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A Free Boundary Problem Arising in Magnetohydrodynamic System

AVNER FRIEDMAN - YONG LIU

1. - The plasma problem

We shall consider a two-dimensional problem arising in magnetohydrodynamic (MHD) system, modeling the plasma confinement in the Tokamak machine.

Let Ω be a bounded domain in $\mathbb{R}^2$, representing the cross-section of perfectly electrically conducting shell. As shown in Figure 1, the plasma fills a subdomain Ω_p of Ω, with boundary Γ_p; the complimentary domain $\Omega\backslash\overline{\Omega}_p$ is the vacuum region Ω_v.

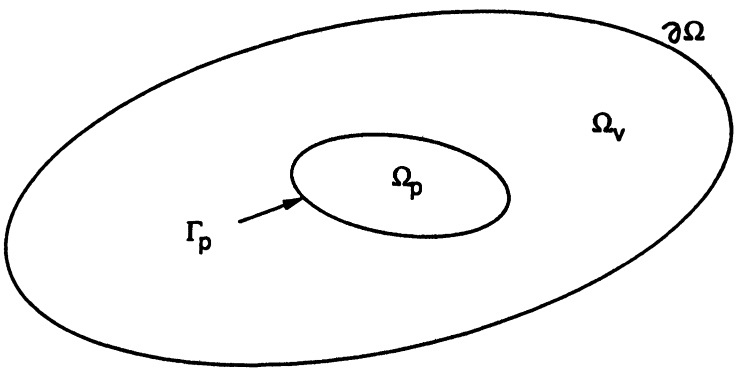

Figure 1

Let $\vec{B}$, $\vec{J}$ and P denote the magnetic induction, the current density and the fluid pressure, respectively. In equilibrium, the triple $(\vec{B}, \vec{J}, P)$ satisfies the MHD system:

$$\operatorname{div} \vec{B} = 0, \quad \operatorname{rot} \vec{B} = \mu_0 \vec{J}, \quad \operatorname{grad} P = \vec{J} \times \vec{B} \text{ in } \Omega_p, \tag{1.1}$$

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$$\operatorname{div} \vec{B} = 0, \ \operatorname{rot} \vec{B} = 0, \ P = 0 \text{ in } \Omega_v, \tag{1.2}$$

$$\vec{B} \cdot \nu = 0 \text{ on } \Gamma_p \text{ and on } \partial\Omega, \tag{1.3}$$

$$P + \frac{1}{2\mu_0} B_p^2 = \frac{1}{2\mu_0} B_v^2 \text{ on } \Gamma_p \tag{1.4}$$

where μ_0 is the magnetic permeability of the vacuum, ν is the outward unit normal, $B^2 = \vec{B} \cdot \vec{B}$, and the subscripts p and v denote, respectively, the subscripted quantities from the interior of the plasma region Ω_p and the vacuum region Ω_v.

The first two equations in (1.1) and (1.2) are Maxwell's equations, the third equation in (1.1) is the equation of plasma motion in the stationary case, and (1.4) is the pressure balance condition.

Since we are considering here the two-dimensional problem (where Ω is viewed as a cross-section $x_3 = 0$ of an infinite cylinder), $\vec{B}$, $\vec{J}$ and P depend only on the variables x_1, x_2.

Since $\operatorname{div} \vec{B} = 0$, we can introduce locally a flux function u by

$$\vec{B} = (u_{x_2}, -u_{x_1}, 0) \tag{1.5}$$

(assuming that the vector $\vec{B}$ lies in the two-dimensional space $\mathbb{R}^2$); since, by (1.3), $\partial u/\partial\tau = 0$ on Γ_p where τ is the unit tangent vector, u is a single-valued function in the whole domain Ω, uniquely determined up to an additive constant. It is easy to verify that

$$\operatorname{rot} \vec{B} = -(\Delta u)\vec{k}, \tag{1.6}$$

and then, by (1.1) and (1.2),

$$\Delta u = -\mu_0 J \text{ in } \Omega_p \text{ where } \vec{J} = (0, 0, J), \tag{1.7}$$

$$\Delta u = 0 \text{ in } \Omega_v. \tag{1.8}$$

Since $\vec{J} \times \vec{B} = J\nabla u$, the third equation in (1.1) becomes

$$\nabla P = J\nabla u. \tag{1.9}$$

Therefore ∇P is parallel to ∇u. This implies that P depends only on u, that is,

$$P = g(u) \tag{1.10}$$

where g is an unknown function. This function cannot be determined by the Maxwell equations; it is a constitutive function for the plasma. Equations (1.9) and (1.10) imply that

$$J = g'(u), \tag{1.11}$$

so that (1.7) can be written in the form

$$\Delta u = -\mu_0 g'(u) \text{ in } \Omega_p. \tag{1.12}$$

The boundary condition (1.3) implies that

$$\frac{\partial u}{\partial \tau} = 0 \text{ on } \Gamma_p \cup \partial\Omega \tag{1.13}$$

where τ is the unit tangent vector. It follows that u is constant on Γ_p and on $\partial\Omega$, and we determine u uniquely by taking

$$u = 0 \text{ on } \Gamma_p. \tag{1.14}$$

We also set

$$u = \gamma \text{ on } \partial\Omega, \tag{1.15}$$

where γ is an unspecified constant which we take to be positive.

Finally, condition (1.4) can be written as

$$|\nabla u_v|^2 - |\nabla u_p|^2 = 2\mu_0 P = 2\mu_0 g(0) \text{ on } \Gamma_p \tag{1.16}$$

and, in general, $g(0) > 0$. This condition formally coincides with Bernoulli's condition for two-fluid flow.

Temam [21, 22] considered the model with

$$P = g(u) = au^2 \quad (a > 0);$$

in this case $P = 0$ on Γ_p and the system (1.8), (1.12), (1.14)-(1.16) reduces to

$$\Delta u - \lambda u_- = 0 \text{ in } \Omega, \quad u = \gamma \text{ on } \partial\Omega, \tag{1.17}$$

where $u_- = -\min\{u, 0\}$.

More realistic models with $P > 0$ on Γ_p were considered by Grad, Kadish and Stevens [15] and by Ushijima [23]. Here we shall assume the constitutive law

$$g(u) = au^2 + b \text{ with } a > 0,\ b > 0, \tag{1.18}$$

so that $P > 0$ on Γ_p.

In [21] $\lambda = 2a\mu_0$ is unspecified positive constant whereas either the constant γ or the flux

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, ds = I \tag{1.19}$$

are specified; furthermore, the condition

$$\int_{\Omega} u_-^2 = K \quad (K \text{ is given}) \tag{1.20}$$

is imposed. In [22] λ is prescribed but the condition (1.20) is dropped.

In our formulation of the plasma problem we shall assume that γ and b are given, and that a is unknown (which means that $\Delta u_- + \lambda u_- = 0$ holds in Ω_p with unknown positive constant λ). We also impose the additional condition (1.20), where K is given.

Other formulations are also possible; for example, one may prescribe λ and drop the condition (1.20). Since we do not prove here uniqueness and continuous dependence of the solution on the parameter K, the two formulations may not be equivalent. Furthermore, when γ is unprescribed and, instead, the condition (1.19) is imposed, we get yet another different version of the problem. We have chosen the present formulation of prescribing γ, b and K because it is technically the easiest: it lends itself more directly to a variational formulation. We hope to consider other formulations in a later work.

In the next section we rewrite the plasma problem with slightly normalized constants, and describe the main results and the structure of the paper. Many of the results of this paper are valid in n-dimensions. For this reason we shall assume that Ω is n-dimensional where $n \geq 2$, unless the contrary is explicitly stated.

2. - Mathematical formulation; the structure of the paper

Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary $\partial\Omega$, and let γ and μ^2 be given positive constants. The *plasma problem* is to find a function u, a closed surface Γ_p lying in Ω and a positive constant λ such that

$$\Delta u + \lambda u = 0 \text{ in } \Omega_p = \operatorname{int}\{x \in \Omega, u(x) \leq 0\}, \tag{2.1}$$

$$\Delta u = 0 \text{ in } \Omega_v = \{x \in \Omega, u(x) > 0\}, \tag{2.2}$$

$$u = \gamma \text{ on } \partial\Omega, \tag{2.3}$$

$$u = 0 \text{ on } \Gamma_p \text{ where } \Gamma_p = \partial\Omega_p, \tag{2.4}$$

$$|\nabla u_+|^2 - |\nabla u_-|^2 = \mu^2 \text{ on } \Gamma_p \tag{2.5}$$

and

$$\int_{\Omega} u_-^2 = 1; \tag{2.6}$$

here $\nabla u_-(\nabla u_+)$ denotes the limit of ∇u from $\Omega_p(\Omega_v)$.

In this paper we shall prove that for $n = 2$ there exists a solution to the plasma problem and that Γ_p is continuously differentiable and, in fact, analytic; the open set Ω_v is connected, but Ω_p may have several components $G_1, \dots, G_N$. There is however only one component, say G_1, in which $u < 0$ everywhere; in all the other components u vanishes identically. For the simpler plasma model with (1.17), it is known [19] (see also [13; pp. 529-531]) that for any variational solution Ω_p has a single component.

For $n \geq 3$ there still exists a solution of the plasma problem and u is Lipschitz continuous, but the free boundary condition (2.5) is satisfied only in a weak sense. (We do not establish here the regularity of the free surface).

Our approach is based on variational methods of Alt and Caffarelli [2], and Alt, Caffarelli and Friedman [3], and the more recent papers of Caffarelli [5] [6] [7].

The structure of the paper is as follows:

In Section 3 we introduce a variational formulation of the plasma problem with a functional $J(u)$ as in [3] but with the integral constraint (2.6). We establish several simple properties of the minimizers, such as subharmonicity, and a weak form of the free boundary condition. However the integral constraint does not allow us to adapt any of the deeper results from [3]. We can partially overcome this difficulty by introducing, in Section 4, another variational problem which does not have any constraints in the admissible classes but, instead, has a new and a bit more complicated functional, $J_\eta(u)$, namely:

$$J_\eta(u) = J(u) - f_\eta\left(\int_\Omega u_-^2\right)$$

where

$$f_\eta(s) = \begin{cases} \dfrac{1}{\eta}(s-1) & \text{if } s < 1 \\ \eta(s-1) & \text{if } s \geq 1 \end{cases}$$

and $\eta \ll 1$. Any minimizer of J_η is shown to be also a minimizer of J, and for the rest of the paper we study only minimizers of J_η.

In Section 5 we derive C^α regularity of any minimizer u, and in Section 6 we prove nondegeneracy. These results are obtained by adapting methods of [3]. The next fundamental result, proved in Section 7, is the monotonicity theorem which asserts that

$$r \longrightarrow \frac{1}{r^4}\int_{B_r} \rho^{2-n}|\nabla u_+|^2 \cdot \int_{B_r} \rho^{2-n}|\nabla u_-|^2 \cdot e^{\kappa\lambda r^{2-\frac{1-\alpha}{\alpha}N}}$$

is monotone increasing in r. This is an extension of a monotonicity theorem proved in [3] in case $\lambda = 0$, and it can be used, precisely as in [3], to establish the Lipschitz continuity of u.

In Section 8 we consider blow-up sequences and identify their limits. In Section 9 we begin the initial study of the free boundary, showing that Δu_+ and $\Delta u_- + \lambda u_-$ are measures supported on the free boundary, absolutely continuous with respect to dH^{n-1}. The results of Sections 8, 9 are similar to those in [3]; see also [2].

In establishing the regularity of the free boundary in [3] the following comparison result (Lemma 8.1 in [3]) was crucial: if u, $\overline{u}$ are minimizers and $u > \overline{u}$ on the boundary then $u > \overline{u}$ in the entire domain. This lemma is no longer true in our case (where $\lambda > 0$). We therefore proceed in entirely different way, adapting the recent Harnack inequality approach of Caffarelli [5] [6] for weak solutions.

We begin, in Section 10, by introducing weak solutions and subsolutions. In [5] Caffarelli constructed a large class of subsolutions which he used to establish the regularity of "flat" free boundaries. In our case, where $\lambda > 0$, we can only construct a much smaller class of subsolutions. However by exploiting additional ideas from [3] we shall be able, nevertheless, to establish (in subsequent sections) the regularity of the free boundary.

In Section 11 we introduce (as in [6]) the concept of ε-monotonicity and prove (Theorem 11.3) that

(i) ε-monotonicity,

(ii) flatness, and

(iii) full monotonicity at the boundary

imply Lipschitz continuity of the free boundary of a weak solution.

In Section 13 we improve Theorem 11.3 by showing that assumption (i) is always satisfied; this is Theorem 13.2. The proof which is an adaptation from Caffarelli's paper [6] requires several auxiliary estimates, some of which are proved in Section 12. (In case $\lambda = 0$ the condition (iii) was not assumed in [6]; it was in fact verified by using the much larger family of subsolutions available in case $\lambda = 0$).

In Section 14 it is shown that minimizers of J_η are weak solutions. At this point we go back to adapt some results from [3] in order to show that the assumption (iii) above is satisfied by any minimizer. Since for $n = 2$ assumption (ii) is also satisfied (as proved in Section 8 of [3]), we conclude that for $n = 2$ the free boundary is Lipschitz continuous and, in fact, it is also continuously differentiable and $\nabla u_\pm$ is continuous up to the boundary. In Section 15 we establish the $C^{1+\alpha}$ smoothness of the free boundary; the analyticity of the free boundary then follows by known techniques.

The sets $\Omega_v \equiv \Omega^+(u) = \{x \in \Omega, u(x) > 0\}$, $\Omega_p \equiv \Omega^-(u) = \text{int}\{x \in \Omega, u(x) \leq 0\}$ are open and $\Omega^+(u)$ contains a neighborhood of $\partial\Omega$. In Section 16 we prove that $\Omega^+(u)$ is connected, whereas $\Omega^-(u)$ consists of a finite number of components $G_1, \dots, G_N$. In just one component, say G_1, $u < 0$ throughout. In all other components G_j, $u \equiv 0$; such components do in fact exist for some domains Ω.

3. - A variational problem

We introduce the admissible class

$$K = \left\{ v \in H^1(\Omega), v = \gamma \text{ on } \partial\Omega, \int_\Omega v_-^2 = 1 \right\}$$

and the functional

$$J(v) = \int_\Omega [|\nabla v|^2 + \mu_-^2 I_{\{v \leq 0\}} + \mu_+^2 I_{\{v>0\}}] \tag{3.1}$$

where μ_-, μ_+ are positive constants satisfying

$$\mu_+^2 - \mu_-^2 = \mu^2$$

and I_A denotes the characteristic function of a set A.

PROBLEM (J). Find $u \in K$ such that

$$J(u) = \min_{v \in K} J(v).$$

This variational problem is a generalization of the one introduced by Temam [21] for the case $\mu_+ = \mu_- = 0$.

THEOREM 3.1. *There exists a solution of problem (J).*

The proof is the same as for Theorem 1.1 of [3]. We denote a solution by u, and proceed to establish some basic properties.

THEOREM 3.2. *The solution u is subharmonic in Ω.*

PROOF. Take any $\xi \in C_0^1(\Omega)$, $\xi \geq 0$ and $\varepsilon > 0$, and set

$$\tilde{u}_\varepsilon = u - \varepsilon\xi \tag{3.2}$$

and

$$u_\varepsilon = -\sqrt{a_\varepsilon}(\tilde{u}_\varepsilon)_- + (\tilde{u}_\varepsilon)_+, \quad a_\varepsilon = 1/\int_\Omega (\tilde{u}_\varepsilon)_-^2. \tag{3.3}$$

Then $u_\varepsilon \in K$ and

$$
\begin{aligned}
0 &\le J(u_\varepsilon) - J(u) \\
&= a_\varepsilon \int_\Omega [|\nabla(\tilde{u}_\varepsilon)_-|^2 + \int_\Omega [|\nabla(\tilde{u}_\varepsilon)_+|^2 + \mu_-^2 I_{\{u_\varepsilon \le 0\}} + \mu_+^2 I_{\{u_\varepsilon > 0\}}] \\
&\qquad - \int_\Omega [|\nabla u|^2 + \mu_-^2 I_{\{u \le 0\}} + \mu_+^2 I_{\{u>0\}}] \\
&= a_\varepsilon \int_{\{\tilde{u}_\varepsilon \le 0\}} (|\nabla \tilde{u}_\varepsilon|^2 - |\nabla u|^2) + (a_\varepsilon - 1) \int_{\{\tilde{u}_\varepsilon \le 0\}} |\nabla u|^2 \\
&\qquad + \int_{\{\tilde{u}_\varepsilon > 0\}} (|\nabla \tilde{u}_\varepsilon|^2 - |\nabla u|^2) + \int_\Omega \mu_-^2 I_{\{\tilde{u}_\varepsilon \le 0\}} \int_\Omega \mu_-^2 I_{\{u \le 0\}} \\
&\qquad + \int_\Omega \mu_+^2 I_{\{\tilde{u}_\varepsilon > 0\}} - \int_\Omega \mu_+^2 I_{\{u>0\}}
\end{aligned}
$$

where we have used the fact that $\operatorname{sgn} u_\varepsilon = \operatorname{sgn} \tilde{u}_\varepsilon$.

Since $a_\varepsilon = 1 + o(1)$ and

$$
\begin{aligned}
\frac{a_\varepsilon - 1}{\varepsilon} &= \frac{1}{\varepsilon} \frac{1}{\int_\Omega (\tilde{u}_\varepsilon)_-^2} \left[\int_\Omega u_-^2 - \int_\Omega (\tilde{u}_\varepsilon)_-^2 \right] \\
&= \frac{1}{\int_\Omega (\tilde{u}_\varepsilon)_-^2} \frac{1}{\varepsilon} \int_\Omega (u_-^2 - (u - \varepsilon\xi)_-^2) \to -2 \int_\Omega u_- \xi \le 0 \text{ as } \varepsilon \to 0,
\end{aligned}
\tag{3.4}
$$

we have

$$
\begin{aligned}
0 &\le (1 + o(1))(-2\varepsilon) \int_{\{\tilde{u}_\varepsilon \le 0\}} \nabla u \cdot \nabla \xi - 2\varepsilon \int_{\{\tilde{u}_\varepsilon > 0\}} \nabla u \cdot \nabla \xi \\
&\qquad + \int_{\{u \le \varepsilon\xi\}} \mu_-^2 - \int_{\{u \le 0\}} \mu_-^2 + \int_{\{u > \varepsilon\xi\}} \mu_+^2 - \int_{\{u>0\}} \mu_+^2 + o(\varepsilon) \\
&= -2\varepsilon \int_\Omega \nabla u \cdot \nabla \xi - \int_{\{0 < u \le \varepsilon\xi\}} \mu^2 + o(\varepsilon) \le -2\varepsilon \int_\Omega \nabla u \cdot \nabla \xi + o(\varepsilon).
\end{aligned}
$$

It follows that

$$\int_{\Omega} \nabla u \cdot \nabla \xi \leq 0 \quad \forall \xi \in C_0^1(\Omega), \ \ \xi \geq 0,$$

and the proof is complete.

For the next result we temporarily assume that u is continuous in $\overline{\Omega}$. Then $\Omega_p = \operatorname{int}\{x \in \Omega, u(x) \leq 0\}$, $\overset{\circ}{\Omega}_p = \{x \in \Omega, u(x) < 0\}$, and $\Omega_v = \{x \in \Omega, u(x) > 0\}$ are nonempty open subsets of Ω and $\Omega_p \subset\subset \Omega$.

THEOREM 3.3. *If u is continuous in $\overline{\Omega}$ then*

$$\Delta u + \lambda u = 0 \ \ in \ \overset{\circ}{\Omega}_p \tag{3.5}$$

where

$$\lambda = \int_{\Omega} |\nabla u_-|^2, \tag{3.6}$$

and

$$\Delta u = 0 \ \ in \ \Omega_v. \tag{3.7}$$

PROOF. For any $\xi \in C_0^1(\overset{\circ}{\Omega}_p)$ take $\varepsilon \gtrless 0$ with $|\varepsilon|$ sufficiently small so that $u - \varepsilon\xi < 0$ in $\overset{\circ}{\Omega}_p$, and define u_ε as before, by (3.3), (3.2). Then $\operatorname{sgn} u_\varepsilon = \operatorname{sgn} u$ and $u_\varepsilon = u$ in $\Omega \backslash \overset{\circ}{\Omega}_p$ and, since $u_\varepsilon \in K$,

$$\begin{aligned} 0 \leq J(u_\varepsilon) - J(u) &= a_\varepsilon \int_{\overset{\circ}{\Omega}_p} [|\nabla(u - \varepsilon\xi)|^2 - |\nabla u|^2] + (a_\varepsilon - 1) \int_{\overset{\circ}{\Omega}_p} |\nabla u|^2 \\ &= -2\varepsilon a_\varepsilon \int_{\overset{\circ}{\Omega}_p} \nabla u \cdot \nabla \xi + \lambda(a_\varepsilon - 1) + o(\varepsilon). \end{aligned}$$

Recalling (3.4) we get

$$-\int_{\overset{\circ}{\Omega}_p} \nabla u \cdot \nabla \xi + \lambda \int_{\overset{\circ}{\Omega}_p} u\xi = 0 \quad \forall \xi \in C_0^1(\overset{\circ}{\Omega}_p),$$

i.e., (3.5) holds. The proof of (3.7) is similar.

The set $\Gamma_p = \partial\Omega_p \cap \partial\Omega_v$ is called the *strict free boundary*. The next theorem shows that u satisfies, in a generalized sense, the equation

$$|\nabla u_+|^2 - |\nabla u_-|^2 = \mu_+^2 - \mu_-^2 = \mu^2 \text{ on } \Gamma_p.$$

provided the set $\{u = 0\}$ has zero measure.

THEOREM 3.4. *Let Ω_0 be any open subset of Ω. If u is continuous in Ω_0 and* $\text{meas}\{x \in \Omega_0, u(x) = 0\} = 0$ *then, for any $\varsigma \in C_0^1(\Omega_0, \mathbb{R}^n)$,*

$$\lim_{\varepsilon \downarrow 0} \int\limits_{\partial\{u<-\varepsilon\}} (|\nabla u|^2 \mu_-^2)\varsigma \cdot \nu + \lim_{\delta \downarrow 0} \int\limits_{\partial\{u>\delta\}} (|\nabla u|^2 - \mu_+^2)\varsigma \cdot \nu = 0 \tag{3.8}$$

where ν is the outward normal.

Here ε and δ are such that $\partial\{u < -\varepsilon\}$ and $\partial\{u > \delta\}$ are continuously differentiable; by Sard's theorem this is true for *a.a.* ε and δ.

PROOF. For simplicity of notation we take $\Omega_0 = \Omega$. Following the proof of Theorem 2.4 in [3] we define, for any $\varepsilon \gtrless 0$ with small $|\varepsilon|$, diffeomorphism

$$\tau_\varepsilon : \Omega \to \Omega$$

by $\tau_\varepsilon(x) = x + \varepsilon\varsigma(x)$ and set $\tilde{u}_\varepsilon(\tau_\varepsilon(x)) = u(x)$. Next we introduce a function

$$u_\varepsilon = -\sqrt{a_\varepsilon}(\tilde{u}_\varepsilon)_- + (\tilde{u}_\varepsilon)_+, \quad a_\varepsilon = 1 \Big/ \int\limits_\Omega (\tilde{u}_\varepsilon)_-^2 .$$

Then $u_\varepsilon \in K$ and

$$\begin{aligned} 0 \le J(u_\varepsilon) - J(u) = a_\varepsilon & \int\limits_\Omega |\nabla u_-(D\tau_\varepsilon)^{-1}|^2 \det(D\tau_\varepsilon) \\ & + \int\limits_\Omega |\nabla u_+(D\tau_\varepsilon)^{-1}|^2 \det(D\tau_\varepsilon) + \int\limits_\Omega \mu_-^2 I_{\{u_\varepsilon(\tau_\varepsilon(x)) \le 0\}} \det(D\tau_\varepsilon) \\ & + \int\limits_\Omega \mu_+^2 I_{\{u_\varepsilon(\tau_\varepsilon(x)) > 0\}} \det(D\tau_\varepsilon) - \int\limits_\Omega |\nabla u|^2 \\ & - \int\limits_\Omega \mu_-^2 I_{\{u \le 0\}} - \int\limits_\Omega \mu_+^2 I_{\{u>0\}} . \end{aligned}$$

Since $(D\tau_\varepsilon)^{-1} = 1 - \varepsilon D\varsigma + o(\varepsilon)$, $\det(D\tau_\varepsilon) = 1 + \varepsilon\nabla \cdot \varsigma + o(\varepsilon)$ and $u_\varepsilon(\tau_\varepsilon(x)) =$

$-\sqrt{a_\varepsilon}u_-(x) + u_+(x)$, we obtain, after simple calculation,

$$0 \le (a_\varepsilon - 1) \int_\Omega |\nabla u_-|^2 - 2\varepsilon a_\varepsilon \int_\Omega \nabla u_- D\varsigma \nabla u_- - 2\varepsilon \int_\Omega \nabla u_+ D\varsigma \nabla u_+$$
$$+ \varepsilon a_\varepsilon \int_\Omega |\nabla u_-|^2 \nabla \cdot \varsigma + \varepsilon \int_\Omega |\nabla u_+|^2 \nabla \cdot \varsigma + \varepsilon \int_\Omega \mu_-^2 I_{\{u\le 0\}} \nabla \cdot \varsigma$$
$$+ \varepsilon \int_\Omega \mu_+^2 I_{\{u>0\}} \nabla \cdot \varsigma + o(\varepsilon).$$

Noting (cf. (3.4)) that

$$a_\varepsilon - 1 = \frac{1}{\int_\Omega (\tilde{u}_\varepsilon)_-^2} \left[\int_\Omega u_-^2 \int_\Omega u_-^2 (1 + \varepsilon \nabla \cdot \varsigma + o(\varepsilon)) \right] = -\varepsilon \int_\Omega u_-^2 \nabla \cdot \varsigma + o(\varepsilon)$$

and that the linear term in ε in the preceding inequality must vanish, we get

$$0 = -\left(\int_\Omega u_-^2 \nabla \cdot \varsigma \right) \int_\Omega |\nabla u_-|^2 - 2 \int_\Omega \nabla u D\varsigma u$$
$$+ \int_\Omega |\nabla u|^2 \nabla \cdot \varsigma + \int_\Omega \mu_-^2 I_{\{u\le 0\}} \nabla \cdot \varsigma + \int_\Omega \mu_+^2 I_{\{u>0\}} \nabla \cdot \varsigma$$
$$= \int_\Omega (|\nabla u|^2 - \lambda u_-^2 + \mu_-^2 I_{\{u\le 0\}} + \mu_+^2 I_{\{u>0\}}) \nabla \cdot \varsigma - 2 \int_\Omega \nabla u D\varsigma \nabla u.$$

We now use the assumption that meas $\{u = 0\} = 0$ to conclude from the last relation that

$$0 = \lim_{\delta \downarrow 0} \int_{\{u>\delta\}} [(|\nabla u|^2 + \mu_+^2) \nabla \cdot \varsigma - 2 \nabla u D\varsigma \nabla u]$$
$$+ \lim_{\varepsilon \downarrow 0} \int_{\{u<-\varepsilon\}} [(|\nabla u|^2 - \lambda u^2 + \mu_-^2) \nabla \cdot \varsigma - 2 \nabla u D\varsigma \nabla u]. \tag{3.9}$$

Notice that, in $\{u > \delta\}$, $\Delta u = 0$ and

$$\nabla \cdot [(|\nabla u|^2 + \mu_+^2)\varsigma - 2(\varsigma \cdot \nabla u)\nabla u]$$
$$= (|\nabla u|^2 + \mu_+^2) \nabla \cdot \varsigma + (\nabla(|\nabla u|^2)) \cdot \varsigma - 2(\nabla(\varsigma \cdot \nabla u)) \cdot \nabla u$$
$$= (|\nabla u|^2 + \mu_+^2) \nabla \cdot \varsigma - 2 \nabla u D\varsigma \nabla u$$

where in the last equation we made use of the identity

$$\nabla u \cdot \nabla(\varsigma \cdot \nabla u) = \frac{1}{2}\varsigma \cdot \nabla(|\nabla u|^2) + \nabla u D\varsigma \nabla u.$$

Similarly, in $\{u < -\varepsilon\}$, $\Delta u + \lambda u = 0$ and

$$\nabla \cdot [(|\nabla u|^2 + \mu_-^2 - \lambda u^2)\varsigma - 2(\varsigma \cdot \nabla u)\nabla u)\nabla u] = (|\nabla u|^2 + \mu_-^2 - \lambda u^2)\nabla \cdot \varsigma - 2\nabla u D\varsigma \nabla u.$$

Therefore we deduce from (3.9) that

$$\begin{aligned} 0 &= \lim_{\delta \downarrow 0} \int\limits_{\{u>\delta\}} \nabla \cdot [(|\nabla u|^2 + \mu_+^2)\varsigma - 2(\varsigma \cdot \nabla u)\nabla u] \\ &\quad + \lim_{\varepsilon \downarrow 0} \int\limits_{\{u<-\varepsilon\}} \nabla \cdot [(|\nabla u|^2 + \mu_-^2 - \lambda u^2)\varsigma - 2(\varsigma \cdot \nabla u)\nabla u] \\ &= \lim_{\delta \downarrow 0} \int\limits_{\partial\{u>\delta\}} (\mu_+^2 - |\nabla u|^2)\varsigma \cdot \nu + \lim_{\varepsilon \downarrow 0} \int\limits_{\partial\{u<-\varepsilon\}} (\mu_-^2 |\nabla u|^2)\varsigma \cdot \nu \end{aligned}$$

where in the last equality we made use of the fact that $|\nabla u| = \mp \nabla u \cdot \nu$ on $\{u = \delta\}$ and on $\{u = -\varepsilon\}$ respectively; this is the assertion (3.8).

REMARK 3.1. If $u \geq 0$ in Ω_0 and $\text{meas}\{x \in \Omega_0, u(x) \neq 0\} > 0$ then we have $|\nabla u_+|^2 = \mu_+^2 \mu_-^2 = \mu^2$ on $\Gamma_p \cap \Omega_0$ in the sense that

$$\lim_{\delta \downarrow 0} \int\limits_{\partial\{u>\delta\}} (|\nabla u|^2 - \mu^2)\varsigma \cdot \nu = 0 \quad \forall \varsigma \in C_0^1(\Omega_0, \mathbb{R}^n).$$

4. - Another variational principle

The constraint $\int\limits_\Omega u_-^2 = 1$ in the definition of the admissible functions poses serious difficulties to establishing further properties for any minimizer u (by trying to extend the analysis of [3]). We need to have a more "flexible" class of admissible functions. This however will require a modification of the functional J.

Let η be a positive real number and let

$$f_\eta(s) = \begin{cases} \dfrac{1}{\eta}(s-1) & \text{if } s < 1 \\ \eta(s-1) & \text{if } s \geq 1. \end{cases} \tag{4.1}$$

We introduce the admissible class

$$K_0 = \{v \in H^1(\Omega), v = \gamma \text{ on } \partial\Omega\}, \tag{4.2}$$

and a new functional

$$\begin{aligned} J_\eta(v) &= \int_\Omega [|\nabla v|^2 + \mu_-^2 I_{\{v\le 0\}} + \mu_+^2 I_{\{v>0\}}] - f_\eta\left(\int_\Omega v_-^2\right) \\ &\equiv J(v) - f_\eta\left(\int_\Omega v_-^2\right). \end{aligned} \tag{4.3}$$

PROBLEM (J_η). Find $u \in K_0$ such that

$$J_\eta(u) = \min_{v\in K_0} J_\eta(v).$$

THEOREM 4.1. *If η is small enough then any continuous function u in $\overline{\Omega}$ which is a solution to problem (J_η), is also a solution to problem (J).*

PROOF. All we need to prove is that any solution to problem (J_η) satisfies

$$\int_\Omega u_-^2 = 1. \tag{4.4}$$

Suppose

$$\int_\Omega u_-^2 > 1. \tag{4.5}$$

Then $\mathring{\Omega}_p \neq \phi$ and, since u is continous, $\mathring{\Omega}_p$ is open. Take $u_\varepsilon = u - \varepsilon\xi$ where $\xi \in C_0^1(\mathring{\Omega}_p)$. If $|\varepsilon|$ is small enough then $\operatorname{sgn} u_\varepsilon = \operatorname{sgn} u$, $\int_\Omega (u_\varepsilon)_-^2 > 1$, and we easily get

$$0 \le J_\eta(u_\varepsilon) - J_\eta(u) = -2\varepsilon \int_\Omega (\nabla u \cdot \nabla\xi + \eta u_- \xi) + o(\varepsilon).$$

It follows that $\int_\Omega (\nabla u \cdot \nabla \xi + \eta u_-\xi) = 0$ and, therefore,

$$\Delta u + \eta u = 0 \text{ and } u < 0 \text{ in } \mathring{\Omega}_p. \tag{4.6}$$

Since $\overset{\circ}{\Omega}_p \subset \Omega$ we conclude that $\eta > \lambda_1(\Omega)$ where $\lambda_1(\Omega)$ is the first eigenvalue of Δ in Ω. Thus if we choose $\eta \leq \lambda_1(\Omega)$ then (4.5) cannot hold.

Suppose next that

$$\int_\Omega u_-^2 < 1. \tag{4.7}$$

Observe that

$$\inf_{v \in K_0} J_\eta(v) \leq C, \quad C \text{ independent of } \eta.$$

Therefore

$$J(u) - \frac{1}{\eta}\left(\int_\Omega u_-^2 - 1\right) = J_\eta(u) \leq \inf_{v\in K_0} J_\eta(v) \leq C. \tag{4.8}$$

Since $J(u) > 0$, we conclude that

$$\int_\Omega u_-^2 \geq 1 - C\eta > \frac{1}{2} \text{ if } C\eta < \frac{1}{2}; \tag{4.9}$$

consequently $\overset{\circ}{\Omega}_p \neq \phi$. Proceeding as before we derive, analogously to (4.6),

$$\Delta u + \frac{1}{\eta} u = 0 \text{ in } \overset{\circ}{\Omega}_p.$$

Multiplying by u and integrating over $\overset{\circ}{\Omega}_p$ results in

$$\frac{1}{\eta} = \frac{\int_\Omega |\nabla u_-|^2}{\int_\Omega u_-^2} \leq \frac{1}{\int_\Omega u_-^2} J_\eta(u), \text{ since } -f_\eta\left(\int_\Omega u_-^2\right) > 0.$$

Recalling (4.8), (4.9), we obtain

$$\frac{1}{\eta} < 2C$$

which is a contradiction if $\eta < 1/(2C)$.

From now on we study only the minimizers of J_η, and assume that η is small enough so that any continuous minimizer is also a solution to problem (J).

5. - C^α regularity

THEOREM 5.1. *There exists a solution to problem* (J_η).

Indeed, the proof is the same as for Theorem 1.1 of [3].

THEOREM 5.2. *Any solution* u *to problem* (J_η) *is subharmonic in* Ω.

The proof is the same as for Theorem 3.2 provided we choose $a_\varepsilon = \left(\int_\Omega u_-^2\right) \Big/ \left(\int_\Omega (\tilde{u}_\varepsilon)_-^2\right)$. Then

$$\int_\Omega (u_\varepsilon)_-^2 = \int_\Omega u_-^2$$

so that

$$J_\eta(u_\varepsilon) \geq J_\eta(u) \text{ implies } J(u_\varepsilon) \geq J(u).$$

By Lemma 10.1 in [13; p. 90] there is a version of the subharmonic function u such that

$$\fint_{B_r(x)} u \downarrow u(x) \text{ if } r \to 0, \quad \forall x \in \Omega,$$

and this implies that u is upper semicontinuous. It follows that the set $\overset{\circ}{\Omega}_p = \{u < 0\}$ is open and, by the proof of Theorem 3.3,

$$\Delta u + \lambda u = 0 \text{ in } \overset{\circ}{\Omega}_p.$$

By the isoperimetric inequality for the principal eigenvalue, the volume of each component D_i of $\overset{\circ}{\Omega}_p$ is bounded from below by a constant depending only on λ. Hence there is only a finite number of such components D_i. Since $\int_{D_i} u_-^2 \leq 1$, by elliptic estimates

$$u_- \leq c_i \text{ in } D_i \cap \{u_- > 1\}$$

where $c_i > 1$. It follows that

$$u_- \leq C_1 \text{ in } \overset{\circ}{\Omega}_p \text{ where } C_1 = \max_i c_i.$$

Since u is subharmonic, by the maximum principle $u \leq \gamma$ in Ω. Thus, altogether,

$$-C_1 \leq u \leq \gamma \text{ in } \Omega. \tag{5.1}$$

THEOREM 5.3. *Any solution* u *to problem* (J_η) *is in* $C^\alpha(\overline{\Omega})$ *for any* $0 < \alpha < 1$.

PROOF. Let $B_r \subset\subset \Omega$ be any ball of radius r and denote by v_r the solution to

$$\Delta v_r = 0 \text{ in } B_r, \quad v_r = u \text{ on } \partial B_r.$$

Extend v_r by u to $\Omega \backslash B_r$. Then, by the minimality of u,

$$\int_{B_r} [|\nabla u|^2 + \mu_-^2 I_{\{u \le 0\}} + \mu_+^2 I_{\{u>0\}}] - f_\eta \left(\int_\Omega u_-^2 \right)$$
$$\le \int_{B_r} [|\nabla v_r|^2 + \mu_-^2 I_{\{v_r \le 0\}} + \mu_+^2 I_{\{v_r>0\}}] - f_\eta \left(\int_\Omega (v_r)_-^2 \right).$$

It follows that

$$\int_{B_r} |\nabla(u - v_r)|^2 = \int_{B_r} [|\nabla u|^2 - |\nabla v_r|^2] \le Cr^n + f_\eta'(\tilde{s}) \left[\int_{B_r} u_-^2 - \int_{B_r} (v_r)_-^2 \right]$$
$$\le Cr^n + C \int_{B_r} u_-^2 \le Cr^n$$

where we have used (5.1).

Proceeding as in [20; Th. 5.3.6] (cf. [2]) one can use the last estimate to establish the bound

$$\int_{B_r} |\nabla(u - v_R)|^2 \le C(R) r^n \left(\log \frac{R}{r} r + 1 \right) \text{ if } 0 < r < R$$

so that

$$\int_{B_r} |\nabla u|^2 \le C(R) r^n \left(\log \frac{R}{r} + 1 \right)$$

and then, by [20; Th. 3.5.2], u is in $C^\alpha(\Omega)$.

To prove that $u \in C^\alpha$ near the boundary, we flatten the boundary, i.e., for a given $x_0 \in \partial\Omega$ and a small ball $B_\rho(x_0)$ we introduce a C^2 diffeomorphism $y = \Phi(x)$ from $B_\rho(x_0) \cap \Omega$ onto the half ball $B_1^+(0)$. Let v_r be harmonic function in $\Phi^{-1}(B_r^+(0))$ with $v_r = u$ on $\partial\Phi^{-1}(B_r^+(0))$. Consider the function $\tilde{v}_r = v_r \circ \Phi$ in $B_r^+(0)$ and extend it to $B_r^-(0)$ by

$$2\gamma - \tilde{v}_r(y_1, \dots, y_{n-1}, -y_n).$$

Similarly define $\tilde{u} = u \circ \Phi$ and extend it in the same way to $B_r^-(0)$. As before we have

$$\int [|\nabla u|^2 - |\nabla v_r|^2] \leq \dot{C}r^n$$

(v_r has been extended into Ω by u), and this leads to

$$\overline{J}(\tilde{u}) \leq \overline{J}(\tilde{v}_r) + Cr^n$$

where

$$\overline{J}(w) = \int\limits_{B_1(0)} \sum a_{ij}(y) w_{y_i} w_{y_j} dy$$

where (a_{ij}) is a positive definite matrix with bounded measurable coefficients. The technique of [20] can the again be used to deduce that $\tilde{u} \in C^\alpha(B_{r_1}(0))$ for some $0 < r_1 < 1$, and therefore $u \in C^\alpha(B_{\rho_1}(x_0) \cap \overline{\Omega})$ for some $\rho_1 > 0$.

REMARK 5.1. The C^α norm of u in any compact subset G of Ω depends only on M, δ and μ^2 where $M \geq \sup\limits_{\Omega} |u|$ and $\delta \leq \operatorname{dist}(G, \partial\Omega)$.

From Theorems 5.3 and 4.1 we have:

COROLLARY 5.4. *Any minimizer of* (J_η) *is also a minimizer of* (J).

NOTATION. In order to indicate the dependence of Ω_v and Ω_p on u, we introduce the notation:

$$\Omega^+(u) = \{x \in \Omega; u(x) > 0\},$$
$$\Omega^-(u) = \operatorname{int}\{x \in \Omega; u(x) \leq 0\}.$$

DEFINITION. The *free boundary* $F(u)$ is defined by

$$F(u) = \partial\Omega^+(u) \cap \Omega. \tag{5.2}$$

REMARK 5.2. We claim:

$$\partial\Omega^-(u) \subset F(u). \tag{5.3}$$

Indeed, if $x_0 \notin F(u)$ while $x_0 \in \Omega \backslash \Omega^+(u)$, then there is a neighborhood N of x_0 such that $u \leq 0$ in N. Since u is subharmonic, either $u < 0$ in N or $u \equiv 0$ in N. In either case $N \subset \Omega^-(u)$ so that $x_0 \notin \partial\Omega^-(u)$. Finally if $x_0 \in \partial\Omega$ then $u(x_0) > 0$ and again $x_0 \notin \partial\Omega^-(u)$.

REMARK 5.3. In Section 15 it will be shown that for some domains Ω the set $\{u = 0\}$ has nonempty interior. This situation does not occur if $\mu^2 = 0$ and $n = 2$; see [19].

6. - Lower and upper estimates on averages

The results of this section are similar to results obtained in [3] in case the admissible class in K_0 (defined in (4.2)) and the variational functional is $J(v)$; the fact that we are dealing here with $J_\eta(v)$ instead of $J(v)$ causes just minor changes. We shall refer to [3] whenever the changes are obvious.

THEOREM 6.1 (Nondegeneracy). *For any $0 < \kappa < 1$ there exists a positive constant c depending only on κ and μ^2 such that if $B_r \subset \Omega$ and*

$$\frac{1}{r} \fint_{\partial B_r} u_+ < c$$

then $u_+ = 0$ in $B_{\kappa r}$.

The proof is similar to the proof of Theorem 3.1 in [3]; the term $f_\eta\left(\int_\Omega u_-^2\right)$ cancels out the term $f_\eta\left(\int_\Omega v_-^2\right)$, where v is the modification of u, as defined in [3].

COROLLARY 6.2. *If $B_r \subset \Omega$ with center in the free boundary $\partial\{u > 0\} \cap \Omega$, then*

$$\frac{1}{r} \fint_{\partial B_r} u_+ \geq c \quad (c > 0) \tag{6.1}$$

where c depends only on μ^2.

THEOREM 6.3. *There exists a positive constant c depending only on μ_-^2 and μ_+^2 such that if $B_r \subset \Omega$ with center in $\{u = 0\}$ then*

$$\frac{1}{r} \left| \fint_{\partial B_r} u \right| \leq c.$$

PROOF. The proof is similar to the proof of Theorem 4.1 in [3]; the only difference occurs in the proof of Lemma 4.2 of [3], which asserts that

$$\Delta u(B_{r/2}) \leq c r^{n-1}. \tag{6.2}$$

To prove (6.2) define v by $v = u$ in $\Omega \backslash B_r$ and

$$\Delta v = 0 \text{ in } B_r, \quad v = u \text{ on } \partial B_r.$$

From $J_\eta(u) \le J_\eta(v)$ and the uniform bound on u (see (5.1)) we obtain

$$\int_{B_r} |\nabla u|^2 - \int_{B_r} |\nabla v|^2 \le cr^n.$$

The left-hand side is equal to

$$\int_{B_r} \nabla(u-v)\cdot\nabla(u+v) = \int_{B_r} \nabla(u-v)\cdot\nabla u = \int_{B_r} (v-u)\Delta u$$

where Δu is a measure supported in $\{u \le 0\}$. Hence

$$\begin{aligned}\int_{B_r\cap\{u=0\}} (v-u)\Delta u &\le cr^n - \int_{B_r\cap\{u<0\}} (v-u)\Delta u \\ &= cr^n + \lambda \int_{B_r\cap\{u<0\}} (v-u)u \le Cr^n,\end{aligned}$$

where we have again used the uniform bound (5.1). We can now complete the proof of (6.2) precisely as in Lemma 4.2 of [3].

7. - The monotonicity theorem

THEOREM 7.1 (The monotonicity theorem). *Set $N = \max\{n-1, 2\}$ and let $\frac{N}{2+N} < \alpha < 1$. Suppose u is a minimizer of (J_η). Take any point x_0 in the free boundary $\partial\{u>0\}\cap\Omega$ and denote by B_r the ball in Ω with center x_0 and radius r. Then there exists a positive constant κ_α depending only on μ^2, α and the $C^\alpha(\overline{\Omega})$-norm of u such that the function*

$$\varphi(r) = \frac{1}{r^4}\int_{B_r} \rho^{2-n}|\nabla u_+|^2 \cdot \int_{B_r} \rho^{2-n}|\nabla u_-|^2 \cdot e^{\kappa_\alpha \lambda r^{2-\frac{1-\alpha}{\alpha}N}} \tag{7.1}$$

where $\lambda = \int_\Omega |\nabla u_-|^2$, is monotone increasing in r.

In the special case where $u_\pm$ are both harmonic, this result is due to Alt, Caffarelli and Friedman [3]; in their case, $\kappa_\alpha = 0$. The main part of the proof of Theorem 7.1 follows closely the proof in [3]; however we shall also need, for the present case, the additional lemma:

LEMMA 7.2. *Under the assumptions of Theorem* 7.1, *there exists a constant* κ'_α, *depending only on* α, μ^2 *and the* α-*Hölder norm of* u, *such that*

$$\int_{B_r} \rho^{2-n}(u_-)^2 \le \kappa'_\alpha r^{2-\frac{1-\alpha}{\alpha}N} \int_{B_r} \rho^{2-n}|\nabla u_-|^2. \tag{7.2}$$

PROOF. Denote by $S_r^\pm$ the support of $u_\pm$ on $S_r = \partial B_r$. By Corollary 6.2,

$$H^{n-1}(S_r^+) \ne 0$$

where $H^{n-1}(S_r^\pm)$ denotes the Hausdorff measure of $S_r^\pm$.

By nondegeneracy there exists a point $y_r \in S_r$ with $u(y_r) > cr$ $(c > 0)$. Since $u \in C^\alpha$ we then have, for any $x \in B_{\kappa r^{1/\alpha}}(y_r)$ $(0 < \kappa < 1)$

$$u(x) \ge u(y_r) - C_\alpha|x - y_r|^\alpha > cr - C_\alpha\kappa^\alpha r \ge 0 \text{ if } \kappa^\alpha \le \frac{c}{C_\alpha}.$$

Therefore

$$H^{n-1}(S_r^+) \ge \chi_{n-1}(\kappa r^{1/\alpha})^{n-1}$$

where χ_m is the volume of the m-dimensional unit ball.

It follows that

$$\frac{H^{n-1}(S_r^-)}{H^{n-1}(S_r)} \le 1 - \frac{H^{n-1}(S_r^+)}{H^{n-1}(S_r)} \le 1 - \frac{\chi_{n-1}\kappa^{n-1}}{n\chi_n} r^{(\frac{1}{\alpha}-1)(n-1)}. \tag{7.3}$$

Denote by $\nabla_\theta v$ the gradient of a function v on S_r and introduce constants (which depend on r)

$$\frac{1}{\alpha_r^\pm} = \inf_{v \in H_0^1(S_r^\pm)} \frac{\int_{S_r^\pm} |\nabla_\theta v|^2}{\int_{S_r^\pm} v^2},$$

and $\beta_r^\pm \in (0,1]$,

$$\frac{1 - (\beta_r^\pm)^2}{\alpha_r^\pm} = (n-2)\frac{\beta_r^\pm}{\sqrt{\alpha_r^\pm}}. \tag{7.4}$$

Then, (cf. [3; p. 441])

$$\frac{\beta_r^\pm}{\sqrt{\alpha_r^\pm}} = \gamma_r^\pm \text{ where } \gamma_r^\pm(\gamma_r^\pm + n - 2) = \frac{1}{\alpha_r^\pm}, \quad \gamma_r^\pm > 0$$

and, by an estimate of Friedland and Hayman [12] on the functions $\gamma_r^\pm$,

$$\frac{\beta_r^\pm}{\sqrt{\alpha_r^\pm}} \ge \psi\left(\frac{H^{n-1}(S_r^\pm)}{H^{n-1}(S_r)}\right) \text{ if } H(S_r^\pm) \ne 0 \tag{7.5}$$

where

$$\psi(s) = \begin{cases} \frac{1}{2}\log\frac{1}{4s} + \frac{3}{2} & \text{if } s \le \frac{1}{4} \\ 2(1-s) & \text{if } \frac{1}{4} < s < 1. \end{cases}$$

Hence, by (7.3),

$$\frac{\beta_r^-}{\sqrt{\alpha_r^-}} \ge \psi\left(1 - \frac{\chi_{n-1}\kappa^{n-1}}{n\chi_n} r^{(\frac{1}{\alpha}-1)(n-1)}\right) = \frac{2\chi_{n-1}\kappa^{n-1}}{n\chi_n} r^{(\frac{1}{\alpha}-1)(n-1)}. \tag{7.6}$$

By the definition of $\alpha_r^\pm$ and of $\beta_r^\pm$ (in (7.4)),

$$\begin{aligned} \int_{S_r} |\nabla u_-|^2 &\ge \frac{1}{r^2}\int_{S_r} |\nabla_\theta u_-|^2 \ge \frac{1-(\beta_r^-)^2}{\alpha_r^-}\frac{1}{r^2}\int_{S_r} u_-^2 \\ &= \frac{(n-2)\beta_r^-}{\sqrt{\alpha_r^-}}\frac{1}{r^2}\int_{S_r} u_-^2 \text{ if } n > 2 \end{aligned} \tag{7.7}$$

so that, by (7.6),

$$\int_{S_r} u_-^2 \le \kappa_\alpha' r^{2-(\frac{1}{\alpha}-1)N} \int_{S_r} |\nabla u_-|^2 \text{ if } H^{n-1}(S_r^-) \ne 0. \tag{7.8}$$

If $n = 2$ then, since $\beta_r^- = 1$, we have

$$\int_{S_r} |\nabla u_-|^2 \ge \frac{(\beta_r^-)^2}{\alpha_r^-}\frac{1}{r^2}\int_{S_r} u_-^2 \ge \kappa_\alpha' r^{2(\frac{1}{\alpha}-1)-2}\int_{S_r} u_-^2$$

where we have again used (7.6); hence (7.8) is valid also if $n = 2$. Finally if $H(S_r^-) = 0$ then (7.8) is trivially true.

Multiplying (7.8) by r^{2-n} and integrating from 0 to ρ, the assertion (7.2) follows.

PROOF OF THEOREM 7.1. Consider the function

$$g(r) = \frac{1}{r^4}\int_{B_r} \rho^{2-n}|\nabla u_+|^2 \cdot \int_{B_r} \rho^{2-n}|\nabla u_-|^2.$$

Then a.e.

$$
\begin{aligned}
g'(r) = &-\frac{4}{r^5}\int\limits_{B_r}\rho^{2-n}|\nabla u_+|^2\cdot\int\limits_{B_r}\rho^{2-n}|\nabla u_-|^2\\
&+\frac{1}{r^4}\int\limits_{S_r}r^{2-n}|\nabla u_+|^2\cdot\int\limits_{B_r}\rho^{2-n}|\nabla u_-|^2\\
&+\frac{1}{r^4}\int\limits_{B_r}\rho^{2-n}|\nabla u_+|^2\cdot\int\limits_{S_r}r^{2-n}|\nabla u_-|^2.
\end{aligned}
\tag{7.9}
$$

We first wish to prove that

$$
g'(r) \geq -\frac{4\lambda}{r^5}\int\limits_{B_r}\rho^{2-n}u_-^2\cdot\int\limits_{B_r}\rho^{2-n}|\nabla u_+|^2 \text{ where } \lambda = \int\limits_{\Omega}|\nabla u_-|^2. \tag{7.10}
$$

Without loss of generality we may assume that

$$
\int\limits_{B_r}\rho^{2-n}(|\nabla u_-|^2 - \lambda u_-^2) > 0 \tag{7.11}
$$

since, otherwise, (7.10) follows immediately from (7.9).

Proceeding as in the proof of (5.2) in [3] we get

$$
2\int\limits_{B_r}\rho^{2-n}|\nabla u_+|^2 \leq r^{2-n}\int\limits_{S_r}2u_+(u_+)_r + (n-2)r^{1-n}\int\limits_{S_r}u_+^2,
$$

$$
2\int\limits_{B_r}\rho^{2-n}|\nabla u_-|^2 \leq r^{2-n}\int\limits_{S_r}2u_-(u_-)_r + (n-2)r^{1-n}\int\limits_{S_r}u_-^2 + 2\lambda\int\limits_{B_r}\rho^{2-n}u_-^2.
$$

The proof actually requires that

$$
\Delta u_+ \text{ and } \Delta u_- + \lambda u_- \text{ are nonnegative measures,} \tag{7.12}
$$

a fact that will be established in Lemma 9.2 (by a proof which is independent of Theorem 7.1). One uses (7.12) and the regularization procedure, as in [3] to establish the above inequalities. (Actually a careful look at the proof shows that equalities hold). Also, as in [3; (5.6)],

$$
\int\limits_{S_r}|\nabla u_\pm|^2 \geq \frac{\beta_r^\pm}{\sqrt{\alpha_r^\pm}}\frac{1}{r}\left\{\int\limits_{S_r}2|u_\pm(u_\pm)_r| + \frac{n-2}{r}\int\limits_{S_r}u_\pm^2\right\}
$$

so that

$$r^{2-n}\int_{S_r}|\nabla u_+|^2 \geq \frac{2}{r}\frac{\beta_r^+}{\sqrt{\alpha_r^+}}\int_{B_r}\rho^{2-n}|\nabla u_+|^2, \tag{7.13}$$

and

$$r^{2-n}\int_{S_r}|\nabla u_-|^2 \geq \frac{2}{r}\frac{\beta_r^-}{\sqrt{\alpha_r^-}}\int_{B_r}\rho^{2-n}(|\nabla u_-|^2\lambda u_-^2). \tag{7.14}$$

Substituting (7.13) and (7.14) into (7.9) results in

$$\begin{aligned} g'(r) \geq &-\frac{4}{r^5}\int_{B_r}\rho^{2-n}|\nabla u_+|^2\cdot\int_{B_r}\rho^{2-n}|\nabla u_-|^2 \\ &+\frac{2}{r^5}\left(\frac{\beta_r^+}{\sqrt{\alpha_r^+}}+\frac{\beta_r^-}{\sqrt{\alpha_r^-}}\right)\int_{B_r}\rho^{2-n}(|\nabla u_-|^2-\lambda u_-^2)\cdot\int_{B_r}\rho^{2-n}|\nabla u_+|^2. \end{aligned}$$

Since, by [3; (5.7)],

$$\frac{\beta_r^+}{\sqrt{\alpha_r^+}}+\frac{\beta_r^-}{\sqrt{\alpha_r^-}} \geq 2$$

and since we have assumed that (7.11) holds, the assertion (7.10) follows.

We now substitute the inequality (7.2) of Lemma 7.2 into (7.10) to deduce that

$$g'(r) \geq -4\lambda\kappa'_\alpha r^{1-\frac{1-\alpha}{\alpha}N}g(r),$$

from which the assertion of Theorem 7.1 follows.

THEOREM 7.3. *Any solution u of problem (J_η) is Lipschitz continuous in $\overline{\Omega}$.*

The proof in the interior of Ω, based on Theorems 7.1 and 6.3, is similar to the second proof of Theorem 5.3 in [3]. (The fact that Theorem 7.1 differs somewhat from the monotonicity formula in [3] causes only minor changes). In that proof the Harnack inequality and a scaling argument were used. Note that the Harnack inequality for positive solutions of $\Delta u + cu = 0$ $(c > 0)$ is valid [14] and the constant in the inequality depends only on the sup-norm of c (and the compact subdomain). Since the scaling changes the equation $\Delta u + \lambda u = 0$ into an equation $\Delta v + \rho^2\lambda v = 0$ with ρ small, we can use the Harnack inequality (as in [3]) with a constant independent of the scaling, and this allows the proof of Theorem 7.3 to proceed as in [3].

To prove the Lipschitz continuity near the boundary of Ω we recall that, by Theorem 5.3, u is continuous in $\overline{\Omega}$. Since $u = \gamma > 0$ on $\partial\Omega$, u is positive in some Ω-neighborhood N of $\partial\Omega$. Hence $\Delta u = 0$ in N and, by elliptic regularity, $u \in C^{1+\alpha}(N \cup \partial\Omega)$ for any $0 < \alpha < 1$.

8. - Blow-up limits

Suppose u is a solution to problem (J_η) and $x_k \to x_0 \in \Omega$, $\rho_k \downarrow 0$, $u(x_k) = 0$. The sequence

$$u_k(x) = \frac{1}{\rho_k} u(x_k + \rho_k x)$$

is called a *blow-up* sequence with respect to $B_{\rho_k}(x_k)$. Since $|\nabla u_k(x)| \leq C$ in any bounded set and $u_k(0) = 0$, any blow-up sequence has a subsequence for which

$$u_k(x) \to u_0(x) \text{ uniformly in bounded sets,}$$

$$\nabla u_k \to \nabla u_0 \text{ weakly star in } L^\infty_{\mathrm{loc}}(\mathbb{R}^n);$$

u_0 is called a *blow-up limit.*

LEMMA 8.1. *As* $k \to \infty$,

$$\partial\{u_k > 0\} \to \partial\{u_0 > 0\} \textit{ locally in the Hausdorff metric,}$$

$$\nabla u_k \to \nabla u_0 \textit{ a.e. in } \mathbb{R}^n.$$

The proof is the same as for [3; Lemma 6.1].

Set

$$J^0_B(v) = \int\limits_B [|\nabla v|^2 + \mu_-^2 I_{\{v\leq 0\}} + \mu_+^2 I_{\{v>0\}}].$$

DEFINITION 8.1. A function u is called a minimizer of (J^0) in $\mathbb{R}^n$, or a *global minimizer*, if for any $B_r \subset \mathbb{R}^n$ and for any $v \in H^1(B_r)$ with $v = u$ on ∂B_r,

$$J^0_{B_r}(u) \leq J^0_{B_r}(v). \tag{8.1}$$

LEMMA 8.2. *Any blow up limit* u_0 *is a minimizer of* (J^0) *in* $\mathbb{R}^n$.

PROOF. Since u is a minimizer of (J_η), $\int\limits_\Omega u_-^2 = 1$; further, for any $\hat{v}$ such that $\hat{v} = u$ outside B_{ρ_k},

$$J^0_{B_{\rho_k}}(u) \leq J^0_{B_{\rho_k}}(\hat{v}) + \frac{1}{\eta}\left|1 - \int\limits_\Omega \hat{v}_-^2\right| = J_{B_{\rho_k}}(\hat{v}) + \frac{1}{\eta}\left|\int\limits_{B_{\rho_k}} (u_-^2 \hat{v}_-^2)\right|,$$

or, by the change of variables $x = x_k + \rho_k y$,

$$J^0_{B_1}(u_k) \leq J^0_{B_1}(\tilde{v}) + \frac{1}{\eta}\rho_k^n\left|\int\limits_{B_1} ((u_k)_-^2 - \tilde{v}_-^2)\right| \qquad (\tilde{v}(y) = \hat{v}(x)). \tag{8.2}$$

Choosing, as in the proof of Lemma 5.4 of [2], $\hat{v} = v + (1-\varsigma)(u_k - u_0)$ where $\varsigma \in C_0^\infty(B_1)$ and letting $k \to \infty$ in (8.2), we then obtain (cf. [2]) the assertion (8.1) for $B = B_1$. The proof for any B_r is the same.

THEOREM 8.3. *Suppose $D \subset\subset \Omega$, $B_r(x^0) \subset D$ with center x^0 in $\partial\{u > 0\} \cap \Omega$. Then*

$$\frac{1}{r} \fint_{\partial B_r(x^0)} u \geq c, \; c > 0$$

where c is a constant depending on the Lipschitz coefficient of u in D.

The proof is the same as for [3; Theorem 6.3].

Consider a blow-up family

$$u_\varepsilon(x) = \frac{1}{\varepsilon} u(x^0 + \varepsilon x), \quad x^0 \in \partial\{u > 0\} \cap \Omega$$

and let

$$\begin{aligned} I_\varepsilon(r) &= \frac{1}{r^4}\left(\int_{B_r(0)} \rho^{2-n}|\nabla(u_\varepsilon)_+|^2 \cdot \int_{B_r(0)} \rho^{2-n}|\nabla(u_\varepsilon)_-|^2 \right) e^{\kappa_\alpha \lambda(\varepsilon r)^{2-\frac{1-\alpha}{\alpha}N}} \\ &= \frac{1}{(\varepsilon r)^4}\left(\int_{B_{\varepsilon r}(x^0)} \rho^{2-n}|\nabla u_+|^2 \cdot \int_{B_{\varepsilon r}(x^0)} \rho^{2-n}|\nabla u_-|^2 \right) e^{\kappa_\alpha \lambda(\varepsilon r)^{2-\frac{1-\alpha}{\alpha}N}} \\ &\equiv \tilde{I}_{\varepsilon r}. \end{aligned}$$

Note that the u_ε satisfy a uniform Lipschitz bound and therefore κ_α is independent of ε. By monotonicity, $\tilde{I}_\rho$ is increasing function of ρ. Consequently there exists a nonnegative constant σ such that

$$I_\varepsilon(r) \downarrow \sigma \text{ if } \varepsilon \downarrow 0. \tag{8.3}$$

For any converging blow-up sequence $u_{\varepsilon_k} \to u_0$ we have that $u_{\varepsilon_k}(x) \to u_0(x)$ uniformly in bounded subsets, $\nabla u_k(x) \to \nabla u_0(x)$ *a.e.*, and by the Lebesgue bounded convergence theorem,

$$I_{\varepsilon_k}(r) \to \frac{1}{r^4} \int_{B_r} \rho^{2-n}|\nabla(u_0)_+|^2 \cdot \int_{B_r} \rho^{2-n}|\nabla(u_0)_-|^2$$

where $B_r = B_r(0)$. Combining this with (8.3) we conclude:

LEMMA 8.4. *For any blow-up limit u_0 of $\{u_\varepsilon\}$ there holds:*

$$\frac{1}{r^4} \int_{B_r} \rho^{2-n}|\nabla(u_0)_+|^2 \cdot \int_{B_r} \rho^{2-n}|\nabla(u_0)_-|^2 = \sigma \textit{ for all } r > 0. \tag{8.4}$$

Since u_0 is a minimizer of (J_B^0), Lemma 6.6 of [3] is then valid, namely:

LEMMA 8.5. *In Lemma* 8.4,

(i) *if* $\sigma = 0$ *then* $u_0 \geq 0$ *in* $\mathbb{R}^n$;

(ii) *if* $\sigma > 0$ *and* $n = 2$ *then* $u_0(x) = \mu_2(x \cdot e)_+ - \mu_1(x \cdot e)_-$ *in* $\mathbb{R}^2$ *where* e *is a constant unit vector,* μ_i *are positive constants, and* $\mu_2^2 - \mu_1^2 = \mu^2 = \mu_+^2 - \mu_-^2$.

9. - Properties of the free boundary

THEOREM 9.1. *There exists a positive constant* $c \in (0,1)$ *such that, for any ball* $B_r \subset \Omega$ *with center in* $\partial\{u > 0\} \cap \Omega$,

$$c \leq \frac{\mathcal{L}^n(B_r \cap \{u > 0\})}{\mathcal{L}^n(B_r)} \leq 1 - c. \tag{9.1}$$

PROOF. The left-hand inequality can be established, as in [3], by the nondegeneracy and Lipschitz continuity of the solution. To obtain the right-hand inequality let v be the solution of

$$\Delta v = 0 \text{ in } B_r,$$

$$v = u \text{ on } \partial B_r.$$

Then $v \geq u$ in B_r and

$$\int_{B_r} |\nabla(u - v)|^2 \leq \mu^2 \int_{B_r} I_{\{u \leq 0 < v\}} + \frac{1}{\eta}\left|\int_{B_r} (u_-^2 - v_-^2)\right|.$$

By Poincare's inequality

$$\frac{c}{r^2}\int_{B_r} |u - v|^2 \leq \int_{B_r} |\nabla(u - v)|^2 \leq \mu^2 \mathcal{L}^n(B_r \cap \{u \leq 0\})$$

$$+ \frac{1}{\eta}\left|\int_{B_r} (u_- - v_-)(u_- + v_-)\right|.$$

Since $u \leq v$, the inequalities $0 \leq u_- - v_- \leq |u - v|$ and $u_- + v_- \leq 2u_-$ hold, so that

$$\frac{c}{r^2}\int_{B_r} |u - v|^2 \leq \mu^2 \mathcal{L}^n(B_r \cap \{u \leq 0\}) + \frac{\delta}{r^2}\int_{B_r} |u - v|^2 + C_\delta r^2 \int_{B_r} u_-^2.$$

Choosing $\delta = \frac{c}{2}$ and using the fact that u_- is bounded, we get

$$\frac{c}{r^2} \int_{B_r} |u - v|^2 \leq \mathcal{L}^n(B_r \cap \{u \leq 0\})$$

with another constant $c > 0$. The remaining part of proof is the same as for Theorem 7.1 in [3].

We introduce the distributions

$$d\Lambda_+ = \Delta u_+ \text{ and } d\Lambda_- = \Delta u_- + \lambda u_-.$$

LEMMA 9.2. *$\Lambda_+ (\Lambda_-)$ is a Radon measure supported in $\partial\{u > 0\} \cap \Omega$ $(\partial\{u < 0\} \cap \Omega)$.*

PROOF. Proceeding similarly to [2; Remark 4.2] we have for any $\varsigma \in C_0^\infty(\Omega)$, $\varsigma \geq 0$ and $\varepsilon > 0$,

$$-\int_{\Omega\cap\{u<-\varepsilon\}} \nabla\varsigma \cdot \nabla u + \lambda \int_{\Omega\cap\{u<-\varepsilon\}} \varsigma u = -\int_{\Omega\cap\partial\{u<-\varepsilon\}} \varsigma \frac{\partial u}{\partial \nu}\, dH^{n-1}.$$

A similar relation holds for the test function

$$\xi \equiv \varsigma \max\left\{\min\left(2 + \frac{u}{\varepsilon}, 1\right), 0\right\}.$$

Since $\xi = \varsigma$ on $\Omega \cap \partial\{u < -\varepsilon\}$, we obtain

$$\begin{aligned}\int_{\Omega\cap\{u<-\varepsilon\}} \nabla\varsigma \cdot \nabla u + \lambda \int_{\Omega\cap\{u<-\varepsilon\}} \varsigma u = \\ &- \int_{\Omega\cap\{u<-\varepsilon\}} \nabla\left(\varsigma \max\left(\min\left(2 + \frac{u}{\varepsilon}, 1\right), 0\right)\right) \cdot \nabla u \\ &+ \lambda \int_{\Omega\cap\{u<-\varepsilon\}} \varsigma \max\left(\min\left(2 + \frac{u}{\varepsilon}, 1\right), 0\right) u.\end{aligned}$$

Noting that

$$\max\left(\min\left(2 + \frac{u}{\varepsilon}, 1\right), 0\right) \equiv 1 \text{ if } u \geq -\varepsilon, \ \equiv 0 \text{ if } u \leq -2\varepsilon,$$

it follows that

$$\int_\Omega \nabla\varsigma\cdot\nabla u_- - \lambda\int_\Omega \varsigma u_- = \int_\Omega \nabla\left(\varsigma\max\left(\min\left(2-\frac{u_-}{\varepsilon},1\right),0\right)\right)\cdot\nabla u_-$$

$$-\lambda\int_\Omega \varsigma\max\left(\min\left(2-\frac{u_-}{\varepsilon},1\right),0\right)u_-$$

$$\leq \int_{\Omega\cap\{u\geq -2\varepsilon\}} |\nabla\varsigma\cdot\nabla u_-| = \int_{\Omega\cap\{-2\varepsilon<u<0\}} |\nabla\varsigma\cdot\nabla u| \to 0 \text{ if } \varepsilon\to 0,$$

by the continuity of u. Hence $\Delta u_- + \lambda u_- \geq 0$ in the sense of distributions. Similarly $\Delta u_+ \geq 0$ in the sense of distributions. Since $u_\pm$ are continuous, the lemma follows (cf. [2]).

THEOREM 9.3. *For any open set $D \subset\subset \Omega$ there exist positive constants C, c such that for any $B_r \subset D$ with center in $\partial\{u>0\}$*

$$0 \leq \int_{B_r} d\Lambda_- \leq Cr^{n-1}, \tag{9.2}$$

$$cr^{n-1} \leq \int_{B_r} d\Lambda_+ \leq Cr^{n-1}. \tag{9.3}$$

PROOF. We proceed as in [2; Theorem 4.3]. To prove the upper bound in (9.2) we first deduce from the proof of Lemma 9.2 with suitable ς's that

$$\int_{B_r} d\Lambda_- \leq \int_{\partial B_r} \nabla u_-\cdot\nu dH^{n-1} + \lambda\int_{B_r} u_-$$

for a.e. r. Using the Lipschitz continuity of u, it follows that $\int_{B_r} d\Lambda_- \leq Cr^{n-1}$. The proof of (9.3) is precisely as in [2; Theorem 4.3]; for the lower bound we need to use Corollary 6.2.

THEOREM 9.4 (Representation Theorem).

(i) *If $D\subset\subset\Omega$ then $H^{n-1}(D\cap\partial\{u>0\})<\infty$;*

(ii) *There exist Borel functions $q_u^\pm$ such that*

$$d\Lambda_\pm = q_u^\pm H^{n-1}\mathcal{L}\partial\{u>0\},$$

that is, for any $\varsigma\in C_0^\infty(\Omega)$,

$$-\int_{\Omega} \nabla u_- \cdot \nabla \varsigma + \lambda \int_{\Omega} u_- \varsigma = \int_{\Omega\cap\partial\{u<0\}} \varsigma q_u^- dH^{n-1},$$

$$-\int_{\Omega} \nabla u_+ \cdot \nabla \varsigma = \int_{\Omega\cap\partial\{u>0\}} \varsigma q_u^+ dH^{n-1};$$

(iii) *For any open set $D \subset\subset \Omega$ there exist positive constants c, C depending on D and the Lipschitz constant of u such that for any ball $B_r(x) \subset D$ with $x \in \partial\{u > 0\} \cap \Omega$,*

$$c \le q_u^+ \le C, \quad 0 \le q_u^- \le C, \tag{9.4}$$

$$cr^{n-1} \le H^{n-1}(B_r(x) \cap \partial\{u > 0\}) \le Cr^{n-1}. \tag{9.5}$$

Indeed this follows as in [3; Th. 7.3]; we use here Theorem 9.3.

Since $\partial\{u > 0\} \cap \Omega$ has finite H^{n-1} measure, the set $A = \Omega \cap \{u > 0\}$ has finite perimeter locally in Ω, that is, $\mu_u \equiv -\nabla I_A$ is a Borel measure and the total variation $|\mu_u|$ is a Radon measure; see [2] and the references therein to [11; 4.5.11, 4.5.5]. We denote by $\partial_{\text{red}}\{u > 0\}$ the reduced boundary of $\partial\{u > 0\} \cap \Omega$, that is, the set of points for which there exists a unique unit normal. Then [2] [11; 4.5.6]

$$\mu_u = \nu_u H^{n-1} \mathcal{L} \partial_{\text{red}}\{u > 0\}.$$

THEOREM 9.5 (Identification theorem). *Let $x_0 \in \partial_{\text{red}}\{u > 0\}$ with*

$$\theta^{*n-1}(H^{n-1}\mathcal{L}\partial\{u > 0\}, x_0) \le 1, \tag{9.6}$$

$$\int_{B_r(x_0)\cap\partial\{u>0\}} |q_u^{\pm} - q_u^{\pm}(x_0)| = o(1) \text{ as } r \to 0. \tag{9.7}$$

(i) *If, in Lemma* 8.5, *$\sigma > 0$ and $n = 2$, then*

$$u(x_0 + x) = \mu_2(x \cdot e(x_0))_+ - \mu_1(x \cdot e(x_0))_- + o(|x|) \text{ as } |x| \to 0$$

where $e(x_0)$ is a unit vector, μ_i are positive constants, and $\mu_2^2 - \mu_1^2 = \mu^2$, $(\mu_2 - \mu_1)e(x_0) = (q_u^+(x_0) - q_u^-(x_0))\nu_u(x_0)$;

(ii) *If, in Lemma* 8.5, *$\sigma = 0$ then*

$$u(x_0 + x) = q_u^+ \max\{-x \cdot \nu_u(x_0), 0\} + o(|x|) \text{ as } |x| \to 0$$

and

$$q_u^+(x_0) = \mu_2^2 - \mu_1^2 = \mu^2.$$

Here $\nu_u(x_0)$ is the outward normal to $\partial\{u > 0\}$ at x_0.
The proof is the same as for Theorem 7.4 in [3].

As in [2], Theorem 9.1 implies that

$$H^{n-1}(\partial\{u>0\}\backslash\partial_{\text{red}}\{u>0\})=0$$

and then, for H^{n-1} a.a. $x_0\in\partial_{\text{red}}\{u>0\}$, the assumptions (9.6), (9.7) are satisfied. Therefore for H^{n-1} a.a. $x\in\partial\{u>0\}$ the free boundary in a neighborhood of x_0 is approximately a straight line if $n=2$.

10. - Weak solutions and subsolutions

In order to study the regularity of the free boundary for solutions u of problem (J_η), we need to introduce the concept of weak solutions and weak subsolutions.

In the sequel we shall denote the dot product $x\cdot y$ also by $\langle x,y\rangle$.

DEFINITION 10.1. Let $G(s)$ be a continuous strictly monotone increasing function of $s\in\mathbb{R}$ with $G(s)\geq s$ if $s\geq 0$, and let Ω be a bounded open set in $\mathbb{R}^n$. Consider a function u continuous in Ω and satisfying:

(i) $\Delta u=0$ in $\Omega^+(u)=\{x\in\Omega; u(x)>0\}$,

(ii) $\Delta u+\lambda u=0$ in $\Omega^-(u)=\text{int}\{x\in\Omega; u(x)\leq 0\}$ where λ is a positive constant.

Then we say that u satisfies the *weak free boundary condition*

$$u_\nu^+=G(u_\nu^-)\text{ along }F(u)=\partial\{u>0\}\cap\Omega$$

if for any $x_0\in F(u)$ for which $F(u)$ has a one-sided tangent ball at x_0 (i.e., there exists a ball $B_\rho(y)$ such that $x_0\in\partial B_\rho(y)$ and $B_\rho(y)$ is contained either in $\Omega^+(u)$ or in $\Omega^-(u)$),

$$u(x)=\alpha\langle x-x_0,\nu\rangle_+-\beta\langle x-x_0,\nu\rangle_-+o(|x-x_0|)\tag{10.1}$$

and $\beta\geq 0$, $\alpha=G(\beta)$, where ν is the unit radial direction of $\partial B_\rho(y)$ at x_0 pointing into $\Omega^+(u)$. The function u is called a *weak solution.* We shall often write $u_\nu^+=\alpha$, $u_\nu^-=\beta$.

DEFINITION 10.2. A function v continuous in Ω is called a *subsolution* if instead of (i), (ii) and (10.1) it satisfies:

(i′) $\Delta v\geq 0$ in $\Omega^+(v)=\{v>0\}$,

(ii′) $\Delta v+\lambda v\geq 0$ in $\Omega^-(v)=\text{int}\{v\leq 0\}$,

and, for any $x_0\in F(v)$ at which $F(v)$ has a tangent ball $B_\rho(y)$ from $\Omega^+(v)$ or from $\Omega^-(v)$,

$$v(x)\geq\alpha\langle x-x_0,\nu\rangle_+-\beta\langle x-x_0,\nu\rangle_-+o(|x-x_0|)\tag{10.2}$$

for some $\beta\geq 0$ and $\alpha\geq G(\beta)$.

DEFINITION 10.3. A point $x_0 \in F(v)$ at which $F(v)$ has a tangent ball from $\Omega^+(v)$ (see Definition 10.2) will be called a *regular* point.

LEMMA 10.1. *Let $v \leq u$ be two continuous functions in Ω, $v < u$ in $\Omega^+(v)$, v a subsolution and u a weak solution. If every point in $F(v)$ is regular, then $F(v) \cap F(u) = \phi$.*

The proof, which easily follows from Definitions 10.1-10.3 and the strong maximum principle, is the same as that of Lemma 6 in [5].

LEMMA 10.2. *Let Ω be a bounded domain and let $\varphi_i \in C^0(\Omega) \cap L^\infty(\Omega)$ $(i = 1, 2)$ satisfy:*

$$\Delta\varphi_1 + \lambda\varphi_1 \geq 0 \text{ in } D'(\Omega),$$

$$\Delta\varphi_2 + \lambda\varphi_2 \geq -\varepsilon \text{ in } D'(\Omega)$$

for some $\lambda \geq 0$, $\varepsilon > 0$. Then the function $\varphi = \max\{\varphi, \varphi_2\}$ satisfies:

$$\Delta\varphi + \lambda\varphi \geq -\varepsilon \text{ in } D'(\Omega).$$

PROOF. Introduce the solution w to

$$\Delta w = -\lambda\varphi - \varepsilon \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega.$$

Then $\Delta(\varphi_i - w) \geq 0$ for $i = 1, 2$, i.e., $\varphi_i - w$ is subharmonic. For any ball $B \subset \Omega$ let h be the harmonic function in B with boundary values $\varphi - w$ on ∂B. Then $h \geq \varphi_i - w$ on ∂B and, since $\varphi_i - w$ is subharmonic, $h \geq \varphi_i - w$ in B; hence $h \geq \varphi - w$. We conclude that $\varphi - w$ is subharmonic, and consequently,

$$\Delta\varphi + \lambda\varphi + \varepsilon = \Delta(\varphi - w) \geq 0 \text{ in } D'(\Omega).$$

We next state a comparison principle for a family of subsolutions.

LEMMA 10.3. *Let $\lambda \leq \lambda_\Omega$ where λ_Ω is the principal eigenvalue of Δ for the Dirichlet problem in a bounded domain Ω. Let v_t, $0 \leq t \leq 1$, be a family of subsolutions in Ω, continuous in $\overline{\Omega} \times [0, 1]$, and let u be a weak solution continuous in $\overline{\Omega}$. Assume that*

(a) *$v_0 \leq u$ in Ω;*

(b) *$\overline{\Omega^+(v_t)} \cap \partial\Omega$ is nonempty, $v_t \leq u$ on $\partial\Omega$, and $v_t < u$ on $\overline{\Omega^+(v_t)} \cap \partial\Omega$, for $0 \leq t \leq 1$;*

(c) *every point $x_0 \in F(v_t)$ is regular, and*

(d) *the family $\Omega^+(v_t)$ is continuous, that is, for any $\varepsilon > 0$, $\Omega^+(v_{t_1}) \subset N_\varepsilon(\Omega^+(v_{t_2}))$ if $|t_1 - t_2| < \delta(\varepsilon)$, where $\delta(\varepsilon)$ is positive and $N_\varepsilon(A)$ denotes the ε-neighborhood of the set A. Then $v_t \leq u$ in Ω for $0 \leq t \leq 1$.*

The case $\lambda = 0$ is Lemma 7 in [5]. The present case of $\lambda > 0$ requires additional analysis.

PROOF. Set $T = \{t; v_t \leq u \text{ in } \Omega\}$. By (a) and the continuity of v_t, T is nonempty and closed, and it remains to show that T is open. Let $t_0 \in T$. By (b) and the strong maximum principle we have that $v_{t_0} < u$ in $\Omega^+(v_{t_0})$. Since every point of $F(v_{t_0})$ is regular, it follows by Lemma 10.1 that $\overline{\Omega^+(v_{t_0})}$ is compactly contained in $\Omega^+(u) \cup \{x \in \partial\Omega, u(x) > v_{t_0}(x)\} \equiv \Omega_0^+(u)$ (we used here assumption (b)). We can therefore choose $\varepsilon > 0$ small enough such that

$$\overline{\Omega} \cap N_\varepsilon(\Omega^+(v_{t_0})) \subset \Omega_0^+(u).$$

By assymption (d), there is then a $\delta > 0$ such that

$$\Omega^+(v_t) \subset \Omega \cap N_\varepsilon(\Omega^+(v_{t_0})) \subset \Omega^+(u) \text{ if } |t - t_0| \leq \delta. \tag{10.3}$$

Thus u is harmonic in $\Omega^+(v_t)$ while v_t is subharmonic, and, by the maximum principle (using (b)), $v_t < u$ in $\Omega^+(v_t)$ if $|t - t_0| \leq \delta$. Since obviously $v_t \leq u$ in $\Omega^-(v_t)\backslash\overline{\Omega^-(u)}$, it remains to show that

$$v_t \leq u \text{ in } \Omega^-(u). \tag{10.4}$$

Observe that (10.3) implies that

$$\Omega^-(u) \subset \Omega\backslash\overline{\Omega^+(u)} \subset \Omega\backslash\overline{\Omega^+(v_t)} = \Omega^-(v_t) \tag{10.5}$$

if $|t - t_0| \leq \delta$. For such t's we introduce the function

$$w_t^\tau = v_t - u - \tau \text{ for any small } \tau > 0.$$

It satisfies (by (10.5))

$$\Delta w_t^\tau + \lambda w_t^\tau \geq -\lambda\tau \text{ in } \Omega^-(u).$$

It will be convenient to work with the functions

$$W_t^\tau = \max\{w_t^\tau, 0\}. \tag{10.6}$$

By Lemma 10.2

$$\Delta W_t^\tau + \lambda W_t^\tau \geq -\lambda\tau \text{ in } \Omega^-(u). \tag{10.7}$$

Also, $v_t \leq u$ on $\partial\Omega^-(u)$ so that $w_t^\tau \leq -\tau$ on $\partial\Omega^-(u)$, and consequently $W_t^\tau = 0$ in some η-neighborhood of $\partial\Omega^-(u)$; hence

$$\operatorname{supp} W_t^\tau \subset\subset \Omega^-(u). \tag{10.8}$$

Without loss of generality we may assume that $\Omega^-(u)$ is connected (otherwise we restrict W_t^τ to each component of $\Omega^-(u)$). Extending W_t^τ by 0 into $\mathbb{R}^n \backslash \Omega^-(u)$ we then have, by (10.7),

$$\Delta W_t^\tau + \lambda W_t^\tau \geq -\lambda\tau \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

Since $\lambda < \lambda_\Omega$, we can choose a subdomain G of Ω such that $\overline{\Omega^-(u)} \subset G$ and $\lambda < \lambda_G$. Denote by $W_t^{\tau,\sigma}$ σ-mollifiers of W_t^τ. Then

$$\Delta W_t^{\tau,\sigma} + \lambda W_t^{\tau,\sigma} \geq -\lambda\tau \text{ in } \mathbb{R}^n \tag{10.9}$$

and, for small enough σ,

$$\operatorname{supp} W_t^{\tau,\sigma} \subset\subset G.$$

Consider the principal eigenfunction φ of Δ in G:

$$\varphi > 0, \ \Delta\varphi + \lambda_G\varphi = 0 \text{ in } G, \ \varphi = 0 \text{ on } \partial G. \tag{10.10}$$

By Sard's theorem, $\partial\{\varphi > \rho\}$ is smooth for almost every $\rho > 0$. We choose ρ small enough so that also

$$\operatorname{supp} W_t^{\tau,\sigma} \subset \{\varphi > \rho\}.$$

Then, by Green's formula and (10.9), (10.10),

$$\begin{aligned} 0 &= \int\limits_{\partial\{\varphi>\rho\}} \left(\varphi \frac{\partial W_t^{\tau,\sigma}}{\partial\nu} - W_t^{\tau,\sigma} \frac{\partial\varphi}{\partial\nu}\right) = \int\limits_{\{\varphi>\rho\}} (\varphi\Delta W_t^{\tau,\sigma} - W_t^{\tau,\sigma}\Delta\varphi) \\ &\geq (\lambda_G - \lambda) \int\limits_{\{\varphi>\rho\}} \varphi W_t^{\tau,\sigma} - \lambda\tau \int\limits_{\{\varphi>\rho\}} \varphi. \end{aligned}$$

Taking $\rho \to 0$ and then $\sigma \to 0$, we obtain

$$\lambda\tau \int\limits_G \varphi \geq (\lambda_G - \lambda) \int\limits_G \varphi W_t^\tau = (\lambda_G - \lambda) \int\limits_{\Omega^-(u)} \varphi W_t^\tau.$$

Letting, finally, $\tau \to 0$ we get

$$0 \geq (\lambda_G - \lambda) \int\limits_{\Omega^-(u)} \varphi(v_t - u)_+$$

and, since $\lambda_G - \lambda > 0$, $(v_t - u)^+ = 0$ in $\Omega^-(u)$, which completes the proof of (10.4).

We shall need later on to work with specific families of subsolutions. A very general construction was given in [5; Lemma 10] in case $\lambda = 0$. However this construction does not extend to the case $\lambda > 0$. We shall therefore have to be content with a much smaller family:

LEMMA 10.4. *Let u be a weak solution (as in Definition 10.1), and let*

$$v_\sigma(x) = \sup_{B_\sigma(x)} u, \ \sigma > 0. \tag{10.11}$$

Then v_σ is a subsolution in the open set $\Omega_\sigma = \{x \in \Omega, \ \mathrm{dist}(x, \partial\Omega) > \sigma\}$, and every point of $F(v_\sigma)$ is regular.

This is a generalization of Lemma 8 of [5], where the case $\lambda = 0$ was considered.

PROOF. Since u is continuous in Ω, v_σ is obviously continuous in Ω_σ. In the open set $\Omega^+(v_\sigma)$, v_σ is locally the supremum of an equicontinuous family of translations of harmonic functions $u(x+e)$; hence v_σ is subharmonic in $\Omega^+(v_\sigma)$, and it vanishes on $\partial\Omega^+(v_\sigma) \cap \Omega$.

Next we proceed analogously to Lemma 10.2 to prove that

$$\Delta v_\sigma + \lambda v_\sigma \geq 0 \text{ in } \Omega^-(v_\sigma). \tag{10.12}$$

Let w be the solution to

$$\Delta w = -\lambda v_\sigma \text{ in } B, \ w = 0 \text{ on } \partial B$$

where B is a ball in $\Omega^-(v_\sigma)$. For any $e \in B_\sigma(0)$, we have

$$\Delta(u_e - w) \geq 0 \text{ in } B, \quad \text{where } u_e(x) = u(x+e).$$

Take any ball $B_0 \subset\subset B$ and let h be the harmonic function in B_0 with boundary values $v_\sigma - w$. Then $h \geq u_e - w$ on ∂B_0 and, since $u_e - w$ is subharmonic, $h \geq u_e - w$ in B_0. It follows that $h \geq v_\sigma - w$ in B_0. Hence $v_\sigma - w$ is subharmonic in B, and

$$\Delta v_\sigma + \lambda v_\sigma = \Delta(v_\sigma - w) \geq 0 \text{ in } B,$$

and the proof of (10.12) is complete. To show that v_σ is a weak subsolution we next need to verify the free boundary condition for subsolutions.

Let $x_0 \in F(v_\sigma)$. We claim that there exists a point $y_0 \in F(u)$ such that $y_0 \in \partial B_\sigma(x_0)$. Indeed, there is sequence $x_m \in \Omega^+(v_\sigma)$ such that $x_m \to x_0$ (and then $v_\sigma(x_m) \to v_\sigma(x_0) = 0$). By definition of v_σ,

$$B_\sigma(x_m) \cap \Omega^+(u) \neq \phi \text{ (since } x_m \in \Omega^+(v_\sigma)).$$

Now, u is harmonic in $B_\sigma(x_m) \cap \Omega^+(u)$, so that by the maximum principle,

$$u > 0 \text{ at some points on } \partial(B_\sigma(x_m) \cap \Omega^+(u))$$

Consequently $\partial B_\sigma(x_m) \cap \Omega^+(u) \neq \phi$. It follows that there exists a sequence $y_m \in \partial B_\sigma(x_m) \cap \Omega^+(u)$ such that $u(y_m) > 0$, $y_m \to y_0$ and, therefore,

$$0 < u(y_m) \leq \sup_{B_\sigma(x_m)} u = v_\sigma(x_m).$$

Letting $m \to \infty$ we see that $u(y_0) = 0$. Hence $y_0 \in F(u) \cap \partial B_\sigma(x_0)$ and the claim is proven.

Since $u(y) \leq \sup_{B_\sigma(x_0)} u = v_\sigma(x_0) = 0$ for all $y \in B_\sigma(x_0)$, it follows that $B_\sigma(x_0)$ is tangent to $F(u)$ at y_0, from $\Omega^-(u)$. By definition of the weak solution u we then have

$$u(y) = \alpha\langle y - y_0, \nu\rangle_+ - \beta\langle y - y_0, \nu\rangle_- + o(|y - y_0|) \tag{10.13}$$

with $\alpha = G(\beta)$, where ν is the outer normal to $\partial B_\sigma(x_0)$ at y_0.

For x near x_0 the point $y = y_0 + x - x_0$ is near y_0 and $|y - x| = |y_0 - x_0| = \sigma$, $y \in \overline{B_\sigma(x)}$. By definition of v_σ and (10.13) we then have

$$v_\sigma(x) \geq u(y) = \alpha\langle x - x_0, \nu\rangle_+ - \beta\langle x - x_0, \nu\rangle_- + o(|x - x_0|).$$

Hence v_σ satisfies the free boundary condition (10.2).

Finally it remains to prove that all points of $F(v_\sigma)$ are regular. Let $x_0 \in F(v_\sigma)$, and choose $y_0 \in F(u)$ with $y_0 \in \partial B_\sigma(x_0)$, as above. For any $x \in B_\sigma(y_0)$ we have $y_0 \in B_\sigma(x)$, and since $y_0 \in F(u) = \partial\Omega^+(u)$, $B_\sigma(x) \cap \Omega^+(u) \neq \phi$. It follows that $v_\sigma(x) > 0$. Hence $B_\sigma(y_0) \subset \Omega^+(v_\sigma)$ and, since $x_0 \in \partial B_\sigma(y_0)$, $B_\sigma(y_0)$ is tangent to $F(v_\sigma)$ at x_0 from $\Omega^+(v_\sigma)$.

11. - ε-monotonicity and flatness imply Lipschitz

DEFINITION 11.1. A function u defined in an open set Ω is said to be *ε-monotone in the direction τ* (τ is a unit vector) if $u(x + h\tau) \geq u(x)$ for all $h \geq \varepsilon$ and x, $x + h\tau$ in Ω.

LEMMA 11.1. *Let u be 1-monotone function in a direction e, in the ball B_M of $\mathbb{R}^n$, satisfying*

$$\Delta u + \lambda u = 0 \textit{ with } \lambda \geq 0.$$

If $M = M(n)$ is large enough then $D_e u(0) \geq 0$, where D_e is the directional derivative.

The case $\lambda = 0$ coincides with Lemma 1 in [6].

PROOF. For any $1 < \mu < \frac{1}{2}M$ the function

$$w_\mu(x) = u(x + \mu e) - u(x)$$

is nonnegative and

$$\Delta w_\mu + \lambda w_\mu = 0 \text{ in } B_{M/2}.$$

We claim that

$$\lambda \leq \frac{C}{M^2}. \tag{11.1}$$

Indeed, if (λ_1, φ) is the principal eigensolution for Δ in $B_{M/2}$ with $\varphi > 0$ in $B_{M/2}$, $\varphi = 0$ on $\partial B_{M/2}$, then

$$0 \leq - \int_{\partial B_{M/2}} w_\mu \frac{\partial \varphi}{\partial \nu} = \int_{B_{M/2}} (\varphi \Delta w_\mu - w_\mu \Delta \varphi) = (\lambda_1 - \lambda) \int_{B_{M/2}} \varphi w_\mu.$$

Since the last integral is positive, we get $\lambda \leq \lambda_1 \leq C/M^2$.

Making use of the estimate (11.1), we can apply Harnack's inequality [14; p. 199] to $w_\mu(Mx)$ in $B_{1/2}$. We then conclude that

$$\frac{1}{c_0} \leq \frac{w_\mu(x)}{w_\mu(y)} \leq c_0 \quad \forall x,\ y \text{ in } B_{M/4}, \tag{11.2}$$

where $c_0 = c_0(n) > 1$.

Set

$$\tilde{w}_\mu(x) = u(x + \mu e) - u(x + (\mu - 1)e).$$

By the same argument as before,

$$\frac{1}{c_0} \leq \frac{\tilde{w}_\mu(x)}{\tilde{w}_\mu(y)} \leq c_0 \quad \forall x,\ y \text{ in } B_{M/4}.$$

Taking, in particular, $0 < \mu < \frac{1}{8} M$, $x = y - (\mu - 1)e$ and $y \in B_{M/8}$, we get

$$\frac{1}{c_0} \leq \frac{\tilde{w}_\mu(y)}{w_1(y)} \leq c_0.$$

Now take μ to be an integer and write

$$w_\mu(y) = \tilde{w}_\mu(y) + \tilde{w}_{\mu-1}(y) + \ldots + \tilde{w}_2(y) + w_1(y).$$

Then, by the previous inequality,

$$\frac{\mu}{c_0} w_1(y) \leq w_\mu(y) \leq \mu c_0 w_1(y) \ \forall y \text{ in } B_{M/8}$$

and, by (11.2),

$$\frac{1}{c_0^2} \leq \frac{w_\mu(x)}{\mu w_1(y)} \leq c_0^2 \text{ if } x,\ y \in B_{M/8}. \tag{11.3}$$

Because of the 1-monotonicity, if $\mu \geq 2$ then

$$w_{[\mu]-1} \leq w_\mu \leq w_{[\mu]+2}$$

and, consequently, (11.3) holds for any μ with $2 \leq \mu \leq \frac{1}{8} M$.

By the gradient estimates for the Poisson equation [14; p. 37] and Harnack's inequality (11.2), we have, in $B_{M/8}$,

$$|D_e w_\mu| \leq \frac{C_1}{M} w_\mu(0) + \lambda C_1 M w_\mu(0) \leq \frac{C}{M} w_\mu(0) \tag{11.4}$$

where C, C_1 are positive constants independent of M.

We can now estimate, by (11.2),

$$\begin{aligned}
\frac{1}{c_0} w_1(0) \leq w_1(2e) = u(3e) - u(2e) &= \int_2^3 D_e u(\mu e) d\mu \\
&= \int_2^3 D_e w_\mu(0) d\mu + D_e u(0) \text{ (since } D_e w_\mu(0) = D_e u(\mu e) - D_e u(0)) \\
&\leq \frac{C}{M} \int_2^3 w_\mu(0) d\mu + D_e u(0) \text{ (by (11.4))} \\
&\leq \frac{C}{M} w_1(0) + D_e u(0) \text{ (by (11.3))}.
\end{aligned}$$

Hence

$$D_e u(0) \geq \left(\frac{1}{c_0} - \frac{C}{M} \right) w_1(0) > 0 \text{ if } M \text{ is large enough,}$$

and the lemma follows.

Denote by $\Gamma(\theta, e)$ the cone with axis in the direction of the unit vector e and opening θ, $0 < \theta < \pi/2$.

DEFINITION 11.2. Let $\varepsilon_0 > 0$. A function u defined in an open set Ω is said to be ε_0-*monotone* in a subset G (of Ω) *in the direction* $\Gamma(\theta, e)$ if

$$u(x) \geq \sup_{|y| \leq \sin\theta} u(x - \varepsilon(e + y)) \ \forall x \in G,$$

for any $\varepsilon \geq \varepsilon_0$ such that the cone with vertex x and base $\{x-\varepsilon(e+y), |y| \leq \sin\theta\}$ is contained in Ω. If in the above definition $\varepsilon_0 = 0$ then we say that u is *fully monotone in the direction* $\Gamma(\theta, e)$.

In case $\theta = 0$ and $G = \Omega$, this definition coincides with Definition 11.1 when $\tau = e$.

Lemma 11.1 yields:

COROLLARY 11.2. *Let u be a weak solution of the free boundary problem, as in Definition* 10.1. *Then there exists a positive constant $M(n)$ such that if u is ε_0-monotone in Ω in the direction $\Gamma(\theta, e)$ then u is monotone in the direction $\Gamma(\theta, e)$ outside $M\varepsilon_0$-neighborhood of the free boundary $F(u)$.*

Indeed, this follows by working with the scaled function $\tilde{u}(x) = u(x_0+\varepsilon_0 x)$, where $\operatorname{dist}(x_0, F(u)) > M\varepsilon_0$.

In the next theorem we show that if u is ε-monotone then "flat" free boundaries are Lipschitz. In the case $\lambda = 0$ (where $u_\pm$ are both harmonic) a more general result was established in Theorem 1 of [6]; that result was based on the construction of subsolutions in Lemma 4 of [6]. Such a construction however does not seem possible in the case $\lambda > 0$ (cf. the remark preceding Lemma 10.4).

We denote by e_n the unit vector in the x_n-direction.

THEOREM 11.3. *Let u be a weak solution, as in Definition* 10.1, *in the cylinder $C_2 = B_2(0) \times (-2, 2)$ in $\mathbb{R}^n$. Assume that $\lambda < \lambda_{C_2}$ and that*

(i) *the free boundary $F(u)$ lies in the strip $\{|x_n| < \varepsilon\}$;*

(ii) *u is ε-monotone in C_2 in the direction $\Gamma(\theta, e_n)$;*

(iii) *u is fully monotone on ∂C^ε in the direction $\Gamma(\theta, e_n)$, where*

$$C^\varepsilon = C_1 \cap \{|x_n| < \sqrt{\varepsilon}\}, \quad C_1 = B_1(0) \times (-1, 1).$$

Then u is fully monotone in C_1 in the direction $\Gamma(\theta, e_n)$; in particular, $F(u)$ is a Lipschitz graph in C_1 with respect to any direction in $\Gamma(\theta, e_n)$.

In Section 13 we shall improve Theorem 11.3, by removing the ε-monotonicity assumption.

PROOF. Note that (i) and (ii) imply that $u > 0$ in $\{x_n > \varepsilon\}$ and $u < 0$ in $\{x_n < -\varepsilon\}$. For any $0 < \sigma < \varepsilon$ let $u_1(x) = u(x - \sigma e_n)$ and introduce

$$v_\sigma(x) = \sup_{B_{\sigma \sin\theta}(x)} u_1(y).$$

By Lemma 10.4 v_σ is a subsolution and every point of $F(v_\sigma)$ is regular.

To show that u is fully monotone in the direction $\Gamma(\theta, e_n)$ in the set C_1 we have to prove that

$$u(x) \geq v_\sigma(x) \text{ in } C_1, \text{ for any } 0 < \sigma < \varepsilon. \tag{11.5}$$

Now, by Corollary 11.2, u is fully monotone in the direction $\Gamma(\theta, e_n)$ in $\{x_n > M\varepsilon\}$ and in $\{x_n < -M\varepsilon\}$, so that

$$u(x) \geq v_\sigma(x) \text{ if } |x_n| \geq M\varepsilon. \tag{11.6}$$

Hence to prove (11.5) it suffices to show that for ε sufficiently small

$$u(x) \geq v_\sigma(x) \text{ in } C^\varepsilon, \text{ for any } 0 < \sigma < \varepsilon. \tag{11.7}$$

We claim that

$$F(v_\sigma) \text{ lies above } F(u) \text{ in } C^\varepsilon, \tag{11.8}$$

that is, if $u(x', x_n) = 0$, $v(x', \tilde{x}_n) = 0$ and both points lie in C^ε then $\tilde{x}_n \geq x_n$. To prove this we note that, by assumption (iii),

$$u(x) \geq v_\sigma(x) \text{ on } \partial C^\varepsilon \tag{11.9}$$

so that

$$F(v_\sigma) \text{ lies above } F(u) \text{ on } \partial C^\varepsilon. \tag{11.10}$$

By the proof of Lemma 10.4, the distance from every point of $F(v_\sigma)$ to $F(u)$ is $\leq \sigma$. Hence $F(v_\sigma) \cap C^\varepsilon$ lies in $|x_n| < 2\varepsilon$.

Suppose now that (11.8) is not true, and introduce translates of u,

$$u_\tau(x) = u(x + \tau e_n) \text{ in } C_2^\tau = C_2 - \tau e_n, \quad \tau \geq 0.$$

Then $F(u_\tau) = F(u) - \tau e_n$ and there exists a smallest $\tau > 0$ such that

$$F(v_\sigma) \text{ lies above } F(u_\tau) \text{ in } C^\varepsilon \tag{11.11}$$

and the two sets intersect at some point $x_0 \in \overline{C^\varepsilon}$. By (11.10), $x_0 \in C^\varepsilon$. Since $F(v_\sigma) \cap C^\varepsilon$ lies in $\{|x_n| \leq 2\varepsilon\}$, we also have

$$0 < \tau < 3\varepsilon, \quad x_0 \in \{|x_n| \leq 2\varepsilon\}.$$

Recalling (11.6), it follows that if ε is small enough then in the τ-neighborhood of $B_1(0) \times \left\{\frac{1}{2}\sqrt{\varepsilon} < |x_n| < 2\sqrt{\varepsilon}\right\}$ u is fully monotone in the direction $\Gamma(\theta, e_n)$. Hence

$$u_\tau(x) \geq u(x) \geq v_\sigma(x) \text{ on } \partial C_\tau^\varepsilon \cup \left\{B_1(0) \times \left\{\frac{1}{2}\sqrt{\varepsilon} < |x_n| < 2\sqrt{\varepsilon}\right\}\right\}, \tag{11.12}$$

where $C_\tau^\varepsilon = C^\varepsilon - \tau e_n$. In particular $u_\tau \geq v_\sigma$ on $\partial C_\tau^\varepsilon$ and, by (11.11),

$$u_\tau(x) \geq v_\sigma(x) \text{ on } \partial(C_\tau^\varepsilon \cap \{v_\sigma > 0\}). \tag{11.13}$$

Since u_τ is harmonic in $\mathcal{C}_\tau^\varepsilon \cap \{v_\sigma > 0\}$ while v is subharmonic, the maximum principle yields

$$u_\tau(x) > v_\sigma(x) \text{ in } \mathcal{C}_\tau^\varepsilon \cap \{v_\sigma > 0\}. \tag{11.14}$$

Indeed, otherwise we have $u_\tau \equiv v_\sigma$ in this set and, by (11.12), $u_\tau \equiv u > 0$ in the same set. By analytic continuation $u_\tau \equiv u > 0$ everywhere in $(B_1(0) \times \{-\sqrt{\varepsilon} < x_n < 2\sqrt{\varepsilon}\}) \cap \{u > 0\}$, which is a contradiction.

From (11.12) we infer that

$$u_\tau(x) \geq v_\sigma(x) \text{ on } \partial\{\mathcal{C}^\varepsilon \cap \{u_\tau < 0\}\}.$$

We can then proceed precisely as in the proof of (10.4) and establish that

$$u_\tau(x) \geq v_\sigma(x) \text{ in } \mathcal{C}^\varepsilon \cap \{u_\tau < 0\}.$$

Combining this with (11.14) we deduce that

$$u_\tau(x) \geq v_\sigma(x) \text{ in } \mathcal{C}_\tau^\varepsilon \cap \mathcal{C}^\varepsilon.$$

Since every point of $F(v_\sigma)$ is regular, Lemma 10.1 can then be applied to conclude that $F(v_\sigma) \cap F(u_\tau) \cap (\mathcal{C}_\tau^\varepsilon \cap \mathcal{C}^\varepsilon) = \phi$, a contradiction since x_0 belongs to this set.

Having proved (11.8) we can now apply the maximum principle (as in the proof of (11.14)) to conclude that

$$u > v_\sigma \text{ in } \Omega^+(v_\sigma) \cap \mathcal{C}^\varepsilon. \tag{11.15}$$

Since u is fully monotone on $\partial\mathcal{C}^\varepsilon$ in the direction $\Gamma(\theta, e_n)$, we have that $u \geq v_\sigma$ on $\partial\mathcal{C}^\varepsilon$. Using this inequality and (11.15), we can argue as in the proof of Lemma 10.3 and show that $v_\sigma - u - \tau \leq 0$ in $\mathcal{C}^\varepsilon \backslash \overline{\Omega^+(v_\sigma)}$, for any $\tau > 0$. (Here we need the assumption that $\lambda < \lambda_{C_2}$.) Hence $u \geq v_\sigma$ in all of $\mathcal{C}^\varepsilon$, i.e., (11.7) is satisfied.

The assertion about the Lipschitz continuity of $F(u)$ is an immediate consequence of the full monotonicity of u in the direction $\Gamma(\theta, e_n)$.

12. - Auxiliary estimates

In the next section we shall improve Theorem 11.3 by dropping the assumption (ii) of ε-monotonicity. To accomplish this we need some auxiliary results. The first lemma shows that "flatness" implies ε-monotonicity for u_+; more precisely:

LEMMA 12.1. *Suppose u is a weak solution (as in Definition 10.1) in $C_2 = B_2(0) \times (-2, 2)$ satisfying:*

(i) *the free boundary $F(u)$ lies in the strip $|x_n| < \varepsilon$,*

(ii) *there exists positive constants α_0 and α_1 such that*

$$\alpha_0 d(x, F) \le u_+(x) \le \alpha_1 d(x, F) \text{ in } \Omega^+(u)$$

where $d(x, F)$ is the distance from x to $F(u)$. Then for any $0 < \theta < \frac{\pi}{2}$ there exist $\delta = \delta(\theta) > 0$ and $c = c(\theta) > 0$ such that u_+ is $c\varepsilon$-monotone in $C_1 \cap \left\{|x_n| < \frac{\delta}{2}\right\}$ in the direction $\Gamma(\theta, e_n)$.

PROOF. Although the proof is contained in the arguments developed in the proof of Theorem 2′ in [6; p. 73], we shall present it here in detail in order to make it clear, later on (in the proof of Lemma 12.2), how the result can be extended to u_- which satisfies $\Delta u_- + \lambda u_- = 0$.

Let $A = \{x_n = -\varepsilon\}$. Denote by v the harmonic function in $C_2 \cap \{x_n > -\varepsilon\}$ with boundary values u_+. Observe that $v = 0$ on $\{x_n = -\varepsilon\}$ By the maximum principle and assumption (ii),

$$v(x) \ge u_+(x) \ge \alpha_0(d(x, A) - \varepsilon). \tag{12.1}$$

By Lemma 5 in [5], for any $0 < \theta < \frac{\pi}{2}$ there exists a $\delta = \delta(\theta)$ such that v is fully monotone in the direction $\Gamma(\theta, e_n)$ in a δ-neighborhood of A which lies in C_1. In particular, if $\delta \ge d(x, A) \ge c_1\varepsilon$ then $D_\tau v \ge 0$ for any $\tau \in \Gamma(\theta, e_n)$. It then follows by Lemma 4 of [5], properly scaled, that

$$\frac{D_\tau v(x)}{v(x)} d(x, A) \ge c_2(\theta) > 0. \tag{12.2}$$

Hence, by (12.1),

$$D_\tau v(x) \ge \alpha_0 c_2 - \frac{\alpha_0 \varepsilon}{d(x, A)} \ge \alpha_0\left(c_2 - \frac{1}{c_1}\right) \equiv c_3 > 0 \tag{12.3}$$

if c_1 is large enough.

On the other hand, by the maximum principle and assumption (ii),

$$v(x) \le \alpha_1 d(x, A) = \alpha_1(x_n + \varepsilon) \text{ in } C_1 \cap \{x_n \ge -\varepsilon\} \tag{12.4}$$

so that $v(x) \le 2\alpha_1\varepsilon$ if $x_n = \varepsilon$. Using the maximum principle, we get

$$v(x) \le u_+(x) + 2\alpha_1\varepsilon \tag{12.5}$$

in $C_1 \cap \{x_n \ge \varepsilon\}$; by (12.4), the inequality (12.5) holds also if $\{-\varepsilon \le x_n \le \varepsilon\}$.

We can now complete the proof of the lemma as follows:

If ε is small enough then for all $x \in C_1 \cap \{-\varepsilon < x_n < \delta\}$ and $\tau \in \Gamma(\theta, e_n)$,

$$u_+(x + c_4\varepsilon\tau) - u_+(x) \geq v(x + c_4\varepsilon\tau) - 2\alpha_1\varepsilon - v(x) \text{ (by (12.1), (12.5))}$$

$$= \int_0^{c_4\varepsilon} D_\tau v(x + t\tau)dt - 2\alpha_1\varepsilon \geq \int_{c_1\varepsilon/\cos\theta}^{c_4\varepsilon} D_\tau v(x + t\tau)dt - 2\alpha_1\varepsilon \text{ (since } D_\tau v \geq 0)$$

$$\geq c_3(c_4 - c_1/\cos\theta)\varepsilon - 2\alpha_1\varepsilon \text{ (by (12.3))}$$

and the right-hand side is positive if c_4 is sufficiently large. If $x \in C_1 \cap \{x_n < -\varepsilon\}$ then $u_+(x + c_4\varepsilon\tau) - u_+(x) = u_+(x + c_4\varepsilon\tau) \geq 0$ for any $c_4 > 0$.

LEMMA 12.2. *Let u be a weak solution in $C_2 = B_2(0) \times (-2, 2)$ satisfying assumption* (i), *of Lemma* 12.1, *and suppose that*

(ii) $u_-(x) \leq \alpha_1 d(x, F)$ *in* $\Omega^-(u)$, *and*

(iii) *there exist positive constants* γ, η *(independent of* ε*) such that*

$$u_-(-\gamma e_n) > \eta.$$

If $\lambda < \lambda_{C_2}$ *then for any* $0 < \theta < \dfrac{\pi}{2}$ *there exist* $\delta = \delta(\theta) > 0$ *and* $c = c(\theta) > 0$ *such that* u_- *is* $c\varepsilon$*-monotone in* $C_1 \times \left\{|x_n| < \dfrac{\delta}{2}\right\}$ *in the direction* $\Gamma(\theta, e_n)$.

PROOF. Proceeding analogously to the proof of Lemma 12.1, we introduce a function v satisfying

$$\Delta v + \lambda v = 0 \text{ in } C_2 \cap \{x_n < \varepsilon\},$$
$$v = u_- \text{ on } \partial(C_2 \cap \{x_n < \varepsilon\}).$$

Since $\lambda < \lambda_{C_2}$ such a v exists and is unique.

In trying to extend the proof of Lemma 12.1 we shall need the following comparison theorem:

(12.6) If $\Delta w + \lambda w \leq 0$ in D, $w \geq 0$ on ∂D then $w \geq 0$ in D,

provided $D \subset C_2$. This is a well known result since $\lambda < \lambda_{C_2} < \lambda_D$. We shall also need the Harnack inequality for nonnegative solutions of $\Delta w + \lambda w = 0$; this, in fact, is true for any real λ (see [14]). From Harnack's inequality and assumption (iii) it follows that

$$u_-(x) \geq c\eta d(x, \{x_n = -\varepsilon\}) \text{ in } C_1 \cap \{x_n < -\varepsilon\}$$

so that, together with assumption (ii), u_- satisfies the same conditions as does u_+ in Lemma 12.1, (ii).

We finally need extensions of Lemma 4 and 5 of [5] to the equation $\Delta w + \lambda w = 0$. These lemmas follow directly from Lemmas 1-3 of [5], so that it only remains to extend these three lemmas to solutions of $\Delta w + \lambda w = 0$. Lemma 1 is a boundary Harnack principle for harmonic functions; it is concerned with estimating the quotient u_1/u_2 of two positive harmonic functions from above and from below near the boundary [10] [8]. Its extension to solution of $\Delta u_i + \lambda u_i = 0$ was established in [9] [24]. Lemma 2 establishes the Hölder continuity for the quotient u_1/u_2 [16] [4] and the proof (as given in [16]) extends to solution of $\Delta u_i + \lambda u_i = 0$; see also [9; p. 192]. Finally Lemma 3 shows how u_1, near the boundary, is dominated by values of u_1 away from the boundary [10] [8]; the proof follows by taking $u_2 = y + \varepsilon$ in Lemma 1 (the bottom boundary is assumed to be $y = -\varepsilon$). To extend Lemma 3 to the case of $\Delta u + \lambda u = 0$ we again take $u_2 = y + \varepsilon$ but use an extension of Lemma 1 (given in [9] [24]) to estimate u_1/u_2 in the case where u_1 is a solution of $\Delta u_1 + \lambda u_1 = 0$.

We next need some gradient estimates.

LEMMA 12.3. *Let u be a nonnegative function in $\Omega \cap B_1$ satisfying*

$$\Delta u + \lambda u = 0 \text{ in } \Omega \cap B_1, \ \lambda \geq 0.$$

Assume that

(i) *u vanishes on $(\partial\Omega) \cap B_1$, and*

(ii) *for any point $x_0 \in (\partial\Omega) \cap B_1$ for which there is a tangent ball from inside Ω,*

$$\liminf_{\varepsilon \downarrow 0} \frac{u(x_0 + \varepsilon\nu)}{\varepsilon} \leq \alpha_0$$

where ν is the inner normal. Then u is Lipschitz continuous in $\Omega \cap B_{1/2}$ and

$$|\nabla u|_{L^\infty(\Omega\cap B_{1/2})} \leq C\alpha_0. \tag{12.7}$$

For $\lambda = 0$ this result is Lemma A2 in [6]. The proof given below is similar except for minor changes.

PROOF. We may assume that $\alpha_0 = 1$. It is enough to prove that $u(x) \leq Cd(x, \partial\Omega)$, for (12.7) then follow from interior gradient estimates, properly scaled. Set $h = d(x, \partial\Omega)$ $\left(0 < h < \frac{1}{2}\right)$. Then the ball $B_h(x)$ is tangent to $\partial\Omega$ at some point x_0. By Harnack's inequality

$$\inf_{B_{h/2}} u \geq c_0 u(x).$$

If v is the harmonic function in $B_h(x)\backslash\overline{B_{h/2}(x)}$ with $v = 0$ on $\partial B_h(x)$ and $v = c_0 u(x)$ on $\partial B_{h/2}(x)$ then, by comparison, $v \leq u$ (since u is supersolution).

It follows that

$$\liminf_{\varepsilon\downarrow 0} \frac{v(x_0+\varepsilon\nu)}{\varepsilon} \le \liminf_{\varepsilon\downarrow 0} \frac{u(x_0+\varepsilon\nu)}{\varepsilon} \le 1.$$

Since $v(y) = v_0(|y-x|)$ (a radially symmetic function), we can easily compute that

$$\lim_{\varepsilon\downarrow 0} \frac{v(x_0+\varepsilon\nu)}{\varepsilon} = c_1 \frac{u(x)}{h}.$$

Consequently $u(x) \le h/C_1 = C\,\mathrm{dist}(x,\partial\Omega)$.

COROLLARY 12.4. *If in Definition* 10.1 *of weak solution the parameters* α, β *are uniformly bounded, then the weak solution is Lipschitz continuous.*

In the next two lemmas we shall use the monotonicity theorem and its proof.

LEMMA 12.5. *Let* u *be a weak solution in* $\mathcal{C}_1 = B_1(0)\times(-1,1)$ *and let* $0\in F(u)$. *Denote points in* $\mathcal{C}_1$ *by* $x=(x',y)$ *where* $x'\in B_1$, $-1<y<1$. *If near* 0

$$u_+(x',y) \ge \alpha y_+ + o(|x|), \tag{12.8}$$

$$u_-(x',y) \ge \beta y_- + o(|x|), \tag{12.9}$$

where $\alpha\ge 0$, $\beta \ge 0$, *then*

$$c(n)\alpha^2\beta^2 \le \liminf_{R\to 0} \phi(R) \tag{12.10}$$

where $c(n)>0$ *and*

$$\phi(R) = \frac{1}{R^4}\int_{B_R} \rho^{2-n}|\nabla u_+|^2 \cdot \int_{B_R} \rho^{2-n}|\nabla u_-|^2 \qquad (B_R = B_R(0)). \tag{12.11}$$

This result is stated without proof in [6; A3].

PROOF. We may assume that $\alpha>0$, $\beta>0$. Denote by $S_r^\pm$ the support of $u_\pm$ on $S_r = \partial B_r$. From (12.8), (12.9) we have

$$H^{n-1}(S_r^\pm) > \frac{1}{4}H^{n-1}(S_r) \text{ for all } r \text{ small,}$$

so that

$$H^{n-1}(S_r^\mp) \le \frac{3}{4}H^{n-1}(S_r).$$

Hence, by (7.5),

$$\frac{\beta_r^\pm}{\sqrt{\alpha_r^\mp}} \ge \psi\left(\frac{3}{4}\right) = \frac{1}{2}.$$

By the proof of (7.7) it then follows that

$$\int_{S_r} |\nabla u_\pm|^2 \geq c(n) \int_{S_r} \left(\frac{u_\pm}{r}\right)^2 \quad (r = |x| = \sqrt{|x'|^2 + y^2}) \tag{12.12}$$

where $c(n) = \dfrac{n-2}{2}$ if $n \geq 3$, $= \dfrac{1}{4}$ if $n = 2$. Substituting (12.8) into the right-hand side of (12.12) (for u_+) yields

$$\int_{S_r} |\nabla u_+|^2 \geq c(n)\alpha^2 \int_{S_r} \left(\frac{y_+}{r}\right)^2 + o(r^{n-1}).$$

If we multiply by r^{2-n} and integrate over r, $0 < r < R$, we get

$$\int_{B_R} r^{2-n}|\nabla u_+|^2 \geq c(n)\alpha^2 R^2 + o(R^2)$$

with another constant $c(n)$. A similar inequality holds for u_- and, taking the product, the assertion (12.10) follows.

LEMMA 12.6. *Let u be a weak solution as in Definition 10.1, with uniformly bounded parameters α, β. Let the assumptions of Lemma* 12.5 *hold for some nonnegative numbers α, β. Then*

$$\alpha^2\beta^2 \leq \frac{c}{R^4} (\sup_{B_{2R}} u_-)^2. \tag{12.13}$$

PROOF. By Corollary 12.4 u is Lipschitz continuous and therefore the monotonicity theorem (which requires u to be Hölder continuous) can be applied. Combining it with the estimate (12.10) we get

$$\alpha^2\beta^2 \leq \frac{C}{R^4} \int_{B_R} \rho^{2-n}|\nabla u_+|^2 \cdot \int_{B_R} \rho^{2-n}|\nabla u_-|^2, \quad 0 < R < 1. \tag{12.14}$$

Denote by u_ε the ε-mollifier of u_-. Since $\Delta u_- + \lambda u_- \geq 0$ in the distribution sense, we have $\Delta u_\varepsilon + \lambda u_\varepsilon \geq 0$ and

$$\Delta u_\varepsilon^2 = 2|\nabla u_\varepsilon|^2 + 2u_\varepsilon \Delta u_\varepsilon \geq 2|\nabla u_\varepsilon|^2 - 2\lambda u_\varepsilon^2. \tag{12.15}$$

Suppose $n \geq 3$ and introduce Green's function in B_{2R} with pole at the origin:

$$G(x) = \frac{1}{(n-2)\omega_n} (|x|^{2-n} - (2R)^{2-n})$$

By Green's formula and (12.15),

$$\int_{\partial B_{2R}} u_\varepsilon^2 \frac{\partial G}{\partial \nu} + u_\varepsilon^2(0) = \int_{B_{2R}} (\Delta u_\varepsilon^2) G$$

$$\geq 2 \int_{B_{2R}} |\nabla u_\varepsilon|^2 G - 2\lambda \int_{B_{2R}} u_\varepsilon^2 G \geq c \int_{B_R} \rho^{2-n} |\nabla u_\varepsilon|^2 - 2\lambda \int_{B_{2R}} u_\varepsilon^2 G$$

where $c > 0$. Letting $\varepsilon \to 0$ and noting that $u_-(0) = 0$, we get

$$\int_{B_R} \rho^{2-n} |\nabla u_-|^2 \leq \frac{1}{c} \left[2\lambda \int_{B_{2R}} (u_-)^2 G - \int_{\partial B_{2R}} (u_-)^2 \frac{\partial G}{\partial \nu} \right] \leq C (\sup_{B_{2R}} u_-)^2.$$

Similarly

$$\int_{B_R} \rho^{2-n} |\nabla u_+|^2 \leq C (\sup_{B_{2R}} u_+)^2.$$

The same estimates hold for $n = 2$ if we take $G = \dfrac{1}{2\pi} \log \dfrac{2R}{|x|}$. Substituting these estimates into (12.14), the assertion (12.13) follows.

13. - Flat free boundaries are Lipschitz

The purpose of this section is to improve Theorem 11.3 by relaxing the condition (ii) of ε-monotonicity. From Lemma 12.1 we already know that u_+ is ε-monotone if u is Lipschitz continuous and nondegenerate; since we shall later on apply the results for weak solutions to the variational solution of problem (J_η), u_+ will indeed be Lipschitz continuous (Theorem 7.3) as well as nondegenerate (Theorem 6.1). If u_- is also nondegenerate then Lemma 12.2 can be used to deduce that also u_- is ε-monotone, and the same is then true of $u = u_+ - u_-$. The following lemma tells us that if u_- is degenerate then the monotonicity of u_+ can be improved:

LEMMA 13.1. *Let u be a weak solution (as in Definition 10.1) in $\mathcal{C}_2 = B_2 \times (-2, 2)$ with uniformly bounded parameters α, β. Assume that*

$$\alpha_0 d(x, F) \leq u_+(x) \leq \alpha_1 d(x, F) \text{ in } \Omega^+(u) \ (0 < \alpha_0 < \alpha_1 < \infty),$$

and that $u_-(x) \leq \alpha_2 d(x, F)$ in $\Omega^-(u)$ $(\alpha_2 > 0)$. Set $u_0 = u(e_n)$. Then there exist θ_0 close to $\frac{\pi}{2}$, $0 < \theta_0 < \frac{\pi}{2}$, $\varepsilon_0 > 0$, $\delta_0 > 0$ and $0 < \sigma < 1$, all independent *of ε, such that if u_+ is ε-monotone in $\Gamma(\theta_0, e_n)$ for $\varepsilon < \varepsilon_0$ then there is a large constant C (independent of ε) such that either u_- is $C\varepsilon$-monotone in $\mathcal{C}_1 \cap \{|x_n| < \delta_0\}$ in the direction $\Gamma(\theta_0, e_n)$ or the following alternative holds:*

(a) *if $u_-(-e_n) \geq C\varepsilon u_0$ then u is $C\varepsilon^{1/8}$-monotone in $\mathcal{C}_1 \cap \{|x_n| < \delta_0\}$, in the direction $\Gamma(\theta, e_n)$ for some $0 < \theta < \frac{1}{2}\theta_0$;*

(b) *if $u_-(-e_n) \leq C\varepsilon u_0$ then u_+ is $\sigma\varepsilon$-monotone in the domain $\mathcal{C}_1 - \varepsilon^{\tau_2}$ in the direction $\Gamma(\theta_0 - \varepsilon^{\tau_1}, e_n)$, for some positive constants τ_1, τ_2 (independent of ε).*

Lemma 13.1 for $\lambda = 0$ coincides with Lemma 6 of [6]. As in [6; p. 68] we can iterate the lemma by taking a decreasing sequence of ε's, say $\varepsilon\gamma^{-j}$ ($\gamma \gg 1$, $j = 1, 2 \ldots$). This allows us to deduce that either u is $C\varepsilon$-monotone in the direction $\Gamma(\theta, e_n)$ for some $0 < \theta < \theta_0$, or the following dichotomy holds: either

(i) u_+ is fully monotone in $\mathcal{C}_1$ in the direction $\Gamma(\theta_1, e_n)$ for some $0 < \theta_1 < \frac{\pi}{2}$, or

(ii) $u_-(-e_n) \geq c\varepsilon$ for some $c > 0$.

In case (i) it follows that $F(u)$ is Lipschitz continuous. In case (ii), if $u_-(x) \leq \alpha_2 d(x, F)$ then (by Lemma 12.2) u_- is $c\varepsilon$-monotone and the assumption (ii) in Theorem 11.3 is satisfied. Observe that the assumptions

$$u_+(x) \leq \alpha_1 d(x, F), \;\; u_-(x) \leq \alpha_2 d(x, F)$$

hold if the parameters α, β in Definition 10.1 are uniformly bounded, whereas the inequality

$$u_+(x) \geq \alpha_0 d(x, F) \;\; (\alpha_0 > 0)$$

is satisfied if $G(0) > 0$, since $\beta \geq 0$ and

$$\alpha \geq G(\beta) \geq G(0) > 0.$$

We thus have:

THEOREM 13.2. *Let u be a weak solution (as in Definition 10.1) in $\mathcal{C}_2 = B_2 \times (-2, 2)$ with uniformly bounded parameters α, β, and assume that $G(0) > 0$ and $\lambda < \lambda_{\mathcal{C}_2}$. If*

(i) *$F(u)$ lies in $\{|x_n| < \varepsilon\}$, and*

(ii) *assumption (iii) in Theorem 11.3 holds,*

then $F(u) \cap B_1(0)$ is Lipschitz graph in any direction $0 \leq \theta \leq \theta_1$ for some $0 < \theta_1 < \frac{\pi}{2}$; here $B_1(0)$ is the unit ball in $\mathbb{R}^n$.

REMARK 13.1. In case $\lambda = 0$ Caffarelli [6] established Theorem 13.2 without making the assumptions (ii). The reason that our result requires the additional assumption (ii) originated from the fact that, when $\lambda > 0$, we have a much smaller class of subsolutions to work with (cf. the remarks preceding Lemma 10.4 and Theorem 11.3). However, it will be shown later on that for variational solutions (i.e., for minimizers of problem (J_η)) both assumptions (i) and (ii) of Theorem 13.2 are satisfied.

PROOF OF LEMMA 13.1. By Lemma 12.6 we have, for $\rho < \frac{1}{4}$,

$$(u_\nu^-)^2(u_\nu^+)^2 \leq \frac{C}{\rho^4}(\sup_{B_\rho} u_-)^2(\sup_{B_\rho} u_+)^2 \tag{13.1}$$

at any point x_0 where $F(u) \cap \mathcal{C}_{2-2\rho}$ has a tangent ball from $\Omega^-(u)$ (or from $\Omega^+(u)$); by Definition 10.1,

$$u(x) = u_\nu^+\langle x - x_0, \nu\rangle_+ - u_\nu^-\langle x - x_0, \nu\rangle_- + o(|x - x_0|)$$

as $x \to x_0$, and $u_\nu^+ = G(u_\nu^-) \geq u_\nu^-$.

We may assume that

$$\sup_{B_{2\rho}(x_0)} u_- \leq \sup_{B_{2\rho}(x_0)} u_+. \tag{13.2}$$

Indeed, otherwise, by nondegeneracy of u_+ and Harnack's inequality (for u_-) there exists a point $p = -\gamma e_n$ $(\gamma > 0)$ such that $u_-(p) \geq \eta > 0$ where both γ and η are independent of ε. Applying Lemma 12.2 we then conclude that u_- is $C\varepsilon$-monotone in the direction $\Gamma(\theta, e_n)$ for any $0 < \theta < \frac{\pi}{2}$, and the lemma follows.

From (13.1), (13.2) and the inequality $u_\nu^+ \geq u_\nu^-$ it follows that for any x_0 as above,

$$u_\nu^- \leq \frac{C}{\rho} \sup_{B_{2\rho}(x_0)} u_+,$$

and by the nondegeneracy and linear growth assumptions on u_+,

$$u_\nu^- \leq \frac{C}{\rho}\frac{\alpha_1}{\alpha_0}u(e_n) \leq \frac{C}{\rho}u_0 \tag{13.3}$$

where C is used to denote a generic constant. Applying Lemma 12.3 we conclude that

$$\|Du_-\|_{L^\infty[\mathcal{C}_{2-4\rho}\cap N_\rho(A)]} \leq \frac{C}{\rho}u_0 \tag{13.4}$$

where $A = \{x_n = f(x') - C\varepsilon\}$ and $F(u)$ lies in $C\varepsilon$-neighborhood of the graph $x_n = f(x')$ (see Remark in Section 3 of [6]). Since $u_- = 0$ if $x_n \geq f(x') + C\varepsilon$, we then have

$$u_- \leq \frac{C\varepsilon u_0}{\rho} \text{ in } \{x_n \geq f(x') - C\varepsilon\}. \tag{13.5}$$

Introduce the solution v to

$$\begin{aligned} \Delta v + \lambda v &= 0 \text{ in } D, \\ v &= 0 \text{ on } A, \\ v &= u_- \text{ on } \partial D \backslash A \end{aligned} \tag{13.6}$$

where $D = \mathcal{C}_{2-2\rho} \cap \{x_n \le f(x') - C\varepsilon\}$; since $\lambda < \lambda_{\mathcal{C}_2}$, v exists and is unique and, by comparison,

$$v \le u_- \text{ in } D. \tag{13.7}$$

Introduce also the solution $\tilde{v}$ to

$$\begin{aligned} \Delta \tilde{v} + \lambda \tilde{v} &= 0 \text{ in } D, \\ \tilde{v} &= C \frac{\varepsilon u_0}{\rho} \text{ on } A, \\ \tilde{v} &= u_- \text{ on } \partial D \backslash A. \end{aligned}$$

Using (13.5) we easily deduce, by comparison, that

$$u_- \le \tilde{v} \le v + \frac{C\varepsilon u_0}{\rho} \text{ in } D. \tag{13.8}$$

We next apply Harnack's inequality ([9; Theorem 5.1] [24]) to v in $\mathcal{C}_{2-4\rho} \cap \{x_n < f(x') - C\varepsilon\}$ to get for some K large, depending only on $\|f\|_{\text{Lip}}$,

$$v|_{\mathcal{C}_{2-4\rho}} \le \frac{C}{\rho^K} v(-e_n) \le \frac{C}{\rho^K} u_-(-e_n) \text{ (by (13.7))},$$

so that, by (13.8),

$$u_-|_{\mathcal{C}_{2-4\rho}} \le \frac{C\varepsilon u_0}{\rho} + \frac{C}{\rho^K} u_-(-e_n).$$

Since u_+ grows like $d(x, F)$,

$$u_+|_{\mathcal{C}_{2-4\rho}} \le C u_0.$$

Using these estimates in (13.1) and recalling that $u_\nu^+ \ge u_\nu^-$, we obtain

$$(u_\nu^-)^4 \le \frac{C}{\rho^4} \sup_{B_{2\rho}} (u_+)^2 \cdot \sup_{B_{2\rho}} (u_-)^2 \le \frac{C}{\rho^4} u_0^2 \left(\frac{\varepsilon u_0}{\rho} + \frac{u_-(-e_n)}{\rho^K} \right)^2,$$

or

$$u_\nu^- \le \frac{C}{\rho} u_0^{1/2} \left[\frac{\varepsilon u_0}{\rho} + \frac{C}{\rho^K} u_-(-e_n) \right]^{1/2}. \tag{13.9}$$

We now proceed to consider separately two cases, (i) and (ii).

CASE (i). Choose $\rho = \dfrac{1}{40}$ so that $\mathcal{C}_{2-8\rho} = \mathcal{C}_{9/5}$. Using (13.9) we obtain, by Lemma 12.3,

$$|Du_-(x)|_{L^\infty} \leq C u_0^{1/2}(u_-(-e_n))^{1/2}$$

and, therefore,

$$u_-(x) \leq C u_0^{1/2}(u_-(-e_n))^{1/2} d(x, F(u));$$

in particular

$$u_-(x) \leq C u_0^{1/2}(u_-(-e_n))^{1/2}\varepsilon \text{ on } A. \tag{13.10}$$

By the arguments used in deriving (13.7) and (13.8) (whereupon we replace (13.5) by (13.10)) we get

$$v \leq u_- \leq v + C u_0^{1/2}(u_-(-e_n))^{1/2}\varepsilon \text{ in } \mathcal{C}_{9/5}; \tag{13.11}$$

in particular, upon using the assumption in (a),

$$\begin{aligned} v(-e_n) &\geq u_-(-e_n) - C u_0^{1/2}(u_-(-e_n))^{1/2}\varepsilon \\ &\geq u_-(-e_n) - C\varepsilon^{1/2} u_-(-e_n) \geq c u_-(-e_n) \ (c > 0) \end{aligned} \tag{13.12}$$

provided ε is small.

By Lemma 2 of [12] for any $0 < \theta_0 < \dfrac{\pi}{2}$ there exists a harmonic function

$$h(z) = r^\alpha g(\sigma) \ (\alpha > 1)$$

defined in the cone K with openning θ_0 such that h vanishes on ∂K, and

$$\alpha(\alpha + n - 2) = \frac{1}{\alpha_1}$$

where

$$\frac{1}{\alpha_1} = \inf_f \frac{\displaystyle\int_{K_0} |\nabla_\theta f|^2}{\displaystyle\int_{K_0} |f|^2};$$

here the infimum is taken over all $f \in H_0^1(K_0)$ where K_0 is the spherical cap of opening θ_0.

Since $\alpha = \gamma_1$, (7.5) becomes

$$\alpha \geq \psi(s_0)$$

where s_0 is the fraction of the surface area of K_0 on S^{n-1}. If $\theta_0 \to \frac{\pi}{2}$ then $\alpha \to 1$. Indeed suppose $\alpha \to \tilde{\alpha}$ for a sequence of α's. Then $\tilde{\alpha} \geq 1$. The function $h = r^{\tilde{\alpha}} g(\sigma)$ (for the limiting g) is harmonic in $\{x_n > 0\}$ and vanishes on $x_n = 0$. It must then be regular up to $x_n = 0$ and, since $\partial h / \partial x_n \neq 0$ at the origin (by the maximum principle), $\tilde{\alpha} = 1$.

Since v is superharmonic in $\{x_n < f(x') - C\varepsilon\}$, by (13.12), Harnack's inequality (for v) and the maximum principle we have, for any $x_0 \in A \cap C_{7/5}$ and x in the cone $x_0 + \Gamma(\theta_0, -e_n)$,

$$v(x) \geq c u_-(-e_n) h(x - x_0).$$

In particular, for $0 < \theta < \theta_0$,

$$v(x) \geq c|x - x_0|^\alpha u_-(-e_n) \text{ for } x_0 \in A \cap C_{7/5},\ x \in x_0 + \Gamma(\theta, -e_n) \tag{13.13}$$

where $c = c(\theta) > 0$.

Recall (see the proof of Lemma 12.2) that Lemma 5 of [5] extends to our case of $\lambda > 0$, so that

$$D_\tau v \geq 0 \text{ in } N_{\delta_0}(A) \cap C_1, \text{ for } \tau \in \Gamma\left(\frac{\theta_0}{2}, e_n\right),\ \delta_0 = \delta_0(\theta_0). \tag{13.14}$$

Now take any two points x_1, x_2 in $N_{\delta_0}(A) \cap C_1$ such that

$$c_1 \varepsilon^{1/8} \leq |x_1 - x_2| \leq c_2 \varepsilon^{1/8} \ (c_1 > 0, c_2 > 0) \tag{13.15}$$

and

$$x_2 - x_1 \in \Gamma(\theta, -e_n),\ 0 < \theta < \frac{\theta_0}{2}.$$

If we prove that

$$u_-(x_2) \geq u_-(x_1), \tag{13.16}$$

then the assertion (a) follows.

Without loss of generality we may assume that $x_1 \in \Omega^-(u)$. Set

$$\tau = \frac{x_2 - x_1}{|x_2 - x_1|}.$$

By (13.14) and Lemma 4 of [5] extended to the case of $\lambda > 0$ (cf. the proof of Lemma 12.2),

$$d_x D_\tau v(x) \geq c v(x) \text{ if } x \in N_{\delta_0}(A) \cap C_1$$

where $d_x = d(x, A)$. Using also Harnack's inequality we find that if $x - x_1 = \mu(x_2 - x_1)$ and $\frac{1}{2} < \mu < 1$ then

$$D_\tau v(x) \geq \frac{c}{\delta_0} v(x_2). \tag{13.17}$$

Hence

$$\begin{aligned} v(x_2) - v(x_1) &= \int_0^1 D_\tau v(x_1 + t(x_2 - x_1))|x_2 - x_1| dt \\ &\geq \int_{1/2}^1 D_\tau v(x_1 + t(x_2 - x_1))|x_2 - x_1| dt \text{ (since } D_\tau v \geq 0) \\ &\geq c \frac{v(x_2)}{\delta_0} \varepsilon^{1/8} \text{ (by (13.15), (13.17)).} \end{aligned} \tag{13.18}$$

If ε is small enough then $x_2 \in \{x_0 + \Gamma(\theta, -e_n)\}$ where $x_0 \in A \cap C_1$ and $|x_2 - x_0| \geq |x_2 - x_1|$. Hence, by (13.13) and (13.15),

$$v(x_2) \geq c\varepsilon^{\alpha/8} u_-(-e_n).$$

Substituting this into the right-hand side of (13.18), we get

$$v(x_2) - v(x_1) \geq \frac{c}{\delta_0} u_-(-e_n)\varepsilon^{\frac{\alpha+1}{8}}.$$

If we now recall (13.11), we deduce that

$$\begin{aligned} u_-(x_2) &\geq v(x_2) \geq v(x_1) + \frac{c}{\delta_0} u_-(-e_n)\varepsilon^{\frac{\alpha+1}{8}} \\ &\geq u_-(x_1) - cu_0^{\frac{1}{2}}[u_-(-e_n)]^{\frac{1}{2}}\varepsilon + \frac{c}{\delta_0} u_-(-e_n)\varepsilon^{\frac{\alpha+1}{8}} \geq u_-(x_1) \end{aligned} \tag{13.19}$$

provided

$$[u_-(-e_n)]^{1/2} \geq cu_0^{1/2}\varepsilon^{\frac{7-\alpha}{8}}.$$

Recall that if $\theta_0 \to \frac{\pi}{2}$ then $\alpha \to 1$; hence $\alpha < 3$ for some $0 < \theta_0 < \frac{\pi}{2}$, and the last inequality is then satisfied for ε small, by the assumption in (i). It follows that (13.16) holds.

CASE (ii). In this case we still have the estimate (13.9) and the proof of [6; p. 72] is valid since we need to work only with u_+.

14. Minimizers are weak solutions

Let u be a minimizer of (J_η). We know that u is Lipschitz continuous and

$$\Delta u = 0 \text{ in } \Omega^+(u),$$

$$\Delta u + \lambda u = 0 \text{ in } \Omega^-(u).$$

To prove that u is a weak solution, with

$$G(\beta) = (\mu^2 + \beta^2)^{1/2}, \tag{14.1}$$

it remains to prove (10.1) at any admissible point x_0 of $F(u) = \partial\Omega^+(u)$, i.e., at any point for which there is a ball $B_\rho(z_0)$ such that either $B_\rho(z_0) \subset \Omega^+(u)$ or $B_\rho(z_0) \subset \Omega^-(u)$, and $x_0 \in \partial B_\rho(z_0)$; for simplicity we take $x_0 = 0$ and e_n in the direction $\overrightarrow{x_0 z_0}$. We shall need the following lemma:

LEMMA 14.1. *Let Ω_i $(i = 1, 2)$ be open sets such that*

$$\Omega_1 \cap B_1 \supset B_\rho(z_0) \cap B_1,$$

$$\Omega_2 \cap B_1 \subset B_1 \backslash B_\rho(z_0)$$

(see Figure 2(a), (b)). Assume that u is Lipschitz continuous in $\overline{\Omega}_i$ and positive in Ω_i, for either $i = 1$ or $i = 2$, and that

$$\Delta u + \lambda u = 0 \text{ in } \Omega_i \ (\lambda > 0),$$

$$u = 0 \text{ on } \partial\Omega_i.$$

Then, near 0, u has the asymptotic behavior

$$u(x) = \frac{\alpha}{\sqrt{\lambda}} \sin \sqrt{\lambda} y + o(|x|) \ (x = (x', y)) \tag{14.2}$$

in $y > 0$; furthermore, $\alpha > 0$ for the case $i = 1$.

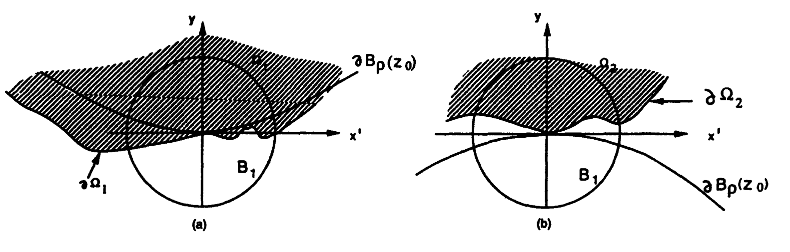

Figure 2

The case $\lambda = 0$ with $(\sin \sqrt{\lambda} y)/\sqrt{\lambda}$ replaced by y coincides with Lemma A1 in [6].

PROOF. Consider first the case $i = 1$ and define an increasing sequence

$$\varepsilon_k = \inf_{B_\rho(z_0) \cap B^+_{2^{-k}}} \frac{u}{\frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda} y} \qquad (B^+_\mu = B_\mu \cap \{y > 0\}).$$

Since u a Lipschitz, $\alpha \equiv \lim \varepsilon_k$ is finite; further,

$$u \geq \frac{\alpha}{\sqrt{\lambda}} \sin \sqrt{\lambda} y + o(|x|) \text{ in } B_1 \cap B_\rho(z_0). \tag{14.3}$$

We want to prove that equality holds in (14.3).

If this is not true then there is a sequence x_k with $r_k = |x_k| \to 0$ such that

$$u(x_k) - \frac{\alpha}{\sqrt{\lambda}} \sin \sqrt{\lambda} y(x_k) > \delta_0 |x_k|$$

for some $\delta_0 > 0$. Since u is Lipschitz continuous,

$$u(x) - \frac{\alpha}{\sqrt{\lambda}} \sin \sqrt{\lambda} y \geq \frac{\delta_0}{2} |x_k| \text{ for } x \in B_{\delta_1 r_k}(x_k)$$

where $\delta_1 > 0$ is independent of k, or

$$\frac{u(r_k x)}{r_k} - \frac{\alpha}{\sqrt{\lambda}} \frac{\sin \sqrt{\lambda} r_k y}{r_k} \geq \frac{1}{2} \delta_0 \text{ for } x \in B_{\delta_1}\left(\frac{x_k}{r_k}\right), \ \frac{x_k}{r_k} \in S^{n-1}. \tag{14.4}$$

Consider the function

$$v(x) = \frac{u(r_k x)}{r_k} - \frac{\alpha}{\sqrt{\lambda}} \frac{\sin \sqrt{\lambda} r_k y}{r_k}.$$

It satisfies

$$\Delta v + \lambda r_k^2 v = 0 \text{ in } B_1 \cap B_\rho(z_0).$$

By Poisson's formula (see [9])

$$v(x) = \int\limits_{\partial(B_1 \cap B_\rho(z_0))} v(y) dW_k^x(y)$$

where

$$\frac{1}{C} \mathcal{P}^x(\xi) dS_\xi \leq dW_k^x \leq C \mathcal{P}^x(\xi) dS_\xi$$

and $\mathcal{P}^x(y)$ is the normal derivative of Green's function for the Laplacian. Using (14.4) and (14.3) we get

$$v(x) \geq \frac{\delta_0}{C} \int\limits_{S^{n-1} \cap B_{\delta_1}\left(\frac{x_k}{r_k}\right)} \mathcal{P}^x(\xi) dS_\xi - C\delta_k \int\limits_{\left(S^{n-1} \setminus B_{\delta_1}\left(\frac{x_k}{r_k}\right)\right) \cup \partial B_\rho(z_0)} \mathcal{P}^x(\xi) dS_\xi$$

where $\delta_k \to 0$ if $k \to \infty$. This yields

$$v(x) \geq 2\delta \operatorname{dist}(x, \partial B_\rho(z_0)) \text{ for some } \delta > 0,$$

and therefore also

$$v(x) \geq \delta y \geq \frac{\delta}{2} \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda} y$$

near the origin. Since this is a contradiction to the definition of α, the proof of (14.2) in $B_\rho(z_0)$ is complete. If $x \notin B_\rho(z_0)$ and $y > 0$, then $y = o(|x|)$ and since $u(x) \geq 0$, (14.2) is again valid. Thus (14.2) holds for all $y > 0$. Finally, by the strong maximum principle, $\alpha > 0$.

Consider next the case $i = 2$, and extend u into $\{y > 0\}$ by 0. The extended function, $\tilde{u}$, satisfies: $\Delta\tilde{u} + \lambda\tilde{u} \geq 0$. Define a decreasing sequence

$$\varepsilon_k = \sup_{B^+_{2-k}\setminus B_\rho(z_0)} \frac{\tilde{u}_\varepsilon}{\frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda} y}.$$

We can now proceed as before to establish the relation (14.2) in $y > 0$, where $\alpha = \lim \varepsilon_k \geq 0$.

LEMMA 14.2. *If u is a minimizer of (J_η) then u satisfies the weak free boundary condition along $F(u) = \partial\Omega^+(u) \cap \Omega$.*

PROOF. Suppose $x_0 \in F(u)$ and there is a tangent ball $B_\rho(y_0)$ to $F(u)$ at x_0, say from $\Omega^+(u)$. By Lemma 14.1, near x_0,

$$u_+(y) = \alpha\langle x - x_0, \nu\rangle_+ + o(|x - x_0|), \quad \alpha \geq 0$$

(case 1 with $\lambda = 0$), and

$$\begin{aligned} u_-(x) &= \beta \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}\langle x - x_0, \nu\rangle_- + o(|x - x_0|) \\ &= \beta\langle x - x_0, \nu\rangle_- + o(|x - x_0|) \ (\beta \geq 0) \end{aligned}$$

(case 2 with $\lambda > 0$), since $\sin t = t + o(t)$. Hence

$$u(x) = u_+(x) - u_-(x) = \alpha\langle x - x_0, \nu\rangle_+ - \beta\langle x - x_0, \nu\rangle_- + o(|x - x_0|).$$

Furthermore, by taking a blow up limit we get

$$U(x) = \alpha\langle x - x_0, \nu\rangle_+ - \beta\langle x - x_0, \nu\rangle_-,$$

where U is a minimizer of the functional

$$J^0(U) = \int [|\nabla U|^2 + \mu_-^2 I_{\{U\leq 0\}} + \mu_+^2 I_{\{U>0\}}].$$

in the sense Definition 8.1. Theorem 9.5 (or Theorem 3.4) yields

$$\left(\frac{\partial U_+}{\partial \nu}\right)^2 - \left(\frac{\partial U_-}{\partial \nu}\right)^2 = \alpha^2 - \beta^2 = \mu_+^2 - \mu_-^2 = \mu^2,$$

so that $\alpha = G(\beta)$.

We now wish to apply Theorem 13.2. We have already proved that any variational solution (i.e., a solution u to problem (J_η)) is a weak solution. Since u is also Lipschitz continuous, the parameters α and β are uniformly bounded. By Lemma 8.5 and by Lemma 8.4 of [3], if $n = 2$ then the flatness condition holds. Corollary 11.2 tells us that u is fully monotone in the direction $\Gamma(\theta, e_n)$ for $x_n = \pm\sqrt{\varepsilon}$. Thus, if we prove that u is also fully monotone in the direction $\Gamma(\theta, e_n)$ on $\partial\mathcal{C}_1 \cap \{|x_n| < \sqrt{\varepsilon}\}$ then the assumption (iii) of Theorem 11.3 (which is the assumption (ii) of Theorem 13.2) is completely satisfied. But this full monotonicity u is the same as the assertion (8.12) (or (8.29)) in [3]. The proof of this inequality in case $\lambda = 0$ is proved in Lemma 8.2 of [3] in case $u\left(0, -\frac{1}{2}\right) < -\eta$ and in Lemma 8.11 of [3] if this nondegeneracy assumption is dropped. The proof of Lemma 8.11 is based on Lemmas 8.2-8.7, 8.9, 8.10 and Corollary 8.8 of [3]. The same proof can be given in case $\lambda > 0$, except for two changes:

(i) In pages 456 and 458 of [3] we need to take G to be Green's function of $\Delta + \lambda$ (not Δ), and

(ii) instead of exploiting the subharmonicity of $w \equiv |\nabla u_-|$ (in Lemma 8.10) we now use the fact that $w^2 = |\nabla u_-|^2$ is subsolution, i.e., $\Delta w^2 + 2\lambda w^2 \geq 0$. We then have, by elliptic estimates,

$$W^2(x) \leq C \int_{B_R \setminus B_{R/2}} W^2 \text{ in } B_{R/4}$$

where $W^2 = \max\{w^2, C_0 u_+(A) u_-(A_*)\}$ in B_R; observe that W^2 is a subsolution, by Lemma 10.2. The constant C depends on R, but since $R = \frac{1}{4}\delta(\gamma_0)$ (see Corollary 8.8 of [3]), C depends on γ_0, as in Lemma 8.10 of [3].

Finally, Theorem 13.2 requires that $\lambda < \lambda_{\mathcal{C}_2}$. However, all the other assumptions of the theorem are valid in any smaller cylinder $\mathcal{C}_\rho$. We choose ρ so small that $\lambda < \lambda_{\mathcal{C}_\rho}$ (the principal eigenvalue of the Laplacian in $\mathcal{C}_\rho$), and then Theorem 13.2 can be applied to $\mathcal{C}_\rho$ (after scaling). We conclude:

THEOREM 14.3. *If $n = 2$ then the free boundary of any minimizer of (J_η) is Lipschitz continuous.*

The proof that Lipschitz continuity implies C^1 continuity is given in [3; Th. 8.12]. That proof extends to the present case with some obvious changes. Hence

THEOREM 14.4. *If $n = 2$ then the free boundary of any minimizer of (J_η) is continuously differentiable.*

By the proof of Theorem 8.13 in [3], for any $x_0 \in F(u)$, u is continuously differentiable as $x \to x_0$ from any angular region $\{(x-x_0)\cdot\nu > c|x-x_0|\}$, $c > 0$.

15. - $C^{1+\alpha}$ regularity of the free boundary

In Sections 10-14 we assumed that $G(s)$ is continuously differentiable and strictly monotone increasing function, and $G(s) \geq s$ if $s > 0$. In this section we shall also assume that $G(s)/s^c$ is monotone decreasing function in $s \in (0,\infty)$, for some $c > 0$.

In this section we prove:

THEOREM 15.1. *Let u be a weak solution, as in Definition* 10.1. *If the free boundary is Lipschitz continuous then it is also in $C^{1+\alpha}$.*

Combining this with Theorem 14.4 we conclude:

THEOREM 15.2. *If $n = 2$ then the free boundary of any minimizer u of problem (J_η) is in $C^{1+\alpha}$.*

We can use $C^{1+\alpha}$ transformation to flatten the free boundary and then apply to u_- and the reflected u_+ the Schauder estimates for elliptic equations in divergence form [1]. This yields $C^{1+\alpha}$ regularity of $u_\pm$ up to the boundary. To obtain additional regularity we use the mapping $y = (x_1, \dots, x_{n-1}, u(x))$ ([17]; see also [13; pp. 135-136]) and define functions $v^\pm(y) = (x_n)_\pm$. Then v^- and the reflected v^+ across $y_n = 0$ satisfy a nonlinear elliptic system with coefficients as smooth as $\nabla v^\pm$, and with "good" nonlinear boundary conditions (cf. [17] [18]). Using elliptic regularity [1] we can derive $C^{2+\alpha}$ regularity and, by bootstrapping, establish C^∞ and, in fact, analyticity, provided $G(s)$ has the same regularity. Thus we have:

COROLLARY 15.3. *If in Theorem* 15.1 *$G(s)$ in C^∞ (analytic) then the free boundary is C^∞ (analytic); in particular, in Theorem* 15.2 *the free boundary is analytic.*

Theorem 15.1 in the case $\lambda = 0$ is due to Caffarelli [5]. Our proof is based on his method; however several new arguments are needed.

Let φ be a bounded, C^2 positive function in Ω satisfying:

$$\Delta\varphi \leq C\,\frac{|\nabla\varphi|^2}{\varphi} \quad \text{for } C \text{ sufficiently large } (C > 1). \tag{15.1}$$

For any continuous function u in Ω, consider the function

$$v(x) = \sup_{B_{\varphi(x)}(x)} u \tag{15.2}$$

defined in $\Omega_\delta = \{x \in \Omega, d(x,\partial\Omega) > \delta\}$ where $\delta = \sup\varphi$ and $B_\rho(x)$ denotes the ball with center x and radius ρ.

LEMMA 15.4. *Suppose* $|\nabla\varphi|_{L^\infty} < 1$. *If* $u < 0$ *and* $\Delta u + \lambda u = 0$ *in* Ω, *where* $\lambda \geq 0$, *then*

$$\Delta v + \lambda(1 - |\nabla\varphi|_{L^\infty})^2 v \geq 0 \text{ in } \Omega_\delta. \tag{15.3}$$

PROOF. For $\lambda = 0$ this is Lemma 9 in [5]. To prove the lemma in case $\lambda > 0$, take any ball $B \subset \Omega_\delta$ and let w be the solution to

$$\Delta w = -\lambda(1 - |\nabla\varphi|_{L^\infty})v \text{ in } B, \ \ w = v \text{ on } \partial B. \tag{15.4}$$

If we prove that the function $V = v - w$ is subharmonic in B then $\Delta v = \Delta V + \Delta w \geq -\lambda(1 - |\nabla\varphi|_{L^\infty})^2 v$, and the lemma follows. But V is subharmonic if and only if

$$\liminf_{r\to 0} \frac{1}{r^2} \fint_{B_r(x_0)} [V(x) - V(x_0)]dx > 0 \quad \forall x_0 \in B, \tag{15.5}$$

so that it suffices to prove (15.5). For simplicity we take $x_0 = 0$.

The function u is subharmonic ($\Delta u = -\lambda u > 0$) and therefore there exists a unit vector ν_0 such that

$$v(0) = u(\varphi(0)\nu_0).$$

We choose a system of coordinates such that $\nu_0 = e_n$, $\nabla\varphi(0) = \alpha e_1 + \beta e_n$. For any $x \in B_r(0)$, set $\nu(x) = \nu^*/|\nu^*|$ where

$$\nu^* = e_n + \frac{\beta x_1 - \alpha x_n}{\varphi(0)} e_1 + \frac{\rho}{\varphi(0)} \sum_{i=2}^{n-1} x_i e_i,$$

where ρ is chosen so that $(1+\rho)^2 = (1+\beta)^2 + \alpha^2$. Finally set

$$y = y(x) = x + \varphi(x)\nu(x) = y^* + O(|x|^2).$$

Proceeding as in [5] we write

$$\begin{aligned}
\fint_{B_r(0)} [V(x) - V(0)] &= \fint_{B_r(0)} [(V(x) - w(x)) - (V(0) - w(0))] \\
&\geq \fint_{B_r(0)} [(u(y(x)) - w(x)) - (u(y(0)) - w(0))] \\
&= \fint_{B_r(0)} [u(y(x)) - u(y^*(x))] \\
&\quad + \fint_{B_r(0)} \{[u(y^*(x)) - w(x)] - [u(y(0)) - w(0)]\} \\
&\equiv I_1(r) + I_2(r)
\end{aligned}$$

and

$$\fint_{B_r(0)} [u(y) - u(y^*)] = |\nabla u(y(0))| \left\{ \frac{r^2}{n+1} \left[\frac{1}{2} \Delta\varphi - \frac{1}{\varphi(0)} (\beta^2 + \alpha^2 + (n-2)\rho^2) \right] + O(r^3) \right\}. \tag{15.6}$$

It follows that

$$\liminf_{r\to 0} \frac{1}{r^2} I_1(r) > 0,$$

and in order to complete the proof of (15.5) it suffices to show that

$$\liminf_{r\to 0} \frac{1}{r^2} I_2(r) \geq 0,$$

that is,

$$\liminf_{r\to 0} \frac{1}{r^2} \fint_{B_r(0)} (u^*(x) - u^*(0)) \geq 0 \tag{15.7}$$

where $u^*(x) = u(y^*(x)) - w(x)$.

As in [5] $y^*(x)$ is a translation by $\varphi(0)e_n$ plus a rotation followed by expansion by $1+\rho$ of x. Hence, by (15.4),

$$\begin{aligned} \Delta u^* &= -\lambda(1+\rho)^2 u(y^*(x)) + \lambda(1 - |\nabla\varphi|_{L^\infty})^2 v \\ &\geq -\lambda(1+\rho)^2 u(y^*(x)) + \lambda(1 - |\nabla\varphi|_{L^\infty})^2 u(y(x)). \end{aligned}$$

Since $u(y^*(x)) < 0$ and

$$(1+\rho)^2 = (1+\beta)^2 + \alpha^2 \geq 1 + |\nabla\varphi|^2_{L^\infty} - 2|\nabla\varphi|_{L^\infty} = (1 - |\nabla\varphi|_{L^\infty})^2,$$

it follows that

$$\Delta u^*(x) \geq \lambda(1 - |\nabla\varphi|_{L^\infty})^2 [u(y) - u(y^*)].$$

Recalling (15.6) we conclude that

$$\int_{B_r(0)} \Delta u^* \geq O(r^{n+3}) \text{ or } \int_{\partial B_r(0)} \frac{\partial u^*}{\partial r} \geq O(r^{n+3}).$$

Multiplying the last inequality by r^{1-n} and integrating with respect to r, then multiplying by r^n, we get

$$\int_{B_r(0)} [u^*(x) - u^*(0)] \geq O(r^{n+5}),$$

from which (15.7) follows.

LEMMA 15.5. *Let u be a weak solution in Ω and define φ and v as in Lemma* 15.4. *Then*

(i) $\Delta v \geq 0$ *in* $\Omega^+(v) \cap \Omega_\delta$ *and* $\Delta v + \lambda(1 - |\nabla\varphi|_{L^\infty})^2 v \geq 0$ *in* $\Omega^-(v) \cap \Omega_\delta$*;*

(ii) *every point of* $F(v)$ *is regular;*

(iii) *at every point* x_0 *of* $F(v)$*,* v *satisfies the inequality:*

$$v(x) \geq \alpha\langle x - x_0, \nu\rangle_+ - \beta\langle x - x_0, \nu\rangle_- + o(|x - x_0|)$$

with

$$\frac{\alpha}{1 - |\nabla\varphi|} = G\left(\frac{\beta}{1 + |\nabla\varphi|}\right);$$

(iv) *if* $F(u)$ *is Lipschitz graph with Lipschitz norm* L *and* $|\nabla\varphi|$ *is small enough (depending on* L*), then* $F(v)$ *is Lipschitz graph with Lipschitz norm* $L' \leq L + C|\nabla\varphi|_{L^\infty}$.

The assertion (i) follows from Lemma 15.4. The assertion (ii), (iii) and (iv) are included in Lemmas 11 and 10 of [5] in case $\lambda = 0$, and the proof for $\lambda > 0$ is precisely the same.

The function v satisfies (15.3) in $\Omega^-(v) \cap \Omega_\delta$. However for the iterative argument to be carried out later on it is imperative that the inequality $\Delta v + \lambda v \geq 0$ be satisfied. Since we cannot establish this inequality, we shall try to remedy the situation by introducing a "corrective" function w, defined by:

$$\begin{aligned} \Delta w + \lambda w &= 2\lambda|\nabla\varphi|_{L^\infty} v \text{ in } \Omega^-(v), \\ w &= 0 \text{ on } \partial\Omega^-(v), \\ w &= 0 \text{ in } \Omega\backslash\Omega^-(v). \end{aligned} \tag{15.8}$$

Here we assume that $F(u)$ is Lipschitz and that $|\nabla\varphi|_{L^\infty}$ is small enough so that also $\partial\Omega^-(v)$ is Lipschitz. We further assume that $\lambda < \lambda_\Omega$ (which is not really a restriction, since we can study the $C^{1+\alpha}$ regularity in just a very small neighborhood of a point). Notice that since every point of $F(v)$ is regular, every point of $\partial\Omega^-(v)$ has the outside ball property and therefore also a barrier. It follows that w is continuous in $\overline{\Omega^-(v)}$.

We claim:

$$0 \leq w \leq \frac{2C|\nabla\varphi|_{L^\infty}}{(1 - |\nabla\varphi|_{L^\infty})^2} v_- \text{ in } \Omega^-(v). \tag{15.9}$$

To prove it we use the representation

$$w = \int_{\Omega^-(v)} 2\lambda|\nabla\varphi|_{L^\infty} G_\lambda v_-$$

where G_λ is Green's function for $\Delta+\lambda$, and the pointwise estimate $G_\lambda \le CG_0$, to get

$$0 \le w \le 2\lambda C|\nabla\varphi|_{L^\infty} V,\ V = \int_{\Omega^-(v)} G_0 v_-. \tag{15.10}$$

Clearly $\Delta V = -v_-$ in $\Omega^-(v)$ and $V = 0$ on $\partial\Omega^-(v)$. On the other hand

$$\Delta \frac{v_-}{\lambda(1-|\nabla\varphi|_{L^\infty})^2} \le -v_- \text{ in } \Omega^-(v)$$

so that, by comparison,

$$V \le \frac{v_-}{\lambda(1-|\nabla\varphi|_{L^\infty})^2}.$$

Using this in (15.10), the assertion (15.9) follows.

We now define

$$\overline{v} = v - w \text{ in } \Omega \tag{15.11}$$

where v is defined as in Lemma 15.5.

THEOREM 15.6. *The function $\overline{v}$ satisfies:*

(i) $\Delta\overline{v} \ge 0$ *in* $\Omega^+(\overline{v})$, $\Delta\overline{v} + \lambda\overline{v} \ge 0$ *in* $\Omega^-(\overline{v})$;

(ii) *every point of* $F(\overline{v})$ *is regular;*

(iii) *at every point* x_0 *of* $F(\overline{v})$, $\overline{v}$ *satisfies:*

$$\overline{v}(x) \ge \alpha\langle x - x_0, \nu\rangle_+ - \beta\langle x - x_0, \nu\rangle_- + o(|x - x_0|)$$

with

$$\frac{\alpha}{1-|\nabla\varphi|} = G\left(\frac{\beta}{(1+|\nabla\varphi|)\cdot\left(1+\dfrac{2C|\nabla\varphi|_{L^\infty}}{(1-|\nabla\varphi|_{L^\infty})^2}\right)}\right). \tag{15.12}$$

PROOF. The proof follows from Lemma 15.5 and (15.8), (15.9), with

$$\frac{\alpha}{1-|\nabla\varphi|} = G\left(\frac{\overline{\beta}}{1+|\nabla\varphi|}\right)$$

instead of (15.12); the relation between $\overline{\beta}$ and β is given by $\beta = \left(1+\dfrac{2C|\nabla\varphi|_{L^\infty}}{(1-|\nabla\varphi|_{L^\infty})^2}\right)\overline{\beta}$ and this easily gives the assertion (15.12).

We introduce the unit cylinder $\mathcal{C}_1 = B_1(0) \times (-1,1)$ in $\mathbb{R}^n$ and set $\partial\mathcal{C}_1^0 = \partial\mathcal{C}_1 \backslash \{x_n = 1\}$, $S_\eta = \{x \in \mathcal{C}_1, d(x, \partial\mathcal{C}_1^0) < \eta\}$ for any $0 < \eta < 1$.

LEMMA 15.7. *For any $\sigma > 0$, $\delta > 0$ and small $\eta > 0$ there exist $\tau = \tau(\eta) > 0$ and C^2 function φ in $\mathcal{C}_1$ such that*

(i) $1 \le \varphi \le 1 + \delta\sigma$,

(ii) $\varphi\Delta\varphi \ge C|\nabla\varphi|^2$, *$C$ as in* (15.1) *(which ensures that Lemma* 15.1 *is valid),*

(iii) $\varphi = 1$ *in* S_η,

(iv) $\varphi|_{\mathcal{C}_{1/2}} \ge 1 + \delta\sigma\tau$,

(v) $|\nabla\varphi| \le \overline{C}\delta\sigma$ *($\overline{C}$ depends only on η),*

(vi) $\dfrac{\partial}{\partial x_n}\varphi \ge 0$.

This lemma differs from Lemma 13 in [5] because we assert here the additional property (vi). Our construction of φ is also different from the construction given in [5].

PROOF. Let ψ_0 be a function satisfying:

$$\begin{aligned} &\Delta\psi_0 = 0 \text{ in } \mathcal{C}_1^\eta = B_{1-\eta}(0) \times (-1+\eta, 2), \\ &\psi_0 = 1 \text{ on } \partial\mathcal{C}_1^\eta \setminus \{x_n = 2\}, \\ &\psi_0 = a|x'|^2 + b \text{ on } \{x_n = 2\} \end{aligned} \tag{15.13}$$

where a, b are positive constants chosen so that ψ_0 is continuous on $\partial\mathcal{C}_1^\eta$.

By the maximum principle

$$\frac{\partial\psi_0}{\partial x_n} \le 0 \text{ in } \mathcal{C}_1^\eta$$

and

$$0 < \psi_0 \le \tilde{\sigma} < 1 \text{ in } \mathcal{C}_{2/3}.$$

Extending ψ_0 by 1 into $\{x_n < 2\} \setminus \mathcal{C}_1^\eta$, we have

$$\Delta\psi_0 \le 0, \ \frac{\partial\psi_0}{\partial x_n} \le 0 \text{ in } \{x_n < 2\}.$$

Let $\psi_1 = \psi_0 * \rho_\eta$ be a mollifier of ψ_0, where $\operatorname{supp}\rho_\eta \subset B_\eta(0)$. Then $\psi_1 \in C^2(\mathcal{C}_1)$, and

$$\Delta\psi_1 \le 0 \text{ in } \mathcal{C}_1, \ \frac{\partial\psi_1}{\partial x_n} \le 0 \text{ in } \mathcal{C}_1, \tag{15.14}$$

$$0 < \psi_1 \le \tilde{\sigma} < 1 \text{ in } \mathcal{C}_{1/2}. \tag{15.15}$$

Let

$$\psi_2 = \left(\frac{\psi_1 + 1}{2}\right)^{1/1-2C} \quad (C \text{ as in } (15.1)). \tag{15.16}$$

Then
$$\begin{aligned} 0 \geq \Delta\psi_2^{1-2C} &= \nabla((1-2C)\psi_2^{-2C}\nabla\psi_2) \\ &= (2C-1)2C\psi_2^{-2C-1}|\nabla\psi_2|^2 - (2C-1)\psi_2^{-2C}\Delta\psi_2 \end{aligned}$$
so that

$$\psi_2 \cdot \Delta\psi_2 \geq 2C|\nabla\psi_2|^2. \tag{15.17}$$

Also

$$1 \leq \psi_2 \leq 2^{\frac{1}{2C-1}} \text{ (since } 0 \leq \psi_1 \leq 1), \tag{15.18}$$

$$\psi_2 = 1 \text{ in } S_\eta, \tag{15.19}$$

$$\psi_2 \geq \left(\frac{1+\tilde{\sigma}}{2}\right)^{\frac{1}{1-2C}} \equiv \overline{\sigma} > 1 \text{ in } C_{1/2}, \tag{15.20}$$

and

$$\frac{\partial\psi_2}{\partial x_n} = \frac{1}{1-2C}\left(\frac{\psi_1+1}{2}\right)^{\frac{1}{1-2C}-1}\frac{\partial\psi_1}{\partial x_n} \geq 0 \text{ in } C_1. \tag{15.21}$$

Now let

$$\varphi = 1 + \delta\sigma\,\frac{\psi_2 - 1}{2^{\frac{1}{2C-1}} - 1}. \tag{15.22}$$

Using (15.17)-(15.21) one can easily verify that φ satisfies the assertions (i)-(vi).

LEMMA 15.8. *Let u be a continuous function in C_1 satisfying $u_{x_n} \geq 0$, and let φ, v be as in* (15.1), (15.2). *If $\varphi_{x_n} \geq 0$ then $v_{x_n} \geq 0$.*

PROOF. For any x in the domain of definition of v there exists a $y \in \overline{B}_{\varphi(x)}(x)$ such that $v(x) = u(y)$. For any small $h > 0$,

$$|(y + he_n) - (x + he_n)| = |y - x| \leq \varphi(x) \leq \varphi(x + he_n)$$

since $\varphi_{x_n} \geq 0$. Hence $y + he_n \in \overline{B}_{\varphi(x+he_n)}(x + he_n)$ and therefore

$$\frac{1}{h}(v(x + he_n) - v(x)) \geq \frac{1}{h}(u(y + he_n) - u(y)).$$

Taking $h \to 0$ we get $v_{x_n} \geq u_{x_n} \geq 0$.

We shall need the following comparison result.

LEMMA 15.9. *Let C_1, S_η be as above and assume that $\lambda < \lambda_{C_1}$. Let u be a weak solution in C_1, continuous in $\overline{C}_1$, with $u_{x_n} \geq 0$ in C_1. Let v be a continuous function in $\overline{C}_1$ satisfying:*

(i) $\Delta v \geq 0$ *in* $\{v > 0\}$, $\Delta v + \lambda v \geq 0$ *in* $\text{int}\{v \leq 0\}$,

(ii) $F(u) \cap \overline{S}_\eta \neq \phi$ *and* $F(v) \cap \overline{S}_\eta$ *lies above* $F(u) \cap \overline{S}_\eta$,

(iii) *every point of* $F(v)$ *is regular,*

(iv) *at each point* $x_0 \in F(v) \backslash \overline{S}_\eta$, v *satisfies:*

$$v(x) \geq \alpha\langle x - x_0, \nu\rangle_+ - \beta\langle x - x_0, \nu\rangle_- + o(|x - x_0|) \text{ with } \alpha \geq G(\beta),$$

(v) $v_{x_n} \geq 0$ *in* $\mathcal{C}_1 \cap \{v > 0\}$.

Assume also that $v \leq u$ *on* $\partial \mathcal{C}_1$ *and* $u \geq 0$ *on* $\{x_n = 1\}$. *Then* $v \leq u$ *in* $\mathcal{C}_1$.

REMARK 15.1. This comparison result will play a crucial role in establishing the $C^{1+\alpha}$ regularity of the free boundary. It replaces Lemma 10.3 (that is the extension of the comparison Lemma 7 in [5] to $\lambda > 0$) which is inadequate for our subsequent needs.

PROOF. We first prove that

$$F(v) \text{ lies above } F(u). \tag{15.23}$$

Suppose this is not true. Then by (ii) there exists a $\tau > 0$ such that $F(u) - \tau e_n$ lies below $F(v)$ and they touch at a point $x_0 \in F(v) \backslash \overline{S}_\eta$. Set

$$u_\tau(x) = u(x + \tau e_n), \quad x \in \mathcal{C}_1 - \tau e_n \equiv \mathcal{C}_1^\tau. \tag{15.24}$$

Then $F(u_\tau) = F(u) - \tau e_n$. We claim that

$$u_\tau \geq v \text{ on } \partial(\mathcal{C}_1^\tau \cap \mathcal{C}_1). \tag{15.25}$$

To prove it notice that $\mathcal{C}_1^\tau \cap \mathcal{C}_1 = B_1(0) \times (-1, 1 - \tau)$, so that $\partial(\mathcal{C}_1^\tau \cap \mathcal{C}_1) = \partial B_1 \times (-1, 1 - \tau) \cup B_1 \times \{-1\} \cup B_1 \times \{1 - \tau\}$ where $B_1 = B_1(0)$. If $x \in \partial B_1 \times (-1, 1 - \tau) \cup B_1 \times \{-1\} \subset \partial \mathcal{C}_1$ then, since $u_{x_n} \geq 0$ and $u \geq v$ on $\partial \mathcal{C}_1$,

$$u_\tau(x) \geq u(x) \geq v(x).$$

On the other hand if $x_0 \in B_1 \times \{1 - \tau\}$ then $x_0 + \tau e_n \in \partial C_1$ so that, if also $x_0 \in \{v > 0\}$,

$$u_\tau(x_0) = u(x_0 + \tau e_n) \geq v(x_0 + \tau e_n) \geq v(x_0) \text{ by } (v).$$

If however $x_0 \in \{v \leq 0\}$ then

$$u_\tau(x_0) = u(x_0 + \tau e_n) \geq 0 \text{ (by assumption)}$$

so that again $u_\tau(x_0) \geq v(x_0)$.

From (15.25) and the fact that $F(u_\tau)$ lies below $F(v)$ we deduce that $u_\tau \geq v$ on $\partial(\{v > 0\} \cap C_1^\tau)$. Since u_τ is harmonic and v is subharmonic in $\{v > 0\} \cap C_1^\tau$, the maximum principle yields

$$u_\tau \geq v \text{ in } \{v > 0\} \cap C_1^\tau.$$

Furthermore, we have strict inequality:

$$u_\tau > v \text{ in } \{v > 0\} \cap C_1^\tau;$$

for otherwise $u_\tau \equiv v$ in $\{v > 0\} \cap C_1^\tau$, and this contradicts (ii), since $\tau > 0$. Similarly we deduce from (15.25) and the fact that $F(u_\tau)$ lies below $F(v)$ that $u_\tau \geq v$ in $\{u_\tau \leq 0\} \cap C_1$.

Setting $\Omega = C_1^\tau \cap C_1$ we then have:

(a) $u_\tau \geq v$ in Ω,

(b) $u_\tau > v$ in $\{v > 0\} \cap \Omega$,

(c) $x_0 \in [F(u_\tau) \cap F(v)] \backslash \overline{S}_\eta$.

By (iii) and (iv)

$$v(x) \geq \alpha\langle x - x_0, \nu\rangle_+ - \beta\langle x - x_0, \nu\rangle_- + o(|x - x_0|), \quad \alpha \geq G(\beta).$$

Since $F(u_\tau)$ lies below $F(v)$ and they touch at x_0, there is a ball in $\Omega^+(u_\tau)$ tangent to $F(u_\tau)$ at x_0; therefore

$$u_\tau(x) = \overline{\alpha}\langle x - x_0, \nu\rangle_+ - \overline{\beta}\langle x - x_0, \nu\rangle + o(|x - x_0|), \quad \overline{\alpha} = G(\overline{\beta}).$$

Comparing the two expansions and using (a), (b) we get $\alpha \leq \overline{\alpha}$, $\overline{\beta} \leq \beta$. Since G is strictly monotone increasing, it follows that $\alpha = \overline{\alpha}$, $\beta = \overline{\beta}$. Hence

$$0 \leq u_\tau - v \leq o(|x - x_0|).$$

On the other hand, by (b) and the strong maximum principle,

$$u_\tau - v \geq \varepsilon_0|x - x_0|, \quad \varepsilon_0 > 0$$

in a small ball in $\Omega^+(v)$ tangent to $F(v)$ at x_0, which is a contradiction.

We have thus completed the proof of (15.23). Applying the maximum principle once more we get $v \leq u$ in C_1, and the assertion of the lemma follows.

We now state the final lemma of this section, which is crucial for the iterative procedure used in proving the $C^{1+\alpha}$ regularity of the free boundary.

LEMMA 15.10. *Let $u_1 \leq u_2$ be two weak solutions in C_1 with Lipschitz free boundaries $F_i = F_i(u)$, and let $0 \in F_2$. Assume that $\lambda < \lambda_{C_1}$ and that*

(i) $v_\varepsilon(x) \equiv \sup\limits_{B_\varepsilon(x)} u_1(y) \le u_2(x)$ *in* $C_{1-\varepsilon}$,

(ii) $v_\varepsilon\left(\frac{3}{4}e_n\right) \le (1-\sigma\varepsilon)u_2\left(\frac{3}{4}e_n\right)$ *for some* $\sigma > 0$,

(iii) $C_\rho\left(\frac{3}{4}e_n\right) \subset \Omega^+(u_1)$, $0<\rho<\frac{1}{8}$ *where* $C_\rho\left(\frac{3}{4}e_n\right) = C_\rho + \frac{3}{4}e_n$.

Then for ε *small enough there exist* $\delta > 0$ *and* $\tau > 0$ *depending on* ρ *such that, on* $C_{\rho/4}$,

$$(v_{(1+\delta\tau\sigma)\varepsilon}(x))_+ \le (u_2(x))_+. \tag{15.26}$$

PROOF. Let $C^\rho = B_{\rho/2} \times \left(-\frac{3}{4},\frac{3}{4}\right)$, $\rho < \frac{1}{16}$. Define

$$v(x) = \sup_{B_{\varepsilon\varphi^\delta_{\sigma,\tau}(x)}(x)} u_1$$

and w by (15.8) with $\Omega = C^\rho$, $\varphi = \varepsilon\varphi^\delta_{\sigma,\tau}$, and set $\overline{v} = v - w$ and

$$\overline{\overline{v}} = \overline{v} + \overline{c}\sigma w^0 = v - w + \overline{c}\sigma w^0;$$

here, $\varphi^\delta_{\sigma,\tau}$ is the function constructed in Lemma 15.7 in C^ρ (instead of C_1) with $\eta < \frac{\rho}{16}$, $\overline{c}$ is still to be determined, and w^0 is the solution of

$$\begin{aligned} &\Delta w^0 = 0 \text{ in } \Omega^+(v)\cap C^\rho,\\ &w^0 = 0 \text{ on } \partial(\Omega^+(v)\cap C^\rho)\setminus\left\{x_n = \frac{3}{4}\right\},\\ &w^0 = u_2\left(\frac{3}{4}e_n\right) \text{ on } x_n = \frac{3}{4},\\ &w^0 = 0 \text{ in } \Omega^-(v)\cap C^\rho. \end{aligned}$$

We wish to apply the comparison lemma 15.9 to the pair u_2, $\overline{\overline{v}}$. We therefore need to check the assumptions (i)-(v) for $\overline{\overline{v}}$.

Assumption (i) follows from Theorem 15.6 (i). Notice that $\{\overline{\overline{v}} > 0\} = \{v > 0\}$, $\{\overline{\overline{v}} \le 0\} = \{v \le 0\}$, so that $F(\overline{\overline{v}}) = F(v)$. By Theorem 15.6 (ii) we then conclude that the assumption (iii) holds for $\overline{\overline{v}}$, i.e., every point of $F(\overline{\overline{v}})$ is regular.

Proof of assumption (ii): If ρ is small enough then $u_2 > 0$ on $x_n = \frac{3}{4}$, and it remains to prove that $F(\overline{\overline{v}}) \cap \overline{S}_\eta$ lies above $F(u_2)$. Since $\varphi^\delta_{\sigma,\tau} = 1$ in $\overline{S}_\eta$, we have $\overline{\overline{v}} = v_\varepsilon - w + \overline{c}\sigma\varepsilon w^0$ and $F(\overline{\overline{v}}) = F(v_\varepsilon)$, and, by assumption (i) of Lemma 15.10, $F(\overline{\overline{v}}) \cap \overline{S}_\eta$ lies above $F(u_2)$.

Proof of assumption (iv): For each $x_0 \in F(\overline{\overline{v}})\backslash\overline{S}_\eta$, $\overline{\overline{v}}$ satisfies

$$\overline{\overline{v}}(x) \geq \alpha\langle x - x_0, \nu\rangle_+ - \beta\langle x - x_0, \nu\rangle_- + o(|x - x_0|), \quad \alpha \geq G(\beta).$$

By Dahlberg's theorem (Lemma 1 in [5]) we have

$$\frac{w^0}{v_+} \geq C > 0 \text{ in } (\mathcal{C}^\rho\backslash\overline{S}_\eta) \cap \Omega^+(v), \quad \text{near } F(v)\backslash\overline{S}_\eta$$

provided $\varepsilon|\nabla\varphi^\delta_{\sigma,\varepsilon}|$ is small enough so that $F(v)$ is Lipschitz; C depends only on ρ and on the Lipschitz norm of $F(u_1)$. Using also (15.9) we get

$$\overline{\overline{v}} \geq (1 + \overline{c}\sigma\varepsilon)v_+ - \left(1 + \frac{2c|\nabla\varphi|_{L^\infty}}{(1 - |\nabla\varphi|_{L^\infty})^2}\right)v_- \text{ where } \varphi = \varepsilon\varphi^\delta_{\sigma,\tau}$$

and by Lemma 15.5 (iii)

$$\overline{\overline{v}} \geq (1 + c\sigma\varepsilon)\overline{\alpha}\langle x - x_0, \nu\rangle_+ - \left(1 + \frac{2c|\nabla\varphi|_{L^\infty}}{(1 - |\nabla\varphi|_{L^\infty})^2}\right)\overline{\beta}\langle x - x_0, \nu\rangle_- + o(|x - x_0|)$$

with

$$\frac{\overline{\alpha}}{1 - |\nabla\varphi|} = G\left(\frac{\overline{\beta}}{1 + |\nabla\varphi|}\right).$$

Setting

$$\alpha = (1 + c\sigma\varepsilon)\overline{\alpha}, \quad \beta = \left(1 + \frac{2c|\nabla\varphi|_{L^\infty}}{(1 - |\nabla\varphi|_{L^\infty})^2}\right)\overline{\beta}$$

and using the fact that $s^{-c}G(s)$ is decreasing for some positive constant c, we can deduce as in [5; p. 153] that $G(\beta) \leq \alpha$ if

$$\frac{1 + C|\nabla\varphi|_{L^\infty}}{1 - |\nabla\varphi|_{L^\infty}}\,\frac{1}{1 + c\sigma\varepsilon} < 1, \quad \varphi = \varepsilon\varphi^\delta_{\sigma,\tau},$$

which is indeed the case if δ is small (independent of ε), since $|\nabla\varphi|_{L^\infty} \leq c\varepsilon\delta$.

We next need to verify the assumption (v) of Lemma (15.9), i.e., that $\overline{\overline{v}}_{x_n} \geq 0$ in $\mathcal{C}^\rho \cap \{\overline{\overline{v}} > 0\}$. But in this set, $\overline{\overline{v}} = v + \overline{c}\sigma\varepsilon w^0$ and, as easily seen, $w^0_{x_n} \geq 0$, whereas by Lemma 15.8, $v_{x_n} \geq 0$.

In order to apply Lemma 15.9 we still need to check that $\overline{\overline{v}} \leq u_2$ on $\partial\mathcal{C}^\rho$. On $\partial\mathcal{C}^\rho\backslash\left\{x_n = \frac{3}{4}\right\}$, $\overline{\overline{v}} = v_\varepsilon \leq u_2$ by assumption (i). On $\partial\mathcal{C}^\rho \cap \left\{x_n = \frac{3}{4}\right\}$,

$$\begin{aligned}\overline{\overline{v}} &\leq v_{(1+\delta\sigma)\varepsilon} + \overline{c}\sigma\varepsilon u_2\left(\frac{3}{4}e_n\right) \quad (\text{since } \varphi^\delta_{\sigma,\tau} \leq 1 + \delta\sigma)\\ &\leq u_2\end{aligned}$$

by Lemma 12 [5] provided $\delta \leq \overline{\delta}(\rho)$. Hence $\overline{\overline{v}} \leq u_2$ on $\partial\mathcal{C}^\rho$.

We can now apply Lemma 15.9 and deduce that

$$\overline{\overline{v}} \leq u_2 \text{ in } \mathcal{C}^{\rho};$$

in particular, on $\mathcal{C}_{\rho/4}$,

$$(v_{(1+\delta\tau\sigma)\varepsilon})_+ \leq \overline{\overline{v}}_+ \leq (u_2)_+.$$

REMARK 15.2. In the special case $\lambda = 0$, Caffarelli [5; Lemma 14] established a stronger result than Lemma 15.10, namely,

$$v_{(1+\delta\sigma)\varepsilon}(x) \leq u_2(x). \tag{15.27}$$

Furthermore, his proof relies on the comparison lemma 10.3 (for $\lambda = 0$). In our case of $\lambda > 0$, we had to introduce a corrective term w (so that $v - w$ becomes a subsolution, i.e., $\Delta(v - w) + \lambda(v - w) \geq 0$). This change (from v to $v - w$) rendered Lemma 10.3 inadequate, and the new comparison lemma that we have used instead was Lemma 15.9.

PROOF OF THEOREM 15.1. Although our result (15.26) is weaker than (15.27), it is still strong enough to establish the $C^{1+\alpha}$ regularity of the free boundary. We first use the proof of Lemma 17 of [5] to deduce from (15.26) that if u is monotone in the direction $\Gamma(\theta, e_n)$ in B_1 then u_+ is monotone in a direction $\Gamma(\overline{\theta}, e)$ with $\overline{\theta} > \theta$ in B_ρ, for some $\rho > 0$, provided $\lambda < \lambda_{B_1}$. This implies that $F(u)$ is a Lipschitz graph in any direction in $\Gamma(\overline{\theta}, e)$. We can therefore use Lemma 5 of [5] extended to the case $\lambda > 0$ (see the proof of Lemma 12.2) to deduce that $D_\tau u_- \geq 0$ for any direction τ in $\Gamma(\overline{\theta}, -e)$, in a smaller ball whose radius depends only on the Lipschitz coefficient of $F(u)$; this radius is invariant under similarity scaling $u(\mu x)/\mu$. We can now apply the iterative argument of [5; p. 157] to deduce that the free boundary is in $C^{1+\alpha}$ in a neighborhood of a free boundary point; finally, the assumption $\lambda < \lambda_{B_1}$ can be dropped, for we can start the iteration in a ball B_{ρ_0} (instead of B_1) where ρ_0 is so small that $\lambda < \lambda_{B_{\rho_0}}$.

16. - Properties of $\Omega^{\pm}(u)$

Let u be a minimizer of (J_η).

THEOREM 16.1. *$\Omega^+(u)$ is a connected open set.*

PROOF. Since $u \in C^\alpha(\overline{\Omega})$ and $u = \gamma > 0$ on $\partial\Omega$, there is an Ω-neighborhood of $\partial\Omega$ which belongs to $\Omega^+(u)$. If $\Omega^+(u)$ is not connected then it has a component G such that $\overline{G} \subset\subset \Omega$. But since $u = 0$ on ∂G, and $\Delta u = 0$ in G, u is then identically zero in G, which is a contradiction.

From now on we consider only the case $n = 2$. Then $F(u) = \partial\Omega^+(u) \cap \Omega$ is locally continuously differentiable and, since $F(u) \subset\subset \Omega$, $F(u)$ consists of a finite number of closed smooth curves $F_i(u)$; each $F_i(u)$ encloses a domain G_i, and in each G_i either $u < 0$ and $\Delta u + \lambda u = 0$ everywhere, or else $u \equiv 0$.

THEOREM 16.2. *There cannot exist two components G_i, G_j such that $u < 0$ in both G_i and G_j.*

PROOF. Suppose the assertion is not true and take for simplicity $i = 1$, $j = 2$. Following [19], consider the function

$$\tilde{u} = \begin{cases} c_1 u \text{ in } G_1 \\ c_2 u \text{ in } G_2 \end{cases}$$

and $\tilde{u} = u$ elsewhere in r, where

$$c_1^2 \int_{G_1} u^2 + c_2^2 \int_{G_2} u^2 = \int_{G_1} u^2 + \int_{G_2} u^2. \tag{16.1}$$

Then

$$\int_\Omega \tilde{u}_-^2 = \int_\Omega u_-^2$$

and, therefore, $J_\eta(u) \le J_\eta(\tilde{u})$ is equivalent to

$$\int_{G_1} |\nabla u|^2 + \int_{G_2} |\nabla u|^2 \le c_1 \int_{G_1} |\nabla u|^2 + c_2 \int_{G_2} |\nabla u|^2. \tag{16.2}$$

By integration by parts

$$\int_{G_i} |\nabla u|^2 = \lambda \int_{G_i} u^2.$$

Using this and (16.1) we conclude that equality holds in (16.2), so that $\tilde{u}$ is also a minimizer. Consequently

$$|\nabla \tilde{u}_+|^2 - |\nabla \tilde{u}_-|^2 = \mu^2 = |\nabla u_+|^2 - |\nabla u_-|^2 \text{ on } \partial G_i,$$

i.e.,

$$|\nabla u_+|^2 - c_i^2 |\nabla u_-|^2 = |\nabla u_+|^2 - |\nabla u_-|^2 \text{ on } \partial G_i.$$

By varying c_1 we deduce that $\nabla u_- = 0$ on ∂G_1, so that $u \equiv 0$ in G_1, which is a contradiction.

We shall now construct domains Ω for which $\Omega^-(u)$ has several components in which $u \equiv 0$. We begin with a dumbell shaped domain consisting of two discs with a long narrow bridge connecting them:

$$\Omega = B_r(0) \cup B_r(z_0) \cup T_\delta$$

where

$$z_0 = (2L, 0), \quad T_\delta = \{(x_1, x_2); 0 < x_1 < 2L, -\delta < x_2 < \delta\}$$

and $L = 1/\sqrt{\delta}$, say.

THEOREM 16.3. *If $\mu_\pm$ are sufficiently large then, whenever δ is small enough, $\Omega^-(u)$ must have at least one component in which $u \equiv 0$.*

PROOF. Suppose the assertion is not true. Then $\Omega^-(u)$ is a domain in which $u < 0$. We claim:

$$\Omega^-(u) \text{ cannot contain points in both } B_r(0) \text{ and } B_r(z_0). \tag{16.3}$$

To prove (16.3) we suppose that it is not true. Since

$$\int\int |\nabla u_-|^2 \leq C_0, \quad C_0 \text{ independent of } \delta,$$

we must have

$$\int |\nabla u_-(\xi_1, x_2)|^2 dx_2 \leq C\delta^{1/2}$$

for some $\xi_1 \in \left(\frac{L}{8}, \frac{L}{4}\right)$ and $C > C_0$ (recall that $L = 1/\sqrt{\delta}$). Hence

$$\begin{aligned} |u_-(\xi_1, y)| &\leq \int |\nabla u_-(\xi_1, x_2)| dx_2 \\ &\leq \left(2\delta \int |\nabla u_-(\xi_1, x_2)|^2\right)^{1/2} \leq C\delta^{3/4}. \end{aligned} \tag{16.4}$$

Similarly

$$|u_-(\xi_2, y)| \leq C\delta^{3/4} \tag{16.5}$$

for some $\left(1 + \frac{3}{4}\right) L < \xi_2 < \left(1 + \frac{7}{8}\right) L$.

Now take half a disc $B_\rho^+(L, -\delta) = B_\rho(L, -\delta) \cap \{x_2 > -\delta\}$ and increase ρ until it touches $\partial\Omega^-(u)$, say at a point x_0; we may assume that $\rho < \delta$, for otherwise we can work with half discs $B_\rho^-(L, \delta) \cap \{x_2 < \delta\}$.

We shall prove that

$$|\nabla u_+(x_0)| \geq \frac{c}{\rho} \quad (c > 0) \tag{16.6}$$

and

$$|\nabla u_-(x_0)| \leq \frac{C\delta}{\rho}. \tag{16.7}$$

To prove (16.6) we compare u_+ with the harmonic function U in $B^+_\rho(L,-\delta)$ which takes the boundary value γ on $\left\{(x_1,-\delta), |x_1-L|<\frac{1}{2}\rho\right\}$ and zero elsewhere. Clearly $u_+ \geq U \geq 0$ in $B^+_\rho(L,-\delta)$ and $u_+ = U = 0$ at x_0, so that $|\nabla u_+| \geq |\nabla U|$ at x_0. On the other hand, by scaling U and applying elliptic estimates, we find that $|\nabla U| \geq c/\rho$ at x_0, and therefore (16.6) holds.

To prove (16.7), consider the function

$$V = \frac{C_1}{\delta^{5/4}}(2\delta - x_2)(x_2 + 2\delta)(e^{\xi_1 - x_1} + e^{x_1 - \xi_2})$$

in

$$S = \{\xi_1 < x_1 < \xi_2, -\delta < x_2 < \delta\} \backslash B^+_\rho(L,-\delta).$$

Since $\lambda \leq C$ (C independent of ρ, δ),

$$\Delta V + \lambda V = \frac{C_1}{\delta^{5/4}}[-2 + (1+\lambda)(2\delta - x_2)(x_2 + 2\delta)](e^{\xi_1 - x_1} + e^{x_1 - \xi_2}) < 0$$

if δ is small. Further, $V > 0 = u_-$ on $\partial\Omega^-(u) \cap S$ and (by (16.4), (16.5)) at $x_1 = \xi_1$ and $x_1 = \xi_2$ if C_1 is large enough. Since λ is smaller than the principal eigenvalue of Δ in $\Omega^-(u) \cap S$, we can use a comparison theorem to deduce that $u_- \leq V$ in S; in particular

$$u_-(x_1, x_2) \leq \frac{C\delta^2}{\delta^{5/4}} e^{-1/\sqrt{\delta}} \leq \delta \text{ if } |x_1 - L| < 1. \tag{16.8}$$

Let W be the solution of $\Delta W + \lambda W = 0$ in

$$S_0 = \{|x_1 - L| < 2\delta, -\delta < x_2 < \delta\} \backslash B^+_\rho(L,-\delta)$$

with $W = u_-$ on ∂S_0. By (16.8) $0 \leq W \leq \delta$ on ∂S_0. Scaling W by $1/\delta$ and using a version of the maximum principle, we easily find that if δ is small enough then

$$0 \leq W \leq C\delta \text{ in } S_0.$$

If we scale W/δ by $1/\rho$ and apply elliptic estimates, we get

$$|\nabla(W/\delta)| \leq \frac{C}{\rho} \text{ at } x_0.$$

Since also $W \geq u_-$ in S_0, $|\nabla W| \geq |\nabla u_-|$ at x_0, and (16.7) follows.

Having proved (16.6), (16.7) we see that the free boundary condition $|\nabla u_+|^2 - |\nabla u_-|^2 = \mu^2$ is contradicted if δ is small enough and this proves the assertion (16.3). It follows that $\Omega^-(u)$ does not intersect both $B_r(0)$ and $B_r(z_0)$; for definiteness we shall suppose that $\Omega^-(u)$ does not intersect $B_r(0)$.

We now take $\delta \to 0$ and obtain a limit function u_0 in $B_r(0)$, which minimizes

$$\tilde{J}(v) \equiv \int_{B_r(0)} [|\nabla v|^2 + \mu_+^2 I_{\{v>0\}}]$$

subject to $v = \gamma$ on $\partial B_r(0)$, and $u_0 > 0$ throughout $B_r(0)$. If however μ_+ is sufficiently large, this is a contradiction since the minimizer must vanish on some disc.

Similarly one can show that for a domain Ω consisting of one disc connected to m disjoint discs by "thin" and "long" bridges, $\Omega^-(u)$ must have at least m components in which $u \equiv 0$ (provided $\mu_\pm$ are large enough).

Actually the above analysis can be carried out if the length of a bridge is just $\gg \log \frac{1}{\delta}$.

REMARK 16.1. The 3-dimensional radially symmetric plasma problem is to find (u, Γ_p, λ) satisfying:

$$u_{rr} + u_{zz} - \frac{1}{r} u_r = 0 \text{ in } \Omega_v = \{(r,z) \in \Omega; u(r,z) > 0\},$$

$$u_{rr} + u_{zz} - \frac{1}{r} u_r + \lambda u = v \text{ in } \Omega_p = \Omega \setminus \overline{\Omega}_v,$$

$$u = \gamma \text{ on } \partial\Omega,$$

$$u = 0 \text{ on } \Gamma_p,$$

$$\frac{|\nabla u_+|^2}{r^2} - \frac{|\nabla u_-|^2}{r^2} = \mu^2 \text{ on } \Gamma_p$$

where Ω is a bounded domain in $\mathbb{R}^2$. If we set $\tilde{\Omega} = \{x = (x', z) \in \mathbb{R}^5; (|x'|, z) \in \Omega\}$ and define a function $w(x)$ by

$$|x'|^2 w(x) = u(|x'|, z),$$

then w satisfies (2.1)-(2.4) in $\tilde{\Omega}$ and

$$|\nabla w_+|^2 - |\nabla w_-|^2 = \frac{\mu^2}{|x'|^2}$$

on the free boundary. All the results of the paper extend to the present case with one important exception: the verification of the full monotonicity condition in Theorem 11.3 (i.e., of assumption (iii)) for a minimizer of J_η. On one hand we cannot show that $\sup_{B_\sigma(x)} u$ is a subsolution in the (r, z) variables. On the other hand if we use the mapping $(|x'|, z) \to (x', z)$ then a rectangle about a free boundary point goes into a shell between two concentric cylinders, and this does not allow us to prove monotonicity in the desired directions e. Since we are unable to verify assumption (iii) of Theorem 11.3, we cannot establish that the free boundary is a Lipschitz graph.

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