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Periods and Duality of p-adic Barsotti-Tate Groups

MAURIZIO CANDILERA - VALENTINO CRISTANTE

Introduction

Let k be a perfect field of characteristic $p \neq 0$, $A = \mathbf{W}(k)$ the ring of Witt vectors with components in k, K the field of fractions of A, C the completion of the algebraic closure of K, A_C the ring of integers of C, G a Barsotti-Tate group over A, and $\tilde{G}$ its dual.

In this paper we propose a new method for computing the periods of the elements of $H^1_{dR}(G)$ against the elements of $T(G)$, the Tate module of G. The construction we use, which is based on the Witt realization of B-T groups, has two pleasant features: first, it provides a natural habitat for the periods; second, it allows us to compare the main theories used until now to treat periods.

Now we shall give a brief description of the main ingredients in our recipe: denote by $\mathfrak{R}_k$ the completion of the direct limit $\mathfrak{R}_k^0 = \varinjlim(R_k \to R_k \to \ldots)$, where R_k is the affine algebra of G_k, the special fiber of G, and the maps are induced by the multiplication by p in G_k. The bialgebra $\mathfrak{R}_k$ represents the Tate space in the following precise sense:

$$V(G)(A_C) \cong \mathrm{Hom}^{\mathrm{cont}}_{A\text{-alg}}(\mathfrak{R}_k, \mathcal{R}),$$

where $\mathcal{R} = \varprojlim(A_C/pA_C \leftarrow A_C/pA_C \leftarrow \ldots)$, and all the maps are defined by $x \to x^p$. Now consider $\mathbf{biv}\,\mathfrak{R}_k$, the K-module of Witt bivectors with components in $\mathfrak{R}_k$ (the definition of $\mathbf{biv}$ is given in 6.1); having fixed a point P of $V(G)(A_C)$ and an element ξ of $\mathbf{biv}\,\mathfrak{R}_k$, the identification given above allows us to define the bivector $\xi(P)$ as $\mathbf{biv}(P)(\xi)$; in this way the elements of $\mathbf{biv}\,\mathfrak{R}_k$ become $\mathbf{biv}\,\mathcal{R}$-valued functions on $V(G)(A_C)$.

Let $\mathcal{U}(G)$ be the universal vectorial extension of G; the Witt realization gives a canonical embedding of $\mathcal{E}(R)$, the affine algebra of $\mathcal{U}(G)$, into $\mathbf{W}(\mathfrak{R}_k)$. After this immersion, all the objects we need can be embedded in $\mathbf{biv}\,\mathfrak{R}_k$,

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or possibly $\mathbf{biv}\,\mathfrak{R}_k\hat{\otimes}\mathbf{W}(\mathcal{R})$ and, once inside $\mathbf{biv}\,\mathfrak{R}_k$, objects that usually are canonically isomorphic become equal. For instance, $t^*_{\mathcal{U}(G)}(K)$, the cotangent space at the identity of $\mathcal{U}(G)$, can be so embedded, and similarly for the Dieudonné module $M(G_k)$, for $H^1_{dR}(G)$, and for $I_2(G)$, the module of integrals of the second kind of G. Inside $\mathbf{biv}\,\mathfrak{R}_k$ we *really* have

$$t^*_{\mathcal{U}(G)}(K) = M(G_k) \otimes K = H^1_{dR}(G) \otimes K.$$

Hence, the map

$$p : (H^1_{dR}(G) \otimes K) \times V(G)(A_C) \to \mathbf{biv}\,\mathcal{R},$$

defined by $(\eta, P) \to \eta(P)$ makes perfect sense. This is our period pairing.

We stress that $\mathbf{biv}\,\mathcal{R}$ is the natural habitat of our periods because, after Witt realization, $H^1_{dR}(G) \otimes K$ lives inside $\mathbf{biv}\,\mathfrak{R}_k$.

We observe that $\mathbf{biv}\,\mathcal{R}$ is a K-module, not a ring; to get a ring we have to put a suitable topology on $\mathbf{biv}\,\mathcal{R}$ and then take the completion; the complete ring so obtained will be denoted by $\mathbf{Biv}\,\mathcal{R}$. Later we will give more details about this point: here, in favor of $\mathbf{biv}\,\mathcal{R}$ we remark also that, to compare Tate and Dieudonné modules, we do not need $\mathbf{Biv}\,\mathcal{R}$; $\mathbf{biv}\,\mathcal{R}$ is quite appropriate (cf. th. 4.1).

At this point it is easy to explain why the method of integration of differential forms of the second kind (as introduced by Coleman ([CO]) and later by Colmez ([COL])) and the method used by Fontaine ([FO1]) give the same results. In fact if $\xi \in \mathbf{biv}\,\mathfrak{R}_k$ is an integral of the second kind, or a lifting of a covector which represents such an integral, and if *pi* denotes the endomorphism of $\mathbf{biv}\,\mathfrak{R}_k$ coming from the multiplication by p, then $\eta = \lim_{n\to\infty} p^n(pi)^{-n}\xi$ is the element of $H^1_{dR}(G)$ represented by ξ. Since both Colmez and Fontaine define $\eta(P)$ by means of $\lim_{n\to\infty}(p^n(pi)^{-n}\xi)(P))$ (cf. Remarks 3.13 and 3.14) it is clear that their results coincide and also coincide with ours.

Now we wish to sketch the relations between p and the Tate pairing. If $\xi \in V(\tilde{G})(A_C)$ and $P \in T(G)(A_C)$, then P is identified with a *multiplicative* element of $\tilde{\mathfrak{R}}_k\hat{\otimes}\mathcal{R}$; so, recalling that

$$V(\tilde{G})(A_C) \cong \mathrm{Hom}^{\mathrm{cont}}_{A\mathrm{alg}}(\tilde{\mathfrak{R}}_k, \mathcal{R}) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{R}\mathrm{alg}}(\tilde{\mathfrak{R}}_k\hat{\otimes}\mathcal{R}, \mathcal{R}),$$

ξ becomes a function on $T(G)(A_C)$, and $\xi(P)$ makes perfect sense: this is the Tate pairing. Since we prove that if $\{\xi(P)\}$ denotes the Teichmüller representative of $\xi(P)$ in $\mathbf{W}(\mathcal{R})$ then the series

$$\log\{\xi(P)\} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\{\xi(P)\} - 1)^n}{n}$$

converges in $\mathbf{biv}\,\mathcal{R}$ (cf. 7.3), we have an additive pairing,

$$\ell : V(\tilde{G})(A_C) \times T(G)(A_C) \to \mathbf{biv}\,\mathcal{R},$$

defined by $(\xi, P) \to \log\{\xi(P)\}$: this is the additive version of the Tate pairing.

The link between ℓ and p passes through the universal vectorial extension. Let us explain this point: we prove that $V(\tilde{G})(A_C) \cong \mathcal{U}(\tilde{G})(\mathbf{W}(\mathcal{R}))$, canonically and functorially with respect to G (cf. 2.5 and 2.7); moreover the duality between G and $\tilde{G}$ gives a canonical pairing between the tangent spaces $t_{\mathcal{U}(\tilde{G})}(\mathbf{W}(\mathcal{R}))$ and $t_{\mathcal{U}(G)}(\mathbf{W}(\mathcal{R}))$, so that we can assume

$$t_{\mathcal{U}(\tilde{G})}(\mathbf{W}(\mathcal{R})) = t^*_{\mathcal{U}(G)}(\mathbf{W}(\mathcal{R})) = H^1_{dR}(G) \otimes \mathbf{W}(\mathcal{R}).$$

Now, by composing these identifications with the logarithm of $\mathcal{U}(\tilde{G})$, we have the map $L : V(\tilde{G})(A_C) \to H^1_{dR}(G) \otimes \mathbf{Biv}\,\mathcal{R}$. The final point of our comparison shows that the following diagram commutes:

$$\begin{array}{ccc} V(\tilde{G})(A_C) \times T(G)(A_C) & \xrightarrow{\ell} & \mathbf{Biv}\,\mathcal{R} \\ \downarrow L \times i & & \| \\ H^1_{dR}(G) \otimes \mathbf{Biv}\,\mathcal{R} \times V(G)(A_C) & \xrightarrow{p} & \mathbf{Biv}\,\mathcal{R}, \end{array}$$

i.e. $[L(\xi)](P) = \log\{\xi(P)\}$.

Another significant point is the following: we prove that the homomorphism from $\mathcal{U}(\tilde{G})(\mathbf{W}(\mathcal{R}))$ to $\tilde{G}(A_C)$ obtained by composing the canonical projection $\mathcal{U}(\tilde{G})(\mathbf{W}(\mathcal{R})) \to \tilde{G}(\mathbf{W}(\mathcal{R}))$ with the map $\Theta_0 : \mathbf{W}(\mathcal{R}) \to A_C$, as defined in [FO3], coincides, after the identification $V(\tilde{G})(A_C) \cong \mathcal{U}(\tilde{G})(\mathbf{W}(\mathcal{R}))$, with the canonical projection $V(\tilde{G})(A_C) \to \tilde{G}(A_C)$, so that Θ_0, or more precisely its extension Θ to $\mathbf{Biv}\,\mathcal{R}$, allows us to pass from our context to the Tate situation. What makes the real difference between the two situations is that, in contradistinction to $G(A_C)$, the universal vectorial extension $\mathcal{U}(G)(\mathbf{W}(\mathcal{R}))$ has no torsion; so L, unlike the logarithm of $G(A_C)$, is injective.

This way of computing periods gives very good control over the results, with particular regard to the Galois action; for instance the Hodge-Tate decomposition comes out immediately from the structure of the map $\xi \to L(\xi)$ (cf. 4.4).

Now let us explain how $\mathbf{Biv}\,\mathcal{R}$ arises in our theory. Although $\mathbf{Biv}\,\mathcal{R}$ is defined as the completion of the ring of the special bivectors, $K \otimes \mathbf{W}(\mathcal{R})$, with respect to a topology coming from a valuation (cf. 5.15), it can be better understood using the results of § 6, where we prove that $\mathbf{biv}\,\mathcal{R}$ is naturally embedded in $\mathbf{Biv}\,\mathcal{R}$. More precisely, we show that the module $\mathbf{biv}\,\mathcal{R}$ is dense in the ring $\mathbf{Biv}\,\mathcal{R}$; so, if we want our periods to live in a ring, on observing that the image of p is in $\mathbf{biv}\,\mathcal{R}$, we conclude that $\mathbf{Biv}\,\mathcal{R}$ is the natural candidate.

We emphasize that the topology on $\mathbf{Biv}\,\mathcal{R}$ that we are using here does not coincide with the topology used by Barsotti, so that our ring $\mathbf{Biv}\,\mathcal{R}$ is different from the object denoted with the same symbol in [MA]. The reason for our choice, is that the map $\Theta : \mathbf{Biv}\,\mathcal{R} \to C$ which we alluded to above, fails to be continuos with respect to the Barsotti topology, while it becomes continuous with respect to this new topology; and the continuity is absolutely essential if we desire that $\mathbf{Biv}\,\mathcal{R}$ possess good properties (cf. 6.10). And in fact, the Galois structure of $\mathbf{Biv}\,\mathcal{R}$ is so clear that the Hodge-Tate decomposition (cf. 4.4) as well the comparison "crystalline-ètale" (cf. 4.1) drops out without any pain.

Perhaps the good properties of $\mathbf{Biv}\,\mathcal{R}$ and $\mathbf{biv}\,\mathcal{R}$ will appear less surprising after 7.12, where we describe the relation between $\mathbf{Biv}\,\mathcal{R}$ and the rings B^+ and B^+_{DR} defined by Fontaine in [FO3]. In fact we prove the following chain of inclusions: $\mathbf{biv}\,\mathcal{R} \subseteq B^+ \subseteq \mathbf{Biv}\,\mathcal{R} \subseteq B^+_{DR}$; moreover, B^+_{DR} is none other than the completion of the localization of $\mathbf{Biv}\,\mathcal{R}$ at $\ker \Theta$.

As a final remark, we observe that the techniques of Witt realization we used are such that all our result apply, without any change, to abelian varieties with good reduction. Soon we hope to have a machine of the same type working in a more general situation.

It's a plesure to conclude this introduction by heartily thanking J.-P. Wintenberger who, during a visit to Padova, patiently initiated us to this circle of ideas regarding periods.

1. - Notations and summary of known results

In this paragraph K' denotes a heterocharacteristic local field, *i.e.* a field of characteristic 0, complete with respect to a discrete valuation, with a perfect residue field k of characteristic $p \neq 0$. A' denotes the ring of integers of K' and $A = \mathbf{W}(k)$ is the ring of Witt vectors with components in k, K is the field of fractions of A, C the completion of the algebraic closure, $\overline{K}$, of K, and finally $\overline{A}$ and A_C are the rings of integers of $\overline{K}$ and C. The action of $\mathcal{G} = \mathrm{Gal}(\overline{K}/K)$ extends to C by continuity.

Let G be a Barsotti-Tate group of height h and dimension g over A'; denote by (G_n, i_n) the inductive system of finite, flat, commutative group schemes over A', which gives G, and by R_n the affine algebra of G_n. Let $p_- : R_{n+1} \to R_n$ denote the map corresponding to i_n; then (R_n, p_-) is a projective system of p-adically complete bialgebras and we will identify G with the formal affine group represented by the profinite bialgebra $R = \varprojlim R_n$. In particular, if S is a topological A'-algebra, the group $G(S)$ of the S-valued points of G will be identified with the group of continuous A'-algebra homorphisms, $\mathrm{Hom}^{\mathrm{cont}}_{A'}(R, S)$.

The A'-dual of R_n, $\tilde{R}_n = \mathrm{Hom}^{\mathrm{cont}}_{A'}(R_n, A')$, has a natural bialgebra structure coming from R_n by adjunction; it represents $\tilde{G}_n$, the Cartier dual of G_n. If $p_+ : R_n \to R_{n+1}$ denotes the map corresponding to the multiplication by p, $[p] : G_{n+1} \to G_n$, and $\tilde{p}_- : \tilde{R}_{n+1} \to \tilde{R}_n$ is the adjoint of p_+, then $(\tilde{R}_n, \tilde{p}_-)$ is a projective system. As before, the Barsotti-Tate group $\tilde{G}$ dual of G, will be identified with the formal group represented by $\tilde{R} = \varprojlim \tilde{R}_n$.

We will denote by $T(G)$ or $T(G)(A_C)$, if we need more precision, the Tate module, *i.e.* $T(G) = \varprojlim G_n(\overline{A})$ and by $V(G)$ the Tate space, *i.e.* $\varprojlim G^{(n)}(\overline{A})$, where $G^{(n)} = G$ for each n; in both cases the connecting maps are induced by the multiplication by p. Finally by $V_0(G)$ we will denote the the set of the elements of $V(G)$ whose components are torsion points. $T(G)$ is a free $\mathbb{Z}_p$-module of rank h (= height of G) and $V_0(G) = T(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Now, let us recall the definitions of integrals of the first and the second kind of G.

DEFINITION 1.1. Let $R_{K'} = R\hat{\otimes}_{A'}K'$ denote the affine algebra of $\hat{t}_G$, the completion at the origin of the tangent space of G, and $d : R_{K'} \to \Omega_{K'}(R_{K'})$ the usual differential; then the *integrals of* G are the elements h of $R_{K'}$ such that $dh \in \Omega = \Omega_{A'}(R)$. If the integral h belongs to $(R_{K'})^+$, the augmentation ideal of $R_{K'}$, then we say it is normalized. An *integral of the first kind* of G is a normalized integral, such that $\mathbf{P}h = h \otimes 1 + 1 \otimes h$, where $\mathbf{P}$ denotes the coproduct. An *integral of the second kind* of G is an integral such that $\mathbf{P}h - h \otimes 1 - 1 \otimes h \in R\hat{\otimes}_{A'}R$. In particular the elements of R are integrals of the second kind: they are called *exact integrals.*

The integrals of the first and second kind of G form two sub-A'-modules of $R_{K'}$ which will be denoted by $I_1(G)$ and $I_2(G)$. The restriction of d to $I_1(G)$ gives an isomorphism $I_1(G) \cong \omega_G$, where ω_G is the module of the invariant differential of G. The restriction of d to $I_2(G)$ induces an isomorphism of filtered modules between $I_1(G) \hookrightarrow I_2(G)/R$ and $\omega_G \hookrightarrow H^1_{dR}(G)$. Let us recall that $I_1(G_{A_C}) = I_1(G) \otimes_{A'} A_C$ and $H^1_{dR}(G_{A_C}) = H^1_{dR}(G) \otimes_{A'} A_C$.

REMARK 1.2. We will denote by $\mathcal{U}(G)$ the universal vectorial extension of G as defined in [MM]. The affine algebra $\mathcal{E} = \mathcal{E}(R)$ of $\mathcal{U}(G)$ can be realized as follows (cf. 5.4.9 and 5.4.10 of [WR]): let $c = h - g$, choose $(h_1, \dots, h_c)$ in $I_2(G)$ in such a way that their image in $H^1_{dR}(G)/\omega_G$ gives a set of generators of this quotient-A'-module. Denote by $(T_1, \dots, T_c)$ a set of indeterminates and by γ_i, $i = 1, \dots, c$, the elements of $R\hat{\otimes}_{A'}R$ defined by $\gamma_i = \mathbf{P}h_i - h_1 \otimes 1 - 1 \otimes h_i$; then $\mathcal{E} = R[T_1, \dots, T_c]$, with the bialgebra structure extending the corresponding structure of R in the following way:

$$\mathbf{P}T_i = T_i \otimes 1 + 1 \otimes T_i + \gamma_i, \quad \varepsilon T_i = 0, \quad \rho T_i = -T_i,$$

where ε and ρ denote the coidentity and the inversion, respectively.

Now we give a summary of the results on Witt realization we need in order to discuss periods. Here G_k denotes the special fiber of G and more generally we will use the subscript k in order to denote the reduction mod m (= maximal ideal of A') of the objects related to G. In particular, $\mathfrak{R}_k$ denotes the completion of the limit $\varinjlim (R_k \xrightarrow{pi} R_k \xrightarrow{pi} \dots)$, where pi is the map corresponding to $[p] : G_k \to G_k$. Observe that $\mathfrak{R}_k$ represents the Tate space $V(G_k)$, in the sense that for any topological k-algebra S, $V(G_k)(S) = \mathrm{Hom}^{\mathrm{cont}}_{k\text{-alg}}(\mathfrak{R}_k, S)$. In what follows we need bivectors with components in $\mathfrak{R}_k$: the definition of $\mathbf{biv}\,\mathfrak{R}_k$ and $\mathbf{Biv}\,\mathfrak{R}_k$ we use here is given in [WR], 0.4.

THEOREM 1.3 (cf. [WR], Th. 4.3.2). *Let B be the sub-A'-algebra of $R_{K'}$ spanned by $I_2(G)$. Denote by $\sigma : R \to R_k$ the reduction map, by $\mathbf{biv}\,\mathfrak{R}_k$ the*

K-module of the bivectors with components in $\mathfrak{R}_k$, and by $\mathbf{Biv}\,\mathfrak{R}_k$ *the ring obtained by completion of* $\mathbf{biv}\,\mathfrak{R}_k$. *Assume the degree of ramification of K'/K is less than $p-1$; then there exists a unique injective A'-algebra homomorphism* $j : B \to \mathrm{biv}(\mathfrak{R}_k) \otimes A'$, *such that* $\mathbf{biv}(\mathbf{P}_{\mathfrak{R}_k}) \circ j = (j \hat{\otimes} j) \circ \mathbf{P}_R$, *which fits into the following commutative diagram:*

$$\begin{array}{ccc} B & \xrightarrow{j} & \mathbf{biv}(\mathfrak{R}_k) \underset{A}{\otimes} A' \\ \uparrow & & \uparrow \\ R & \xrightarrow{j'} & \mathbf{W}(\mathfrak{R}_k) \underset{A}{\otimes} A' \\ \downarrow \sigma & & \downarrow \rho \\ R_k & \xrightarrow{i} & \mathfrak{R}_k \end{array}$$

Here the vertical arrows without a name are the natural inclusions, ρ is the canonical projection, j' the restriction of j, i the immersion of R_k in $\mathfrak{R}_k$ (which is fixed once and for all).

Now we shall give some information on the embedding of $\mathcal{E}(R)$, the affine algebra of $\mathcal{U}(G)$, the universal vectorial extensions of G. This embedding allows a very natural identification of $I_1(\mathcal{U}(G))$ with $H^1_{dR}(G)$ and $M(G_k) \otimes K'$, where $M(G_k)$ is the Dieudonné module of G_k.

THEOREM 1.4 (cf. [WR], Th. 5.4.9). *Let the notations of* 1.3 *maintain their meanings, then:*

i) *the embedding j' can be uniquely extended to an A'-bialgebra injective map of $\mathcal{E}(R)$ in* $\mathbf{W}(\mathfrak{R}_k) \otimes A'$; *and $\rho(j(\mathcal{E}(R)))$ is the affine algebra of $\mathcal{U}(G_k)$;*

ii) *the extension $\mathcal{E}(R)$, identified with its j'-*image *splits in* $\mathbf{biv}\,\mathfrak{R}_k \otimes A'$;

iii) $\mathbf{biv}\,R_k \hat{\otimes} A'$ *contains $I_1(\mathcal{U}(G))$, the module of the integrals of first kind of $\mathcal{U}(G)$; this A'-module coincides with $M(G_k) \otimes A'$, and it is filtered by $I_1(G)$;*

iv) *identify $I_2(G)$ with its j-image, then the map*

$$h \to \lim_{n \to \infty} (pi)^{-n}(p^n h),$$

defined on $I_2(G)$, leaves $I_1(G)$ fixed, has image $I_1(\mathcal{U}(G))$, and its kernel is R: so it identifies the filtered module $\omega_G \hookrightarrow H^1_{dR}(G)$ with $j(I_1(G)) \hookrightarrow M(G_k) \otimes A'$.

REMARK 1.5. The map pi, which appears in iv), means $\mathbf{biv}(pi)$, where pi is the endomorphism of $\mathfrak{R}_k$ corresponding to the multiplication by p. As a consequence $\eta = \lim_{n \to \infty} (pi)^{-n}(p^n h)$ is the unique canonical (= additive) bivector whose components with negative indices coincide with the corresponding components of h.

We complete this summary with the following result on the duality:

THEOREM 1.6 (cf. [MA], th. 5.41). *There exists a sub-K'-algebra of* $\mathbf{Biv}\,\mathfrak{R}_k \otimes K'$, $\mathcal{D}$, *containing both* $M(G_k) \otimes K'$ *and* $\mathcal{E}(R)$, *which satisfies the following properties: there is defined a natural action*

$$b : (M(\tilde{G}_k) \otimes K') \times \mathcal{D} \to K',$$

such that, if $t_{\mathcal{U}(G)}(K')$ *denotes the tangent space of* $\mathcal{U}(G)$ *at the identity, then there exists an isomorphism*

$$j^* : M(\tilde{G}_k) \otimes K' \to t_{\mathcal{U}(G)}(K'),$$

such that $b(x,y) = j^*(x)y$, *for each* $x \in M(\tilde{G}_k) \otimes K'$ *and* $y \in (\mathcal{E}(G)\hat{\otimes}K') \cap \mathcal{D}$. *Moreover, the restriction*

$$b : (M(\tilde{G}_k) \otimes A') \times (M(G_k) \otimes A') \to A'$$

is a (non-degenerate) pairing of A'*-modules compatible with Frobenius and Verschiebung.*

2. - The Tate space

Our aim in this section is to discuss those realizations of the Tate space of G, which we need later to compute periods.

Since the results of the previous section are proved in the case of tame ramifcation, *i.e.* e $< p-1$, all our results could be proved under the same hypothesis, however, in order to render our exposition more transparent we will assume $A' = A$, *i.e.* e $= 1$; it is straightforward to extend every thing to the case e $< p-1$.

In general, to represent $V(G)$, the Tate space of G, we will use the bialgebra $\mathfrak{R}$ obtained as completion of $\mathfrak{R}^0 = \varinjlim(R \xrightarrow{pi} R \xrightarrow{pi} \cdots)$; we mean that, if S is a (complete) topological A algebra, then $V(G)(S) = \mathrm{Hom}^{\mathrm{cont}}_{A'\text{-alg}}(\mathfrak{R}, S)$. In particular, the Tate module, $T(G)$, is represented by the limit of the inductive system (R_n, p_+), so

$$T(G)(A_C) = \{(P_n)_{n\in\mathbb{N}} \,|\, P_n \in \mathrm{Hom}^{\mathrm{cont}}_{A'\text{-alg}}(R_n, A_C), P_{n+1} \circ p_+ = P_n\},$$

or, after Cartier duality,

$$T(G) = \{P \in \tilde{R}\hat{\otimes}_A A_C | P \equiv 1, \mathrm{mod}(\tilde{R}\hat{\otimes}_A A_C)^+, \mathbf{P}f = f \otimes f\},$$

where $\tilde{R} = \varprojlim \tilde{R}_n$ denotes the affine algebra of $\tilde{G}$, the dual of G, $(\tilde{R}\hat{\otimes}_A A_C)^+$ denotes the augmentation ideal and $\mathbf{P}$ the extension of the coproduct of R

to $R\hat{\otimes}_A A_C$ (these notations will maintain their meaning for each bialgebra throughout this paper).

Let us connect these descriptions of the Tate module. Let $P = (P_n)_{n\in\mathbb{N}} \in T(G)$; then $P_n : R_n \to \overline{A}$ is a continuous A-algebra homomorphism, so:

$$P_n \in \tilde{R}_n \otimes_A \overline{A}, \ \ P_n \equiv 1 \operatorname{mod}(\tilde{R}_n \otimes_A \overline{A})^+, \ \ \mathbf{P}P_n = P_n \otimes P_n.$$

Since $\tilde{p}_-(P_{n+1}) = P_n$, this sequence defines an element $P \in \varprojlim \tilde{R}_n \otimes_A \overline{A}$, where the last limit can be identified with $\tilde{R}_{A_C} = \tilde{R}\hat{\otimes}_A A_C$. Namely, if P'_n denotes an element of $\tilde{R} \otimes_A \overline{A}$ whose image in $\tilde{R}_n \otimes_A \overline{A}$ is P_n, the sequence (P'_n) satisfies the Cauchy condition in $\tilde{R} \otimes_A \overline{A}$, so that the limit $P = \lim_{n\to\infty} P'_n$ exists in $\tilde{R}_{A_C}$.

Here is a picture of the situation which will be useful later on:

$$(2.1)\qquad \begin{array}{ccccc}
\tilde{R}_0 \otimes A_C & \xleftarrow{p_-} & \tilde{R}_1 \otimes A_C & \cdots & \tilde{R}\hat{\otimes} A_C \\
\downarrow p_+ & & \downarrow p_+ & & \downarrow pi \\
\tilde{R}_1 \otimes A_C & \xleftarrow{p_-} & \tilde{R}_1 \otimes A_C & \cdots & \tilde{R}\hat{\otimes} A_C \\
\downarrow p_+ & & \downarrow p_+ & & \downarrow pi \\
\tilde{R}_2 \otimes A_C & \xleftarrow{p_-} & \tilde{R}_3 \otimes A_C & \cdots & \tilde{R}\hat{\otimes} A_C \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
\varinjlim(\tilde{R}_i \otimes A_C) & \xleftarrow{pi} & \varinjlim(\tilde{R}_i \otimes A_C) & \cdots & \varinjlim(\tilde{R}\hat{\otimes} A_C) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Hom}_A^{\text{cont}}(R, A_C) & \xleftarrow{pi} & \operatorname{Hom}_A^{\text{cont}}(R, A_C) & \cdots & \varprojlim \operatorname{Hom}_A^{\text{cont}}(R, A_C)
\end{array}$$

the diagram is commutative, and the symbols not yet defined have the following meanings: the ring $\tilde{R}\hat{\otimes}_A A_C$ is identified with the inverse limit $\varprojlim \tilde{R}_n \otimes_A A_C$, and then it is the inverse limit of the corresponding row. Each element of the row above the last is obtained by a direct limit from the sequence in the corresponding column. Finally the maps connecting the last two rows are obtained by composing the elements of $\tilde{R}_n \otimes_A A_C \cong \operatorname{Hom}_A^{\text{cont}}(R_n, A_C)$ with the natural projections $R \to R_n$.

REMARK 2.2. Given a bialgebra S, let denote by *mult*(S) the group of the multiplicative elements of S, more precisely:

$$mult(S) = \{s \in S | s \equiv 1 \operatorname{mod} S^+, \mathbf{P}s = s \otimes s\}.$$

Then 2.1 reveals the key relations among the following groups:

$$mult(\varinjlim \tilde{R}_n \otimes_A A_C) = G(A_C)^{\text{tor}}, \quad mult(\operatorname{Hom}_A^{\text{cont}}(R, A_C)) = G(A_C,),$$

$$mult(\varprojlim \operatorname{Hom}_A^{\text{cont}}(R, A_C)) = V(G), \ \text{ and } mult(\varinjlim \tilde{R} \otimes_A A_C) = T(G) \otimes \mathbb{Q}_p.$$

The following proposition explains how the integrals of the first kind of G can be approximated by a sequence (ξ_i') such that $d\xi_i'/\xi_i'$ is regular on the finite group G_i.

Paragraph III. of [CO] develops a more sophisticated method to do a process of the same type.

PROPOSITION 2.3. *Let $\xi \in T(\tilde{G})$, then $\xi \in mult(R\hat{\otimes}_A A_C)$ and*

$$\log \xi = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\xi - 1)^n}{n}$$

exists in $R\hat{\otimes}_A C$. Thus $\log \xi$ is an integral of the first kind of G, i.e. $d\xi/\xi \in \omega_G$. Moreover, if $(\xi_i')_{i\in\mathbb{N}}$ is any sequence in $R\hat{\otimes}_A A_C$ such that $\xi_i' \equiv \xi \operatorname{mod}(pi)^i(R\hat{\otimes}_A A_C)^+$, the differential $d\xi_i'/\xi_i'$ is regular on the subgroup G_i.

PROOF. Let $(\xi_i)_{i\in\mathbb{N}}$, where $\xi_i \in mult(R_i \otimes_A A_C,)$, represent ξ; then after identifying $\varprojlim R_i \otimes_A \overline{A}$ with $R\hat{\otimes}_A A_C$, we have $\xi \in mult(R\hat{\otimes}_A A_C,)$; so the logarithmic series converges because $\xi \equiv 1 \operatorname{mod}(R\hat{\otimes}_A A_C,)^+$, while $\log \xi$ is additive because $\mathbf{P}\xi = \xi \otimes \xi$. Since $\xi_i \in mult(R_i \otimes_A A_C,)$, we see that $d\xi_i/\xi_i \in \omega_{G_i}$. Remarking that ξ_i is the canonical image of ξ_i' in $R_i \otimes_A A_C$, the last claim follows directly. □

Remark that, since the sequence $(\xi_i')_{i\in\mathbb{N}}$ is Cauchy in $R\hat{\otimes}_A A_C$, the same holds for the sequence $d\xi_i'/\xi_i'$ with respect to the p-adic topology of $\Omega_{A_C}(R\hat{\otimes}_A A_C,)$: in fact

$$d((pi)^n(R\hat{\otimes}_A A_C)) \subseteq p^n \Omega(R\hat{\otimes}_A A_C).$$

Now we introduce other realizations of $V(G)$.

Let $\mathcal{R} = \mathcal{R}(A_C)$ be the ring defined in 5.1; let $\mathbf{W}(\mathcal{R})$ be the ring of Witt vectors with components in $\mathcal{R}$ and $\Theta_0 : \mathbf{W}(\mathcal{R}) \to A_C$ the map defined in 5.8. Here we will use the following notations (cf. also diagram 2.1) D denotes the A-module $\operatorname{Hom}_A^{\text{cont}}(R, A)$, endowed with the topology whose fundamental system of neighbourhoods is given by the family of sub-A-modules $(V_n = \{f | v(f(R^+)) > n\})_{n\in\mathbb{N}}$, and $\mathcal{D}_{\mathbf{W}(\mathcal{R})} = \varprojlim D\hat{\otimes}_A \mathbf{W}(\mathcal{R})$.

PROPOSITION 2.4. *Let $V(G)(\mathbf{W}(\mathcal{R}))$ be the $\mathbb{Q}_p$-space of $\mathbf{W}(\mathcal{R})$-valued points of the Tate space of G. Then the natural map from $V(G)(\mathbf{W}(\mathcal{R}))$ to $V(G)(A_C)$ induced by Θ_0 is an isomorphism.*

PROOF. Let $\hat{P} = (\hat{P}_0, \hat{P}_1, \ldots) \in V(G)(A_C)$; this means that each $\hat{P}_i$ is an element of $mult(D\hat{\otimes}_A A_C)$. Since R is a topologically free A-algebra, there exists $\tilde{P}_i' \in mult(D\hat{\otimes}_A \mathbf{W}(\mathcal{R}))$ such that $\hat{P}_i = (i \otimes \Theta_0)\tilde{P}_i'$, and we set

$\tilde{P}_i = \lim_{j\to\infty} (\tilde{P}_{i+j})^{p^j}$. Once we prove that $\tilde{P}_i$ exists, it is immediate to check that $\tilde{P} = (\tilde{P}_0, \tilde{P}_1, \ldots) \in V(G)(\mathbf{W}(\mathcal{R}))$ and that $\hat{P} = (i \otimes \Theta_0)\tilde{P}$.

In order to show the existence, first observe that the map $\tilde{P}'_i \to (i \otimes \Theta_0)\tilde{P}'_i$ gives an isomorphism between $G^{\text{ét}}(\mathbf{W}(\mathcal{R}))$ and $G^{\text{ét}}(A_C)$.

In fact, this implies that the restriction of $(\tilde{P}'_{i+j+1})^{p^{j+1}} - (\tilde{P}'_{i+j})^{pj}$ to $R^{\text{ét}}$ is the coidentity; therefore if $x \in R^+$, when we compute

$$[(\tilde{P}'_{i+j+1})^{p^{j+1}} - (\tilde{P}'_{i+j})^{p^j}](x) = [\tilde{P}_{i+j+1} \circ (pi)^{j+1} - \tilde{P}'_{i+j} \circ (pi)^j](x)$$

we can assume that $x \in (R^0)^+$ (where R^0 is the affine algebra of the connected component of G). Now for such an x we have:

$$(\tilde{P}_{i+j+1} \circ (pi)^{j+1} - \tilde{P}_{i+j} \circ (pi)^j)(x) = F_j[(\tilde{P}_{i+j+1} \circ pi - \tilde{P}'_{i+j})(x)],$$

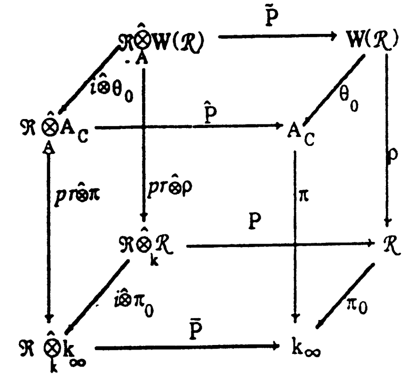
where F_j is the series with coefficients in A such that $F_j(x) = (pi)^j(x)$. As explained in § 5, the ring $\mathbf{W}(\mathcal{R})$ is complete with respect to the topology for which a fundamental system of neighbourhoods of zero is given by the ideals $(W_n)_{n\in\mathbf{N}}$, and W_n admits the following decomposition:

$$W_n = (\ker \Theta_0)^n + p(\ker \Theta_0)^{n-1} + \ldots + p^n\mathbf{W}(\mathcal{R}) \quad \text{(cf. 5.10).}$$

By definition of the points $\tilde{P}'_j$ one has $(\tilde{P}'_{i+j+1} \circ pi - \tilde{P}'_{i+j})(x) \in \ker \Theta_0$; then, in view of the form of the series F_j, one deduces that $(\tilde{P}'_{i+j+1} \circ (pi)^{j+1} - \tilde{P}'_{i+j} \circ (pi)^j)(x) \to 0$, when $j \to \infty$; *i.e.*, the limit exists. □

The following result puts together different realizations of the Tate space, and it shows how to pass from one to another.

PROPOSITION 2.5. *The $\mathbb{Q}_p[\mathcal{G}]$-modules $V(G)(A_C)$, $V(G)(\mathbf{W}(\mathcal{R}))$, $V(G_k)(\mathcal{R})$, $V(G_k)(A_C/pA_C)$ are all canonically isomorphic to one another. The correspondences are defined as follows: given $P \in V(G_k)(\mathcal{R})$, there exist $\tilde{P} \in V(G)(\mathbf{W}(\mathcal{R}))$, $\hat{P} \in V(G)(A_C)$, $\overline{P} \in V(G_k)(A_C/pA_C)$, which fit into the following commutative diagram:*



where k_∞ denotes A_C/pA_C, and the arrows not yet defined, have the following meanings: ρ sends any vector to its 0^{th}-component; π is the canonical map from A_C onto A_C/pA_C; π_0 is the canonical map from the the projective limit $\mathcal{R}$ to the first factor of the sequence by which it is defined; i denotes the identity map.

Observe that the points $\hat{P}$, $\overline{P}$, P and $\tilde{P}$ appearing in the diagram, are extensions by linearity and continuity of those in the claim, so that they are uniquely determined by them.

PROOF. The commutativity of the right vertical square is an immediate consequence of the definition of Θ_0 (cf. 5.8): in fact if $x = (x_0, x_1, \ldots)$ is an element of $\mathcal{R}$, then $\pi(\Theta_0(\{x\})) = \pi(\hat{x}_0) = x_0$. The commutativity of the left vertical square follows from that of the one on the right. Now we define $\hat{P}$ by observing that the homorphism $(\Theta_0 \circ \tilde{P})$ can be factorized through $\mathfrak{R}\hat{\otimes} A_C$, and we can do the analogous factorizations to get P and $\overline{P}$. By the previous remark on the vertical squares, the cube constructed in this way is commutative.

At this point is remains to show that the correspondences built up between the various groups are isomorphisms. This can be done in the same way we used in 2.4 to go from $\hat{P}$ to $\tilde{P}$. We repeat once again these arguments by showing how to invert the map $P \to \overline{P}$. Again we denote by $\hat{P} = (\hat{P}_i)$ the restriction of $\hat{P}$ to $\mathfrak{R}^0 = \varinjlim(R \xrightarrow{pi} R \xrightarrow{pi} \cdots)$, where $\hat{P}_i$ is the map induced on the i^{th}-element of the sequence defining the direct limit, so that $\hat{P}_i = \hat{P}_{i+1} \circ pi$. We do the same for $\overline{P}$, P and $\tilde{P}$. Since R_k is formally smooth, for each i, there exists a k-algebra homomorphism $P_i' : R_k \to \mathcal{R}$, such that $\pi \circ P_i' = \overline{P}_i$. We set $P_i = \lim_{j \to \infty} P_{i+j}' \circ (pi)^j$. It is clear that, once we prove that such limit exists, the sequence (P_i) gives a point $P \in \mathrm{Hom}^{\mathrm{cont}}_{k\text{-alg}}(\mathfrak{R}_k, \mathcal{R})$, and the map $\overline{P} \to P$ is the desired inverse.

In order to show the existence of the above limit, first note that the restriction of P_i' to $R_k^{\text{ét}}$ coincides with the same restriction of $\overline{P}_i$ (of course the copy of $\overline{k}$ inside k_∞ is identified, by means of π_0, with the copy of the same field inside $\mathcal{R}$); as a consequence the restriction of the difference $[P'_{i+j+1} \circ (pi)^{j+1} - P'_{i+j} \circ (pi)^j]$ to $R_k^{\text{ét}}$ is the coidentity; therefore if $x \in R_k^+$, when we compute $[P'_{i+j+1} \circ (pi)^{j+1} - P'_{i+j} \circ (pi)^j](x)$, we can assume $x \in (R_k^0)^+$ (where R_k^0 is the affine algebra of the connected component of G_k). Now for such x we have:

$$[P'_{i+j+1} \circ (pi)^{j+1} - P'_{i+j} \circ (pi)^j](x) = F_j([P'_{i+j+1} \circ (pi) - P'_{i+j}](x)),$$

where F_j is the series with coefficients in k such that $F_j(x) = (pi)^j(x)$. Since, by construction, $[P'_{i+j+1} \circ (pi) - P'_{i+j}](x) \in \ker(\pi_0)$ and since $\mathcal{R}$ is a complete valuation ring in which $v(y) \geq 1$ for any element $y \in \ker(\pi_0)$ (cf. 5.5 for the definition of the valuation v), one has

$$\lim_{j \to \infty} [P'_{i+j+1} \circ (pi)^{j+1} - P'_{i+j} \circ (pi)^j](x) = 0$$

and this implies that the limit exists. □

REMARKS 2.6. i) Let us stress the following immediate consequence of 2.5: given $P \in V(G_k)(\mathcal{R})$, there exists $\mathbf{biv}(P) \in \mathrm{Hom}^{\text{cont}}_{k\text{-alg}}(\mathbf{biv}\,\mathfrak{R}_k, \mathbf{biv}\,\mathcal{R})$, and $\tilde{P}$ is just the restriction of $\mathbf{biv}(P)$ to $\mathfrak{R}$ (canonically embedded in $\mathbf{W}(\mathfrak{R}_k)$).

ii) Since, after Witt realization, $\mathcal{E}(R)$ is identified with its image in $\mathbf{W}(\mathfrak{R}_k)$ (cf. 1.3 and 1.4), the restriction of $\mathbf{biv}(P)$ to $\mathcal{E}(R)$ gives a $\mathbf{W}(\mathcal{R})$-valued point $\check{P}$ of the universal vectorial extension $\mathcal{U}(R)$.

Now we prove that the map $P \to \check{P}$ is an isomorphism of $\mathbb{Q}_p[\mathcal{G}]$-modules.

PROPOSITION 2.7. i) *Let $V(\mathcal{U}(G))(\mathbf{W}(\mathcal{R}))$ be the Tate space of the group of $\mathbf{W}(\mathcal{R})$-valued points of $\mathcal{U}(G)$, then the canonical map $\mathcal{U}(G)(\mathbf{W}(\mathcal{R})) \to G(\mathbf{W}(\mathcal{R}))$ induces an isomorphism*

$$\phi : V(\mathcal{U}(G))(\mathbf{W}(\mathcal{R})) \to V(G)(\mathbf{W}(\mathcal{R})).$$

ii) *The projection $\psi : V(\mathcal{U}(G))(\mathbf{W}(\mathcal{R})) \to \mathcal{U}(G)(\mathbf{W}(\mathcal{R}))$ of the inverse limit on its first factor is an isomorphism. Moreover, with the notation of* 2.6, *$\psi(\phi^{-1}(\tilde{P})) = \check{P}$.*

PROOF. i) First we show how to get the inverse of the above map ϕ. This can be done as follows: let $\tilde{P} = (\tilde{P}_0, \tilde{P}_1, \ldots) \in V(G)(\mathbf{W}(\mathcal{R}))$; fix a section $\sigma : G \to \mathcal{U}(G)$, and use it to define the points $\sigma\tilde{P}_i = P'_i \in \mathcal{U}(G)(\mathbf{W}(\mathcal{R}))$, and then take the limit $P''_i = \lim_{n\to\infty}([p]^j P'_{i+j})$, where, as usual, we denote by $[p]$ the multiplication by p in the group of points.

We will limit ourselves to checking that this limit exists in $\mathcal{U}(G)(\mathbf{W}(\mathcal{R}))$, leaving to the reader the easy verifications that $P'' = (P''_0, P''_1, \ldots) \in V(\mathcal{U}(G))(\mathbf{W}(\mathcal{R}))$ and that the map $\tilde{P} \to P''$ inverts ϕ.

To this end, recall that we can choose $t_1, \ldots, t_c$ in $\mathcal{E}(R)$ with the following two properties: $\mathcal{E} = R[t_1, \ldots, t_c]$, and $(pi)^r(t_i) \equiv p^r t_i \bmod R^+$ (cf. Theorem 1.4). Now, let $x \in \mathcal{E}(R)^+$. Since, by construction, $R^+ \subseteq \ker([p](P'_{i+j+1}) - (P'_{i+j}))$, one gets

$$([p]^{j+1} P'_{i+j+1}) - [p]^j P'_{i+j})x = ([p]P'_{i+j+1} - P'_{i+j})(pi)^j x \in p^j \mathbf{W}(\mathcal{R});$$

and this means that the sequence $j \to [p]^j P'_{i+j}$ satisfies the Cauchy condition; finally, since $\mathrm{Hom}^{\mathrm{cont}}_{A'}(\mathcal{E}(R), \mathbf{W}(\mathcal{R}))$ is complete, the existence is proved.

ii) Now remark that, once $\mathcal{E}(R)$ is canonically embedded in $\mathbf{W}(\mathfrak{R}_k)$, then the point $P'' = (P''_0, P''_1, \ldots)$ can be obtained as follows: $P''_i = \mathbf{biv}(P) \circ \mathbf{W}((pi)^{-j})$, where P is the image of $\tilde{P}$ in $V(G_k)(\mathcal{R})$ by the isomorphism described in 2.5. In particular $P''_0 = \psi(\phi^{-1}(\tilde{P}))$ is the restriction of $\mathbf{biv}(P)$ to $\mathcal{E}(R)$, *i.e.* $P''_0 = \check{P}$.

Since ϕ is an isomorphism, we get the conclusion if we prove that $\psi \circ \phi^{-1}$ is an isomorphism. This can be obtained by observing the following facts:

a) let $P''_0 : \mathcal{E}(R) \to \mathbf{W}(\mathcal{R})$ be a point of $\mathcal{U}(G)$, then it can be extended to a K-linear map, compatible with Frobenius and Verschiebung, defined on $I_1(\mathcal{U}(G))$ $(= M(G_k) \otimes K)$ with values in $\mathbf{biv}\,\mathcal{R}$;

b) any K-linear map, compatible with Frobenius and Verschiebung, defined on $M(G_k) \otimes K$ with values in $\mathbf{biv}\,\mathcal{R}$ is induced by a point P of $V(G_k)(\mathcal{R})$.

To prove a), note that the elements of $I_1(\mathcal{U}(G))$ are series of logarithmic type whose arguments are vectors, so the same holds for their images under P''_0; therefore, to show they converge in $\mathbf{biv}\,\mathcal{R}$, we can use the arguments of Theorem 6.8. To prove the compatibility with F and V, it's sufficient to observe that each canonical bivector $x = (x_i)$, *i.e.* each element of $M(G_k) \otimes K$ (cf. [WR], § 0.5), satisfies the equation $V^i x_j = x_{j-i}$, where V is the Verschiebung of $\mathfrak{R}_k$.

To prove b), first observe that any K-linear map, compatible with Frobenius and Verschiebung, defined on $M(G_k)$ with values in $\mathbf{biv}\,\mathcal{R}$, can be extended as a k-algebra homorphism from the closed sub-algebra of $\mathfrak{R}_k$ generated by the components of the elements of $M(G_k)$ and with values in $\mathcal{R}$; but it's well known (cf. [MA] Chp. IV) that this sub-algebra coincides with $\mathfrak{R}_k$, and then our extended map is an element of $V(G_k)(\mathcal{R})$. □

Now, the following results are direct consequences of 2.5 and 2.7.

COROLLARY 2.8. $\mathcal{U}(G)(\mathbf{W}(\mathcal{R}))^{\mathrm{tor}} = 0$.

COROLLARY 2.9. *There is a natural exact sequence of groups ($\mathbb{Z}_p$-modules)*

$$0 \to T(G)(A_C) \to \mathcal{U}(G)(\mathbf{W}(\mathcal{R})) \to G(A_C) \to 0,$$

where the map $\mathcal{U}(G)(\mathbf{W}(\mathcal{R})) \to G(A_C)$ *is the composition of the natural projection* $\mathcal{U}(G)(\mathbf{W}(\mathcal{R})) \to G(\mathbf{W}(\mathcal{R}))$ *with the map* $G(\mathbf{W}(\mathcal{R})) \to G(A_C)$ *induced by* Θ_0, *and the inclusion* $T(G)(A_C) \to \mathcal{U}(G)(\mathbf{W}(\mathcal{R}))$ *is the composition of the natural inclusion* $T(G)(A_C) \to V(G)(A_C)$ *with the isomorphisms* $V(G)(A_C) \cong V(G)(\mathbf{W}(\mathcal{R})) \cong V(\mathcal{U}(G))(\mathbf{W}(\mathcal{R})) \cong \mathcal{U}(G)(\mathbf{W}(\mathcal{R}))$ *given above.*

We end this section by describing one more way to lift the points of $T(G)(A_C)$ to $V(G)(\mathbf{W}(\mathcal{R}))$; this will simplify our further computations.

PROPOSITION 2.10. *Let* $\hat{P} \in T(G)(A_C)$, *denote by* P' *an element of* $\tilde{R}\hat{\otimes}\mathbf{W}(\mathcal{R})$, *such that* $P' \equiv 1 \operatorname{mod}(\tilde{R}\hat{\otimes}\mathbf{W}(\mathcal{R}))^+$ *and whose image in* $\tilde{R}\hat{\otimes}A_C$ *is* $\hat{P}$. *Then the limit*

$$\tilde{P} = \lim_{n\to\infty}((pi)^{-n}P')^{p^n}$$

exists in $mult(\tilde{\mathfrak{R}}\hat{\otimes}\mathbf{W}(\mathcal{R}))$ *(notation of* 2.2*), and the map* $\hat{P} \to \tilde{P}$ *is the inverse of the isomorphism appearing in* 2.4.

PROOF. Since $\tilde{\mathfrak{R}}\hat{\otimes}\mathbf{W}(\mathcal{R})$ is complete, to prove that the limit exists, we will check that $((pi)^{-n}P')^{p^n}$ is a Cauchy sequence. Observing that pi and its inverse are ring homomorphisms, one gets

$$[(pi)^{-n}P']^{p^n} - [(pi)^{-(n+1)}P']^{p^{n+1}} = (pi)^{-(n+1)}[(piP')^{p^n} - (P')^{p^{n+1}}].$$

Now recalling that P' lifts $\hat{P}$, we have $piP' = P'^p + x$, where $x \in \ker(i\hat{\otimes}\Theta_0) = \tilde{R}\hat{\otimes}\ker\Theta_0$ thus:

$$[(piP')^{p^n} - (P')^{p^{n+1}}] = \sum_{i=1}^{p^n}\binom{p^n}{i}P'^{p^n-i}x^i. \tag{2.11}$$

The decomposition of the ideals W_n given in 5.10(b) tells us that $p^{n-v(i)}x^{v(i)} \in \tilde{R}\hat{\otimes}W_n$, for any $i \le p^n$, so from equation 2.11 and the obvious remark that $i \ge v(i)$, we conclude that

$$\begin{aligned}[(pi)^{-n}P']^{p^n} - [(pi)^{-(n+1)}P']^{p^{n+1}} &= (pi)^{-(n+1)}[(piP')^{p^n} - (P')^{p^{n+1}}]\\ &\in ((pi)^{-n-1}\tilde{R})\hat{\otimes}W_n,\end{aligned}$$

which means that the sequence $((pi)^{-n}P')^{p^n}$ satisfies the Cauchy condition in $\tilde{\mathfrak{R}}\hat{\otimes}\mathbf{W}(\mathcal{R})$. Since the image of each $((pi)^{-n}P')^{p^n}$ is $\hat{P}$ the same holds for $\tilde{P}$. Finally, it's straightforward to check that $\tilde{P}$ is multiplicative, so this verification is left to the reader. □

3. - Pairings

In this section, since we don't need to distinguish among the different realizations of the Tate space (cf. Proposition 2.5), we shall simply write $V(G)$ to denote that space. Let $P \in V(G)$ and $\xi \in \mathbf{biv}\,\mathfrak{R}_k$, then by $\xi(P)$ we shall denote the element $(\mathbf{biv}\,P)(\xi) \in \mathbf{biv}\,\mathcal{R}$; in this way *the elements of* $\mathbf{biv}\,\mathfrak{R}_k$ *can be computed at a point P of the Tate space, in this way they are functions defined on $V(G)$ with values* in $\mathbf{biv}\,\mathcal{R}$.

The first result we shall prove, shows that the functions given by the elements of $M(G_k) \otimes K$, or equivalently by $H^1_{dR}(G) \otimes K$ (recall that by 1.4, once embedded in $\mathbf{biv}\,\mathfrak{R}_k$, these modules coincide) are linear functions and they supply a "system of coordinates" on $V(G)$.

THEOREM 3.1. *The pairing $p : M(G_k) \times V(G) \to \mathbf{biv}\,\mathcal{R}$, defined by setting $p(\eta, P) = \eta(P)$, is non-degenerate, and compatible with Frobenius, Verschiebung and the action of Galois group.*

PROOF. Let $\eta = (\dots, x_{-1}, x_0, x_1, \dots)$ be an element of $M(G_k) \subseteq \mathbf{biv}\,\mathfrak{R}_k$, this means that $x_i \in (R_k)^+$ for $i < 0$, and $V\eta = (\dots, V_{\mathfrak{R}_k}x_{-1}, V_{\mathfrak{R}_k}x_0, V_{\mathfrak{R}_k}x_1, \dots)$, or likewise $p\eta = (\dots, (pi)x_{-1}, (pi)x_0, (pi)x_1, \dots)$ (cf. [WR] 0.5.3). As a consequence, one gets

$$[\mathbf{biv}(V_{\mathfrak{R}_k})\eta](P) = V(\eta(P)) \text{ and } [\mathbf{biv}(F_{\mathfrak{R}_k})\eta](P) = F(\eta(P)) :$$

these are the claimed compatibilities with Frobenius and Verschiebung. With regard to the action of Galois group, it follows from the above definition that $s[\eta(P)] = \eta(s(P))$ for any $s \in \mathcal{G}$ and any $P \in V(G)$, because any point P acts on the components of a canonical bivector η and s acts on the components of the elements of $\mathbf{biv}\,\mathcal{R}$ and on P by composition.

Now we will prove the non-degeneracy of p First, if $0 \neq \eta = (\dots, x_{-1}, x_0, x_1, \dots) \in M(G_k)$, then $x_{-1} \neq 0$; so in order to get a point $P \in V(G)$ such that $\eta(P) \neq 0$ it suffices to exhibit a $P' : R_k \to k_\infty$ for which $x_{-1}(P') \neq 0$. Recalling that $x_{-1} = x_{-1}(t_1, \dots, t_g)$ is a power series without constant term with coefficients in $R_k^{\text{ét}}$, and that the étale points have values in $\overline{k}$; we are reduced to proving the following claim: *let $X_{-1} = X_{-1}(t_1, \dots, t_g)$ be a power series without constant term and with invertible coefficients in $\mathcal{R}$, then there exists $a = (a_1, \dots, a_g) \in (m_{\mathcal{R}})^g$ such that* $v(X_{-1}(a)) < 1$. Assuming that result, which will be proved in Lemma 3.2, we conclude the proof.

If $P \in V(G)$ and $\eta(P) = 0$ for each $\eta \in M(G_k)$ then $P = 0$. In fact, $M(G_k) \otimes K = t^*_{\mathcal{U}(G)}(K)$, and the condition $\eta(P) = 0$ for each $\eta \in M(G_k)$ implies that the restriction of $\mathbf{biv}(P)$ to $\mathcal{E}(R)$ is 0. This, in view of 2.7, implies $P = 0$. □

LEMMA 3.2. *Let $X_{-1} = X_{-1}(t_1, \dots, t_g)$ be a power series without constant term and with invertible coefficients in $\mathcal{R}$, then there exists $a = (a_1, \dots, a_g) \in (m_{\mathcal{R}})^g$ such that $v(X_{-1}(a)) < 1$.*

PROOF. Let φ_{n_0} be the initial homogeneus form of X_{-1} and n_0 be its degree. If $\nu = (\nu_1, \dots, \nu_g)$ is the greatest multi-index with respect to the lexicographic order for which there exists a monomial $b_\nu t^\nu$ with non-zero coefficient in φ_{n_0}, we'll show that there exists $(a_1, \dots, a_g) \in (m_{\mathcal{R}})^g$, whose values are an increasing sequence of rational numbers $0 < c_1 < c_2 < \dots < c_g$ such that the value of the monomial $b_\nu t^\nu(a)$ will be strictly less than the value of any other monomial which appears in the series $X_{-1}(t)$ calculated in a.

By hypothesis the coefficients of the series are invertible so they don't give any contribution to the value of the monomials. This means that this estimate is only a matter of exponents.

The condition $0 < c_g < 1/n_0$ is sufficient to guarantee that any monomial appearing in φ_{n_0} assumes a value less than 1 in $X_{-1}(a)$; hence we make that assumption. In particular, this condition allows us to conclude in the case $g = 1$, and then we can invoke a recursive hypothesis on the number g of variables involved.

If one sets the conditions

$$\frac{n_0}{n_0+1} c_g < c_1 < \dots < c_g < \frac{1}{n_0}, \tag{3.3}$$

then any monomial of degree greater than n_0 has value strictly greater than the value of the monomials in φ_{n_0}; this means that we are restricted to comparing the value of $b_\nu t^\nu(a)$ with the value of the other monomials in φ_{n_0} calculated in a. We leave to the reader the easy verification that 3.3 gives the conclusion when $g = 2$, and we go on with the proof of the general case.

Let $\nu' \neq \nu$, be a multindex corresponding to a monomial appearing with non-zero coefficient in φ_{n_0}; and suppose $\nu'_1 \neq \nu_1$. The hypothesis that $\nu = (\nu_1, \dots, \nu_g)$ is the greatest multi-index with respect to the lexicographic order for which there exists a monomial $b_\nu t^\nu$ with non-zero coefficient in φ_{n_0}, implies that $\nu'_1 < \nu_1$, and then the value of $b_{\nu'} t^{\nu'}(a)$ satisfies

$$\nu'_1 c_1 + \nu'_2 c_2 + \dots + \nu'_g c_g \geq (\nu_1 - 1)c_1 + (n_0 - \nu_1 + 1)c_2.$$

On the other hand, the value of $b_\nu t^\nu(a)$ satisfies

$$\nu_1 c_1 + \nu_2 c_2 + \dots + \nu_g c_g \leq \nu_1 c_1 + (n_0 - \nu_1)c_g.$$

Thus the value of $b_{\nu'} t^{\nu'}(a)$ is strictly greater than the value of $b_\nu t^\nu(a)$ when the following inequality holds

$$\nu_1 c_1 + (n_0 - \nu_1)c_g < (\nu_1 - 1)c_1 + (n_0 - \nu_1 + 1)c_2;$$

which is equivalent to

$$\frac{(n_0 - \nu_1)c_g + c_1}{n_0 - \nu_1 + 1} < c_2. \tag{3.4}$$

The easy remark that

$$c_1 < \frac{(n_0 - \nu_1)c_g + c_1}{n_0 - \nu_1 + 1} < c_g$$

shows that the choice of c_2 is compatible with the given conditions.

Now, if ν' is a multindex corresponding to a monomial appearing with non-zero coefficient in φ_{n_0}, and $\nu'_1 = \nu_1$, then one has $\nu_2 + \ldots + \nu_g = n_0 - \nu_1 = \nu'_2 + \ldots + \nu'_g$, and

$$\frac{(n_0 - \nu_1)c_g}{n_0 - \nu_1 + 1} < c_2 < \ldots < c_g;$$

this means that we are reduced to a problem of $g-1$ variables and then, by the recursive hypothesis, we can suppose the existence of $c_2 < \ldots < c_g$ such that $\nu_2 c_2 + \ldots + \nu_g c_g$ is strictly less than $\nu'_2 c_2 + \ldots + \nu'_g c_g$ for any other multi-index ν' corresponding to a monomial with non-zero coefficient in φ_{n_0}, and with $\nu'_1 = \nu_1$. This concludes our proof. □

REMARK 3.5. Our next goal is to perform a comparison between the pairing $p' : M(G_k) \times T(G) \to \mathbf{biv}\,\mathcal{R}$ obtained by restriction of p, and the Tate pairing $\mathbf{t} : V(\tilde{G}) \times T(G) \to \mathcal{R}$ (cf. [TA]) In order to do that, we begin by recalling how $\mathbf{t}$ works: if $\xi \in V(\tilde{G})$, then, with the notation of 2.1, it is an element of $\mathcal{D}_{A_C}$, *i.e.* $\xi = (x_0, x_1, \ldots)$, where $x_i : \tilde{R} \to A_C$ is a continuous algebra homomorphism, and $x_{i+1} \circ pi = x_i$. On the other hand, as explained at the beginning of § 2, any point $P \in T(G)$ can be identified with an element of $mult(\tilde{R}\hat{\otimes}_A A_C)$. This implies that, after an extension of the map x_i by linearity and continuity, the element $x_i(P) \in A_C$ makes perfectly sense, and one has $x_{i+1}(P)^p = x_i(P)$; so that $(x_0(P), x_1(P), \ldots) \in \mathcal{R}$. The pairing $\mathbf{t}$ is defined by setting

$$\mathbf{t}(\xi, P) = (x_0(P), x_1(P), \ldots).$$

We will also use the notation $\mathbf{t}(\xi, P)$ when $\xi \in \tilde{G}(A_C)$: in this case it simply means $\xi(P)$.

Now we shall give an additive version of $\mathbf{t}$: remark that $\mathbf{t}(\xi, P) \equiv 1 \bmod \mathcal{R}^+$, so that the Teichmüller representative $\{\mathbf{t}(\xi, P)\}$ of $\mathbf{t}(\xi, P)$ in $\mathbf{W}(\mathcal{R})$ satisfies the condition of Corollary 7.3, therefore there exists $\log\{\mathbf{t}(\xi, P)\}$ in $\mathbf{biv}\,\mathcal{R}$.

DEFINITION 3.6. From now on we will denote by

$$\ell : V(\tilde{G}) \times T(G) \to \mathbf{biv}\,\mathcal{R},$$

the map defined by $\ell(\xi, P) = \log\{\mathbf{t}(\xi, P)\}$; this map is called the *logarithmic Tate pairing*.

REMARK 3.7. Assume $\xi \in T(\tilde{G})$; with the above notations, this implies $x_i(P)^{p^i} = 1$ and then $\ell(T(\tilde{G}) \times T(G)) = \mathbb{Z}_p\tau$, where $\tau = \log\{\varepsilon\}$ is the element of $\mathbf{biv}\,\mathcal{R}$ defined in Remark 7.4; namely, $\mathbb{Z}_p\tau \cong \mathbb{Z}_p(\chi)$, where $\chi : \mathcal{G} \to \mathbb{Z}_p^*$ is the cyclotomic character and $\mathbb{Z}_p(\chi)$ denotes the Tate twist (Th. 3 of [TA]).

In what follows we consider the ring $\mathrm{Hom}^{\mathrm{cont}}_{A\mathrm{mod}}(\mathcal{E}(\tilde{R}), \mathbf{W}(\mathcal{R}))$, where the multiplicative structure is induced by duality from the comultiplicative structure of $\mathcal{E}(\tilde{R})$. We suppose such ring endowed with the topology having the family $(V_n = \{f | f(\mathcal{E}(\tilde{R}) \subseteq W_n\})_{n\in\mathbb{N}}$ as a fundamental system of neighbourhoods of zero (cf. § 5 for the definition of the ideals W_n of $\mathbf{W}(\mathcal{R})$).

Clearly $\mathrm{Hom}^{\mathrm{cont}}_{A\text{-mod}}(\mathcal{E}(\tilde{R}), \mathbf{W}(\mathcal{R}))$ is separated and complete with respect to this topology.

PROPOSITION 3.8. *Let $\tilde{\xi} : \mathcal{E}(\tilde{R}) \to \mathbf{W}(\mathcal{R})$ be a $\mathbf{W}(\mathcal{R})$-valued point of $\mathcal{U}(\tilde{G})$, and denote by ϵ the coidentity of $\mathcal{E}(\tilde{R})$; then the series*

$$\mathrm{Log}\,\tilde{\xi} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\tilde{\xi} - \varepsilon)^n}{n} = \lim_{n\to\infty} \frac{\tilde{\xi}^{p^n} - \varepsilon}{p^n},$$

converges in $\mathrm{Hom}^{\mathrm{cont}}_{A\text{-mod}}(\mathcal{E}(\tilde{R}), \mathbf{Biv}\,\mathcal{R})$. *More precisely,* $\mathrm{Log}\,\tilde{\xi}$ *is an element of* $t_{\mathcal{U}(\tilde{G})}(K) \otimes \mathbf{Biv}\,\mathcal{R}$.

PROOF. Once the convergence of the series is proved, it is standard to check that its sum coincides with the above limit. To see that the operator $\mathrm{Log}\,\tilde{\xi}$ is a derivation is a purely formal matter.

Now we prove that the above series converges. Since $\tilde{\xi}$ is continuous, there exists a real number $a > 0$ such that $\tilde{\xi}(\mathcal{E}(\tilde{R})^+) \subseteq W_a$. Now, let $x \in \mathcal{E}(\tilde{R})$, then $(\tilde{\xi} - \varepsilon)(x) = \tilde{\xi}(x - \varepsilon(x)) \in W_a$; as a consequence,

$$\begin{aligned}\frac{(\tilde{\xi} - \varepsilon)^n}{n}(x) &= \frac{1}{n}\mu[(\tilde{\xi} - \varepsilon) \otimes \ldots \otimes (\tilde{\xi} - \varepsilon)(\mathbf{P} \otimes \ldots \otimes i) \circ \ldots \circ (\mathbf{P} \otimes i) \circ \mathbf{P}x] \\ &\in \frac{1}{n} W_{na} = W_{na - v(n)}.\end{aligned}$$

This implies that the sequence

$$\left(\frac{(\tilde{\xi} - \varepsilon)^n}{n}\right)_{n\in\mathbb{N}}$$

satisfies the Cauchy condition. Since $\mathrm{Hom}^{\mathrm{cont}}_{A\mathrm{mod}}(\mathcal{E}(\tilde{R}), \mathbf{Biv}\,\mathcal{R})$ is complete, our verification is concluded. □

From now on we will denote by L the map from $V(\tilde{G})$ to $M(G_k) \otimes \mathbf{Biv}\,\mathcal{R}$ obtained by composition of the map defined in the above Proposition 3.4 with the canonical identifications:

$$V(\tilde{G}) \cong \mathcal{U}(\tilde{G}(\mathbf{W}(\mathcal{R}))) \qquad \text{(cf. 2.7)},$$

$$t_{\mathcal{U}(\tilde{G})}(K) \otimes \mathbf{Biv}\,\mathcal{R} \cong M(G_k) \otimes \mathbf{Biv}\,\mathcal{R} \qquad \text{(cf. 1.4 and 1.6)}.$$

The next theorem is the main result on the comparison between the Tate pairing and the pairing obtained *via* Witt realization.

THEOREM 3.9. *Let* $\ell : V(\tilde{G}) \times T(G) \to \mathbf{Biv}\,\mathcal{R}$ *be the logarithmic Tate pairing; let*

$$p : (M(G_k) \otimes \mathbf{Biv}\,\mathcal{R}) \times V(G) \to \mathbf{Biv}\,\mathcal{R} \text{ and}$$

$$b : (M(\tilde{G}_k) \otimes \mathbf{Biv}\,\mathcal{R}) \times (M(G_k) \otimes \mathbf{Biv}\,\mathcal{R}) \to \mathbf{Biv}\,\mathcal{R}$$

be the extension by linearity of the pairings defined in 3.1, *and* 1.6, *respectively; then:*

$$\ell(\xi, P) = p(L(\xi), P) = b(L(\xi), L(P)).$$

The proof of 3.9 is given in three steps: 3.10, 3.11, and 3.12. We start here by giving some definitions. As remarked in the proof of Proposition 2.7, a point $\tilde{\xi}$ of $\mathcal{U}(\tilde{G})(\mathbf{W}(\mathcal{R}))$ determines an element of $\mathrm{Hom}^{\mathrm{cont}}_{\mathrm{Amod}}(\tilde{\mathfrak{R}}\hat{\otimes}_A \mathbf{W}(\mathcal{R}), \mathbf{W}(\mathcal{R}))$; and in view of Proposition 2.10, to a point $P \in T(G)$ there corresponds an element of $mult(\tilde{\mathfrak{R}}\hat{\otimes}\mathbf{W}(\mathcal{R}))$. In the above notations the element $\tilde{\xi}(P) \in \mathbf{W}(\mathcal{R})$ is well-defined and this procedure gives rise to a map

$$\alpha : \mathcal{U}(\tilde{G})(\mathbf{W}(\mathcal{R})) \to \mathrm{Hom}_{\mathbb{Z}_p}(T(G), \mathbf{W}(\mathcal{R})),$$

defined by setting: $[\alpha(\tilde{\xi})](P) = \tilde{\xi}(P)$. In particular, from the definition of the set $mult(\tilde{\mathfrak{R}}\hat{\otimes}\mathbf{W}(\mathcal{R}))$, it follows that $\tilde{\xi}(P) \in U_{\mathbf{W}(\mathcal{R})}$ and then $\log(\tilde{\xi}(P)) \in \mathbf{biv}\,\mathcal{R}$ is well-defined (cf. Theorem 7.1).

Our first step is to clarify the relations between the above map α and the Tate pairing $\mathbf{t} : V(\tilde{G}) \times T(G) \to \mathcal{R}$ described in Remark 3.5.

LEMMA 3.10. *Notations as above and denote by* $\rho : \mathbf{W}(\mathcal{R}) \to \mathcal{R}$ *the map sending a vector into its zero component. Set* $\overline{P} = (i\hat{\otimes}\rho)P$, *and* $\xi = \rho \circ \tilde{\xi}$; *then*

$$\tilde{\xi}(P) = \{\mathbf{t}(\xi, \overline{P})\}.$$

PROOF. We use the notation of 2.5. Since $\tilde{\xi} \in \mathrm{Hom}^{\mathrm{cont}}_{A\text{-alg}}(\tilde{\mathfrak{R}}\hat{\otimes}_A \mathbf{W}(\mathcal{R}), \mathbf{W}(\mathcal{R}))$, it makes sense to consider ξ as an element of $\mathrm{Hom}^{\mathrm{cont}}_{k\text{-alg}}(\tilde{\mathfrak{R}}_k\hat{\otimes}_k \mathcal{R}, \mathcal{R})$. On the other hand, since $P \in mult(\tilde{\mathfrak{R}}\hat{\otimes}_A \mathbf{W}(\mathcal{R}))$, then $\overline{P} \in mult(\tilde{\mathfrak{R}}_k\hat{\otimes}_k \mathcal{R})$. The commutativity of the diagram of 2.5, implies that $\rho(\tilde{\xi}(P)) = \xi(\overline{P})$, namely this means $\tilde{\xi}(P) = (\mathbf{t}(\xi, \overline{P}), \ldots)$.

It remains to prove that this vector is in fact a Teichmüller representative, *i.e.*

$$\tilde{\xi}(P) = (\mathbf{t}(\xi, \overline{P}), 0, 0, \ldots).$$

Now recall how to construct the point P starting from $\overline{P}$, as explained in Proposition 2.10; $P = \lim_{n\to\infty} ((pi)^{-n} P')^{p^n}$, where P' is any element of $\tilde{R}\hat{\otimes}\mathbf{W}(\mathcal{R})$, such that $\overline{P} = (i\hat{\otimes}\rho)P'$ and $P' \equiv 1 \bmod (\tilde{R}\hat{\otimes}\mathbf{W}(\mathcal{R}))^+$.

By the continuity of the map $\tilde{\xi}$, we have $\tilde{\xi}(P) = \lim_{n\to\infty} ((pi)^{-n} \tilde{\xi}(P'))^{p^n}$; this implies that the components with positive indices of the Witt vector $\tilde{\xi}(P)$ are all equal to zero. □

LEMMA 3.11. *Let* $\alpha : \mathcal{U}(\tilde{G})(\mathbf{W}(\mathcal{R})) \to \mathrm{Hom}_{\mathbf{Z}_p}(T(G), \mathbf{W}(\mathcal{R}))$ *be the map defined above and denote by* $\beta : t_{\mathcal{U}(\tilde{G})}(K) \otimes \mathbf{Biv}\,\mathcal{R} \to \mathrm{Hom}_{\mathbf{Z}_p}(T(G), \mathbf{Biv}(\mathcal{R}))$, *the map defined by setting* $\beta(\delta)(P) = p(\delta, P)$*; then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{U}(\tilde{G})(\mathbf{W}(\mathcal{R})) & \xrightarrow{\mathrm{Log}} & t_{\mathcal{U}(\tilde{G})}(K) \otimes \mathbf{biv}\,\mathcal{R} \\ \downarrow \alpha & & \downarrow \beta \\ \mathrm{Hom}(T(G), U_{W(\mathcal{R})}) & \xrightarrow{\log} & \mathrm{Hom}(T(G), \mathbf{biv}(\mathcal{R})) \end{array}$$

where the map log *in the bottom row is defined by composition with the logarithm map (cf. Theorem* 7.1*).*

PROOF. Since $P \in mult(\mathfrak{R} \hat{\otimes} \mathbf{W}(\mathcal{R}))$, one deduces $\tilde{\xi}(P) = \{\mathbf{t}(\xi, \overline{P})\} \in U_{\mathbf{W}(\mathcal{R})}$. Now, in view of the continuity of the map "evaluation in P"

$$P : \mathrm{Hom}^{\mathrm{cont}}_{A\text{-alg}}(\mathfrak{R} \hat{\otimes}_A \mathbf{W}(\mathcal{R}), \mathbf{W}(\mathcal{R})) \to \mathbf{W}(\mathcal{R}),$$

defined by $\tilde{\xi} \to \tilde{\xi}(P)$, one deduces

$$\log[\tilde{\xi}(P)] = \lim_{n\to\infty} \frac{\tilde{\xi}(P)^{p^n} - 1}{p^n} = \left(\lim_{n\to\infty} \frac{\tilde{\xi}^{p^n} - \varepsilon}{p^n}\right)(P) = [\mathrm{Log}\,\tilde{\xi}](P), \qquad \square$$

REMARK 3.12. Let $\eta = (\eta_1, \eta_2, \dots, \eta_h)$ be a basis of $t_{\mathcal{U}(\tilde{G})}(K)$ and $(\delta_1, \delta_2, \dots, \delta_n)$ the basis of $t^*_{\mathcal{U}(\tilde{G})}(K)$ dual of η. Then, taking the Taylor expansion of $\mathrm{Log}\,\tilde{\xi}$, we have:

$$\mathrm{Log}\,\tilde{\xi} = \sum_{i=1}^{h} \delta_i(\tilde{\xi})\eta_i,$$

and so (cf. 1.6):

$$[\mathrm{Log}\,\tilde{\xi}](P) = \sum_{i=1}^{h} \delta_i(\tilde{\xi})\eta_i(P) = b(L(\tilde{\xi}), L(P)).$$

This verification concludes the proof of Theorem 3.5 giving the required comparison between the above pairings.

REMARK 3.13. Let h be an integral of the second kind of G, as explained in 1.4 (iv); after the "canonical embedding" in $\mathbf{biv}\,\mathfrak{R}_k$, the limit $\eta = \lim_{n\to\infty} (pi)^{-n}(p^n h)$ exists and it belongs to the submodule $M(G_k) \otimes K \cong H^1_{dR}(G) \otimes K$. Such limit is the cohomology class represented by h. Since, for every P, the map of evaluation in P is continuous on $\mathbf{biv}\,\mathfrak{R}_k$, one deduces by the above limit that

$$p(\eta, P) = \lim_{n\to\infty} p^n((pi)^{-n}h)(P).$$

In view of the comparison between the rings $\mathbf{B}_{DR}^{+}$ and $\mathbf{Biv}\,\mathcal{R}$ given in Remark 7.12, the limit in the right hand term is the quantity denoted by $\int_P dh$ by Colmez (cf. [COL]).

REMARK 3.14. Notation as in 3.13. Let $\eta' = \eta^{(<0)}$, *i.e.* the bivector whose components are defined as follows:

$$(\eta')_i = \begin{cases} \eta_i \text{ if } i < 0 \\ 0 \text{ if } i \geq 0 \end{cases},$$

then for $\eta \in M(G_k)$, one has $\eta = \lim_{n\to\infty} p^n((pi)^{-n}\eta')$.

Again, by using continuity arguments, it can be proved that $p(\eta, P) = \lim_{n\to\infty} p^n((pi)^{-n}\eta')\cdot(P)$. This limit is the key to the method of computing periods which makes use of the *Relèvement des covecteurs*, as developed by Fontaine in [FO1].

4. - Some consequences

The first consequence is a very easy description of the relations between $V_0(G) = T(G) \otimes \mathbb{Q}_p$ and the filtered Dieudonné module of the special fiber (cf. [FO1], ch. V, § 1, Thm. 1 and Remarque 1.8).

THEOREM 4.1. *The pairing p is compatible with the following filtrations:* $I_1(G) \hookrightarrow M(G_k)$, $T(G) \hookrightarrow V(G)$, $\ker\Theta \hookrightarrow \mathbf{Biv}\,\mathcal{R}$; *more precisely it gives the following isomorphisms:*

(i) $$V_0(G) \cong \mathrm{Hom}^{\mathrm{fil}}_{A[F,V]}(M(G_k), \mathbf{biv}\,\mathcal{R});$$

(ii) $$M(G_k) \otimes K \cong \mathrm{Hom}_{\mathbb{Q}_p[\mathcal{G}]}(T(G), \mathbf{biv}\,\mathcal{R}).$$

In particular (i) *is an isomorphism of* $\mathbb{Q}_p[\mathcal{G}]$*-modules, and* (ii) *is an isomorphism of* $K[F,V]$*-modules.*

PROOF. i) As remarked during the proof of Theorem 3.1 the bialgebra $\mathfrak{R}_k$ is generated by the components of the elements of $M(G_k)$, so

$$\mathrm{Hom}_{A[F,V]}(M(G_k), \mathbf{biv}\,\mathcal{R}) \cong \mathrm{Hom}^{\mathrm{cont}}_{k\mathrm{alg}}(\mathfrak{R}_k, \mathcal{R}) = V(G).$$

The map $\log : T(G) \otimes C \to I_1(G_{A_C}) \cong I_1(G) \otimes C$ is surjective because it is the cotangent map of the map $G(A_C) \to \mathrm{Hom}(T(\tilde{G}), \mathbf{G}_m(A_C))$, induced by the Tate pairing, where $\mathbf{G}_m$, as usual, denotes the formal multiplicative group (cf. [TA], § 4). Then, there exists points $\xi_1, \ldots, \xi_g \in T(\tilde{G})$, whose image gives a C-basis of $I_1(G) \otimes C$. As a consequence, any element of $I_1(G) \otimes \mathbf{Biv}\,\mathcal{R}$ is congruent mod $I_1(G) \otimes \ker \Theta$ to a linear combination of $\operatorname{Log} \xi_j$'s, $j = 1, \dots, g$. This means that for any $P \in V(G)$ and any $\eta \in I_1(G)$, $p(\eta, P)$ is congruent mod $\ker \Theta$ to a linear combination with coefficients in $\mathbf{Biv}\,\mathcal{R}$ of $\log\{\mathbf{t}(\xi_j, P)\}$ for $j = 1, \dots, g$ (cf. 3.6).

Now it suffices to remark that $\log\{\mathbf{t}(\xi, P)\} \in \ker \Theta$ if, and only if, $\Theta(\mathbf{t}(\xi, P))$ is a root of unity (cf. Theorem 7.1); in fact this happens if, and only if, $P \in V_0(G)$.

ii) We recall here that, after Proposition 7.11, one has:

$$B(i) = \{\xi \in \mathbf{Biv}\,\mathcal{R} \,|\, s(\xi) = \chi(s)^i \xi \;\; \forall s \in \mathcal{G}\} = \tau^i K.$$

Now we recall that, by [TA] (cf. also the following Theorem 4.4), the Tate module is a Hodge-Tate representation of $\mathcal{G}$, which means that $T(G) = T_0 \oplus T_1$, where $T_i = \{P \in T(G) | s(P) = \chi(s)^i P \;\; \forall s \in \mathcal{G}\}$, $i = 0, 1$.

Theorem 3.1 gives an injective homomorphism of K-vector spaces:

$$M(G_k) \otimes K \to \operatorname{Hom}_{\mathbb{Q}_p[\mathcal{G}]}(T(G), \mathbf{Biv}\,\mathcal{R}) \cong \operatorname{Hom}_{\mathbb{Q}_p}(T_0, B(0)) \oplus \operatorname{Hom}_{\mathbb{Q}_p}(T_1, B(1)),$$

which is an isomorphism, because both spaces have the same dimension h.

We remark here that, after the proof of part (ii) of Theorem 4.1 and Remark 7.4, which implies that the Bivector τ belongs to the K-module $\mathbf{biv}\,\mathcal{R}$, we can state, more precisely,

$$M(G_k) \otimes K \cong \operatorname{Hom}_{\mathbb{Q}_p[\mathcal{G}]}(T(G), \mathbf{biv}\,\mathcal{R}), \qquad \square$$

We are interested in another consequence of our constructions, namely the Hodge-Tate decomposition of $T(G) \otimes C$. We start by proving a simple lemma; recall that L is the map defined on $V(\tilde{G})(A_C)$ with values in $M(G_k) \otimes \mathbf{Biv}\,\mathcal{R}$ obtained by composing the map defined in 3.4 with the following canonical identifications:

$$V(\tilde{G})(A_C) \cong \mathcal{U}(\tilde{G}(\mathbf{W}(\mathcal{R})), \quad t_{\mathcal{U}(\tilde{G})}(K) \otimes \mathbf{Biv}\,\mathcal{R} \cong M(G_k) \otimes \mathbf{Biv}\,\mathcal{R}.$$

LEMMA 4.2. *Notations as above and denote by* $\pi : M(G_k) \to t_{\tilde{G}}(K)$ *the canonical map which arises after the identification of* $t_{\tilde{G}}(K)$ *with* $M(G_k)/I_1(G)$. *Then the following diagram commutes:*

$$\begin{array}{ccc} V(\tilde{G}) & \xrightarrow{L} & M(G_k) \otimes \mathbf{Biv}\,\mathcal{R} \\ pr_0 \downarrow & & \downarrow \pi \otimes \Theta \\ \tilde{G}(A_C) & \xrightarrow{\log} & t_{\tilde{G}}(C) \end{array}$$

PROOF. Let $P \in V(\tilde{G})(A_C)$. Then, in view of 2.6 and 2.7, its projection onto the first term of the sequence $(\tilde{G}(A_C) \leftarrow \tilde{G}(A_C) \leftarrow \dots)$, $pr_0(P)$, can be obtained by composing the following maps:

$$\tilde{R} \xrightarrow{j} \mathcal{E}(\tilde{R}) \xrightarrow{P'} \mathbf{W}(\mathcal{R}) \xrightarrow{\Theta} A_C,$$

where $j : \tilde{R} \to \mathcal{E}(\tilde{R})$ is the immersion which corresponds to the canonical projection $p : \mathcal{U}(\tilde{G}) \to \tilde{G}$, and P' is the point of $\mathcal{U}(\tilde{G})$ which corresponds to P in the identification $V(\tilde{G})(A_C) \cong \mathcal{U}(\tilde{G})(\mathbf{W}(\mathcal{R}))$.

Since, as is well known, in the pairing b defined in 1.6, the submodule $I_1(\tilde{G})$ is orthogonal to $I_1(G)$, the map $\pi : M(G_k) \to M(G_k)/I_1(G)$ is the adjoint of the cotangent map $t^*p : I_1(\tilde{G}) \to M(\tilde{G}_k)$; then π is the tangent map to p and the commutativity of our diagram is a standard fact of Lie theory. □

COROLLARY 4.3. *Let $N \subseteq M(G_k)$ be a sub-A-module such that $M(G_k) = I_1(G) \oplus N$, then*

$$L(T(\tilde{G})(A_C)) \subseteq (I_1(G) \otimes \mathbf{Biv}\,\mathcal{R}) \oplus (N \otimes \ker\Theta).$$

PROOF. Let $P \in T(\tilde{G})(A_C)$; since $pr_0(P) = 0$, by 4.2 it follows that $(\pi \otimes \Theta)(LP) = 0$. Now, since $t_{\tilde{G}}(C) \cong (M(G_k)/I_1(G)) \otimes G$, we conclude that

$$(I_1(G) \otimes \mathbf{Biv}\,\mathcal{R}) \oplus (N \otimes \ker\Theta) = \ker(\pi \otimes \Theta). \qquad \square$$

THEOREM 4.4 (Tate). *There is the following decomposition of Galois modules:*

$$\mathrm{Hom}(TG, C) \cong t^*_{G}(C) \otimes C(-1) \oplus t_{\tilde{G}}(C).$$

PROOF. Let consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}(TG, \ker\Theta) & \xleftarrow{\phi_p} & & & I_1(G) \otimes \mathbf{Biv}\,\mathcal{R} \oplus N \otimes \ker\theta \\
 & & & \nearrow{\scriptstyle L} & \\
f\downarrow & & T\tilde{G}(A_C) & & \downarrow h \\
 & \swarrow{\scriptstyle \phi_\ell} & & & \\
\mathrm{Hom}(TG, C(1)) & \xleftarrow[g]{} & & & I_1(G) \otimes C \oplus N \otimes C(1)
\end{array}$$

where the maps not yet defined have the following meanings: f sends ξ to $\varphi_1 \circ \xi$, where $\varphi_1 : \ker\Theta \to \ker\Theta/(\ker\Theta)^2 \cong C(1)$ is the map of Corollary 7.10; ϕ_p is the extension by $\mathbf{Biv}\,\mathcal{R}$-linearity of the map induced by the pairing p of 3.1 (observe that, by 4.1, $\phi_p(I_1(G)) \subseteq \mathrm{Hom}(TG, \ker\Theta)$, so $\mathrm{Im}\,\phi_p \subseteq \mathrm{Hom}(TG, \ker\Theta)$). Finally, ϕ_ℓ is the map induced by the logarithmic Tate pairing (cf. 3.6).

The C-vector space $(I_1(G) \otimes C) \oplus (N \otimes C(1))$ is identified with $\dfrac{(I_1(G) \otimes \mathbf{Biv}\,\mathcal{R}) \oplus (N \otimes \ker \Theta)}{(I_1(G) \otimes \ker \Theta) \oplus (N \otimes (\ker \Theta)^2)}$ and h is the canonical projection; finally, since

$$\ker(f \circ \phi_p) \supseteq (I_1(G) \otimes \ker \Theta) \oplus (N \otimes (\ker \Theta)^2),$$

it makes sense to define g by requiring that $f \circ \phi_p = g \circ h$; clearly g is C-linear and compatible with the actions of $\mathcal{G}$.

By Theorem 3.9 we know that $\phi_\ell = f \circ \phi_p \circ L$; then, by the nondegeneration of ℓ, it follows that

$$\phi_\ell(T(\tilde{G})(A_C)) \otimes C = \mathrm{Hom}(T(G)(A_C), C(1)).$$

As a consequence g is a surjective morphism between C-vector spaces of the same dimension, therefore it is an isomorphism. □

5. - The ring Biv $\mathcal{R}(A_C)$

Let $A = \mathbf{W}(k)$, $\overline{K}$, $\overline{A}$, C, A_C, $\mathcal{G}$ maintain the meanings stated in § 1. As usual, we will denote by p a uniformizing parameter of A, and suppose that the valuation of C is normalized by setting $v(p) = 1$. In this section, following [FO3], we begin by describing some properties of the rings

$$\mathcal{R} = \mathcal{R}(A_C) = \varprojlim(A_C/pA_C \xleftarrow{x^p} A_C/pA_C \xleftarrow{x^p} \ldots), \tag{5.1}$$

and $\mathbf{W}(\mathcal{R})$, the ring of Witt vectors with components in $\mathcal{R}$; then we outline the construction of the ring $\mathbf{Biv}\,\mathcal{R}$ of Witt Bivectors with components in $\mathcal{R}$. We'll give a definition of $\mathbf{Biv}\,\mathcal{R}$ and of its topology slightly different from the classical definition of Barsotti, such new definition appears to be more appropriate for what follows.

If the elements x of $\mathcal{R}$ are represented as sequences, $x = (x_n)_{n \in \mathbf{N}}$, of elements of A_C/pA_C, where $x_{n+1}^p = x_n$, then the ring operations are defined by operating elementwise on the sequences. The description 5.1 shows that $\mathcal{R}$ *is a perfect ring of characteristic p.*

We observe that the elements of $\mathcal{R}$ can be represented as sequences of elements of A_C: precisely $x = (x_n)_{n \in \mathbf{N}} \in \mathcal{R}$ is represented by the sequence $(\hat{x}_n)_{n \in \mathbf{Z}}$ obtained as follows: for each $n \in \mathbb{N}$, denote by $x_n^\#$ a lifting of x_n to A_C and define

$$\hat{x}_n = \lim_{m \to \infty} x_{n+m}^{\# p^m}, \text{ if } n \geq 0 \text{ and } \hat{x}_n = \hat{x}_0^{p^{-n}}, \text{ if } n < 0. \tag{5.2}$$

In fact it is straightforward to see that the limits in 5.2 exist in A_C and do not depend on the choice of the elements $x_n^\#$, that $\hat{x}_{n+1}^p = \hat{x}_n$, and finally that the map

$(x_n)_{n\in\mathbb{N}} \to (\hat{x}_n)_{n\in\mathbb{Z}}$ is a set bijection between $\mathcal{R}$ and $\varprojlim(A_C \xleftarrow{x^p} A_C \xleftarrow{x^p} \ldots)$. The operations on the new set are defined by requiring that the previous bijection becomes a ring isomorphism:

$$(5.3) \qquad \widehat{(xy)}_n = \hat{x}_n\hat{y}_n \text{ and } \widehat{(x+y)}_n = \lim_{m\to\infty}(\hat{x}_{n+m} + \hat{y}_{n+m})^{p^m}.$$

It's easy to see that $\mathcal{R}$ *is an integral domain.* In fact since A_C is an integral domain,

$$(5.4) \qquad \hat{x}_n = 0 \text{ for some } n, \text{ implies that } \hat{x}_n = 0 \text{ for each } n.$$

The map v defined on $\mathcal{R}$ by setting

$$(5.5) \qquad v(x) = v_C(\hat{x}_0) \in \mathbb{Q}_{\geq 0} \cup \{\infty\},$$

is a valuation. In fact:

$$\begin{aligned} v(x+y) = v_C(\widehat{(x+y)}_0) &= \lim_{m\to\infty} v_C[(\hat{x}_m + \hat{y}_m)^{p^m}] \\ \geq \lim_{m\to\infty} \min\{p^m v_C(\hat{x}_m), p^m v_C(\hat{y}_m)\} &= \min\{v_C(\hat{x}_0), v_C(\hat{y}_0)\} \\ &= \min\{v(x), v(y)\}; \end{aligned}$$

and

$$v(xy) = v_C(\widehat{(xy)}_0) = v_C(\hat{x}_0) + v_C(\hat{y}_0) = v(x) + v(y);$$

and finally, as a consequence of 5.4,

$$v(x) = \infty \Leftrightarrow v_C(\hat{x}_0) = \infty \Leftrightarrow \hat{x}_0 = 0 \Leftrightarrow x = 0.$$

Since the elements of $\mathcal{R}$ can be represented by sequences of elements of A_C, the elements of its field of fractions can be represented by sequences of elements of C. As a consequence v extends to $\operatorname{Frac}\mathcal{R}$ and $\mathcal{R} = \{x \in \operatorname{Frac}\mathcal{R} : v(x) \geq 0\}$.

Let us summarize the properties of $\mathcal{R}$ in a final statement:

PROPOSITION 5.6. *Notations as above. The ring $\mathcal{R}(A_C)$ is a perfect ring of characteristic p. It is a complete valuation ring with respect to the (non-discrete) valuation defined in 5.5. Its residue field is isomorphic to $\overline{k} = A_C/m$, where m is the maximal ideal of A_C.*

Moreover, there is a natural action of $\mathcal{G}$ on $\mathcal{R}$.

PROOF. To prove the completeness of $\mathcal{R}$, it suffices to observe that, if x, $y \in \mathcal{R}$ and $v(x-y) > a > p^j$, then $x_i - y_i = 0$ in A_C/pA_C, for any $i \leq j$. In fact this implies that a Cauchy sequence in $\mathcal{R}$ converges componentwise.

Now consider the map $\mathcal{R} \to A_C$ which sends x to $\hat{x}_0$. This map is surjective and sends the valuation ideal, $\mathcal{R}^+ = \{x \in \mathcal{R} | v(x) > 0\}$, to the maximal ideal m of A_C. Thus it induces a surjective ring homomorphism from the residue field of $\mathcal{R}$ to $\overline{k}$.

As any element of $\mathcal{R}$ can be represented by a sequence of elements of A_C, it is natural to define the action of $\mathcal{G}$ as follows: $s(x) = (s(\hat{x}_n))_{n\in\mathbb{N}}$, for any $x = (x_n)_{n\in\mathbb{N}} \in \mathcal{R}$ and $s \in \mathcal{G}$. Since s is continuous, in view 5.3 we conclude that it acts as a ring homomorphism. □

Now let us consider the ring $\mathbf{W}(\mathcal{R})$, and denote an element $\xi \in \mathbf{W}(\mathcal{R})$ either as the sequence of its components or as the (infinite) sum of its *scomponents*:

$$\xi = (x_n)_{n\in\mathbb{N}} = \sum_{n=0}^{\infty} p^n\{x_n^{p^{-n}}\}, \tag{5.7}$$

where $\{y\} = (y, 0, 0 \ldots)$ denotes the Teichmüller representative in $\mathbf{W}(\mathcal{R})$ of an element $y \in \mathcal{R}$.

Let us consider the following families of ideals of $\mathbf{W}(\mathcal{R})$:

$$U_k(a) = \{\xi \in \mathbf{W}(\mathcal{R}) | n < k \Rightarrow v(x_n^{p^{-n}}) \geq a\},$$
$$V_k(a) = \{\xi \in \mathbf{W}(\mathcal{R}) | n < k \Rightarrow v(x_n) \geq a\},$$
$$W_a = \{\xi \in \mathbf{W}(\mathcal{R}) | n < a \Rightarrow v(x_n^{p^{-n}}) \geq a - n\},$$

where a is a non-negative real number and k is a non-negative integer. The linear topologies on $\mathbf{W}(\mathcal{R})$, which have one of the previous families as a fundamental system of neighbourhoods of 0, coincide. In fact one has

$$V_m(p^{m-1}a) \subseteq U_m(a) \subseteq V_m(a) \subseteq U_m(p^{-m}a);$$

and, putting $\langle a \rangle = \min\{n \in \mathbb{Z} \,|a \leq n\}$ for any positive real number a,

$$U_{\langle a \rangle}(a) \subseteq W_a \quad \text{and} \quad W_{m+b} \subseteq U_m(b).$$

Observe that, for this topology, the series 5.7 converges to ξ in $\mathbf{W}(\mathcal{R})$.

In what follows, we'll make use only of the ideals W_a's, thus, for the sake of completeness, we'll prove in the following Proposition 5.10 that these subsets are in fact ideals of $\mathbf{W}(\mathcal{R})$.

PROPOSITION 5.8 (Fontaine, cf. [FO3]). *Let* $\Theta_0 : \mathbf{W}(\mathcal{R}) \to A_C$ *the map defined by setting*

$$\Theta_0(\xi) = \sum_{n=0}^{\infty} p^n \hat{x}_{n,n},$$

where $\xi = (x_n)_{n\in\mathbb{N}}$, *and* $x_n = (\hat{x}_{n,i}))_{i\in\mathbb{Z}}$. *Then:*

a) Θ_0 *is a continuous and surjective homomorphism of rings, which commutes with the natural actions of* $\mathcal{G}$.

b) $\ker \Theta_0$ *is a principal ideal in* $\mathbf{W}(\mathcal{R})$.

PROOF. (a) The fact that Θ_0 is an homomorphism follows from the definition of Witt vectors. The continuity is proved by observing that, for any positive real number a, $\xi \in W_a$ implies that $v_C(\Theta_0(\xi)) > a$. Finally, the surjectivity can be estabilished as follows: if $a' \in A_C$, there exists an element $a \in \mathcal{R}$ such that $\hat{a}_0 = a'$ (cf. 5.2); then $\Theta_0(\{a\}) = a'$. The action of $\mathcal{G}$ on $\mathbf{W}(\mathcal{R})$ is defined by applying the action of $\mathcal{G}$ on $\mathcal{R}$ componentwise, then the equivariancy of Θ_0 follows directly from the definitions.

(b) Let π be an element of $\mathcal{R}$ such that $\hat{\pi}_0 = -p$; then, by the definition of Θ_0, the vector $\alpha = \{\pi\} + p$ belongs to $\ker \Theta_0$. Now we will prove that

$$\text{a vector } \eta = \sum_{n=0}^{\infty} p^n\{y_n\} \text{ spans } \ker \Theta_0 \text{ if, and only if, } v(y_1) = 0.$$

In fact if $\eta \in \ker \Theta_0$ is such that $v(y_1) > 0$, and $\xi = \sum_{n=0}^{\infty} p^n\{x_n\}$, then the scomponent z_1 of the vector $\xi\eta = \sum_{n=0}^{\infty} p^n\{z_n\}$ is equal to $z_1 = x_1y_0 + x_0y_1$. By observing that $v(y_1) > 0$ implies $v(y_0) > 0$, we conclude that $v(z_1) \geq \min\{v(y_1), v(y_0)\} > 0$. This implies that α does not belong to the ideal generated by η, hence that η cannot span $\ker \Theta_0$.

Now, let $\eta \in \ker \Theta_0$ be such that $v(y_1) = 0$. After observing that this implies $v(y_0) = 1$, we will show that given $\beta \in \ker \Theta_0$ and an integer $m \geq 0$, there exist $x_0, \dots, x_{m-1} \in \mathcal{R}$ and $\gamma_m \in \mathbf{W}(\mathcal{R})$, such that:

$$(5.9_m) \qquad \beta = \eta \left[\sum_{j=0}^{m-1} p^j\{x_j\} \right] + p^m \gamma_m.$$

5.9_1 follows by remarking that if $\beta \in \ker \Theta_0$ then $v(\beta_0) \geq 1$; in fact, $v(\beta_0) \geq 1$ implies that there exists $x_0 \in \mathcal{R}$ such that $\beta_0 = y_0x_0$, so that $\beta = \eta\{x_0\} + p\gamma_1$, for some $\gamma_1 \in \mathbf{W}(\mathcal{R})$. Now assume 5.9_{m-1} is true; this implies that $\Theta_0(\gamma_{m-1}) = 0$, so that $v(\gamma_{m-1,0}) \geq 1$, and therefore there exists $x_{m-1} \in \mathcal{R}$, such that $\gamma_{m-1,0} = y_0x_{m-1}$. Finally we obtain

$$\beta - \eta \left[\sum_{j=0}^{m-1} p^j\{x_j\} \right] = p^{m-1}(\gamma_{m-1} - \eta\{x_{m-1}\}) \in p^m\mathbf{W}(\mathcal{R}),$$

which is the content of 5.9_m. □

In particular, looking at the previous proof we conclude that

$$\ker \Theta_0 = \alpha\mathbf{W}(\mathcal{R}), \text{ where } \alpha = \{\pi\} + p$$

We give some further information about the topology of $\mathbf{W}(\mathcal{R})$ by proving the following:

PROPOSITION 5.10. *Notation as above.*

(a) *The submodules W_a are ideals of $\mathbf{W}(\mathcal{R})$ with the following property: $W_aW_b \subseteq W_{a+b}$.*

(b) *The ideal W_n can be decomposed as follows:*

$$W_n = (\ker \Theta_0)^n + p(\ker \Theta_0)^{n-1} + \ldots + p^n\mathbf{W}(\mathcal{R}).$$

PROOF. (a) First we prove that the subsets W_a are in fact subgroups of the additive group $\mathbf{W}(\mathcal{R})$. Let $\xi = (x_0, x_1, \ldots)$ and $\eta = (y_0, y_1, \ldots)$ be elements of W_a. In order to prove that $\xi + \eta \in W_a$, let recall that, if we give the weight p^i to x_i and y_i, for $i = 0, \ldots, m$, then the Witt polynomial giving the m-th component of $\xi + \eta$ is isobaric with weight p^m. In fact, since $v(x_i) \geq p^i(a-i) \leq v(y_i)$, for each $i < a$, given a monomial $x_0^{\mu_0}y_0^{\nu_0} \ldots x_m^{\mu_m}y_m^{\nu_m}$ of weight p^m, i.e. such that $\sum_{j=0}^{m} p^j(\mu_j + \nu_j) = p^m$, assuming $m \leq a$, we get:

$$\begin{aligned} v(x_0^{\mu_0}y_0^{\nu_0} \ldots x_m^{\mu_m}y_m^{\nu_m}) &\geq \sum_{j=0}^{m} p^j(a-j)(\mu_j + \nu_j) \\ &\geq \sum_{j=0}^{m} p^j(a-m)(\mu_j + \nu_j) = p^m(a-m). \end{aligned} \tag{5.11}$$

This inequality, in view of the isobaricity, implies that $\xi + \eta \in W_a$. Since the arguments to prove that $-\xi \in W_a$ are essentially the same, we pass to study the product of two vectors.

Now represent the vectors by means of their scomponents, i.e. $\xi = \sum_{n=0}^{\infty} p^n\{x'_n\}$. We assume $\xi \in W_a$, and we will prove that

$$\xi\varsigma = \sum_{i=0}^{\infty} p^i \left(\sum_{j=0}^{i} \{x'_j z_{i-j}\} \right) \in W_a,$$

for any $\varsigma = \sum_{n=0}^{\infty} p^n\{z_n\} \in \mathbf{W}(\mathcal{R})$. Our assumption is $v(x'_n) \geq a - n$ for any $n < a$; and, denoting by y_n the n^{th} scomponent of $\xi\varsigma$, we have to check that $v(y_n) \geq a - n$, for $n < a$.

First, $v(y_0) = v(x'_0z_0) = v(x'_0) + v(z_0) \geq v(x'_0) \geq a$: this is the expected inequality for $i = 0$. Then we will use an inductive argument: assuming that $i < a$, and the claim be true for any index less than i, we will prove it's true for i. By the inductive hypothesis and by the recalled isobaricity of the Witt polynomials giving the components of a sum,

$$y_i = \sum_{j=0}^{i} x_j z_{i-j} + \text{ (terms with value } \geq a - i + 1).$$

Since $v(x'_j z_{i-j}) = v(x'_i) + v(z_{i-j}) \geq v(x_i) \geq a - j \geq a - i$, the proof of the first claim is concluded.

Now we will check the inclusion $W_a W_b \subseteq W_{a+b}$. With the notations as above, assume $\xi \in W_a$ and $\varsigma \in W_b$. Observe that it suffices to prove the relations

$$(5.12) \qquad v(x'_j z_{h-j}) \geq a + b - n,$$

for $h < a+b$ and $j = 0, \dots, h$. In fact, if 5.12 is true, then the 0-th component of the vector $\eta_h = \sum_{j=0}^{j} \{x'_j z_{h-j}\}$ has the correct value; since the i-th component of η_h is a sum of monomials with degree p^i in the arguments $x'_j z_{h-j}$, 5.12 implies that $p^h \eta_h \in W_{a+b}$. Finally, since W_{a+b} is a group, $\xi\varsigma \in W_{a+b}$.

To check 5.12 we assume $a \leq b$ and distinguish three cases:

i) $h < a$; then $j < a$ and $h - j < a \leq b$, so that

$$v(x'_j z_{h-j}) = v(x'_j) = v(z_{h-j}) \geq a - j + b - h + j = a + b - h.$$

ii) $a \leq h < b$; then $h - j < b$, so either $0 \leq j < a$ and then the result follows as above, or $a \leq j$ and then $v(x'_j z_{h-j}) \geq v(z_{h-j}) \geq b - h + j \geq a + b - h$.

iii) $b \leq h < a + b$; then either $0 \leq j \leq h - b < a$ so that $v(x'_j z_{h-j}) \geq a - j \geq a+b-h$; or $h-b < j < a$, in which case $b > h-j > i-a$ and we are done; or, finally, $a \leq j \leq h$ then $b > h-j$, and then $v(x'_j z_{h-j}) \geq b-h+j \geq a+b-h$. This concludes the proof of part (a).

(b) It is immediate to check that $\ker \Theta_0 + p\mathbf{W}(\mathcal{R}) \subseteq W_1$. On the other hand, by definition, a vector $\xi = \sum_{n=0}^{\infty} p^n \{x'_n\}$ belongs to W_1 if and only if $v(x'_0) \geq 1$, i.e. $x'_0 = \pi y$ for some $y \in \mathcal{R}$; this implies $\xi - \alpha\{y\} \in p\mathbf{W}(\mathcal{R})$, which proves our claim for $n = 1$. Now, using the last claim of part (a), we deduce that

$$(\ker \Theta_0)^n + p(\ker \Theta_0)^{n-1} + \ldots + p^n \mathbf{W}(\mathcal{R}) \subseteq W_n.$$

Conversely, let $\xi \in W_n$, then $x'_0 = \pi^n y$ for some $y \in \mathcal{R}$; as a consequence $\xi - \alpha^n\{y\} \in p\mathbf{W}(\mathcal{R}) \cap W_n$. Finally, since $W_n \cap p\mathbf{W}(\mathcal{R}) \subseteq pW_{n-1}$, we can conclude by an inductive argument. □

In what follows it is useful to know that the topology on $\mathbf{W}(\mathcal{R})$ is induced by a valuation.

DEFINITION 5.13. Let $w \cdot \mathbf{W}(\mathcal{R}) \to \mathbb{Q} \cup \{\infty\}$ be the map defined as follows: $w(0) = \infty$ and

$$w(\alpha) = \min\{p^{-i} v(a_i) + i | i \geq 0\}$$

when $0 \neq \alpha = (a_0, a_1, \dots) \in \mathbf{W}(\mathcal{R})$.

REMARKS 5.14. i) In 5.13 the minimum exists, because the sequence $i \to p^{-i} v(a_i) + i$ diverges with i;

ii) $$W_b = \{\alpha \in \mathbf{W}(\mathcal{R}) : w(\alpha) \geq b\};$$

in fact, $\alpha = (a_0, a_1, \ldots) \in W_b$ if and only if $i < b \Rightarrow p^{-i}v(a_i) \geq b - i$, i.e., if and only if $b \leq w(\alpha)$.

The next step is to prove the following

PROPOSITION 5.15. *Notation as above. The map w is a valuation on $\mathbf{W}(\mathcal{R})$.*

PROOF. From 5.13 it is clear that $\xi \neq 0 \Rightarrow w(\xi) \neq \infty$. Now, given the vectors $\xi = (x_0, x_1, \ldots)$ and $\eta = (y_0, y_1, \ldots)$ such that $w(\xi) = r$ and $w(\eta) = s$, we want to compute $w(\xi + \eta)$. If $r \leq s$, then ξ and η belong both to the ideal W_r, so $\xi + \eta \in W_r$; by 5.14 ii) this implies $w(\xi + \eta) \geq r = \min\{w(\xi), w(\eta)\}$. Now assume $r < s$. Let consider z_{i_0}, the i_0-th component of the sum $\xi + \eta$, where i_0 is the smallest index such that $r = w(\xi) = p^{-i_0}v(x_{i_0}) + i_0$:

$$z_{i_0} = x_{i_0} + y_{i_0} + \begin{pmatrix} \text{monomials of weight } p^{i_0} \\ \text{in } x_i \text{ and } y_j \\ \text{with } i \text{ and } j \text{ less than } i_0 \end{pmatrix}.$$

Since $i_0 \leq r < s$, one has $v(y_{i_0}) \geq p^{i_0}(s - i_0) > p^{i_0}(r - i_0) = v(x_{i_0})$; moreover for i, $j < i_0$, $v(x_i) > p^i(r - i)$ and $v(y_j) \geq p^j(s - j) > p^j(r - j)$. These two inequalities, together with the isobaricity, imply that each monomial in the sum inside the brackets has value strictly greater than $p^{i_0}(r - i_0)$ (cf. 5.11). This implies that $v(z_{i_0}) = p^{i_0}(r - i_0) = v(x_{i_0})$, and finally that $w(\xi + \eta) = r$.

Now, we will check that $w(\xi\eta) = r + s$. By 5.10 (a), and 5.14, we know that $\xi \in W_r$ and $\eta \in W_s$, so $\xi\eta \in W_{r+s}$, which implies $w(\xi\eta) \geq r + s$. It remains to prove that the equality actually holds. Let i_0 and j_0 be the smallest indices such that $r = w(\xi) = p^{-i_0}v(x_{i_0}) + i_0$, and $s = w(\eta) = p^{-j_0}v(y_{j_0}) + j_0$; and consider the following decomposition:

$$\xi = \xi^{(<i_0)} + \xi^{(\geq i_0)} \text{ and } \eta = \eta^{(<j_0)} + \eta^{(\geq j_0)},$$

where $\xi^{(<i_0)} = (x_0, \ldots, x_{i_0-1}, 0, \ldots)$, $\xi^{(\geq i_0)} = (0, \ldots, 0, x_{i_0}, x_{i_0+1}, \ldots)$ and $\eta^{(<j_0)} = (y_0, \ldots, y_{j_0-1}, 0, \ldots)$, $\eta^{(\geq j_0)} = (0, \ldots, 0, y_{j_0}, y_{j_0+1}, \ldots)$. Then

$$\begin{aligned}\xi\eta &= (\xi^{(<i_0)} + \xi^{(\geq i_0)})(\eta^{(<j_0)} + \eta^{(\geq j_0)}) \\ &= \xi^{(<i_0)}\eta^{(<i_0)} + \xi^{(<i_0)}\eta^{(\geq j_0)} + \xi^{(\geq i_0)}\eta^{(<j_0)} + \xi^{(\geq i_0)}\eta^{(\geq j_0)},\end{aligned}$$

so noting that $w(\xi^{(<i_0)}) > r = w(\xi^{(\geq i_0)})$ and $w(\eta^{(<j_0)}) > s = w(\eta^{(\geq j_0)})$, we conclude that the first three summands have values strictly greater than $r + s$; while $w(\xi^{(\geq i_0)}\eta^{(\geq j_0)}) = r + s$, in fact the first non-zero component of $\xi^{(\geq i_0)}\eta^{(\geq j_0)}$ has index $i_0 + j_0$ and it is equal to $x_{i_0}^{p^{j_0}} y_{j_0}^{p^{i_0}}$. Finally, the computation of the value of a sum gives $w(\xi\eta) = w(\xi^{(\geq i_0)}\eta^{(\geq j_0)}) = r + s$. $\square$

REMARK 5.16. Observe that $w(p) = 1$, so that $p\mathbf{W}(\mathcal{R})$ is contained in $W^+ = \{\xi \in \mathbf{W}(\mathcal{R}) | w(\xi) > 0\}$, the ideal of the valuation; this implies that the quotient of $\mathbf{W}(\mathcal{R})/W^+$ has positive characteristic and is isomorphic to a quotient of $\mathbf{W}(\mathcal{R})/p\mathbf{W}(\mathcal{R}) \cong \mathcal{R}$; in fact, it is isomorphic to $\bar{k} \cong \mathcal{R}/m_{\mathcal{R}}$. This can be seen by considering the equivalence classes of the 0^{th}-components with respect to the valuation of $\mathcal{R}$

Now we define the ring $\mathbf{Biv}\,\mathcal{R}$.

DEFINITION 5.17. Let $\mathcal{K}$ be the completion of Frac($\mathbf{W}(\mathcal{R})$), the field of fractions of $\mathbf{W}(\mathcal{R})$, endowed with the topology induced by the extension of w, the valuation of $\mathbf{W}(\mathcal{R})$ defined in 5.13. The elements of the subring $\mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right]$ of Frac($\mathbf{W}(\mathcal{R})$) are called *special bivectors*. The ring of *Witt Bivectors* is the subring of $\mathcal{K}$ obtained by taking the the completion of $\mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right]$ with respect to the induced topology:

$$\mathbf{Biv}\,\mathcal{R} = \mathbf{W}(\mathcal{R})\widehat{\left[\frac{1}{p}\right]}$$

(Note the capital letter, which distinguishes this ring from its sub-$\mathbf{W}(\mathcal{R})$-module $\mathbf{biv}\,\mathcal{R}$ cf. § 6):

6. Some remarks about $\mathbf{biv}\,\mathcal{R}$

Let us begin with a definition:

DEFINITION 6.1. A *bivector* with components in $\mathcal{R}$ is a sequence of elements of $\mathcal{R}$, $\xi = (x_n)_{n\in\mathbf{Z}}$, which satisfies the following condition:

$$\lim_{n\to\infty} \inf\{v(x_{-k})|k > n\} > 0.$$

The set of the bivectors with components in $\mathcal{R}$, will be denoted by $\mathbf{biv}\,\mathcal{R}$.

This definition is the same as the one in chpt. 2 of [MA], or in § 0 of [WR]; so, for instance, we know that $\mathbf{biv}\,\mathcal{R}$ is in fact a K-module.

The goal of the first part of this section is to describe the relations between $\mathbf{biv}\,\mathcal{R}$ and $\mathbf{Biv}\,\mathcal{R}$.

We start with some remarks about special bivectors. Any element ξ of $\mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right]$ is of the form $p^{-k}\alpha$ for some k and some vector $\alpha \in \mathbf{W}(\mathcal{R})$, so that any special bivector ξ can be represented (as usual) by a sequence $(x_n)_{n\in\mathbf{Z}}$ satisfying the following condition:

$$\exists n_0 \text{ such that } n > n_0 \Rightarrow x_{-n} = 0.$$

Such representation does not depend on the choices of k and α as above; the elements $x_n \in \mathcal{R}$ are the components of the special bivector ξ. In particular, the elements of $\mathbf{W}(\mathcal{R})$ are the special bivectors for which each component with negative index is equal to zero. Moreover $\mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right] \subseteq \mathbf{biv}\,\mathcal{R}$.

Frobenius and Verschiebung, as defined on $\mathbf{W}(\mathcal{R})$, extend naturally to $\mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right]$:

$$F\xi = (x_n^p)_{n\in\mathbf{Z}} \text{ (Frobenius) and } V\xi = (x_{n-1})_{n\in\mathbf{Z}} \text{ (Verschiebung).}$$

for each $\xi = (x_n)_{n\in\mathbf{Z}} \in \mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right]$. Moreover $VF = FV = p$.

About the extension of w to $\mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right]$, observe that if $\xi = (x_n)_{n\in\mathbf{Z}} = p^{-k}\alpha$, then

$$w(\xi) = w(\alpha) - k = \min\{p^{-n}v(x_n) + n \mid n \in \mathbb{Z}\}. \tag{6.2}$$

In fact, if $\xi = (x_n)_{n\in\mathbf{Z}} = p^{-k}\alpha$, where $\alpha = (a_n)_{n\in\mathbf{Z}}$ and $a_n = 0$ for $n < 0$; then $x_n = a_{n+k}^{p^{-k}}$ for any n, so $p^{-n}v(x_n) + n = p^{-n-k}v(a_{n+k}) + (n+k) - k$.

This means that the topology of $\mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right]$ induced by the valuation admits as a fundamental set of neighbourhoods of zero the family of sub-$\mathbf{W}(\mathcal{R})$-modules

$$W_c = \left\{\xi \in \mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right] \mid n < c \Rightarrow v(x_n^{p^{-n}}) \geq c - n\right\}, \tag{6.3}$$

indexed by $c \in \mathbb{R}$.

As a consequence, just in the same way as for vectors, $\xi \in W_c$ if and only if $c \leq w(\xi)$ (cf. 5.14).

REMARK 6.4. The topology of $\mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right]$ induced by w is finer than the topology for which a fundamental set of neighbourhoods of zero is given by the family of submodules

$$U_k(a) = \left\{\xi \in \mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right] \mid n < k \Rightarrow v(x_n^{p^{-n}}) \geq a\right\},$$

indexed by $k \in \mathbb{N}$ and $a \in \mathbb{R}_{>0}$. More precisely, $W_{m+b} \subseteq U_m(b)$, for any $b \in \mathbb{R}_{>0}$, $m \in \mathbb{N}$.

This last topology, which can be extended naturally to $\mathbf{biv}\,\mathcal{R}$, was introduced by Barsotti. So, from now on, it will be called the *Barsotti topology*.

Frobenius, Verschiebung and their inverse produce *continuous* maps with respect to both topologies. In particular:

$$FW_a \subseteq W_a,\ VW_{p(a-1)} \subseteq W_a \text{ and } F^{-1}W_{pa} \subseteq W_a,\ V^{-1}W_{a+1} \subseteq W_a. \tag{6.5}$$

The following notation are useful: if $\xi = (x_n)_{n\in\mathbf{Z}}$ is a bivector, then $\xi^{(\leq r-1)} = \xi^{(<r)}$ denotes the bivector $(\ldots, x_{r-2}, x_{r-1}, 0, \ldots)$ and $\varsigma^{(\geq r)} = \xi^{(>r-1)}$ denotes $(\ldots, 0, x_r, x_{r+1}, \ldots)$; $\xi^{(<r)}$ and $\varsigma^{(\geq r)}$ are called the *r-tail* and the *r-head* of ξ, respectively.

If $\xi \in \mathbf{biv}\,\mathcal{R}$, then $\xi^{(\geq -r)} \in \mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right]$ and the sequence $(\xi^{(\geq -r)})_{r\in\mathbf{N}}$ is Cauchy, so that

$$\lim_{r\to\infty} \xi^{(\geq -r)}$$

exists in $\mathbf{Biv}\,\mathcal{R}$. The Cauchy condition can be checked as follows: let $\xi = (x_n)_{n\in\mathbf{Z}}$, then 6.1 implies that there exists a positive real number b and an integer n_0, such that $n \geq n_0 \Rightarrow v(x_{-n}) \geq b$ i.e. $v(x_{-n}^{p^n}) \geq p^n b$. Since for any fixed positive real number c, there exists an integer $r_0 \geq n_0$, such that $r \geq r_0 \Rightarrow p^r b \geq c + r$, we have $w(\xi^{(\geq -r)} - \xi^{(>-r)}) = w(V^{-r}\{x_{-r}\}) = v(x_{-r}^{p^r}) - r \geq c$, i.e. $\xi^{(\geq -r)} - \xi^{(>-r)} \in W_c$: this is the desired condition.

The map $i : \xi \to \lim_{r\to\infty} \xi^{(\geq -r)}$ defined on $\mathbf{biv}\,\mathcal{R}$ with values in $\mathbf{Biv}\,\mathcal{R}$ is injective. Moreover if $\xi, \eta \in \mathbf{biv}\,\mathcal{R}$, looking at Theorem 0.3.6 of [WR] it can easily be verified that

$$\lim_{r\to\infty} (\xi + \eta)^{(\geq -r)} = \lim_{r\to\infty} (\xi^{(\geq -r)} + \eta^{(\geq -r)})$$

and more precisely that i is an injective K-module homomorphism.

From now on, $\mathbf{biv}\,\mathcal{R}$ *will be identified with its image in* $\mathbf{Biv}\,\mathcal{R}$.

After this identification we will write $\xi = \lim_{r\to\infty} \xi^{(\geq -r)}$; and the natural relations among heads and tails of a bivector,

$$\xi^{(<r)} + \xi^{(\geq r)} = \xi = \xi^{(\leq r)} + \xi^{(>r)},$$

hold in $\mathbf{Biv}\,\mathcal{R}$.

The restriction to $\mathbf{biv}\,\mathcal{R}$ of the valuation of $\mathbf{Biv}\,\mathcal{R}$ is defined by

$$w(\xi) = \min\{p^{-n} v(x_n) + n \mid n \in \mathbb{Z}\},$$

for each $\xi = (x_n)_{n\in\mathbf{Z}} \in \mathbf{biv}\,\mathcal{R}$, i.e. by the same formula used for special bivectors. In fact 6.1 implies that the set of rational numbers $\{p^{-n} v(x_n) + n \mid n \in \mathbb{Z}\}$ has a minimum and, of course, this minimum is equal to $\lim_{r\to\infty} w(\xi^{(\geq -r)})$. So ξ belongs to (the closure of) W_b if and only if $b \leq w(\xi)$.

Let us remark that, by 6.1, the series $\sum_{n\in\mathbf{Z}} p^n \{x_n^{p^{-n}}\}$ converges in $\mathbf{Biv}\,\mathcal{R}$;

$$\xi = (x_n)_{n\in\mathbf{Z}} = \sum_{n\in\mathbf{Z}} p^n \{x_n^{p^{-n}}\}, \tag{6.6}$$

is called the *Teichmüller representation of* ξ.

We stress that **biv** $\mathcal{R}$ is strictly contained in **Biv** $\mathcal{R}$; in fact, as is shown in the following example,

(6.7) **biv** $\mathcal{R}$ *is not complete for the topology of the valuation.*

Let y be an element of $\mathcal{R}$ with positive value. Since $(p^{-n}\{y^{n^2}\})_{n\in\mathbf{N}}$ converges to zero in **Biv** $\mathcal{R}$, one gets $\sum_{n=0}^{\infty} p^{-n}\{y^{n^2}\} \in$ **Biv** $\mathcal{R}$. On the other hand, the finite sums of this series define a sequence of special bivectors which cannot converge to an element satisfying the condition 6.1.

The following theorem, which gives a (sufficient) condition in order that a Cauchy sequence of bivectors converges in **biv** $\mathcal{R}$, was originally proved for the Barsotti topology (cf. [MA] Thm. 2.1 or [WR] Lemma 4.2.4). The present proof shows that it holds, in the same way, for the topology of the valuation.

THEOREM 6.8. *A Cauchy sequence of bivectors* $(\xi_n)_{n\in\mathbf{N}}$ *converges in* **biv** $\mathcal{R}$ *if there exists a positive real number* b *such that:*

$$\lim_{k\to\infty} \inf\{v(x_{n,-j})|j > k, n \in \mathbb{N}\} > b,$$

where $\xi_n = (x_{n,j})_{j\in\mathbf{Z}}$.

PROOF. We begin by showing that, for any fixed $j \in \mathbb{Z}$, the sequence of the j-components, $(x_{n,j})_{n\in\mathbf{N}}$, is Cauchy, and then converges in $\mathcal{R}$. We have to show that, for any fixed positive real number c, there exists an integer n_0 such that $n,\ m \geq n_0 \Rightarrow v(x_{n,j} - x_{m,j}) > c$.

By hypothesis, for any fixed positive real number a, there exist an integer n_0 such that $n \geq n_0 \Rightarrow \xi_n - \xi_{n_0} = \psi_{n,n_0} \in W_a$. Suppose n fixed and let $\psi_{n,n_0} = (u_j)_{j\in\mathbf{Z}}$. From the relation $\xi_n = \xi_{n_0} + \psi_{n,n_0}$ one has

$$x_{n,j} = \lim_{h\to\infty} S_h(x_{n_0,j-h}, \ldots, x_{n_0,j}; u_{j-h}, \ldots, u_j),$$

where S_h denotes the h-th Witt polynomial for the sum. From the uniform limitation of the values of components of all the ξ_n's with very negative index, it follows that there exists a positive integer h such that

$$x_{n,j} \equiv S_h(x_{n_0,j-h}, \ldots, x_{n_0,j}; u_{j-h}, \ldots, u_j) \bmod_{b_c},$$

where $b_c = \{z \in \mathcal{R}|v(z) > c\}$; let observe that the bound h does not depend on n because the condition above on the values of the components is independent of n.

From the definition of the Witt polynomial S_h, one has:

$$S_h(x_{n_0,j-h}, \ldots, x_{n_0,j}; u_{j-h}, \ldots, u_j) = x_{n_0,j} + \begin{pmatrix} \text{monomials} \\ \text{divisible by} \\ \text{some of the } u_i\text{'s} \end{pmatrix};$$

Now observe that if $a > p^{h-j}c + j$ and $j - h \leq i \leq j$, then $v(u_i) > p^i(a-i) > c$. Hence $x_{n,j} \equiv x_{n_0,j} \bmod b_c$, as we wanted to conclude.

Set $z_j = \lim_{n\to\infty} x_{n,j}$; then the limit z_j exists in $\mathcal{R}$ because this ring is complete, and the above condition $\lim_{k\to\infty} \inf\{v(x_{n,-j}) | j > k, n \in \mathbb{N}\} > b$ implies that the sequence $\varsigma = (z_j)_{j\in\mathbf{Z}}$ satisfies 6.1, i.e., ς is a bivector.

It remains to prove that $\varsigma = \lim_{n\to\infty} \xi_n$ in $\mathbf{biv}\,\mathcal{R}$. For any fixed positive real a, the hypothesis implies that there exist an integer $r << 0$, such that all the r-tails $\varsigma^{(<r)}$ and $\xi_n^{(<r)}$, for each n, are in W_a. Then looking at the decomposition

$$\varsigma - \xi_n = (\varsigma^{(<r)} - \xi_n^{(<r)}) + (\varsigma^{(\geq r)} - \xi_n^{(\geq r)}),$$

it is clear that it suffices to prove $\varsigma^{(\geq r)} - \xi_n^{(\geq r)} \in W_a$, if $n >> 0$; this is an easy consequence of the already proved componentwise convergence. □

By means of 6.8 we will prove that $\mathbf{biv}\,\mathcal{R}$ is a sub-$\mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right]$-module of $\mathbf{Biv}\,\mathcal{R}$; let $\alpha = (a_0, a_1, \ldots) \in \mathbf{W}(\mathcal{R})$ and $\xi = (x_n)_{n\in\mathbf{Z}} \in \mathbf{biv}\,\mathcal{R}$, then the sequence $(\xi^{(\geq -r)})_{r\in\mathbf{N}}$ is Cauchy and then the same condition holds for the sequence $(\alpha\xi^{(\geq -r)})_{r\in\mathbf{N}}$, because the topology is defined by a valuation. So, to prove that $\alpha\xi \in \mathbf{biv}\,\mathcal{R}$, it suffices to check that the sequence $(\alpha\xi^{(\geq -r)})_{r\in\mathbf{N}}$ satisfies the condition of 6.8. After writing $\alpha\xi^{(\geq -r)}$ explicitly as follows:

$$\alpha\xi^{(\geq -r)} = \sum_{h=-r}^{\infty} p^h \sum_{j=0}^{h+r} \{a_j^{p^{-j}} x_{h-j}^{p^{j-h}}\} = \sum_{h=-r}^{\infty} V^h \sum_{j=0}^{h+r} \{a_j^{p^{h-j}} x_{h-j}^{p^j}\}.$$

it becomes clear that 6.8 for $(\alpha\xi^{(\geq -r)})_{r\in\mathbf{N}}$ is a consequence of 6.1 for ξ.

Finally, we remark that there is a continuous action of $\mathcal{G}$ on the components of the elements of $\mathbf{biv}\,\mathcal{R}$; this action is compatible with the structure of $\mathbf{W}(\mathcal{R})$-module.

Our goal now is to extend to $\mathbf{Biv}\,\mathcal{R}$ the map Θ_0 defined in 5.8, and then to study the main features of the kernel of this extension. First, if $\xi = (x_n)_{n\in\mathbf{Z}}$ is a special bivector, we will define

$$\Theta(\xi) = \sum_{n\in\mathbf{Z}} p^n \hat{x}_{n,n} \tag{6.9}$$

The map $\xi \to \Theta(\xi)$ is a *continuous, surjective homomorphism of rings from special bivectors onto C*. All these properties of Θ are immediate consequences of the corresponding properties of Θ_0, except, maybe, the continuity, which becomes clear after observing that

$$v(\Theta(\xi)) \geq w(\xi),$$

for any (special) bivector ξ.

From now on $\Theta : \mathbf{Biv}\,\mathcal{R} \to C$ will denote the extension by continuity of the map defined by 6.9.

Let remark that $\xi \to \Theta(\xi)$ fails to be continuous for the Barsotti topology. This is the reason why we have chosen the topology of the valuation to define $\mathbf{Biv}\,\mathcal{R}$.

Here we collect the properties of Θ which will be used later on.

THEOREM 6.10. *a) The map $\Theta : \mathbf{Biv}\,\mathcal{R} \to C$ is a continuous, surjective homomorphism, of rings, equivariant with respect to the actions of $\mathcal{G}$. In particular, if $\xi = (x_n)_{n\in\mathbf{Z}} \in \mathbf{biv}\,\mathcal{R}$ and $s \in \mathcal{G}$, then $\Theta(\xi) = \sum_{n\in\mathbf{Z}} p^n \hat{x}_{n,n}$ and $s\xi = (sx_n)_{n\in\mathbf{Z}}$.*

b) $\ker\Theta = \alpha\,\mathbf{Biv}\,\mathcal{R}$, *where α is the vector defined in part (b) of the proof of* 1.8

c)
$$\bigcap_{n=1}^{\infty} (\ker\Theta)^n = (0).$$

PROOF. (a) It remains to check the equivariance of Θ with respect to the Galois action, and the properties of its restriction to $\mathbf{biv}\,\mathcal{R}$. The equivariance follows by observing that $\mathcal{G}$ acts componentwise on $\mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right]$ and that this action is continuous. Moreover, the sequence $(s\xi^{(\geq r)})$ satisfies the condition of 6.8, so that:
$$s\xi = \lim_{r\to-\infty} s\xi^{(\geq r)} = (sx_n)_{n\in\mathbf{Z}}.$$

Finally, we show that the series $\sum_{n\in\mathbf{Z}} p^n \hat{x}_{n,n}$ converges in C: let a be a positive real number; if $n > a$, then $v_C(p^n \hat{x}_{n,n}) > a$. On the other hand, since there exists $b > 0$ such that $\lim_{n\to\infty} \inf\{v(x_{-k}) | k > n\} > b$, there also exists a positive integer n_0 such that $n \geq n_0 \Rightarrow p^n b - n > a$ and $v(x_{-n}) > b$; as a consequence, for such n's $v_C(p^{-n}\hat{x}_{-n,-n}) = p^n v(x_{-n}) - n > a$. As C is complete, this concludes the proof of the convergence.

(b) Let $\alpha = \{\pi\}+p$ as in the proof of 5.8; we have to show that α generates $\ker\Theta$. First let observe that 5.8 implies $\mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right] \cap \ker\Theta = \alpha\mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right]$; and that the general result is a consequence of this and of the following

CLAIM 6.11. *Given a Bivector $\eta \in \ker\Theta$, there exists a sequence of special bivectors $(\varsigma_n)_{n\in\mathbf{N}}$, such that $\varsigma_n \in \ker\Theta$ and $\eta = \lim_{n\to\infty} \varsigma_n$.*

In fact from 5.8, it follows that $\varsigma_r = \alpha\xi_r$ for a sequence of special bivectors $(\xi_r)_{r\in\mathbf{N}}$. Since $(\alpha\xi_r)_{r\in\mathbf{N}}$ converges, the sequence $(\xi_r)_{r\in\mathbf{N}}$ is a Cauchy sequence of special bivectors, and therefore there exists $\xi = \lim_{r\to\infty}$ in $\mathbf{Biv}\,\mathcal{R}$. Finally, by the continuity of the product, we conclude that $\eta = \alpha\xi$.

PROOF OF THE CLAIM. Given a Bivector $\eta \in \ker \Theta$, for any fixed positive integer r, there exists a special bivector η_r, such that $\eta - \eta_r \in W_r$. This implies that $v_C(\theta(\eta - \eta_r)) \geq r$, so, by 5.8, there exists a vector $\gamma_r \in \mathbf{W}(\mathcal{R})$, with $w(\gamma_r) \geq r$, such that $\Theta(\eta - \eta_r) = \Theta(\gamma_r)$. Since $(\gamma_r)_{r\in\mathbf{N}}$ converges to 0 in $\mathbf{W}(\mathcal{R})$, $(\eta_r + \gamma_r)_{r\in\mathbf{N}}$ converges to η in $\mathbf{Biv}\,\mathcal{R}$; finally, observing that

$$\Theta(\eta_r + \gamma_r) = \Theta(\gamma_r) + \Theta(\eta_r) = \Theta(\eta - \eta_r) + \Theta(\eta_r) = \Theta(\eta) = 0,$$

we conclude that $(\eta_r + \gamma_r)_{r\in\mathbf{N}}$ is a sequence as requested in the Claim.

(c) To check this point, we will prove that $\xi \in \bigcap_{n=1}^{\infty} (\ker \Theta)^n$ implies $\xi \in W_c$, for each $c > 0$.

Let $\xi \in (\ker \Theta)^n$, for each positive integer n, and assume there exists a positive real number c, such that $\xi \notin W_c$; then given a special bivector $\varsigma = (z_n)_{n\in\mathbf{N}}$ such that $\xi - \varsigma$ belongs to W_c, there exists

$$(6.12) \qquad r \text{ such that } \varsigma^{(<r)} \in W_c \quad \text{and} \quad v(z_r^{p^{-r}}) < c - r.$$

So we can assume $\varsigma = \varsigma^{(\geq r)}$ and $v(z_r^{p^{-r}}) < c - r$. Now, observe that $\xi \in (\ker \Theta)^n$ implies that there exists a special bivector ξ_n such that $\xi \equiv \alpha^n \xi_n \bmod W_c$, so $\varsigma \equiv \alpha^n \xi_n \bmod W_c$. From the last congruence, through the following sequence of implications we will get a contradiction with the assumption about the value of z_r given in 6.12. For each positive integer n we have:

$$\varsigma \equiv \alpha^n \xi_n \bmod W_c \overset{\text{i)}}{\Rightarrow} (\alpha^n \xi_n^{(<r-n)})^{(<r-n)} \in W_c \overset{\text{ii)}}{\Rightarrow} \alpha^n \xi_n^{(<r-n)} \in W_c \overset{\text{iii)}}{\Rightarrow}$$

$$\overset{\text{iii)}}{\Rightarrow} \varsigma \equiv \alpha^n \xi_n^{(\geq r-n)} \bmod W_c \overset{\text{iv)}}{\Rightarrow} v(x_{n,r-j}) \geq p^{r-j}(c - r - n + j), \text{ for } j = 1, \ldots, n \overset{\text{v)}}{\Rightarrow}$$

$$\overset{\text{v)}}{\Rightarrow} z_r = x_{n,r}\pi^{np^r} + \begin{pmatrix} \text{terms of value} \\ \geq p^r(c-r) \end{pmatrix} \overset{\text{vi)}}{\Rightarrow} v(z_r) > p^r(c - r).$$

Now we prove the successive implications:

i) Of course $\alpha^n \xi_n - \varsigma \in W_c$ implies $(\alpha^n \xi_n - \varsigma)^{(<r-n)} \in W_c$, and noting that $\varsigma = \varsigma^{(\geq r)}$, one deduces $(\alpha^n \xi_n^{(<r-n)})^{(<r-n)} = (\alpha^n \xi_n - \varsigma)^{(<r-n)}$.

ii) To prove this implication we observe that $\xi_n^{(<r-n)}$ is a special bivector; so, after a multiplication by a suitable power of p it becomes a vector of finite length. Therefore our implication follows from the following lemma 6.14.

iii) Follows immediately from $\varsigma \equiv \alpha^n \xi_n = \alpha^n \xi_n^{(<r-n)} + \alpha^n \xi_n^{(\geq r-n)} \bmod W_c$.

iv) Let $\alpha^n \xi^{(\geq r-n)} = (\ldots, 0, y_{r-n}, \ldots, y_{r-1}, \ldots)$; since $\varsigma = \varsigma^{(\geq r)}$, $(\alpha^n \xi_n^{(\geq r-n)} - \varsigma)^{(<r)} = (\ldots, 0, y_{r-n}, \ldots, y_{r-1}, 0 \ldots)$; so, if $s < r$, $v(y_s) \geq p^s(c - s)$. This implies that the components of ξ_n satisfy the condition

$$(6.13) \qquad v(x_{n,r-j}) \geq p^{r-j}(c - r - n + j) \text{ for } j = 1, \ldots, n.$$

First we show that 6.13 is true for $j = n$: in fact $y_{r-n} = x_{n,r-n}\pi^{np^{r-n}}$, therefore $np^{r-n} + v(x_{n,r-n}) = v(y_{r-n}) \geq p^{r-n}(c - r + n)$ and then $v(x_{n,r-n}) \geq$

$p^{r-n}(c-r)$. Then for $1 \le j < n$, we will use an inductive argument: assuming that 6.13 holds for the indices greater than j, and recalling the structure of the product of (special) bivectors, we get

$$y_{r-j} = x_{n,r-j}\pi^{np^{r-j}} + \begin{pmatrix} \text{terms of value} \\ \ge p^{r-j}(c-r+j) \end{pmatrix};$$

since $v(y_{r-j}) \ge p^{r-j}(c-r+j)$, we conclude that

$$v(x_{n,r-j}\pi^{np^{r-j}}) = np^{r-j} + v(x_{n,r-j}) \ge p^{r-j}(c-r+j),$$

i.e. 6.13, for $j = 1, \dots, n$.

v) From 6.13, recalling that $\varsigma \equiv \alpha^n \xi_n^{(\ge r-n)}$, the relation

$$z_r = x_{n,r}\pi^{np^r} + \begin{pmatrix} \text{terms of value} \\ \ge p^r(c-r) \end{pmatrix}$$

comes out from the usual arguments on the isobaricity of Witt polynomials;

vi) The last relation implies $v(z_r) \ge np^r + v(x_{n,r})$, for each n, so that $v(z_r) > p^r(c-r)$. □

LEMMA 6.14. *If η is a vector of length k and M is a real number, then*

$$\alpha^n\eta \in W_M \Leftrightarrow (\alpha^n\eta)^{(<k)} \in W_M.$$

PROOF. The claim is obviously true if $M \le k$; so we suppose that $M > k$ and, by 5.14, it suffices to prove that $(\alpha^n\eta)^{(<k)} \in W_M \Rightarrow \eta \in W_{M-n}$.

Let

$$\eta = \sum_{j=0}^{k-1} p^j\{y'_j\} \text{ and } \alpha^n = \sum_{i=0}^{n} \binom{n}{i} p^i\{\pi^{n-i}\},$$

then:

$$\alpha^n\eta = \sum_{h=0}^{n+k-1} p^h\varsigma_h \text{ where } \varsigma_h = \sum_{i+j=h} \binom{n}{i} \{y'_j\pi^{n-i}\}.$$

Since $\varsigma_0 = \{y'_0\pi^n\}$, then $v(y'_0) + n \ge M$, which gives the correct estimate for the 0-th component of η. Now we can use an inductive argument. Let $\alpha^n\eta = (g_0, g_1, \dots)$ and $\varsigma_h = (z_{h,0}, \dots)$, then if we assume $v(y'_r) + n \ge M - r$, when $r < h < k$, we have

$$g_h = z_{h,0} + \begin{pmatrix} \text{terms of} \\ \text{value } \ge M-h \end{pmatrix}.$$

Since $M - h \le v(g_h)$, then $M - h \le v(z_{h,0})$, so looking at the sum producing the component $z_{h,0}$, we conclude that

$$M - h \le v(y'_h) + n, \qquad \square$$

REMARK. If $\eta \in \ker \Theta$ is a bivector, one can choose the sequence of heads $\eta^{(\geq -r)}$ to approximate η and get a sequence of special bivectors $(\varsigma_r)_{r\in\mathbb{N}}$ as in 6.11 with the additional property that any component $z_{r,j}$ of ς_r with index $-r \leq j \leq -1$ is equal to the component of η having the same index. From this remark and a direct check on the value of the components of bivectors ξ_r such that $\varsigma_r = \alpha\xi_r$, one can deduce that the sequence $(\xi_r)_{r\in\mathbb{N}}$ satisfies the condition of uniform convergence of Theorem 6.8. Hence one gets $\eta = \alpha\xi$ for a bivector ξ.

We don't give the details of the proof of the above remark.

Let us prove some easy properties of $\mathbf{biv}\,\mathcal{R}$.

PROPOSITION 6.16. *Notations as above; then:*

a) $$(\mathbf{biv}\,\mathcal{R})^{\mathcal{G}} = \mathbf{biv}\,k = K.$$

b) $$\{\xi \in \mathbf{biv}\,\mathcal{R} | F\xi = \xi\} = \mathbf{biv}\,\mathbb{F}_p = \mathbb{Q}_p.$$

c) $$\mathbf{biv}\,\overline{k} \subseteq \mathbf{biv}\,\mathcal{R}.$$

PROOF. (a) The group $\mathcal{G}$ acts on the components of bivectors, so if $\xi = (x_j)_{j\in\mathbb{Z}}$ is invariant under the action of $\mathcal{G}$, all its components x_j in $\mathcal{R}$ are invariants under the action of $\mathcal{G}$. This means that any such x_j, can be represented by a sequence of elements of A_C invariant under the action of $\mathcal{G}$, i.e. by a sequence of elements of A. This means that x_j belongs to the copy of k inside $\mathcal{R}$.

(b) By the definition of the Frobenius map F, a bivector $\xi = (x_j)_{j\in\mathbb{Z}}$ satisfies the condition $F\xi = \xi$ if, and only if, any of its components x_j satisfies the equation $x_j^p - x_j = 0$ in $\mathcal{R}$. But $\mathcal{R}$ is an integral domain of characteristic p, so that this equation is satisfied only by the elements of the copy of $\mathbb{F}_p$ inside $\mathcal{R}$.

(c) As we have seen in 5.6, $\overline{k}$ is the residue field of the equicharacteristic valuation ring $\mathcal{R}$ and then $\mathcal{R}$ contains a copy of $\overline{k}$. This gives the inclusion of $\mathbf{biv}\,\overline{k}$ into $\mathbf{biv}\,\mathcal{R}$. □

7. - Logarithm of Witt vectors

In this section we will discuss the properties of the logarithm of Witt vectors on $\mathcal{R}$ and its relations with the map Θ defined in the previous section. We start with the following result.

THEOREM 7.1. *Let $U_{\mathbf{W}(\mathcal{R})}$ be the subgroup of the multiplicative group of $\mathbf{W}(\mathcal{R})$ whose elements are the vectors $\xi = (x_0, x_1, \ldots)$ such that: $w(\xi - 1) > 0$. The logarithm map* $\log : U_{\mathbf{W}(\mathcal{R})} \to \mathbf{Biv}\,\mathcal{R}$, *defined as usual by setting*

$$\log(\xi) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\xi - 1)^n}{n},$$

is a continuous and injective homomorphism of groups with values in the subgroup **biv** $\mathcal{R}$.

Moreover, this map commutes with the naturals actions of $\mathcal{G} = \mathrm{Gal}(\overline{K}/K)$ *on* $U_{\mathbf{W}(\mathcal{R})}$ *and* **biv** $\mathcal{R}$ *and it also commutes with the logarithm map of* C*; i.e. there is a commutative diagram of* $\mathcal{G}$*-equivariant maps:*

$$\begin{array}{ccc} U_{\mathbf{W}(\mathcal{R})} & \xrightarrow{\log} & \mathbf{biv}\,\mathcal{R} \\ \downarrow \Theta_0 & & \downarrow \Theta \\ U_{A_C} & \xrightarrow{\log} & C \end{array}$$

where $U_{A_C} = \{x \in A_C | v(x-1) > 0\}$.

PROOF. Let us remark that, for any vector ξ, the conditions $w(\xi - 1) > 0$ and $v(x_0 - 1) > 0$ are equivalent. If ξ is a vector in $U_{\mathbf{W}(\mathcal{R})}$, then $\xi - 1 \in W_a$, for some $a > 0$, and so it follows that

$$w\left(\frac{(\xi-1)^n}{n}\right) = nw(\xi - 1) - v_p(n) \geq na - v_p(n); \tag{7.2}$$

since $\lim\limits_{n\to\infty}(na - v_p(n)) = \infty$, this means that the series of $\log(\xi)$ satisfies the Cauchy condition and thus it converges in **Biv** $\mathcal{R}$. To be more precise, note that for any Witt vector $\eta = (y_0, y_1, \ldots)$ one has $\eta^{p^n} \equiv \{y_0^{p^n}\} \bmod p^{n+1}\mathbf{W}(\mathcal{R})$. By a direct check we can see that this implies that all the components with negative index of any finite sum of the series $\log(\xi)$ are indeed polynomials in $x_0 - 1$ without constant terms, and then recalling that $v(x_0 - 1) \geq a$, we conclude that all these components have value greater or equal than a. The criterion stated in Theorem 6.8 allows us to conclude that $\log(\xi)$ is in **biv** $\mathcal{R}$.

Let b be a fixed positive number, if a is a positive real number such that $na - v_p(n) \geq b$ for each $n \geq 1$; then by 7.2 it follows that $\xi \in 1 + W_a$, implies $\log(\xi) \in W_b$: this shows that the logarithm map is continuous.

The injectivity can be proved as follows: if $\log(\xi) = 0$, then $\log(\xi^{p^n}) = 0$ for any $n \geq 0$; as a consequence we can suppose $a = w(\xi - 1) > \dfrac{1}{p-1}$. If $\xi - 1 \neq 0$, this implies that $na - v(n) > a$, for $n \geq 2$; hence $w(\log(\xi)) = a$, which is a contradiction.

The last claims now follow easily. □

COROLLARY 7.3. *Let* $U_{\mathcal{R}} = \{a \in \mathcal{R} | v(a-1) > 0\}$. *If* $a = (a_0, a_1, \ldots)$ *is a sequence of elements of* A_C, *such that* $a_{i+1}^p = a_i$ *and* $v_C(a_i - 1) > 0$ $\forall i$, *then* $a \in U_{\mathcal{R}}$ *and* $\log\{a\}$ *exists in* **biv** $\mathcal{R}$.

PROOF. By 7.1, the logarithm of $\{a\}$ exists in **biv** $\mathcal{R}$ if $w(\{a\} - 1) > 0$. Since $v(a-1) > 0$ implies $w(\{a\} - 1) > 0$, it suffices to check that, under our hypothesis, the sequence $a = (a_0, a_1, \ldots)$ belongs to $U_{\mathcal{R}}$. In fact we give a precise calculation of the value of $a - 1$ in $\mathcal{R}$.

Let $a_j = 1 + b_j$, where $v_C(b_j) > 0$ and recall that by 5.5 and 5.2

$$v(a-1) = \lim_{j\to\infty} v_C((a_j-1)^{p^j}).$$

By the relation $a_{i+1}^p = a_i$, we get

$$b_i = (1+b_{i+1})^p - 1 = \sum_{j=1}^{p} \binom{p}{j} b_{i+1}^j.$$

Since the above summands, with the exception of b_{j+1}^p, have a value strictly greater than 1 because of the binomial coefficient, if we suppose $v_C(b_i) \leq 1$ for some i, then $v_C(b_i) = v_C(b_{j+1}^p) = pv_C(b_{i+1})$. By iteration one gets that $v_C(b_{i+k}) = p^{-k}v_C(b_i)$, so that

$$v(a-1) = \lim_{j\to\infty} v_C(a_j-1)^{p^j} = \lim_{j\to\infty} p^j v_C(b_j) = p^i v_C(b_i) > 0.$$

On the contrary, if we have $v_C(b_i) > 1$ for any i; then, by the relation $a_{i+1}^p = a_i$, we get $v_C(b_i) = 1 + v_C(b_{i+1})$ for any i, and this is impossible if $a \neq 1$. □

REMARK 7.4. Now we consider the special case of the vector $\{\varepsilon\}$, where $\varepsilon = (1, \varepsilon_1, \dots) \in \mathcal{R}(A_C)$ is a sequence of elements of A_C and ε_1 is a primitive p-root of unity. The choice of such an ε in $\mathcal{R}$ gives a basis of the $\mathbb{Z}_p$-module $T\mathbf{G}_m = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty})$ (the Tate module of the multiplicative group) embedded into the multiplicative group of $\mathcal{R}$. The action of $\mathcal{G}$ on $\{\varepsilon\}$ is determined by a character $\chi : \mathcal{G} \to \mathbb{Z}_p^\times$, i.e. for any $s \in \mathcal{G} = \mathrm{Gal}(\overline{K}, K)$, one has: $s\{\varepsilon\} = \{\varepsilon\}^{\chi(s)}$, where the definition makes sense because ε is a sequence of p^n-roots of unity in A_C.

By an elementary calculation of the value in A_C of $\varepsilon_1 - 1$ and the arguments in the above proof of Corollary 7.3, one gets that $v(\varepsilon - 1) = \dfrac{p}{p-1}$, and so $\log\{\varepsilon\}$ exists in $\mathbf{biv}\,\mathcal{R}$. The module $\mathbf{biv}\,\mathcal{R}$ is in particular a $\mathbb{Q}_p$-module, and this means that one can realize the Tate space $V\mathbf{G}_m = T\mathbf{G}_m \otimes \mathbb{Q}_p$ in the additive group of $\mathbf{biv}\,\mathcal{R}$. In particular, the Galois action now becomes:

$$s(\log\{\varepsilon\}) = \chi(s)\log\{\varepsilon\} \text{ for any } s \in \mathcal{G},$$

and this means that we have in fact embedded $\mathbb{Q}_p(1)$ into $\mathbf{biv}\,\mathcal{R}$.

Let $(t_j)_{j\in\mathbb{Z}} = \tau = \log\{\varepsilon\}$; by the continuity of the Frobenius map, one gets:

$$F\tau = \log F\{\varepsilon\} = \log\{\varepsilon^p\} = p\log\{\varepsilon\} = p\tau, \tag{7.5}$$

and then $\tau = V\tau$, which means $t_j = t_{j+1}$, for each j. As we have seen in the proof of Theorem 7.1, the negative components of the logarithm of a vector ξ are power series without constant term in the 0-th component of $\xi - 1$. In particular, $v(t_j) \geq v(\varepsilon - 1) = \dfrac{p}{p-1}$.

We want to describe more thoroughly the bivectors of type of $\log\{\varepsilon\}$ as ε varies in $\mathcal{R}$. Let us start by giving a sufficient condition for the convergence of a series of exponential type in $\mathbf{Biv}\,\mathcal{R}$.

LEMMA 7.6. *Let $\eta = (y_j)_{j\in\mathbf{Z}}$ be a bivector and $(\beta_n)_{n\in\mathbf{N}}$ be a sequence of vectors. If $w(\eta) = a > \frac{1}{p-1}$, then the series*

$$\sum_{n=0}^{\infty} \beta_n \frac{\eta^n}{n!},$$

converges in $\mathbf{Biv}\,\mathcal{R}$.

PROOF. Since $w\left(\beta_n \frac{\eta^n}{n!}\right) \geq na - v(n!) \geq na - \frac{n-1}{p-1}$, and $\lim_{n\to\infty}\left[na - \frac{n-1}{p-1}\right] = \infty$, if $a > \frac{1}{p-1}$, the above series satisfies the Cauchy condition. □

Let $\mathcal{E}$ the sub-$\mathbf{W}(\mathcal{R})$-module of $\mathbf{Biv}\,\mathcal{R}$ defined as follows:

$$\mathcal{E} = \left\{\eta \,|\, w(\eta) > \frac{1}{p-1}\right\};$$

we define the continuous function $\exp : \mathcal{E} \to \mathbf{Biv}\,\mathcal{R}$ by setting:

$$\exp(\eta) = \sum_{n=0}^{\infty} \frac{\eta^n}{n!},$$

for any $\eta \in \mathcal{E}$. The continuity follows easily by observing that $w(\eta) = b > \frac{1}{p-1}$ implies that $w(\exp(\eta) - 1) = b$.

In this situation, we can state a characterization of the bivectors of type $\log\{\varepsilon\}$ analogous to the one given by Barsotti when he was working with his topology (cf. [MA] Theorem 2.9).

PROPOSITION 7.7. *Let $\mathcal{L} = \{\eta \in \mathcal{E} \cap \mathbf{biv}\,\mathcal{R} \,|\, V\eta = \eta\}$. If a bivector $\eta \in \mathcal{L}$ then $\eta = \log\{x\}$ for some $x \in \mathcal{R}$. More precisely, if y denotes a component of η, then $x = E(y)$, where $E(t) = \exp\left(\sum_{j=0}^{\infty} p^{-j} t^{p^j}\right) \in \mathbb{Z}_p[[t]]$ is the Artin-Hasse function.*

PROOF. If $\eta \in \mathcal{L}$, then $\eta = \lim_{n\to\infty} V^n(\eta^{(\leq 0)})$. By the continuity of the exponential map, we have:

$$\exp(\eta) = \lim_{n\to\infty} \exp(V^n(\eta^{(\leq 0)})),$$

and, by the continuity of the Frobenius map and of its inverse (cf. Remark 6.5) and elementary properties of exponential, one has

$$\exp(V^n(\eta^{(\leq 0)})) = F^{-n}\exp(p^n(\eta^{(\leq 0)})) = F^{-n}[\exp(\eta^{(\leq 0)})]^{p^n}.$$

If we denote by y a component of η (they are all equal), we can write the Teichmüller representation of $\eta^{(\leq 0)}$ (cf. 6.6):

$$\eta^{(\leq 0)} = \sum_{j=0}^{\infty} p^{-j}\{y^{p^j}\}.$$

Now observing that, since η is a bivector, the vector $\{y\}$ has positive w-value (cf. 6.1). Thus, since $E(t) \in \mathbb{Z}_p[[t]]$, we conclude that $\exp(\eta^{(\leq 0)}) = E(\{y\}) \in \mathbf{W}(\mathcal{R})$. As a consequence:

$$\exp(\eta) = \lim_{n\to\infty} \exp(V^n(\eta^{(\leq 0)})) = \lim_{n\to\infty} F^{-n}[E(\{y\})]^{p^n} = \{E(y)\},$$

because, for any Witt vector $\xi = (x_0, x_1, \ldots)$ one has $\xi^{p^n} \equiv F^n\{x_0\} \bmod p^{n+1}\mathbf{W}(\mathcal{R})$, for each n. □

Now we can explain the relations between the bivector $\tau = \log\{\varepsilon\}$ and the map Θ of Theorem 6.10. The result and the arguments of the following proof are in some sense similar to the ones of Proposition 2.17 of [FO3].

PROPOSITION 7.9. *Let $\tau = \log\{\varepsilon\}$. Then τ belongs to* $\ker\Theta$ *and its image generates the one-dimensional C-vector space* $\dfrac{\ker\Theta}{(\ker\Theta)^2}$.

PROOF. From 7.1 it follows that

$$\Theta(\tau) = \Theta(\log\{\varepsilon\}) = \log(\Theta_0(\{\varepsilon\})) = \log(1) = 0.$$

Then $\tau \in \ker\Theta$ so that, by 6.10 (b), $\tau = \alpha\beta$ for some $\beta \in \mathbf{biv}\,\mathcal{R}$. Our aim is to prove that $\Theta(\beta) \neq 0$.

Let $\gamma = (c_0, c_1, \ldots)$, where $\gamma = \{\varepsilon\} - 1$, so $c_0 = \varepsilon - 1$ and, as remarked in 7.4, $v(c_0) = \dfrac{p}{p-1}$. We will prove that in fact $w(\gamma) = \dfrac{p}{p-1}$, and then deduce from this our assertion.

The component c_0 of the vector γ satisfies the required condition, so if $p > 2$ it suffices to prove that $p^{-1}v(c_1) + 1 \geq \dfrac{p}{p-1}$. The case $p = 2$, will be discussed separately.

Suppose $p \neq 2$. After a direct calculation of c_1 we find:

$$c_1 = -\sum_{j=1}^{p-1} \frac{1}{p}\binom{p}{j}(-1)^{p-j}\varepsilon^j = -\sum_{j=1}^{\frac{p-1}{2}} \frac{1}{p}\binom{p}{j}(-1)^{p-j}\varepsilon^j(1-\varepsilon^{p-2j}),$$

which shows that $(1-\varepsilon)$ divides all terms $(1-\varepsilon^{p-2j})$ in the above sum, so that the value of c_1 is greater than or equal to the value of $\varepsilon - 1 = c_0$.

This implies that all terms $\frac{\gamma^n}{n}$, for $n \geq 2$, belong to $(\ker \Theta)^2 \cap W_2$; so by Theorem 6.10

$$\tau = \log(1+\gamma) = \gamma + \alpha^2\mu,$$

for some μ with $w(\mu) \geq 0$. In particular, $\gamma \in \ker \Theta_0$, and this implies $\gamma = \alpha\lambda$, where λ is a vector with $w(\lambda) = w(\gamma) - 1 = \frac{1}{p-1}$; then

$$\tau = \alpha(\lambda + \alpha\mu).$$

This implies $\Theta(\beta) = \Theta(\lambda) \neq 0$, because the vectors in $\ker \Theta_0$ have w-value greater or equal to 1.

Suppose $p = 2$. By a direct calculation of the Witt polynomial that gives the component c_1 of $\gamma = \{\varepsilon\} - 1 = \{\varepsilon\} + (1, 1, \dots)$, one gets: $c_1 = 1 - \varepsilon$, which allows us to conclude that $w(\gamma) = \frac{p}{p-1} = 2$.

This implies that all the terms $\frac{\gamma^n}{n}$, for $n \geq 2$, belong to $(\ker \Theta)^2 \cap W_3$ and, then, by Theorem 6.10, this implies

$$\tau = \log(1+\gamma) = \gamma + \alpha^2\mu,$$

with $w(\mu) \geq 1$. In particular, $\gamma \in \ker \Theta_0$, and this implies $\gamma = \alpha\varsigma$, where ς is a vector with $w(\varsigma) = w(\gamma) - 1 = 1$; hence

$$\tau = \alpha(\varsigma + \alpha\mu),$$

and

$$\varsigma = (z_0, z_1, \dots), \text{ where } z_0 = \frac{\varepsilon - 1}{\pi} \quad \text{and} \quad z_1 = \frac{\varepsilon - 1 - z_0^2}{\pi^2}.$$

It suffices to remark that $\Theta_0(\varsigma) \equiv \hat{z}_{0,0} + 2\hat{z}_{1,1} \bmod 4A_C$ with

$$v(\hat{z}_{0,0}) = 1 \quad \text{and} \quad v(\hat{z}_{1,1}) > 0,$$

to get $v(\Theta(\beta)) = 1$ and then $\Theta(\beta) \neq 0$. □

The usual estimate on the value of the finite sums of the logarithmic series, shows that $w(\{\varepsilon\} - 1) > \frac{1}{p-1}$ implies $w(\log\{\varepsilon\}) = w(\gamma)$; then under our hypothesis we get $w(\tau) = \frac{p}{p-1}$.

COROLLARY 7.10. *For any* $n \geq 0$, *there is a natural homomorphism* $\varphi_n : (\ker \Theta)^n \to C$, *which induces an isomorphism of Galois modules*

$$\varphi_n : \frac{(\ker \Theta)^n}{(\ker \Theta)^{n+1}} \to C(n),$$

where, as usual, $C(n) = C \otimes \mathbb{Z}_p(n)$ is the Tate twist of C by the cyclotomic character.

PROOF. The map φ_0 is the homomorphism Θ of Theorem 6.10, so that it commutes with the natural Galois actions on **Biv** $\mathcal{R}$ and C. Denote by $\alpha = \{\pi\}+p$ the generator of ker Θ (cf. Proposition 5.8 and Theorem 6.10). By 7.9, one has $\tau = \alpha\beta$, with $\Theta(\beta) \neq 0$, so it is well defined the map

$$\varphi_n : (\ker \Theta)^n \to C,$$

which sends $\xi = \alpha^n \mu \in (\ker \Theta)^n$ to $\Theta(\mu)\Theta(\beta)^{-n}$. It is easy to check that this map is **Biv** $\mathcal{R}$-linear, surjective and that its kernel is exactly $(\ker \Theta)^{n+1}$; so φ_n determines a C-linear isomorphism between the quotient $\dfrac{(\ker \Theta)^n}{(\ker \Theta)^{n+1}}$ and C. It remains only to verify the claim regarding the Galois action.

For any $s \in \mathcal{G}$, one has $s(\alpha) \in \ker \Theta$ and $w(\alpha) = w(s(\alpha))$, so that $s(\alpha) = \alpha\lambda_s$ for some vector λ_s, with $w(\lambda_s) = 0$. Moreover,

$$\chi(s)\alpha\beta = \chi(s)\tau = s(\tau) = s(\alpha)s(\beta) = \lambda_s\alpha s(\beta); \qquad \text{(cf. Remark 7.4)}$$

which, because **Biv** $\mathcal{R}$ is an integral domain, implies $\chi(s)\beta = \lambda_s s(\beta)$. By applying Θ we deduce:

$$\chi(s)\Theta(\beta) = \Theta(\lambda_s)\Theta(s(\beta))$$

which gives

$$\Theta(\lambda_s)^n\Theta(\beta)^{-n} = \chi(s)^n s(\Theta(\beta))^{-n},$$

for any $n \geq 1$. Then, for any $\xi = \alpha^n\mu \in (\ker \Theta)^n$, we get

$$\varphi_n(s(\xi)) = \varphi_n(s(\alpha^n)s(\mu)) = \varphi_n(\alpha^n\lambda_s^n s(\mu)) =$$
$$= \Theta(\lambda_s)^n\Theta(s(\mu))(\Theta(\beta))^{-n} = \Theta(s(\mu))\chi(s)^n s(\Theta(\beta))^{-n} =$$
$$= \chi(s)^n s[\Theta(\mu)\Theta(\beta)^{-n}] = \chi(s)^n s[\varphi_n(\xi)]. \qquad \square$$

We will describe another result about the action of the Galois group $\mathcal{G}$ on the ring **Biv** $\mathcal{R}$.

PROPOSITION 7.11. *Notations as above. Then* $(\mathbf{Biv}\,\mathcal{R})^{\mathcal{G}} = \mathbf{biv}\,k = K$. *More generally, for any* $i \geq 0$,

$$B(i) = \{\xi \in \mathbf{Biv}\,\mathcal{R} \mid s(\xi) = \chi(s)^i\xi \ \forall s \in \mathcal{G}\} = \tau^i K.$$

PROOF. Let we start with $B(0) = (\mathbf{Biv}\,\mathcal{R})^{\mathcal{G}}$. If $s(\xi) = \xi$ for any $s \in \mathcal{G}$, then $\Theta(\xi) \in C^{\mathcal{G}} = K$ and this means that there exists an element $\xi' \in \mathbf{biv}\,k = K$

such that $\xi - \xi' = \eta \in \ker \Theta$. As a difference of two $\mathcal{G}$-invariants elements, η is $\mathcal{G}$-invariant and this implies that

$$\varphi_1(\eta) = \varphi_1(s(\eta)) = \chi(s)s(\varphi_1(\eta)) \qquad \text{(cf. Corollary 7.10)}$$

Since $\varphi_1(\eta)$ is an element of C satisfying the condition $s(\varphi_1(\eta)) = \chi(s)^{-1}\varphi_1(\eta)$, for any s, we deduce that $\varphi_1(\eta) = 0$ and then $\eta \in (\ker \Theta)^2$. The analogous arguments applied to $\varphi_2(\eta)$, imply that $\eta \in (\ker \Theta)^3$, and repeating such arguments one gets $\eta \in (\ker \Theta)^n$ for each n. In view of 6.10 (c) we conclude that $\eta = 0$ and then $\xi = \xi' \in \mathbf{biv}\, k = K$.

Now suppose $n \geq 1$ and $\xi \in B(n)$. The arguments used above give $\varphi_j(\xi) = 0$ for $j < n$ and $\xi \equiv \tau^n \xi' \operatorname{mod}(\ker \Theta)^{n+1}$ for some $\xi' \in K$. This implies that $\xi - \tau^n \xi' = \eta_n \in (\ker \Theta)^{n+1} \cap B(n)$ and then that η_n satisfies the condition

$$\chi(s)^n \varphi_{n+1}(\eta_n) = \varphi_{n+1}(s(\eta_n)) = \chi(s)^{n+1} s(\varphi_{n+1}(\eta_n))$$

for any $s \in \mathcal{G}$. Now, $s(\varphi_{n+1}(\eta_n)) = \chi(s)^{-1}\varphi_{n+1}(\eta_n)$, for any $s \in \mathcal{G}$, implies $\varphi_{n+1}(\eta_n) = 0$, so that $\eta_n \in (\ker \Theta)^{n+2}$. By applying the above arguments, one deduces that $\eta_n \in (\ker \Theta)^k$ for each k and then $\eta_n = 0$, and this means $\xi = \tau^n \xi'$ for some $\xi' \in K$. □

REMARK 7.12. Now we are able to explain the relationships existing among our ring $\mathbf{Biv}\,\mathcal{R}$ and the rings B^+_{DR} and B^+ of Fontaine (cf. [FO3]). Precisely we'll show that there are the following canonical injections:

$$\text{(7.13)} \qquad \mathbf{biv}\,\mathcal{R} \to B^+ \to \mathbf{Biv}\,\mathcal{R} \to (\mathbf{Biv}\,\mathcal{R})\hat{}_{\ker\Theta} \cong B^+_{DR},$$

where $(\mathbf{Biv}\,\mathcal{R})\hat{}_{\ker\Theta}$ denotes the completion of the localization of the ring $\mathbf{Biv}\,\mathcal{R}$ at the ideal $\ker \Theta$.

i) Let us start by describing $(\mathbf{Biv}\,\mathcal{R})\hat{}_{\ker\Theta} \cong B^+_{DR}$. As defined in 2.8 of [FO3], the ring B^+_{DR} is the completion of the localization of the ring of special bivectors $\mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right]$ with respect to the kernel of the restriction of Θ to such ring. Thus there is a natural map from B^+_{DR} into $(\mathbf{Biv}\,\mathcal{R})\hat{}_{\ker\Theta}$. This map is an isomorphism; in fact, we will check that, given a Bivector ξ and an integer $n \geq 0$, there exists a special bivector ς such that $\xi - \varsigma \in (\ker \Theta)^n$. By 5.17 it follows that, for the given n, there exists a special bivector η such that $\xi - \eta \in W_n$, and as remarked in the proof of Theorem 6.10, $v(\Theta(\xi - \eta)) \geq w(\xi - \eta) \geq n$. As a conseguence there exists a vector $\lambda_0 \in \mathbf{W}(\mathcal{R})$ such that $\Theta(\xi - \eta) = p^n\Theta(\lambda_0) = \Theta(p^n\lambda_0)$. This implies that

$$\xi - \eta - p^n\lambda_0 \in \ker \Theta \cap W_n,$$

where $\eta + p^n\lambda_0$ is a special bivector. Then, we can write $\xi - \eta - p^n\lambda_0 = \alpha\xi_1$, for some $\xi_1 \in W_{n-1}$: and by the above arguments there exists a vector $\lambda_1 \in \mathbf{W}(\mathcal{R})$ such that $\Theta(\xi_1) = p^{n-1}\Theta(\lambda_1) = \Theta(p^{n-1}\lambda_1)$. This implies that

$$\xi - \eta - p^n\lambda_0 - p^{n-1}\alpha\lambda_1 \in (\ker \Theta)^2 \cap W_n,$$

where $\eta + p^n\lambda_0 + p^{n-1}\alpha\lambda_1$ is a special bivector. Now, observe that we can write $\xi - \eta - p^n\lambda_0 - p^{n-1}\alpha\lambda_1 = \alpha^2\xi_2$, for some $\xi_2 \in W_{n-2}$ and repeat the previous arguments until we get

$$\xi - \eta - p^n\lambda_0 - \ldots - p\alpha^{n-1}\lambda_{n-1} - \alpha^n\lambda_n \in (\ker \Theta)^n \cap W_n,$$

where $\eta + p^n\lambda_0 + \ldots + p\alpha^{n-1}\lambda_{n-1} + \alpha^n\lambda_n$ is a special bivector. In this way we get the required approximation of ξ.

ii) The injectivity of the map $\mathbf{Biv}\,\mathcal{R} \to (\mathbf{Biv}\,\mathcal{R})\hat{}_{\ker\Theta}$ follows immediately from 6.10 (c).

iii) Before describing the embeddings $\mathbf{biv}\,\mathcal{R} \to B^+ \to \mathbf{Biv}\,\mathcal{R}$, we recall the construction of the ring B^+ given in [FO3].

The notations are the following:
for any proper ideal a of $\mathcal{R}$, let

$$S_a = \left\{(x_n)_{n\in\mathbb{N}} \in \mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right] : x_{-n}^{p^n} \in a^n, \text{ for each } n > 0\right\}.$$

S_a is a subring of $\mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right]$ whose completion, with respect to the p-adic topology, will be denoted by $\hat{S_a}$; finally $B_a^+ = \hat{S_a} \otimes K$.

Fontaine proves that if $a \subseteq a'$ are two ideals of $\mathcal{R}$, then $S_a \subseteq S_{a'}$, and that this inclusion induces a continuous embedding of B_a^+ in $B_{a'}^+$; then, he shows that if $a \subseteq a_0 = \{x \in \mathcal{R} : v(x) \geq v(p)\}$, the inclusion $S_a \subseteq \mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right]$ induces a continuous embedding, $\phi_a : B_a^+ \to B_{DR}^+$. Finally, he defines B^+ as the intersection, inside B_{DR}^+, of all the embedded B_a^+.

Our embeddings will be a consequence of the following

CLAIMS. *Let $b_c = \{x \in \mathcal{R} | v(x) \geq c\}$, where $c \in \mathbb{N}_{>0}$, then*

$$B^+ = \bigcap_{c=1}^{\infty} B_{b_c}^+.$$

The inclusion $S_{b_c} \subseteq \mathbf{W}(\mathcal{R})\left[\frac{1}{p}\right]$ induces an embedding, $\psi_{b_c} : B_{b_c}^+ \to$ $\mathbf{Biv}\,\mathcal{R}$, which, composed with the embedding of $\mathbf{Biv}\,\mathcal{R}$ in B_{DR}^+ defined above, gives the map ϕ_{b_c}.

Finally, $\mathbf{biv}\,\mathcal{R} \subseteq B_{b_c}^+$, *for each c.*

PROOF. Since the ideals b_c are all principal, it's easy to see that any ideal of $\mathcal{R}$ contains an ideal of the family $\{b_c | c \in \mathbb{N}_{>0}\}$; then the first claim follows immediately by the results of [FO3] mentioned above.

A special bivector $\xi = (x_n)_{n \in \mathbf{N}}$ belongs to S_{b_c} if, and only if, $p^n v(x_{-n}) \geq nc$ for each $n > 0$; since $c \geq 1$, this implies that $w(\xi) \geq 0$ and then $p^n S_{b_c} \subseteq W_n$. This last inclusion shows that any sequence of elements of S_{b_c}, which converges for the p-adic topology of S_{b_c}, converges also for the w-topology, therefore its limit is in $\mathbf{Biv}\,\mathcal{R}$. As a consequence, the completion of S_{b_c} with respect to the p-adic topology, is contained in $\mathbf{Biv}\,\mathcal{R}$ and, recalling that p is invertible in $\mathbf{Biv}\,\mathcal{R}$, the same holds for $B^+_{b_c}$. This explains how ψ_{b_c} is defined; the factorization of ϕ_{b_c} along ψ_{b_c} follows immediately by the previous remarks i) and ii).

It remains to show that $\mathbf{biv}\,\mathcal{R} \subseteq B^+_{b_c}$; we will prove that, if $\varsigma = (z_n)_{n \in \mathbf{N}}$ is a bivector, then there exists an integer n_0 such that any element of the sequence $(p^{n_0} \varsigma^{(\geq -r)})_{r \in \mathbf{N}}$ is in S_{b_c} and that such sequence satisfies the Cauchy condition with respect to the p-adic topology of S_{b_c}.

In fact, the special bivector $p^{n_0} \varsigma^{(\geq -r)}$ belongs to S_{b_c} if and only if,

$$p^{n+n_0} v(z_{-n-n_0}) \geq nc \text{ for } 0 < n \leq r - n_0. \tag{7.14}$$

As a consequence of the condition 6.1 defining a bivector, the above condition holds for any r and for a suitable n_0. In fact 6.1 implies that there exists a positive real number b such that $v(z_{-m}) \geq b$ for each $m >> 0$, so it's easy to observe that there exists n_0 such that $p^{n_0} b p^n \geq nc$, for any n; and then 7.14 holds.

The sequence $(p^{n_0} \varsigma^{(\geq -r)})_{r \in \mathbf{N}}$ satisfies the Cauchy condition if, and only if, for any given $k \geq 0$, there exists an index r_0, such that $r > r_0 \Rightarrow p^{n_0} \varsigma^{(\geq -r)} - p^{n_0} \varsigma^{(\geq -r+1)} \in p^k S_{b_c}$. Now we have

$$p^{n_0} \varsigma^{(\geq -r)} - p^{n_0} \varsigma^{(\geq -r+1)} = V^{-r+n_0}\{z_{-r}^{p^{n_0}}\}$$

and then the condition $V^{-r+n_0}\{z_{-r}^{p^{n_0}}\} \in p^k S_{b_c}$ can be rewritten as follows:

$$p^{-k} V^{-r+n_0}\{z_{-r}^{p^{n_0}}\} = V^{-r-k+n_0}\{z_{-r}^{n_0-k}\} \in S_{b_c}.$$

This last condition is equivalent to $p^r v(z_{-r}) \geq (r+k-n_0)c$, which, again in view of the condition 6.1 defining a bivector, is true for $r >> 0$. This concludes the proof of the inclusion $\mathbf{biv}\,\mathcal{R} \subseteq B^+_{b_c}$, and thus $\mathbf{biv}\,\mathcal{R} \subseteq B^+$.

A final remark: the graded ring associated to the filtration

$$\ldots \subseteq (\ker \Theta)^{n+1} \subseteq (\ker \Theta)^n \subseteq \ldots \subseteq \ker \Theta \subseteq \mathbf{Biv}\,\mathcal{R}$$

is isomorphic to the ring B^+_{HT} as defined in 1.2 of [FO3]; this follows immediately from Corollary 7.10.

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