy exercise to show that if $t_1, \dots, t_n \in (0,1)$ are such that $\sum_{i=0}^{n} t_i < 1$ then at most one t_i can be replaced by $1 - t_i$ with the sum remaining less than 1. Let

$$t_i = \frac{1+\epsilon_i}{2} - \frac{\epsilon_i(x_i' - x_i)}{\alpha_i}$$

so that $\sum_{i=0}^{n} t_i = n_+ - \sum_{i=0}^{n} \epsilon_i(x_i' - x_i)/\alpha_i$ and changing the sign of ϵ_i simply sends t_i to $1 - t_i$. The observation applies to show that this sum can be less than 1 for at most two vectors (ϵ_i) and these cannot have n_+ of the same parity. Hence the count in (4b) is at most 1, as claimed. □

COROLLARY 4.3. *The moduli space $\mathcal{M}$ is non-compact except in the case $g = 0$ and $n - n_0 = 3$ when it is a point - and connected and simply-connected.*

PROOF. The non-compactness follows from the fact that the critical manifolds cannot be maxima except if $g = 0$ and $n - n_0 = 3$. This is because the critical manifolds have index $i = 2\{2m - l + g - 1 + n_+\}$ and (real) dimension $2r = 2\{l - 2m + 2g - 2 + n_-\}$ and $2r + i = 6g - 6 + 2(n - n_0)$, which is exactly half the (real) dimension of the moduli space. The connectedness and simple-connectedness follow from the analogous facts for the unique critical manifold of index 0 (Lemma 4.2) and the fact that the other Morse indices are all even and strictly positive.

5. - The Determinant Map

Recall that M is an orbifold Riemann surface with negative Euler characteristic, with $E \to M$ a fixed $U(2)$-V-bundle. We assume that E admits no reducible Yang-Mills-Higgs pairs so that the moduli space is smooth.

Thinking of the moduli space as a space of stable Higgs V-bundles, there is a holomorphic gauge-invariant map $(A, \phi) \mapsto \det(\phi)$ which descends to a holomorphic map $\det\colon \mathcal{M}(E, A_\Lambda) \to H^0(K^2)$. Hitchin showed that in the smooth case this map is proper, surjective and makes $\mathcal{M}$ a completely integrable Hamiltonian system. Moreover he showed that when $q \in H^0(K^2)$ has simple zeros the fibre $\det^{-1}(q)$ is biholomorphic to the Prym variety of the double covering determined by $\sqrt{-q}$ [14, Theorem 8.1]. We will see that things are similar but a little more involved in the orbifold case: the first significant observation is that $h^0(K^2) = 3g - 3 + n$ - this is half the dimension of the moduli space exactly when $n_0 = 0$. For this reason it will be useful to suppose that $n_0 = 0$. (In § 5B we will show that the image of the determinant map is contained in a canonical $(3g - 3 + n - n_0)$-dimensional subspace of $H^0(K^2)$ and thus all cases can be reduced to the case $n_0 = 0$.) In addition, there are two special cases which we exclude: when $g = 0$, $n = 3$ the determinant map is identically zero, and when $g = 1$, $n = 1$ we have a special case which leads to a breakdown in our methods - this case is dealt with separately in § 5D.

We summarise our results in the following theorem (proofs are for the most part discussed in the remainder of this section; the details which have been omitted are exactly as in [14, § 8]). We believe that a similar result was obtained by Peter Scheinost.

THEOREM 5.1. *Let E be a fixed rank 2 Hermitian V-bundle over an orbifold Riemann surface of negative Euler characteristic, with $n - n_0 > 3$ if $g = 0$. Suppose further that E admits no reducible Yang-Mills-Higgs pairs. Then the determinant map on the moduli space of Yang-Mills-Higgs pairs on E with fixed determinants*

$$\deg\colon \mathcal{M}(E, A_\Lambda) \to H^0(K^2)$$

has the following properties:

1. det *is proper;*
2. *the image of* det *lies in a canonical $(3g-3+n-n_0)$-dimensional subspace $H^0(\check{M}; K^2_{\check{M}}) \subseteq H^0(K^2)$ and* det *surjects onto $H^0(\check{M}; K^2_{\check{M}})$;*
3. *with respect to* $\det\colon \mathcal{M}(E, A_\Lambda) \to H^0(\check{M}; K^2_{\check{M}})$, *$\mathcal{M}(E, A_\Lambda)$ is a completely integrable Hamiltonian system;*
4. *for a generic q in the image of* det, *the fibre* $\det^{-1}(q)$ *is biholomorphic to a torus of dimension $3g-3+n-n_0$ - this can be identified with the Prym variety of the covering determined by q except when $g = n - n_0 = 1$, when it is identified with the Jacobian;*
5. *$\mathcal{M}(E, A_\Lambda)$ is a fibrewise compactification of $T^*\mathcal{N}(E, A_\Lambda)$ with respect to the map* $\det\colon T^*\mathcal{N}(E, A_\Lambda) \to H^0(\check{M}; K^2_{\check{M}})$, *where $\mathcal{N}(E, A_\Lambda)$ is the moduli space of Yang-Mills connexions on E with fixed determinants.*

It seems possible to obtain results arguing using orbifold methods but it is often simpler to translate this orbifold problem into one about parabolic bundles; we review the necessary results in the next subsection.

5A. - *Parabolic Higgs bundles*

Recall the basic facts concerning the correspondence between V-bundles over M and parabolic bundles over $\tilde{M}$ [10]. Let $\tilde{E}$ be a rank 2 holomorphic vector bundle over $\tilde{M}$. A QUASI-PARABOLIC STRUCTURE on $\tilde{E}$ is, for each marked point $p \in \{p_1, \ldots, p_n\}$, a flag in $\tilde{E}_p$ of the form

$$\tilde{E}_p = \mathbb{C}^2 \supset \mathbb{C} \supset 0, \quad \text{or} \quad \tilde{E}_p = \mathbb{C}^2 \supset 0.$$

A flag of the second form is said to be DEGENERATE. A quasi-parabolic bundle $\tilde{E}$ is a PARABOLIC BUNDLE if to each flag of the first form there is attached a pair of weights, $0 \leq \lambda < \lambda' < 1$ and to each of the second form there is a single (multiplicity 2) weight $0 \leq \lambda = \lambda' < 1$. There is a notion of parabolic degree involving the degree of $\tilde{E}$ and the weights. A basis $\{e, e'\}$ for the fibre at a parabolic point is said to RESPECT THE QUASI-PARABOLIC STRUCTURE if either the flag is degenerate or e' spans the intermediate subspace in the flag. An endomorphism of a parabolic bundle ψ is a PARABOLIC ENDOMORPHISM if for each p, with respect to a basis which respects the quasi-parabolic structure, ψ_p satisfies $(\psi_p)_{12} = 0$ whenever $\lambda < \lambda'$.

Let E be a rank 2 holomorphic V-bundle over M. Recall that by convention $x \leq x'$ (if we assume that $n_0 = 0$ then there is strict inequality). For a line V-bundle L, we can consider the passage $L \mapsto \tilde{L}$ (§ 1B) as a smoothing process and the construction of parabolic bundles follows similar lines: for a marked point p we consider

$$(E|_{M\setminus\{p\}}) \cup_\Psi D^2 \times \mathbb{C}^2,$$

with clutching function Ψ given, in local coordinates, by its $\mathbb{Z}_\alpha$-equivariant lifting

$$\text{(5a)} \qquad \begin{aligned} \hat{\Psi}\colon (D^2 \setminus \{0\}) \times \mathbb{C}^2 &\to D^2 \times \mathbb{C}^2 \\ (z, (z_1, z_2)) &\mapsto (z^\alpha, (z^{-x} z_1, z^{-x'} z_2)). \end{aligned}$$

Now a holomorphic section of $(D^2 \times \mathbb{C}^2)/(\sigma \times \tau)$ is given by holomorphic maps $s_j \colon D^2 \to \mathbb{C}$, for $j = 1, 2$, invariant under the action of $\mathbb{Z}_\alpha$. As with (1a), Taylor's theorem implies that $s_j(z) = z^{x_j} \tilde{s}_j(z^\alpha)$, where $\tilde{s}_j$ is a holomorphic function $D^2 \to \mathbb{C}$ and we use the temporary notations $x_1 = x$ and $x_2 = x'$. Under the map Ψ defined by (5a) we simply get a section of $(D^2 \setminus \{0\}) \times \mathbb{C}^2$ which is given by the functions $\tilde{s}_j(w)$ and hence extends to a holomorphic section of $D^2 \times \mathbb{C}^2$. In other words the map Ψ is an isomorphism between the sheaves of germs of holomorphic sections. Repeating this construction about each marked point, we get a holomorphic bundle $\tilde{E} \to \tilde{M}$ corresponding to the holomorphic V-bundle $E \to M$.

In fact $\tilde{E}$ has a natural parabolic structure as follows: working in our local coordinates about a particular marked point (which respect the V-structure) we define weights $\lambda = x/\alpha$ and $\lambda' = x'/\alpha$. Define a flag in $\mathbb{C}^2$ so that the smallest proper flag space is the subspace of $\mathbb{C}^2$ on which τ acts like $\sigma^{x'}$. The corresponding quasi-parabolic structure on $\tilde{E}_p$ is then given by the image of this flag - notice that this is degenerate if and only if $x = x'$. With the weights λ, λ' it is clear that $\tilde{E}$ is a parabolic bundle. (Whilst it is not true in general that $\Lambda^2\tilde{E} = \tilde{\Lambda}$, the bundle $\Lambda^2\tilde{E}$ is determined by Λ and the isotropy so that our determinant-fixing condition on E translates to one on $\tilde{E}$.) We quote the following result of [10].

PROPOSITION 5.2 (Furuta-Steer). *For a fixed orbifold Riemann surface M, the correspondence $E \mapsto \tilde{E}$ gives a bijection between isomorphism classes of rank 2 holomorphic V-bundles and those of rank 2 parabolic bundles over $\tilde{M}$ with rational weights of the form x/α. Moreover, the induced map $\mathcal{O}(E) \mapsto \mathcal{O}(\tilde{E})$ is an isomorphism of analytic sheaves.*

Now consider what happens to Higgs fields under the passage $E \mapsto \tilde{E}$: we use a local uniformising coordinate z, centred on a given marked point, and let $w = z^\alpha$ be the local holomorphic coordinate on $\tilde{M}$. There is a Taylor series expansion as before: if ϕ is a Higgs field on E then in our local coordinates

$$\phi_{ij}dz = \begin{cases} z^{x_i-x_j-1}\tilde{\phi}_{ij}(z^\alpha)dz & \text{if } x_i > x_j \text{ and} \\ z^{\alpha+x_i-x_j-1}\tilde{\phi}_{ij}(z^\alpha)dz & \text{if } x_i \le x_j, \end{cases} \tag{5b}$$

where $\tilde{\phi}_{ij}$ are holomorphic functions and we again use the temporary notations $x_1 = x$ and $x_2 = x'$.

To transfer this across to $\tilde{E}$ simply notice that away from the marked point the clutching function Ψ defined by (5a) is a bundle isomorphism and so acts on the Higgs field by conjugation. Conjugating by Ψ we obtain

$$\begin{aligned} \phi^\Psi_{ij}dz &= z^{x_j-x_i}\phi_{ij}dz \\ &= \begin{cases} \tilde{\phi}_{ij}(w)\,\dfrac{dw}{\alpha w} & \text{if } x_i > x_j \text{ and} \\ \tilde{\phi}_{ij}(w)\,\dfrac{dw}{\alpha} & \text{if } x_i \le x_j, \end{cases} \end{aligned} \tag{5c}$$

with $x_1 = x$ and $x_2 = x'$. We take this to define a PARABOLIC HIGGS FIELD. Denote the parabolic Higgs field constructed in this way by $\tilde{\phi}$. In Simpson's language [27] $\tilde{\phi}$ is just a filtered regular Higgs field.

This defines a correspondence between Higgs V-bundles and parabolic Higgs bundles (with appropriate parabolic weights). In order to make this a correspondence between the stable objects we simply have to check that the invariant subbundles correspond - this is easy. Thus we can apply many of our preceding results to spaces of stable parabolic Higgs bundles.

5B. - *Reduction to the case* $n_0 = 0$

Suppose that at some marked points the V-bundle E has $x = x'$ so that $n_0 > 0$. Number the marked points so that these are the last n_0. We can twist by a line V-bundle to make the isotropy zero at such points. Thus, as far as E is concerned, the orbifold structure at these points is irrelevant and we suppose that M only has $n - n_0$ marked points. More precisely, we can construct $\breve{M}$ from M using the smoothing process that gives $\tilde{M}$ but only at the last n_0 marked points. We write $\breve{E}$ for E considered as a V-bundle over $\breve{M}$.

We also have to consider the canonical V-bundle K. Notice that $K = K_{\breve{M}} \otimes_{i=n-n_0+1}^{n} L_i^{\alpha_i - 1}$ so that there is a natural inclusion $H^0(K^2_{\breve{M}}) \hookrightarrow H^0(K^2_M)$ given by $s \mapsto s \otimes_{i=n-n_0+1}^{n} s_i^{2\alpha_i - 2}$. (Here the L_i are point V-bundles and s_i are the canonical sections, as in § 1B.) We identify $H^0(K^2_{\breve{M}})$ with its image in $H^0(K^2_M)$. From (5b) it is clear that $\det(\phi)$ vanishes to order $2\alpha - 2$ in z at the last n_0 marked points (since $x = x'$ there). It follows that $\det(\phi) \in H^0(K^2_{\breve{M}})$ for all Higgs fields ϕ on E. Moreover, if we pass from ϕ to $\breve{\phi}$ by applying the smoothing process for Higgs fields at the last n_0 marked points, then it is clear that $(\breve{E}, \breve{\phi})$ is a Higgs V-bundle over $\breve{M}$. Notice that by (5c) $\breve{\phi}$ is holomorphic at the last n_0 marked points because there we have $x = x'$.

The process outlined above is invertible. For the proofs in the remainder of this section therefore, although we will be careful to state results for $q \in H^0(\breve{M}, K^2_{\breve{M}})$ and $n_0 \geq 0$, we can assume that $n_0 = 0$ without loss of generality.

5C. - *Generic fibres of the determinant map*

We assume that $2g + n - n_0 > 3$. Let $q \in H^0(K^2_{\breve{M}})$ and consider the corresponding section $\tilde{q} \in H^0(\widetilde{K^2_{\breve{M}}})$. We want to suppose that $\tilde{q}$ has simple zeros and that none of the zeros of $\tilde{q}$ occurs at a marked point (of $\breve{M}$) but first we would like to know that such behaviour is generic.

LEMMA 5.3. *The generic section $\tilde{q} \in H^0(\widetilde{K^2_{\breve{M}}})$ has simple zeros, none of which is at a marked point of $\breve{M}$, provided $2g + n - n_0 > 3$.*

PROOF. We can assume that $n_0 = 0$. Notice that $\widetilde{K^2} = K^2_{\tilde{M}} \otimes_{i=1}^{n} L_{p_i}$, where $L_{p_i} = L_i^{\alpha_i}$ is the point bundle associated to a marked point p_i. We know that the $\tilde{q}$ with simple zeros form a non-empty Zariski-open set in the complete linear system $|K^2_{\tilde{M}} \otimes_{i=1}^{n} L_{p_i}|$. The extra condition that none of the zeros is at a marked point is obviously also an open condition, so we only need to check that the resulting set is non-empty.

If $n = 1$ then we only need to show that the marked point is not a base-point of the linear system. Similarly, if there are several marked points then it suffices to show that none is a base point, because then the sections vanishing at a given marked point cut out a hyperplane in the projective space $|K^2_{\tilde{M}} \otimes_{i=1}^{n} L_{p_i}|$. Using [13, IV, Proposition 3.1], this is equivalent to showing that $h^0(K^2_{\tilde{M}} L^*_{p_j} \otimes_{i=1}^n L_{p_i}) = h^0(K^2_{\tilde{M}} \otimes_{i=1}^n L_{p_i}) - 1$ for each $j = 1, \ldots, n$ - this follows from an easy Riemann-Roch calculation, provided $2g + n > 3$. □

LEMMA 5.4. *Let ϕ be a Higgs field on E with* $\det(\phi) = q$ *and $\tilde{q}$ generic in the sense of Lemma* 5.3. *Then $\tilde{q}$ has simple zeros at each marked point where $x = x'$. Moreover, at every marked point of M we have $\tilde{\phi}_{21} \neq 0$ and $\tilde{\phi}_{12} \neq 0$, where $\tilde{\phi}_{21}$ and $\tilde{\phi}_{12}$ are as in* (5b).

PROOF. Using (5b) we have that, in our local coordinates around a marked point,

$$\phi = \begin{pmatrix} z^{\alpha-1}\tilde{\phi}_{11}(z^\alpha) & z^{\alpha+x-x'-1}\tilde{\phi}_{12}(z^\alpha) \\ z^{x'-x-1}\tilde{\phi}_{21}(z^\alpha) & -z^{\alpha-1}\tilde{\phi}_{11}(z^\alpha) \end{pmatrix} dz, \tag{5d}$$

assuming that $x \neq x'$. If $x' = x$ then the $(2,1)$-term is $z^{\alpha-1}\tilde{\phi}_{21}(z^\alpha)dz$. Here the $\tilde{\phi}_{ij}$ are holomorphic functions. If $\tilde{q}$ is generic then it is non-zero at a marked point of $\tilde{M}$ and has at most a simple zero at a marked point where $x = x'$ - in fact there will be a zero at such a point. It follows that we must have that $\det(\phi) = q$ vanishes exactly to order $\alpha - 2$ in z in the first case and order $2\alpha - 2$ in the second. Hence $\tilde{\phi}_{21}(0) \neq 0$ and $\tilde{\phi}_{12}(0) \neq 0$ at each marked point of M. □

Henceforth we assume that $\tilde{q}$ is a generic section, as in Lemma 5.3, and construct $\det^{-1}(q)$. For the purposes of exposition we also assume that $n_0 = 0$. We face two problems in defining the spectral variety of ϕ or $\tilde{\phi}$ - the first is that $\tilde{\phi}$ has simple poles at the marked points and the second is that $\tilde{q}$ is not the determinant of $\tilde{\phi}$. Let $s_{p_i} = s_i^{\alpha_i}$ be the canonical section of the point-bundle L_{p_i} associated to a marked point p_i and let $s_0 = \otimes_{i=1}^n s_{p_i}$ be the corresponding section of $\otimes_{i=1}^n L_{p_i}$. Define

$$\overline{q}' = \tilde{q}s_0 \in H^0(K^2_{\tilde{M}} \otimes_{i=1}^n L^2_{p_i}) \text{ and } \overline{\phi}' = \tilde{\phi}s_0 \in \mathrm{ParEnd}_0(\tilde{E}) \otimes K_{\tilde{M}} \otimes_{i=1}^n L_{p_i}. \tag{5e}$$

It is clear that $\det(\overline{\phi}') = \overline{q}'$ and that $\overline{q}'$ has simple zeros (including one at each marked point). Eventually we will need to reverse the construction of $\overline{\phi}'$ from ϕ; this can be done for a given $\overline{\phi}' \in \mathrm{ParEnd}_0(\tilde{E}) \otimes K_{\tilde{M}} \otimes_{i=1}^n L_{p_i}$ provided $\overline{\phi}'$ obeys the obvious vanishing conditions at each marked point.

The square root $\sqrt{-\overline{q}'}$ defines a smooth Riemann surface $\hat{M}$ with double-covering $\pi : \hat{M} \to \tilde{M}$ and branched at the zeros of $\overline{q}$. Therefore there are $4g - 4 + 2n$ branch-points and the Riemann-Hurwitz formula gives the genus of $\hat{M}$ as $\hat{g} = 4g - 3 + n$. We set $s = \sqrt{-\overline{q}'}$ - a section of $\pi^*(K_{\tilde{M}} \otimes_{i=1}^n L_{p_i})$ - and $\hat{\phi} = \pi^*\overline{\phi}'$. Moreover, if σ is the involution interchanging the leaves of $\hat{M}$ then $\sigma^* s = -s$ and $\hat{\phi}$ is σ-invariant.

In order to reverse the passage from E to $\tilde{E}$ we have to keep track of the quasi-parabolic data. The following lemma is useful here. (Applying the involution σ, the same result holds for $\sigma^* L = \ker(\hat{\phi} - s)$.)

LEMMA 5.5. *If ϕ is a Higgs field on E with* $\det(\phi) = q$ *and $\tilde{q}$ generic in the sense of Lemma* 5.3, *then the kernel of $\hat{\phi} + s$ (with s, $\hat{\phi}$ defined as* *above) is a line subbundle L of $\pi^*\tilde{E}$ and, at a marked point (of $\check{M}$) p, $0 \subseteq L_{\pi^{-1}(p)} \subseteq \pi^*\tilde{E}_{\pi^{-1}(p)} = \tilde{E}_p$ describes the quasi-parabolic structure.*

PROOF. At a marked point, using (5c) and (5e), we write

$$\overline{\phi}' = \begin{pmatrix} w\tilde{\phi}_{11}(w) & w\tilde{\phi}_{12}(w) \\ \tilde{\phi}_{21}(w) & -w\tilde{\phi}_{11}(w) \end{pmatrix} \frac{dw}{\alpha}, \tag{5f}$$

with, from Lemma 5.4, $\tilde{\phi}_{21}(0) \neq 0$ and $\tilde{\phi}_{12}(0) \neq 0$. This means that $\overline{\phi}'$ is not zero at a marked point. Similarly, using the fact that $\tilde{q}$ has simple zeros, $\overline{\phi}'$ is non-zero at *every* branch point. Now consider $\hat{\phi} + s$: since $\det(\hat{\phi} + s) \equiv 0$ this mapping has nullity 1 or 2 at every point. Because $\hat{\phi}$ is trace-free and s is scalar it follows that zeros of $\hat{\phi} + s$ can only occur at zeros of s i.e. at the ramification points. However, since $\overline{\phi}'$ is non-zero at a branch point p it is impossible for $\hat{\phi} + s$ to be zero at $\pi^{-1}(p)$. So $\hat{\phi} + s$ is nowhere zero and the kernel is a line bundle. Finally, if p is a marked point it is clear from (5f) that $\ker(\hat{\phi} + s)_{\pi^{-1}(p)}$ is spanned by $(0, 1)^T$ in our local coordinates. The result about the quasi-parabolic structure follows. □

THEOREM 5.6. *Suppose that $2g + n - n_0 > 3$. Given $q \in H^0(\check{M}, K^2_{\check{M}})$ such that $\tilde{q}$ is generic in the sense of Lemma 5.3 the fibre of the determinant map $\det^{-1}(q)$ is biholomorphic to the Prym variety of the covering $\pi : \hat{M} \to \tilde{M}$, determined by q (via $\overline{q}'$).*

PROOF. Since the proof is familiar [14, Theorem 8.1] we only sketch it. We assume $n_0 = 0$. Fix q such that $\tilde{q}$ is generic and $\hat{M}$ as constructed above and also a line bundle P over $\hat{M}$ such that $P\sigma^*P = \pi^*(K^*_{\tilde{M}}\Lambda^2\tilde{E} \otimes_{i=1}^n L^*_{p_i})$.

Suppose that (E, ϕ) is a Higgs V-bundle over $\check{M}$ with $\det(\phi) = q$. Consider the parabolic bundle $\tilde{E}$ and $\overline{\phi}' \in \mathrm{ParEnd}_0(\tilde{E}) \otimes K_{\tilde{M}} \otimes_{i=1}^n L_{p_i}$ with determinant $\overline{q}'$ defined as above. Now set $L = \ker(\hat{\phi} + s)$ and notice that $L\sigma^*L \cong \pi^*(K^*_{\tilde{M}}\Lambda^2\tilde{E} \otimes_{i=1}^n L^*_{p_i})$. Since P was chosen to have the same property LP^* is an element of the Prym variety.

Conversely, we consider L such that LP^* is a given point in the Prym variety. The push-forward sheaf $\pi_*\mathcal{O}(L)$ is locally free analytic of rank 2 and so defines a rank 2 holomorphic vector bundle W over $\tilde{M}$. There is a natural quasi-parabolic structure on W^* at a branch point p because $W_p = (J_1L)_{\pi^{-1}(p)}$ and there is a natural filtration of jets $0 \subset L^*_{\pi^{-1}(p)} \subset (J_1L)^*_{\pi^{-1}(p)}$. The Hecke correspondence for quasi-parabolic bundles defines a rank 2 holomorphic bundle W'^*: that is, the quasi-parabolic structure on W^* defines a natural surjective map $\mathcal{O}(W^*) \twoheadrightarrow \mathcal{S}$, where $\mathcal{S}$ is a sheaf supported at the branch points, and the kernel of this map is locally free analytic of rank 2 and so defines W'^*.

This construction of W' actually recovers $\tilde{E}$: there is a natural map $\mathcal{O}(W) \to \mathcal{O}(W')$ which induces an inclusion $L \hookrightarrow \pi^*W'$. Similarly there is an inclusion $\sigma^*L \hookrightarrow \pi^*W'$. As subbundles of π^*W', L and σ^*L coincide precisely on the ramification points so that there is a map $L \oplus \sigma^*L \to \pi^*W'$ which is an isomorphism away from the ramification points. It follows that

$\Lambda^2 W' = \Lambda^2 \tilde{E}$ and that $W' = \tilde{E}$. Moreover, at a marked point p the inclusion $L_{\pi^{-1}(p)} \hookrightarrow \pi^* \tilde{E}_{\pi^{-1}(p)} = \tilde{E}_p$ gives the quasi-parabolic structure and so we recover the original V-bundle E (see Proposition 5.2 and Lemma 5.5). We recover the Higgs field simply by defining $\hat{\phi}\colon \pi^*\tilde{E} \to \pi^*(\tilde{E} \otimes K_{\tilde{M}} \otimes_{i=1}^n L_{p_i})$ by $\hat{\phi}(e) = \mp se$ according as $v \in L$ or $v \in \sigma^* L$. Since this is σ-invariant it descends to define $\overline{\phi}'$ on $\tilde{M}$ - this is trace-free with determinant $\overline{q}'$ and recovers the old $\overline{\phi}'$. At a marked point p, we have $\ker(\hat{\phi}_{\pi^{-1}(p)}) = L_{\pi^{-1}(p)}$ and hence, in coordinates which respect the quasi-parabolic structure, the $(1,2)$-, $(2,2)$- and $(1,1)$-components of $\overline{\phi}'$ vanish at p to first order in w. Of course this is exactly the condition for $\overline{\phi}'$ to define $\tilde{\phi}$ via (5e) and to $\tilde{\phi}$ there corresponds a Higgs field ϕ on the V-bundle E.

Finally note that if there was an $\overline{\phi}'$-invariant subbundle L' then there would be a section $t \in H^0(K_{\tilde{M}} \otimes_{i=1}^n L_{p_i})$ such that for any $l \in L'$, $\overline{\phi}'(l) = tl$. Since $\overline{\phi}'$ is trace-free it would follow that $\overline{q}' = \det(\overline{\phi}') = -t^2$ - contradicting the assumption that $\overline{q}'$ has simple zeros. So $\overline{\phi}'$ has no invariant subbundles and the same is therefore true of $\tilde{\phi}$ and ϕ. □

Notice that this shows that a Higgs field in the generic fibre of det leaves no sub-V-bundle invariant (compare § 2B).

5D. - *The case* $g = n - n_0 = 1$

We briefly indicate how the preceding arguments can be modified to identify the generic fibre of the determinant map when $g = n - n_0 = 1$. We outline the argument working with V-bundles although the proofs again require translation to the parabolic case. As before we simplify the exposition by supposing that $n_0 = 0$ so that there is a single marked point $p = p_1$.

LEMMA 5.7. *If $g = n - n_0 = 1$ then every Higgs field has an invariant sub-V-bundle.*

PROOF. Since $h^0(K^2) = 1$ the natural squaring map $H^0(K) \to H^0(K^2)$ is surjective. Thus, given any Higgs field ϕ, $\det(\phi) = -s^2$ for some $s \in H^0(K)$. Consider $\theta_\pm = \phi \pm s$: if $\phi \not\equiv 0$ this is non-zero (if $\phi = 0$ then there is nothing to prove) but has determinant zero and so we have line V-bundles $L_\pm \hookrightarrow E$ with $L_\pm \subseteq \ker \theta_\pm$. Clearly $L_\pm$ are invariant, with ϕ acting on $L_\pm$ by multiplication by $\mp s$. □

Since the squaring map is surjective, Lemma 5.3 certainly can't hold in this case - we now consider any non-zero determinant to be 'generic'. Using Lemma 5.7 we see that any Higgs field with a generic (i.e. non-zero) determinant has two invariant sub-V-bundles L_+ and L_-.

Notice that $K = L_1^{\alpha_1 - 1}$ and so sections of K are multiples of the canonical section $s_1^{\alpha_1 - 1}$ and those of K^2 are multiples of $s_1^{2\alpha_1 - 2}$. Thus in (5d) $\tilde{\phi}_{11}(z^{\alpha_1})$ and exactly one of $\tilde{\phi}_{12}(z^{\alpha_1})$ and $\tilde{\phi}_{21}(z^{\alpha_1})$ are non-zero at the marked point, while the other must vanish to first order in $w = z^{\alpha_1}$. A small local calculation using (5d) shows that L_+ and L_- have the same isotropy; it is x if $\tilde{\phi}_{21}(0) = 0$ and x' if $\tilde{\phi}_{12}(0) = 0$. Hence $L_+L_- \cong \Lambda L_1^{x-x'}$ or $\Lambda L_1^{x'-x-\alpha}$, where the isotropy of $L_\pm$ is x in the first case and x' in the second. Using these and stability, we calculate that $c_1(L_\pm) = r/2 + x/\alpha$ or $(r-1)/2 + x'/\alpha$, respectively, where $c_1(E) = r + (x+x')/\alpha$. Notice that the parity of r determines the isotropy of $L_\pm$. Thus a point in the generic fibre gives a point not of a Prym variety but of the Jacobian $\mathrm{Jac}_0 M \cong T^2$ corresponding to L_+. Reversing the correspondence as in Theorem 5.6 yields the following result.

PROPOSITION 5.8. *If $g = n - n_0 = 1$ then for $q \in H^0(\check{M}, K^2_{\check{M}}) \setminus \{0\}$ the fibre $\det^{-1}(q)$ is biholomorphic to the Jacobian torus.*

5E. - *Non-stable V-Bundles in Fibres of the Determinant Map*

We have a natural inclusion of the cotangent bundle to the moduli of stable V-bundles in to the moduli of stable Higgs V-bundles and we would like to show, following [14, § 8], that in fact we have a fibrewise compactification with respect to the determinant map. Thus we need to analyse the fibres of the determinant map and check that, generically, the non-stable V-bundles form subvarieties of codimension at least 1. We wish to adapt Hitchin's argument here but there are additional complications and a new variant of the argument is needed in the special case $g = n - n_0 = 1$.

PROPOSITION 5.9. *Suppose that $2g + n - n_0 > 3$. For fixed, generic, $q \in H^0(\check{M}, K^2_{\check{M}})$ let* Prym$(\hat{M})$ *be the Prym variety which is the fibre of the determinant map (Theorem 5.6). Then the points of* Prym$(\hat{M})$ *corresponding to non-stable V-bundles form a finite union of subvarieties of codimension at least 1.*

PROOF. Suppose $n_0 = 0$ and consider $L_E \hookrightarrow E$ a destabilising sub-V-bundle, with $\widetilde{L_E} \hookrightarrow \tilde{E}$ *parabolically* destabilising (see [10] and § 5A) and $L' = \pi^* \widetilde{L_E}$. The outline of the argument is similar to that of [14, § 8] - with which we assume familiarity - but there are two problems. Firstly, a sufficient condition for lifts from $H^0(L'^* L^* \pi^* \Lambda^2 \tilde{E})$ to $H^0(L'^* \pi^* \tilde{E})$ to be unique is $H^0(L'^* L) = 0$ but this is not always the case if $g = 0$. However, *invariant* lifts will still be unique because $H^0(L'^* L) \hookrightarrow H^0(L'^* \pi^* \tilde{E})$ is moved by the involution σ. Secondly, because $\widetilde{L_E}$ is parabolic destabilising we can't fix the degree of $L'^* L^* \pi^* \Lambda^2 \tilde{E}$ in the same way that Hitchin does. Let the isotropy of L_E be specified by an isotropy vector (ϵ_i). A small computation with the stability condition shows

$$c_1(L'^* L^* \pi^* \Lambda^* \tilde{E}) \leq \sum_{i=1}^{n} \frac{\epsilon_i (x'_i - x_i)}{\alpha_i} + 2g - 2.$$

Since $L_{\pi^{-1}(p)}$ gives the flag which describes the quasi-parabolic structure at a marked point p, by Lemma 5.5, the subset of $\pi^{-1}(\{p_1, \ldots, p_n\})$ at which our section of $L'^*L^*\pi^*\Lambda^2\tilde{E}$ vanishes is just $\pi^{-1}(\{p_i\colon \epsilon_i = 1\})$. Hence, for given (ϵ_i), it is more natural to consider sections of $(\otimes_{\{i:\epsilon_i=1\}}L_i^*)L'^*L^*\pi^*\Lambda^2\tilde{E}$ and these correspond to divisors of degree less than or equal to $\sum_{i=1}^{n}(\epsilon_i(x_i' - x_i)/\alpha_i) - n_+ + 2g - 2$. For each (ϵ_i) (a finite number) we obtain a subvariety of the variety of effective divisors and correspondingly a subvariety of the Prym variety of codimension at least 1. □

PROPOSITION 5.10. *If $g = n - n_0 = 1$ then for $q \in H^0(K_{\tilde{M}}^2) \setminus \{0\}$ there are only a finite number of points in the fibre* $\det^{-1}(q)$ *corresponding to non-stable V-bundles.*

PROOF. Again, we consider a destabilising sub-V-bundle $L_E \hookrightarrow E$ and the corresponding parabolic bundle $\widetilde{L_E}$. Since $\widetilde{L_E}$ is parabolic destabilising $2c_1(\widetilde{L_E}) \geq c_1(\tilde{E}) + 1$ or $2c_1(\widetilde{L_E}) \geq c_1(\tilde{E})$, according to whether L_E has isotropy x or x'. Recall (from § 5D) that E has two ϕ-invariant sub-V-bundles $L_\pm$ and so is an extension $0 \to L_\pm \to E \to L_\pm^*\Lambda \to 0$. Set $r = c_1(\tilde{E})$. The discussion in § 5D also shows that if r is even then $c_1(\widetilde{L_\pm}) = r/2$, $L_\pm$ have isotropy x and $\widetilde{L_+}\widetilde{L_-} \cong \Lambda^2\tilde{E}$, while if r is odd then $c_1(\widetilde{L_\pm}) = (r-1)/2$, $L_\pm$ have isotropy x' and $\widetilde{L_+}\widetilde{L_-} \cong \Lambda^2\tilde{E}L_{p_1}^*$.

Consider the sequence of bundles

$$0 \to \widetilde{L_E}^*\widetilde{L_\pm} \to \widetilde{L_E}^*\tilde{E} \to \widetilde{L_E}^*\widetilde{L_\pm}^*\Lambda^2\tilde{E} \to 0.$$

and the first three terms of the associated cohomology long exact sequence. By assumption $H^0(\widetilde{L_E}^*\tilde{E})$ is non-zero so at least one of $\widetilde{L_E}^*\widetilde{L_+}$ and $\widetilde{L_E}^*\widetilde{L_+}^*\Lambda^2\tilde{E}$ must have a non-zero section and the same is true with L_- in place of L_+. If we had that $H^0(\widetilde{L_E}^*\widetilde{L_\pm}) \neq 0$ and $H^0(\widetilde{L_E}^*\widetilde{L_\pm}^*\Lambda^2\tilde{E}) = 0$ then the inclusion of $\widetilde{L_E}$ in $\tilde{E}$ would have to factor through that of $\widetilde{L_\pm}$, which is impossible as $\widetilde{L_\pm}$ does not destabilise. So $\widetilde{L_E}^*\widetilde{L_+}^*\Lambda^2\tilde{E}$ and $\widetilde{L_E}^*\widetilde{L_-}^*\Lambda^2\tilde{E}$ must have non-zero sections. However, considering cases according to the parity of r and the isotropy of L_E, we see that $c_1(\widetilde{L_E}^*\widetilde{L_\pm}^*\Lambda^2\tilde{E}) \leq 0$. It follows that a non-stable V-bundle occurs only if $\widetilde{L_E} \cong \widetilde{L_\pm}^*\Lambda^2\tilde{E}$. Since $\widetilde{L_+}\widetilde{L_-} \cong \Lambda^2\tilde{E}$ or $\widetilde{L_+}\widetilde{L_-} \cong \Lambda^2\tilde{E}L_{p_1}^*$, it follows that $\widetilde{L_+}^2 \cong \Lambda^2\tilde{E}$ or $\widetilde{L_+}^2 \cong \Lambda^2\tilde{E}L_{p_1}^*$. Hence, if a non-stable V-bundle occurs then $\widetilde{L_+}$ is one of the $2^{2g} = 4$ possible square roots of a given line bundle. □

6. - Representations and Higgs V-bundles

Throughout this section $E \to M$ is a complex rank 2 V-bundle over an orbifold Riemann surface of negative Euler characteristic. We also suppose that a fixed metric and Yang-Mills connexion, A_Λ, are given on Λ.

6A. - *Stable Higgs V-bundles and Projectively Flat Connexions*

Suppose that E is given a Higgs V-bundle structure with Higgs field ϕ, compatible with A_Λ. Given a Hermitian metric on E inducing the fixed metric on Λ, there is a unique Chern connexion A compatible with the holomorphic and unitary structures and inducing A_Λ on Λ. The metric also defines an adjoint of ϕ, ϕ^*. Set

$$D = \partial_A + \phi + \overline{\partial}_A + \phi^*;$$

this is a (non-unitary) connexion with curvature $F_D = F_A + [\phi, \phi^*]$ and D is projectively flat if and only if the pair (A, ϕ) is Yang-Mills-Higgs. The determinant-fixing condition on D is simply that it induces the fixed (*unitary*) Yang-Mills connexion A_Λ in Λ.

Conversely, given a connexion D (with fixed determinant) and a Hermitian metric on E, inducing the fixed metric on Λ, we can decompose D into its $(1,0)$- and $(0,1)$-parts; $D = \partial_1 + \overline{\partial}_2$. There are then uniquely defined operators $\overline{\partial}_1$ and ∂_2 (of types $(0,1)$ and $(1,0)$ respectively) such that $d_1 = \partial_1 + \overline{\partial}_1$ and $d_2 = \partial_2 + \overline{\partial}_2$ are unitary connexions. Define $\phi = (\partial_1 - \partial_2)/2$ and $d_A = (d_1 + d_2)/2$ so that $\overline{\partial}_A = (\overline{\partial}_1 + \overline{\partial}_2)/2$. Clearly (A, ϕ) is a Higgs pair if and only if $\overline{\partial}_A(\phi) = 0$, i.e. ϕ is holomorphic; if we define $D'' = \overline{\partial}_A + \phi$ then this condition becomes $D''^2 = 0$. Here D'' is a first order operator which satisfies the appropriate $\overline{\partial}$-Leibniz rule. Moreover, if $D''^2 = 0$ then (A, ϕ) is Yang-Mills-Higgs if and only if D has curvature $-\pi i\, c_1(\Lambda)\Omega I_E$.

From now on suppose that D has curvature $-\pi i\, c_1(\Lambda)\Omega I_E$. We call a Hermitian metric (with fixed determinant) TWISTED HARMONIC with respect to D if the resulting D''-operator satisfies $D''^2 = 0$. Using the fact that the curvature of D is $-\pi i\, c_1(\Lambda)\Omega I_E$, a small calculation shows that the condition for the metric to be twisted harmonic is $F_1 = F_2$, where F_i is the curvature of d_i, for $i = 1, 2$. If the metric is twisted harmonic then D'' defines a Higgs V-bundle with respect to which the metric is Hermitian-Yang-Mills-Higgs. Clearly the processes of passing from a Higgs V-bundle to a projectively flat connexion and vice-versa are mutually inverse and respect the determinant-fixing conditions.

We prove an existence result for twisted harmonic metrics, following [6]. The connexion D on E comes from a projectively flat connexion in the corresponding principal $GL_2(\mathbb{C})$ V-bundle P with $E = P \times_{GL_2(\mathbb{C})} \mathbb{C}^2$. Hence D determines a holonomy representation $\rho_D \colon \pi_1^V(M) \to PSL_2(\mathbb{C})$. Let Herm_2^+ denote the 2×2 positive-definite Hermitian matrices (with the metric described in [20, § VI.1]). The corresponding V-bundle of Hermitian metrics on E is just $H' = P \times_{GL_2(\mathbb{C})} \mathrm{Herm}_2^+$. Here $GL_2(\mathbb{C})$ acts on Herm_2^+ by $h \mapsto \overline{g}^T h g$, for $h \in \mathrm{Herm}_2^+$ and $g \in GL_2(\mathbb{C})$. This is an action of $PSL_2(\mathbb{C})$ and so H' is flat and can be written as $H' = H'_{\rho_D} = \mathcal{H}^2 \times_{\rho_D} \mathrm{Herm}_2^+$ (where $\mathcal{H}^2$ is the universal cover of M). A choice of Hermitian metric on E is a section of H'_{ρ_D} or a $\pi_1^V(M)$-equivariant map $\mathcal{H}^2 \to \mathrm{Herm}_2^+$ - is this map harmonic in the sense that it minimises energy among such maps?

Using the determinant-fixing condition, we suppose that the map to Herm_2^+ has constant determinant 1. We identify the subspace of $\mathrm{Herm}_2^+ \cong GL_2(\mathbb{C})/U(2)$

in which the image of the map lies with $SL_2(\mathbb{C})/SU(2) \cong \mathcal{H}^3$. So we consider sections of the flat $\mathcal{H}^3$ V-bundle $H_{\rho_D} = \mathcal{H}^2 \times_{\rho_D} \mathcal{H}^3$: the sections of H_{ρ_D} are precisely the types of map considered by Donaldson in [6]. The condition that a metric h be twisted harmonic will then be precisely that it is given by a harmonic $\pi_1^V(M)$-equivariant map $\hat{h}: \mathcal{H}^2 \to \mathcal{H}^3$.

Donaldson shows that the Euler-Lagrange condition for the map $\hat{h}$ to be harmonic is just $d_A^*(\phi+\phi^*) = 0$ and moreover that, at least in the smooth case and when ρ_D is irreducible, such a harmonic map always exists. This Euler-Lagrange condition agrees with our definition of a twisted harmonic metric. For the existence of such harmonic maps we either follow Donaldson's proof directly or argue equivariantly, as in § 3D, obtaining the following results.

PROPOSITION 6.1. *Let $\rho_D: \pi_1^V(M) \to PSL_2(\mathbb{C})$ be an irreducible representation and s_0 a section of the flat $\mathcal{H}^3$ V-bundle $H_{\rho_D} = \mathcal{H}^2 \times_{\rho_D} \mathcal{H}^3$. Then H_{ρ_D} admits a twisted harmonic section homotopic to s_0.*

COROLLARY 6.2. *Let Λ have a fixed Hermitian metric and compatible Yang-Mills connexion. Given an irreducible $GL_2(\mathbb{C})$-connexion D on E with curvature $-\pi \mathrm{i}\, c_1(\Lambda)\Omega I_E$ and fixed determinant, E admits a Hermitian metric of fixed determinant which is twisted harmonic with respect to D. Hence D determines a stable Higgs V-bundle structure on E with fixed determinant, for which the metric is Hermitian-Yang-Mills-Higgs.*

COROLLARY 6.3. *Let E have a fixed Hermitian metric and let Λ have a compatible Yang-Mills connexion. Let D be an irreducible $GL_2(\mathbb{C})$-connexion on E with curvature $-\pi \mathrm{i}\, c_1(\Lambda)\Omega I_E$ and fixed determinant. Then there is a complex gauge transformation $g \in \mathcal{G}^c$, of determinant 1, such that the fixed metric is twisted harmonic with respect to $g(D)$. Hence $g(D)$ determines a stable Higgs V-bundle structure on E with fixed determinant.*

To identify the space of such projectively flat connexions modulo gauge equivalence with our moduli space of Higgs V-bundles we have to consider the actions of the gauge groups and the question of irreducibility. We have the following result adapted from [14, Theorem 9.13 & Proposition 9.18].

PROPOSITION 6.4. *Let $E \to M$ be a complex rank 2 V-bundle with a fixed Hermitian metric and compatible Yang-Mills connexion on the determinant line V-bundle Λ. Then the following hold.*

1. *A Yang-Mills-Higgs pair (A, ϕ) (with fixed determinant) is irreducible if and only if the corresponding projectively flat $GL_2(\mathbb{C})$-connexion $D = \partial_A + \overline{\partial}_A + \phi + \phi^*$ is irreducible.*
2. *Two irreducible $GL_2(\mathbb{C})$-connexions on E with curvature $-\pi \mathrm{i}\, c_1(\Lambda)\Omega I_E$ (and fixed determinant), D and D', are equivalent under the action of $\mathcal{G}^c$ if and only if the corresponding Yang-Mills-Higgs pairs (A, ϕ) and (A', ϕ') are equivalent under the action of $\mathcal{G}$.*

6B. - *Projectively Flat Connexions and Representations*

In the smooth case projectively flat connexions are described by representations of a universal central extension of the fundamental group (see [14], also [1, § 6]). However over an orbifold Riemann surface there is in general no *one* central extension which will do [10, § 3] but the determinant-fixing condition tells us that the appropriate central extension to use is the fundamental group of the circle V-bundle $S(\Lambda)$. Let (y_i) $(0 \leq y_i \leq \alpha_i - 1)$ denote the isotropy of a line V-bundle L and let $b = c_1(L) - \sum_{i=1}^{n} (y_i/\alpha_i)$. The orbifold fundamental group of $S(L)$ is well-known (see, for instance, [10, § 2] and has presentation

$$\begin{aligned} \pi_1^V(S(L)) = \langle a_1, b_1, \ldots, a_g, b_g, q_1, \ldots, q_n, h \quad | \\ [a_j, h] = 1,\ [b_j, h] = 1,\ [q_i, h] = 1, \\ q_i^{\alpha_i} h^{y_i} = 1,\ q_1 \ldots q_n [a_1, b_1] \ldots [a_g, b_g] h^{-b} = 1 \rangle . \end{aligned} \tag{6a}$$

PROPOSITION 6.5. *Let $\Lambda \to M$ be a line V-bundle with a fixed Hermitian metric and compatible Yang-Mills connexion. Let $S(\Lambda)$ be the corresponding circle V-bundle. Then there is a bijective correspondence between*

1. *conjugacy classes of irreducible representations $\pi_1^V(S(\Lambda)) \to SL_2(\mathbb{C})$ such that the generator h in* (6a) *is mapped to $-I_2 \in SL_2(\mathbb{C})$ and*
2. *isomorphism classes of pairs (E, D), where E is a $GL_2(\mathbb{C})$ V-bundle with $\Lambda^2 E = \Lambda$ and D is an irreducible $GL_2(\mathbb{C})$ connexion on E with curvature $-\pi \mathrm{i}\, c_1(\Lambda)\Omega I_E$ and inducing the fixed connexion on Λ.*

PROOF. The proof can be carried over from [10, Theorem 4.1] (compare also [1, Theorem 6.7]) except that we need to replace $U(2)$ with $GL_2(\mathbb{C})$ at each stage - only the unitary structure on the determinant line V-bundle is necessary for the proof. □

Since Proposition 6.5 insists that h maps to $-I_2$, it is sufficient to consider a central $\mathbb{Z}_2$-extension rather than the central $\mathbb{Z}$-extension of $\pi_1^V(M)$ given by the presentation (6a) - this is equivalent to adding the relation $h^2 = 1$ to that presentation. Then it is only the *parity* of the integers y_i and b that matters. Something a little subtler is true. Recall Remark 3.6: it is sufficient to consider topological Λ's modulo the equivalence $\Lambda \sim \Lambda L^2$. Moreover, the topology of Λ is specified by the y_i's and b (Proposition 1.1) - write $\Lambda = \Lambda_{(b,(y_i))}$ to emphasise this. Clearly if $(b, (y_i)) \equiv (b', (y_i')) \pmod 2$ (meaning that the congruence holds componentwise) then $\Lambda_{(b,(y_i))} \sim \Lambda_{(b',(y_i'))}$. However, if α_i is odd then L can be chosen so that tensoring by L^2 brings about a change $y_i \mapsto y_i + 1$; if any α_i is even then a change $b \mapsto b+1$ is possible similarly. These equivalences correspond to group isomorphisms between the corresponding presentations (6a), with the added relation $h^2 = 1$. Thus we normalise the y_i's and b to find exactly one representative of each class, supposing that

$$\text{(6b)}\qquad y_i = \begin{cases} 0 & \text{if } \alpha_i \text{ is odd;} \\ 0,1 & \text{if } \alpha_i \text{ is even;} \end{cases} \qquad b = \begin{cases} 0 & \text{if at least one } \alpha_i \text{ is even;} \\ 0,1 & \text{if no } \alpha_i \text{ is even.} \end{cases}$$

This is equivalent to considering only the following SQUARE-FREE topological Λ's:

$$\text{(6c)}\qquad \Lambda \in \begin{cases} \{\otimes_{\alpha_i \,\mathrm{even}} L_i^{\delta_i} : \delta_i = 0, 1\} & \text{if at least one } \alpha_i \text{ is even;} \\ \{L^{\delta_0} : \delta_0 = 0, 1\} & \text{if no } \alpha_i \text{ is even,} \end{cases}$$

where L has no isotropy with $c_1(L) = 1$ and the L_i are the point V-bundles of § 1B.

An alternative way to understand these $\mathbb{Z}_2$-extensions of the fundamental group is as follows. Since $SL_2(\mathbb{C})$ double-covers $PSL_2(\mathbb{C})$ any representation $\rho_D : \pi_1^V(M) \to PSL_2(\mathbb{C})$ induces a central $\mathbb{Z}_2$-extension of $\pi_1^V(M)$:

$$\text{(6d)}\qquad 0 \to \mathbb{Z}_2 \to \Gamma \to \pi_1^V(M) \to 0.$$

Since the group of central $\mathbb{Z}_2$-extensions of $\pi_1^V(M)$ is discrete, the Γ thus induced is constant over connected components of the representation space.

So, given any ρ_D, we obtain an extension Γ: what invariants $(b, (y_i))$ characterise these Γ's and thus the central $\mathbb{Z}_2$-extensions of $\pi_1^V(M)$? The answer is that $(b, (y_i))$ can be supposed to have one of the normalised forms given by (6b) and so these parameterise the central $\mathbb{Z}_2$-extensions of $\pi_1^V(M)$. This is because the image of each generator of (6a) has exactly two possible lifts to $SL_2(\mathbb{C})$ except that h must map to $-I_2$: choosing lifts at random, the relations $q_i^{\alpha_i} h^{y_i} = 1$ and $q_1 \dots q_n [a_1, b_1] \dots [a_g, b_g] h^{-b} = 1$ of (6a) will be satisfied for exactly one choice of normalised $(b, (y_i))$. By our previous discussion, this is exactly equivalent to considering only the square-free Λ's of (6c).

As well as topological types of determinant line V-bundles we need to consider topological types of rank 2 V-bundles with the *same* determinant line V-bundle - Proposition 6.5 deals with all topological types of V-bundles with the same determinant line V-bundle simultaneously. These types can be determined following the ideas of [10, § 4], as follows. The various topological types are distinguished by the rotation numbers associated to the images of the elliptic generators q_i of the presentation (6a). By this we mean that the image of q_i has conjugacy class described by the roots of its characteristic polynomial, necessarily of the form $e^{\pi \mathrm{i} r_i/\alpha_i}$, $e^{-\pi \mathrm{i} r_i/\alpha_i}$, for $0 \le r_i \le \alpha_i$; these r_i are the ROTATION NUMBERS. Notice that the relation $q_i^{\alpha_i} h^{y_i} = 1$ means that r_i has the same parity as y_i and this is the only *a priori* restriction on r_i. Call an abstract set of rotation numbers (r_i) COMPATIBLE WITH Λ if r_i has the same parity as y_i. The result is the following and the proof, using Proposition 1.1, is easy.

LEMMA 6.6. *The topological types of $GL_2(\mathbb{C})$ V-bundles E with fixed determinant constructed in Proposition 6.5 correspond to the rotation numbers r_i associated to the images of the elliptic generators q_i of the presentation* (6a).

Denote the space of representations of $\pi_1^V(S(\Lambda))$ into $SL_2(\mathbb{C})$, sending the generator h of (6a) to $-I_2$, by $\mathrm{Hom}^{-1}(\pi_1^V(S(\Lambda)), SL_2(\mathbb{C}))$ and the *irreducible* representations by $\mathrm{Hom}^{*,-1}(\pi_1^V(S(\Lambda)), SL_2(\mathbb{C}))$, for a fixed line V-bundle Λ. For any set of rotation numbers (r_i) (with $0 \leq r_i \leq \alpha_i$ and $r_i \equiv y_i \pmod 2$) we have a corresponding subset $\mathrm{Hom}_{(r_i)}^{-1}(\pi_1^V(S(\Lambda)), SL_2(\mathbb{C}))$ and, by Proposition 6.5 and the results of § 6A, a bijection between $\mathrm{Hom}_{(r_i)}^{*,-1}(\pi_1^V(S(\Lambda)), SL_2(\mathbb{C}))/SL_2(\mathbb{C})$ and the moduli space of stable Higgs V-bundles (with fixed determinants) on the topological E corresponding to the rotation numbers (Lemma 6.6).

The representation space $\mathrm{Hom}_{(r_i)}^{*,-1}(\pi_1^V(S(\Lambda)), SL_2(\mathbb{C}))/SL_2(\mathbb{C})$ can be thought of as the quotient of a set of $2g+n$ matrices subject to conditions corresponding to the relations of (6a) and so has a natural topology; whether this description makes it into a smooth manifold is by no means immediate. Therefore we use the bijection with the moduli space of stable Higgs V-bundles, which is easily seen to be a homeomorphism, to define a manifold structure on this representation space. In summary we have the following theorem.

THEOREM 6.7. *Let M be an orbifold Riemann surface with negative Euler characteristic. Let Λ be a fixed line V-bundle over M and (r_i) a set of rotation numbers compatible with Λ. Then the representation space* $\mathrm{Hom}_{(r_i)}^{*,-1}(\pi_1^V(S(\Lambda)), SL_2(\mathbb{C}))/SL_2(\mathbb{C})$ *is a complex manifold of dimension* $6(g-1)+2(n-n_0)$, *where n_0 is the number of rotation numbers congruent to* 0 $\pmod{\alpha}$.

REMARK 6.8. In Remark 3.6 we noted that twisting by a non-trivial topological root L induces a map $\mathcal{M}(E, A_\Lambda) \leftrightarrow \mathcal{M}(E \otimes L, A_\Lambda)$, preserving the topology of Λ but altering that of E. On the level of representations there is an equivalent map. Given any element $\hat{\rho}_D \in \mathrm{Hom}_{(r_i)}^{-1}(\pi_1^V(S(\Lambda)), SL_2(\mathbb{C}))$ we can obtain a representation with different rotation numbers and covering the same $PSL_2(\mathbb{C})$-representation, by altering the signs of the images of certain of the generators of (6a). We can change the sign of $\hat{\rho}_D(q_i)$ (bringing about a change of rotation number $b_i \mapsto \alpha_i - b_i$) provided α_i is even and provided an even number of such changes is made - these conditions preserve the relations $q_i^{\alpha_i} h^{y_i} = 1$ and $q_1 \dots q_n [a_1, b_1] \dots [a_g, b_g] h^{-b} = 1$.

When there are no reducible points we can apply, among other results, Proposition 3.7 and Corollary 4.2. By Lemma 6.6 we can discuss the existence of reducible points in terms of the rotation numbers. (Either Λ or a specific set of rotation numbers may provide an obstruction to the existence of reductions.) The discussion in § 3C shows that reductions exist if and only if there exists an isotropy vector (ϵ_i) such that

$$\sum_{i=1}^{n} \frac{\epsilon_i(x_i' - x_i) + (x_i' + x_i)}{\alpha_i} \equiv c_1(\Lambda) \pmod 2.$$

A small calculation expresses this in terms of the rotation numbers. Thus we obtain the following result.

PROPOSITION 6.9. *Let M be an orbifold Riemann surface with negative Euler characteristic. Let Λ be a fixed line V-bundle over M with isotropy (y_i) and $c_1(\Lambda) = b+\sum_{i=1}^{n}(y_i/\alpha_i)$. Let (r_i) be a compatible set of rotation numbers. Then the representation space $\mathrm{Hom}^{-1}_{(r_i)}(\pi_1^V(S(\Lambda)), SL_2(\mathbb{C}))/SL_2(\mathbb{C})$ contains reducible points if and only if there exists an isotropy vector (ϵ_i) such that*

$$\sum_{i=1}^{n} \frac{\epsilon_i r_i}{\alpha_i} \equiv b \pmod 2.$$

When no reducible points exist the complex manifold $\mathrm{Hom}^{-1}_{(r_i)}(\pi_1^V(S(\Lambda)), SL_2(\mathbb{C}))/SL_2(\mathbb{C})$

1. *admits a complete hyper-Kähler metric and*
2. *is connected and simply-connected.*

6C. - *Real Representations*

In the previous subsection we discussed $SL_2(\mathbb{C})$-representations of central extensions of the orbifold fundamental group. Here we study the submanifold of $SL_2(\mathbb{R})$-representations. First notice that any irreducible representation into $SL_2(\mathbb{C})$ can fix at most one disk $\mathcal{H}^2 \subset \mathcal{H}^3$ because the intersection of two fixed disks would give a fixed line and hence define a reduction of the representation. Moreover, any representation which does fix a disk can be conjugated to a real representation and the conjugation action of $SL_2(\mathbb{C})$ then reduces to that of $SL_2(\mathbb{R})$.

Now consider the action of complex conjugation on a representation. Recall that, via Proposition 6.5 and Corollary 6.2, irreducible representations correspond to stable Higgs V-bundles. Note that $\pi_1^V(S(\Lambda))$ and $\pi_1^V(S(\emptyset\Lambda))$ are isomorphic via the map $h \mapsto h^{-1}$: the following proposition follows, exactly as in [27].

PROPOSITION 6.10. *Let E be a complex rank 2 V-bundle such that Λ has a fixed Hermitian metric and compatible Yang-Mills connexion. Let $\hat{\rho}_D: \pi_1^V(S(\Lambda)) \to SL_2(\mathbb{C})$ be an irreducible representation, sending h to $-I_2$, with corresponding stable Higgs V-bundle structure on E, (E_A, ϕ). Then the complex conjugate representation (thought of as a representation of $\pi_1^V(S(\emptyset\Lambda))$) determines a Higgs V-bundle structure on $\overline{E}$, isomorphic to $(E_A, -\phi)^*$.*

COROLLARY 6.11. *Let E be a complex rank 2 V-bundle such that Λ has a fixed Hermitian metric and compatible Yang-Mills connexion. Let $\hat{\rho}_D$ be an irreducible real representation $\hat{\rho}_D: \pi_1^V(S(\Lambda)) \to SL_2(\mathbb{R})$, sending h to $-I_2$, with corresponding Higgs V-bundle structure (E_A, ϕ). Then there is an isomorphism of Higgs V-bundles $(E_A, \phi) \cong (E_A, -\phi)$.*

Consider the involution on the moduli space of stable Higgs V-bundles (with fixed unitary structure and determinants) defined by $\sigma:(E,\phi)\mapsto(E,-\phi)$, where now E denotes a holomorphic V-bundle and (E,ϕ) is a stable Higgs V-bundle. The fixed points of σ can be determined much as the fixed points of the circle action were in the proof of Theorem 4.1. If (E,ϕ) is itself fixed then $\phi=0$ and E is a stable V-bundle. Suppose now that $\phi\not\equiv 0$. If (E,ϕ) is only fixed up to complex gauge-equivalence then we have an element $g\in\mathcal{G}^c$ such that $g(E,\phi)=(E,-\phi)$. Since g fixes E it must fix the Chern connexion A and since g cannot be a scalar it leads to a reduction of A to a direct sum of $U(1)$-connexions. Hence we have a holomorphic decomposition $E=L\oplus L^*\Lambda$, where, without loss of generality, we may suppose that $2c_1(L)-c_1(\Lambda)\geq 0$. Since (A,ϕ) is an irreducible pair, g must have order 2 in $\mathcal{G}^c$ and fix A. It follows that with respect to this decomposition (or, if A has stabiliser $SU(2)$, *choosing* a decomposition) we can write

$$g=\pm\begin{pmatrix} \mathrm{i} & 0\\ 0 & -\mathrm{i}\end{pmatrix} \quad\text{and}\quad \phi=\begin{pmatrix} t & u\\ v & -t\end{pmatrix}.$$

(Since our Higgs V-bundle is stable, we must have v non-zero.) Calculating the conjugation-action of g on ϕ we find that $t=0$.

Recall that we chose L with $2c_1(L)-c_1(\Lambda)\geq 0$ but to avoid semi-stable points (when $u=0$) we suppose that there is strict inequality. Exactly as in the proof of Theorem 4.1 we consider the topological possibilities $L=L_{(m,(\epsilon_i))}$: we can have any $(m,(\epsilon_i))$ such that $2c_1(L)>c_1(\Lambda)$ and $c_1(\widetilde{KL^{-2}\Lambda})=r\geq 0$. Then the possible holomorphic structures and the values of v, modulo the $\mathbb{C}^*$ automorphism group, are given by the effective (integral) divisors of order r and taking square roots. A difference from Theorem 4.1 is that u needn't be zero; indeed, u can take any value in $H^0(KL^2\Lambda^*)$. We obtain the following result, where l is defined as in § 1A.

PROPOSITION 6.12. *Let M be an orbifold Riemann surface of negative Euler characteristic and suppose that $E\to M$ admits no reducible Yang-Mills-Higgs pairs. Then the fixed points of the involution induced on $\mathcal{M}(E,A_\Lambda)$ by the mapping $(A,\phi)\mapsto(A,-\phi)$ consist of complex $(3g-3+n-n_0)$-dimensional submanifolds $\mathcal{M}_0$ and $\mathcal{M}_{(m,(\epsilon_i))}$, for every integer m and isotropy vector (ϵ_i) such that*

$$l<2m+\sum_{i=1}^{n}\frac{\epsilon_i(x_i'-x_i)}{\alpha_i}\leq l+2g-2+\sum_{i=1}^{n}\frac{\epsilon_i(x_i'-x_i)}{\alpha_i}+n_-.$$

The manifold $\mathcal{M}_0$ is the moduli space of stable V-bundles with fixed determinants, while $\mathcal{M}_{(m,(\epsilon_i))}$ is a rank $(2m-l+g-1+n_+)$ vector-bundle over a 2^{2g}-fold covering of $S^r\tilde{M}$, where $r=l-2m+2g-2+n_-$.

We interpret this as a result about $PSL_2(\mathbb{R})$-representations of $\pi_1^V(M)$. Again, a representation ρ_D of $\pi_1^V(M)$ into $PSL_2(\mathbb{R})$ induces a central $\mathbb{Z}_2$-extension Γ of $\pi_1^V(M)$, as in (6d), which is just $\pi_1^V(S(\Lambda))$ with the added relation $h^2 = 1$, for some square-free Λ. Consider the points of $\mathrm{Hom}^{-1}(\pi_1^V(S(\Lambda)), SL_2(\mathbb{R}))$ covering ρ_D. On the level of representations there are 2^{2g+n_2-1} (or 2^{2g} if $n_2 = 0$) choices of sign for the images of certain generators and these correspond to twisting a stable Higgs V-bundle by any of the 2^{2g+n_2-1} (or 2^{2g}) holomorphic roots of the trivial line V-bundle. In particular, if $n_2 \geq 1$ then the topology of the associated E is only determined up to twisting by the 2^{n_2-1} non-trivial topological roots (see Remark 6.8).

Excluding the topologically non-trivial roots, we have an action of $\mathbb{Z}_2^{2g}$ on the fixed point submanifolds of Proposition 6.12 which is easily seen to be free if E admits no reducible Yang-Mills-Higgs pairs. Moreover, even when E admits reducibles there will be fixed submanifolds $\mathcal{M}_{(m,(\epsilon_i))}$ with

$$l \leq 2m + \sum_{i=1}^{n} \frac{\epsilon_i(x_i' - x_i)}{\alpha_i} \leq l + 2g - 2 + \sum_{i=1}^{n} \frac{\epsilon_i(x_i' - x_i)}{\alpha_i} + n_-,$$

exactly as in Proposition 6.12, and the actions of $\mathbb{Z}_2^{2g}$ on these will be free provided the first inequality is strict.

The quantity $2m - l + \sum_{i=1}^{n} \{\epsilon_i(x_i' - x_i)/\alpha_i\} = 2c_1(L_{(m,(\epsilon_i))}) - c_1(\Lambda)$ is just the Euler class of the flat $\mathbb{RP}^1$ V-bundle $S(\rho_D) = S(L^2_{(m,(\epsilon_i))}\Lambda^*)$ associated to the $PSL_2(\mathbb{R})$-representation (this is well-defined as it is invariant under twisting E by non-trivial topological roots). Note that, just as it is possible to have topologically distinct line V-bundles with the same Chern class, it is possible to have topologically distinct $\mathbb{RP}^1$ V-bundles with the same Euler class - they are distinguished by their isotropy. The central $\mathbb{Z}$-extensions of $\pi_1^V(M)$ induced by the universal covering $\widetilde{PSL_2}\mathbb{R} \to PSL_2\mathbb{R}$ are just the (orbifold) fundamental groups of the flat $\mathbb{RP}^1$ V-bundles $S(\rho_D)$ (see [17]). Using the above discussion and the method of Propositon 6.12, we obtain the following result (compare [17]) and, as a corollary, a Milnor-Wood inequality.

PROPOSITON 6.13. *Let M be an orbifold Riemann surface of negative Euler characteristic. For ρ_D a $PSL_2(\mathbb{R})$)-representation of $\pi_1^V(M)$ let $\mathrm{Hom}_{\rho_D}(\pi_1^V(M), PSL_2(\mathbb{R}))$ denote the corresponding connected component. Let (y_i) be the isotropy and $b + \sum_{i=1}^{n}(y_i/\alpha_i)$ the Euler class of the associated flat $\mathbb{RP}^1$ V-bundle $S(\rho_D)$. Provided $b + \sum_{i=1}^{n}(y_i/\alpha_i) > 0$,*

$\mathrm{Hom}_{\rho_D}(\pi_1^V(M), PSL_2(\mathbb{R}))/PSL_2(\mathbb{R})$ *is a smooth complex* $(3g-3+n-n_0)$*-dimensional manifold, diffeomorphic to a rank* $(g-1+b+n-n_0)$ *vector-bundle over* $S^{2g-2-b}\tilde{M}$.

COROLLARY 6.14. *Let* M *be an orbifold Riemann surface of negative Euler characteristic. Then the Euler class* $b+\sum_{i=1}^{n}(y_i/\alpha_i)$ *of any flat* $PSL_2(\mathbb{R})$ V*-bundle satisfies*

$$\left|b+\sum_{i=1}^{n}\frac{y_i}{\alpha_i}\right| \leq 2g-2+n-\sum_{i=1}^{n}\frac{1}{\alpha_i}.$$

PROOF. In Proposition 6.13 we must have $b \leq 2g-2$. The result follows since $y_i \leq \alpha_i - 1$. □

6D. - *Teichmüller Space for Orbifold Riemann Surfaces*

Assume, as usual, that M is an orbifold Riemann surface of negative Euler characteristic. For a Fuchsian group such as $\pi_1^V(M)$, Teichmüller space, denoted $\mathcal{T}(M)$, is the space of faithful representations onto a discrete subgroup of $PSL_2\mathbb{R}$ modulo conjugation (see Bers's survey article [3]). Our previous results allow us to identify Teichmüller space with a submanifold of the moduli space.

Let $\mathcal{T}_{-4}(M)$ denote the space of orbifold Riemannian metrics of constant sectional curvature -4, modulo the action of the group of diffeomorphisms homotopic to the identity, $\mathcal{D}_0(M)$. There is a bijection between $\mathcal{T}_{-4}$ and $\mathcal{T}$ as each metric of constant negative curvature determines an isometry between the universal cover of M and $\mathcal{H}^2$ and hence a faithful representation of $\pi_1^V(M)$ onto a discrete subgroup of $PSL_2\mathbb{R}$ and conversely each such representation realises M as a geometric quotient of $\mathcal{H}^2$.

The results of [17], as well as those of [14, § 12], suggest that Teichmüller space is the component of the real representation space taking the extreme value in the Milnor-Wood inequality, Corollary 6.14. Working with the holomorphic description, the results of the previous subsection show that the extreme is achieved when $E = L \oplus L^*\Lambda$ with $L^2\Lambda^*$ having the topology of K and a holomorphic structure such that $\widetilde{KL^{-2}}\Lambda$ has sections: in other words we must have $L^2\Lambda^* = K$ (holomorphically). We suppose then that $E = K \oplus 1$ ($\Lambda^2 E$ can be normalised to be square-free but this is not necessary). The corresponding Higgs field is just

$$\phi = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix},$$

where $u \in H^0(K^2)$ and $v \in \mathbb{C}\setminus\{0\}$. There is a $\mathbb{C}^*$-group of automorphisms so that we can normalise with $v = 1$.

Exactly as in [14, Theorem 11.2], we can identify Teichmüller space with the choices of u i.e. with $H^0(K^2)$. The two preliminaries which we need are the strong maximum principle for orbifolds (the proof is entirely local and generalises immediately; see [16]) and the following orbifold version of a theorem of Sampson [7].

PROPOSITION 6.15. *Given two orbifold Riemannian metrics of constant sectional curvature -4 on M, h and h', there is a unique element of $\mathcal{D}_0$ which is a harmonic map between (M,h) and (M,h').*

PROOF. This is a reformulation of Proposition 6.1. The metrics h and h' give two discrete, faithful representations of $\pi_1^V(M)$ into $PSL_2\mathbb{R}$, one of which we consider fixed and the other we denote ρ'. The identity map on M lifts to an orientation-preserving diffeomorphism g of $\mathcal{H}^2$ which is equivariant with respect to the actions of the two representations. Taking this g as an initial section of the V-bundle $H_{\rho_D} = \mathcal{H}^2 \times_{\rho'} \mathcal{H}^3$ of Proposition 6.1 (via the inclusion $\mathcal{H}^2 \subset \mathcal{H}^3$) we obtain a harmonic section g' homotopic to g. This is real and defines a harmonic diffeomorphism between (M,h) and (M,h'). As g' is homotopic to g the resulting harmonic diffeomorphism is homotopic to the identity. Uniqueness follows either by a direct argument or from uniqueness over $\hat{M}$, where $\hat{M}$ is as in Corollary 3.9. □

We obtain the following theorem, which agrees with classical results due to Bers and others [3].

THEOREM 6.16. *Let M be an orbifold Riemann surface of negative Euler characteristic. Let $\mathcal{T}(M)$ be the Teichmüller space of the Fuchsian group $\pi_1^V(M)$ and $\mathcal{T}_{-4}(M)$ the space of orbifold Riemannian metrics on M of constant sectional curvature -4, modulo the action of the group of diffeomorphisms homotopic to the identity. Then $\mathcal{T}(M)$ and $\mathcal{T}_{-4}(M)$ are homeomorphic to $H^0(K^2)$, the space of holomorphic (orbifold) quadratic differentials on M. Hence Teichmüller space is homeomorphic to $\mathbb{C}^{3g-3+n}$.*

We conclude by considering orbifold Riemannian metrics in greater detail. Considered as a metric on the underlying Riemann surface, $\tilde{M}$, an orbifold Riemannian metric h on M has 'conical singularities' at the marked points. To see this recall that locally M is like $D^2/\mathbb{Z}_\alpha$ with h a $\mathbb{Z}_\alpha$-equivariant metric on D^2. If $c_h(r)$ denotes the circumference of a geodesic circle of radius r about the origin in D^2 (with respect to h), then $\lim_{r\to 0}(c_h(r)/r) = 2\pi$. Since this circle covers a circle in $D^2/\mathbb{Z}_\alpha$ exactly α times the metric on the quotient has a CONICAL SINGULARITY at the origin, with CONE ANGLE $2\pi/\alpha$.

Consider a Riemannian metric on M which, near a marked point $D^2/\mathbb{Z}_\alpha$, is compatible with the complex structure and so has the form $h(z)dz \otimes d\bar{z}$. If we set $w = z^\alpha$, then w is a local holomorphic coordinate on $\tilde{M}$. We find that the resulting 'Riemannian metric' on $\tilde{M}$ is given by

$$\frac{h(w^{1/\alpha})}{\alpha^2|w|^{2(1-1/\alpha)}}\, dw \otimes d\overline{w}.$$

Notice that $h(w^{1/\alpha})$ is well-defined by the $\mathbb{Z}_\alpha$-equivariance of h. This 'Riemannian metric' has a singularity like $|w|^{-2(1-1/\alpha)}$ at the origin and is compatible with the complex structure away from there. Hence we obtain a compatible 'singular Riemannian metric' on $\tilde{M}$: the induced metric on $\tilde{M}$ is continuous and induces the standard topology.

How does such a singular Riemannian metric compare with a (smooth) Riemannian metric on $\tilde{M}$? Suppose that g is a fixed Riemannian metric on $\tilde{M}$, compatible with the complex structure. Since $\tilde{M}$ is compact any two Riemannian metrics give metrics on $\tilde{M}$ which are mutually bounded and so will be equivalent for our purposes - we may as well use the Euclidean metric in any local chart. Now, h and g will give mutually bounded metrics on any compact subset of $M \setminus \{p_1, \dots, p_n\}$. However, for small Euclidean distance r from p, the singular metric has distance like $r^{1/\alpha}$. These are exactly the types of singularities of metrics considered by McOwen and Hulin-Troyanov in [21, 15]: they consider metrics which satisfy $h/g = O(r_g^{2k})$ as $r_g(z) = d_g(0, z) \to 0$, for some $k \in (-1, \infty)$. As McOwen points out, our 'singular Riemannian metrics' have exactly this form with $k = -1 + 1/\alpha$. Interpreting Corollary 3.4 in the light of this discussion we obtain the following result. (Our result is weaker than McOwen's since we consider only $k = -1 + 1/\alpha$ but the case of general $k \in (-1, \infty)$ can be obtained by a limiting argument as in [23].)

THEOREM 6.17 (McOwen, Hulin-Troyanov). *Let $\tilde{M}$ be a Riemann surface with marked points $\{p_1, \dots, p_n\}$ with orders of isotropy $\{\alpha_1, \dots, \alpha_n\}$. If the genus g and orders of isotropy satisfy*

$$2 - 2g - n + \sum_{i=1}^{n} 1/\alpha_i < 0$$

then $\tilde{M} \setminus \{p_1, \dots, p_n\}$ admits a unique compatible Riemannian metric h of constant sectional curvature -4 such that, for $i = 1, \dots, n$, h has a conical singularity at p_i with cone angle $2\pi/\alpha_i$.

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