ANNALI DELLA
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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 22, nº 3 (1995), p. 363-373
<http://www.numdam.org/item?id=ASNSP_1995_4_22_3_363_0>

Proper Mappings between Reinhardt Domains with an Analytic Variety on the Boundary

M. LANDUCCI - S. PINCHUK

Introduction

This paper gives a contribute to the study of proper holomorphic mappings

$$F : D \to D' \tag{1}$$

between bounded domains in $\mathbb{C}^2$, when D and D' are pseudoconvex Reinhardt domains. Many results have been achieved in this area when D and D' have a particular shape (e.g. $D = D' =$ ball see [1], D and D' pseudoellipsoids see [4], [3]) or when, assuming the smoothness of the domains, some restrictions on the set of the weakly pseudoconvex points are made (see [5], [8]).

In all the previous cases, in particular, the boundary of D and D' are not allowed to contain complex analytic varieties and the common statement is that, when $D = D'$, the map F is an automorphism.

Here, assuming the existence of a complex analytic variety on ∂D and without any smoothness assumption for the boundaries, it is proved that any $F = (F_1, F_2)$ (like in (1)) has both components which depend only on a single variable and when $D = D' \neq$ polydisc, F is of the form $F_1 = c_1 z$, $F_2 = c_2 w$ (this, in particular, generalizes the theorems in [6]). The precise statement is the following:

MAIN THEOREM. *Let D and D' be bounded pseudoconvex complete Reinhardt domains in $\mathbb{C}^2$ and*

$$F = (F_1, F_2) : D \to D'$$

a holomorphic proper mapping. Assume, furthermore, that there exist a complex analytic variety V and an open neighbourhood U of some $P \in \partial D$ with $V \cap U \subset \partial D$. Then:

Pervenuto alla Redazione il 16 Febbraio 1994.

a) F_1 *and* F_2 *depend only on a single variable* (*say* $F_1 = F_1(z)$, $F_2 = F_2(w)$);
b) *if* $D = D'$ *is not a polydisc, then*

$$F_1 = c_1 z, \quad F_2 = c_2 w$$

with $|c_1| = |c_2| = 1$.

The scheme of the proof is the following: first it is shown in Section 1 that the maximal boundary holomorphic foliation, generated by V, has a particular form and its boundary consists of one or two tori (disc or corona foliation); then (Section 2) the restriction of F to the leaves of the foliation is analysed and by a result on inner functions (Section 3), it is shown the existence of a polydisc $\Delta \subseteq D$ and a polydisc $\Delta' \subseteq D'$ such that F is a proper mapping from Δ onto Δ' (Section 4, Claim 4.1′) and part a) of the theorem is proved; finally, the second part of Section 4 is devoted to prove part b). The results of the paper have been announced at Convegno GNSAGA, Lecce October 1993, by the first author. At last the authors want to thank prof. G. Patrizio for his kindness in discussing the material of this paper.

1. - Holomorphic foliations on the boundary

Let D be a bounded complete Reinhardt domain in $\mathbb{C}^2$, that is a domain such that

$$(z, w) \in D \Rightarrow (tz, \tau w) \in D \text{ for any } |t|, \ |\tau| \leq 1.$$

Assume that:

$\mathcal{A}$: *there is a connected holomorphic variety* V *and a neighbourhood* U *of* $P \in \partial D$ *such that*

$$V \cap U \subset \partial D$$

Then the boundary set $e^{i\theta}V = \{(e^{i\theta_1}z, e^{i\theta_2}w), \ (z, w) \in V\}$ is a Levi flat set on ∂D; if, in addition, D is a pseudoconvex set or equivalently (see [7], pag. 120) its logarithmic image

$$\log|D| = \{(x, y) \in \mathbb{R}^2 : (e^x, e^y) \in D\}$$

is (in $\mathbb{R}^2$) a convex set, more can be said about the foliation in holomorphic curves that the boundary Levi flat set $e^{i\theta}V$ produces.

PROPOSITION 1.1. *Let* D *be a bounded complete pseudoconvex Reinhardt domain in* $\mathbb{C}^2$ *which fulfills* $\mathcal{A}$. *If* $V \cap U$ *consists of regular points of* V *not intersecting the coordinate hyperplanes, then* $\log|V \cap U|$ *is contained, in the* $(\log|z|, \log|w|)$ *plane, in a straight-line segment* (*of lenght not necessarily finite*).

PROOF. By contradiction suppose the existence of $P_0 \in V$ and a neighbourhood U of P_0 such that $\log|V \cap U|$ is not contained in a straight-line segment. Since $\log|D|$ is convex, there would exist a straight-line L such that

$$L \cap \overline{\log|D|} = \{\log|P_0|\}.$$

Take then the linear defining function of L, compose it by log and restrict it to V: call the resulting harmonic function g. The function g violates the maximum principle being negative and vanishing only at P_0. □

The above proposition has the following corollary:

COROLLARY 1.2. *Let D be as in Proposition* 1.1. *Then $e^{i\theta}V \cap \partial D$ is, for suitable real constants h, k, a, b, c, d, e, the set*

$$V(h,k) = \{|z|^h|w|^k = e,\ \ |z| \in [a,b],\ \ |w| \in [c,d]\}. \tag{1.3}$$

PROOF. By Proposition 1.1, any $P_0 \in \operatorname{Reg} V$ admits a boundary neighbourhood such that $\log|V \cap U|$ is contained in a straight-line segment: thus, locally, the foliation $e^{i\theta}V$ has the form (1.3). The unicity of a foliation implies, then, that it must hold globally. □

The above corollary reveals that, if $P \in V \subset \partial D$, then close to P the boundary of D is foliated by complex curves of the type (1.3). Call this foliation $\mathcal{F}(P)$; then:

DEFINITION 1.4. *A Foliation $\mathcal{F}(P)$ is called maximal if it is connected and no other connected foliation $\mathcal{G}(P)$ contains $\mathcal{F}(P)$.*

It is not worthless to observe that only two (topological) different foliations may exist:

a. if $hk = 0$, then the foliation is in *discs* (i.e. every leaf of the foliation is a disc);

b. if $hk \neq 0$, then the foliation is in *annuli* (i.e. every leaf of the foliation is an annulus).

This classification gives a splitting of the boundary $\partial\mathcal{F}$ of a maximal foliation:

case a.: $\partial\mathcal{F}$ is a *single torus*;

case b.: $\partial\mathcal{F}$ is composed by *two tori.*

2. - Proper mappings

Let D and D' be two complete bounded pseudoconvex Reinhardt domains in $\mathbb{C}^2$ and let

$$F : D \to D' \tag{2.1}$$

be a holomorphic proper mapping. Then (see [2]) F has a holomorphic extension to a neighbourhood $\mathcal{U}$ of $\overline{D}$ and it is, then, well defined on ∂D. Denote by $J(F)$ the holomorphic jacobian determinant of F and by

$$Z(J) = \{(z,w) \in \mathcal{U} : J(F) = 0\}.$$

PROPOSITION 2.2. *Let D satisfy condition $\mathcal{A}$ of Section* 1, *and let F be like in* (2.1). *If $\mathcal{F}$ is a maximal foliation on ∂D and $\mathcal{L}$ is one of its leaves not contained in $Z(J)$, then there exists a foliation $\mathcal{F}' \subset \partial D'$ and a leaf $\mathcal{L}' \in \mathcal{F}'$ such that*

$$F : \mathcal{L} \to \mathcal{L}'$$

is a proper mapping.

PROOF. Take $P_0 = (z_0, w_0) \in \mathcal{L} - Z(J)$; then there exists a neighbourhood U of P_0 such that F is biholomorphic in U and thus $F(U \cap \mathcal{L})$ is a complex variety V' on $\partial D'$: let $\mathcal{F}'$ be the relative maximal foliation associated to V' and $\mathcal{L}'$ be the leaf containing V'.

As $\mathcal{L} \cap Z(J)$ cannot disconnect $\mathcal{L}$ ($\mathcal{L} \not\subseteq Z(J)$, by hypothesis), we get that $F(\mathcal{L}) \subseteq \mathcal{L}'$. By absurd, assume, now, the existence of $P \in \partial\mathcal{L}$ such that $F(P)$ is an interior point of $\mathcal{L}$. If $P \notin Z(J)$ this would contradict the maximality of $\mathcal{F}$ because F is invertible close to P and hence $\mathcal{L}$ would extend beyond P. If $P \in Z(J) \cap \mathcal{L}$ there would exist at least one point $Q \in \partial\mathcal{L}$, arbitrarly close to P, such that $Q \notin Z(J)$ (otherwise $\mathcal{L} \equiv Z(J)$). The above argument applied to Q leads, in this case too, to an absurd. The thesis is proved. □

Proposition 2.2 allows to describe the behaviour of the proper mapping F on the boundary of a maximal foliation $\mathcal{F}$. We have:

PROPOSITION 2.3. *Let D satisfy condition $\mathcal{A}$ of Section* 1, *and let F be like in* (2.1). *If $\mathcal{F}$ is a maximal foliation on ∂D and $\mathcal{F}'$ is the maximal foliation given by Proposition* 2.2, *then*

$$F(\partial\mathcal{F}) \subseteq \partial\mathcal{F}'.$$

PROOF. Let $(z_0, w_0) \in \mathcal{L} \subseteq \mathcal{F}$ be such that

$$z_0 w_0 \neq 0, \quad \mathcal{L} \neq Z(J), \quad (z_0, w_0) \notin Z(J) \tag{2.4}$$

and $\mathcal{F}'$ be the maximal foliation containing $\mathcal{L}' = F(\mathcal{L})$. Call $\mathcal{L}_\theta$ the leaf passing throught $P_\theta = (e^{i\theta_1} z_0, e^{i\theta_2} w_0)$. It follows that, for θ_1 and θ_2 sufficiently small, condition (2.4) is still satisfied by P_θ and $\mathcal{L}_\theta$, so that, by the Proposition 2.2 we get

$$F(\partial\mathcal{L}_\theta) \subseteq \partial\mathcal{F}'. \tag{2.5}$$

As the boundary of the maximal foliation consists of two tori (see the end of Section 1) and F is holomorphic, condition (2.5) is sufficient to guarantee that $F(\partial \mathcal{F}) \subseteq \partial \mathcal{F}'$.

PROPOSITION 2.6. *Let F be like in* (2.1). *If $\mathcal{F}'$ is a foliation on $\partial D'$ and $F(z_0, w_0) \in \partial \mathcal{F}'$ then*

$$F(e^{i\theta_1} z_0, e^{i\theta_2} w_0) \in \partial \mathcal{F}'$$

for any θ_1, θ_2.

PROOF. Let $\mathcal{L}' \subset \mathcal{F}'$ be such that $F(z_0, w_0) \in \partial \mathcal{L}'$ and U an open neighbourhood of $\overline{D}$ where F extends holomorphically. Denote by V the analytic set given by

$$V = \{(z, w) \in U : F(z, w) \in \overline{\mathcal{L}}'\}$$

and by $\tilde{V}$ a component of $V \cap \overline{D}$ passing through $P_0 = (z_0, w_0)$. We have

$$F(\tilde{V}) = \overline{\mathcal{L}}'$$

because, since F is proper, $F(\partial D) = \partial D'$. The 1- Hausdorff measure of $F(\tilde{V})$ is not finite and this implies that (see [10], page 297) the 1-Hausdorff measure of $\tilde{V}$ too is not finite.

The consequence is that $\tilde{V}$ is a component of V and hence an analytic set lying on ∂D: this gives rise to a maximal foliation $\mathcal{F}$, containing (z_0, w_0) on its boundary (in particular $z_0 w_0 \neq 0$). The choice of a leaf $\mathcal{L} \in \mathcal{F}$ not contained in $Z(J)$, makes the proposition 2.3 applicable: the thesis then follows recalling that, for any θ_1, θ_2, $\left(e^{i\theta_1} z_0, e^{i\theta_2} w_0\right) \in \partial \mathcal{F}$. □

3. - Inner functions on the polydisc

Let us start recalling the following definition of inner function:

DEFINITION 3.1. *Let Δ^2 be the unit polydisc and let S be its Silov boundary. A holomorphic function on Δ^2, continuous on $\overline{\Delta}$, is an inner function for the polydisc if*

$$|h(z, w)| = 1, \text{ for } (z, w) \in S.$$

The class of inner functions can be described in a very precise way (see [9], chapter 5).

PROPOSITION 3.2. *Any inner function h for the polydisc has the following form:*

$$h(z, w) = z^h w^k \frac{\sum \alpha_{ij} z^{-i} w^{-j}}{\sum \overline{\alpha}_{ij} z^i w^j}$$

where $\sum \overline{\alpha}_{ij} z^i w^j$ is a polynomial not vanishing on $\overline{\Delta}^2$ and h and k are positive integers.

The shape of a holomorphic function with constant modulus on two different tori (roughly speaking inner for two different polydiscs) can be discovered by an application of the above proposition.

PROPOSITION 3.3. *Let $\tilde{\Delta}$ and $\tilde{\Delta}'$ be two different polydiscs in $\mathbb{C}^2$, with Silov boundaries $\tilde{S}$ and $\tilde{S}'$:*

$$\tilde{S} = \{|z| = a, \ |w| = b\}, \ \ \tilde{S}' = \{|z| = a', \ |w| = b'\},$$

and suppose that a holomorphic function g, continuous on $\overline{\Delta} \cup \overline{\Delta}'$, satisfies

$$|g(z,w)| = \alpha, \ for \ (z,w) \in \tilde{S}, \ \ |g(z,w)| = \alpha', \ for \ (z,w) \in \tilde{S}'$$

Then, assuming $a \neq a'$, there exist constants C, β, r, k and a holomorphic function G such that

$$g = Cz^{\beta}(z^r w)\,G(z^r w)$$

and $\beta = 0$ if and only if $\alpha = \alpha'$.

An analogous statement holds if we assume $b \neq b'$.

PROOF. Define $h(\varsigma, w) := \dfrac{1}{\alpha}\, g(a\varsigma, b\omega)$: then h is inner for Δ so that (by Proposition 3.2)

$$h(\varsigma,\omega) = \varsigma^h w^k \frac{\sum \alpha_{ij}\varsigma^{-i}\omega^{-j}}{\sum \overline{\alpha}_{ij}\varsigma^i\omega^j}.$$

This implies that

$$g(z,w) = \alpha\, \frac{1}{a^h b^k}\, z^h w^k \frac{\sum \alpha_{ij} a^i b^j z^{-i} w^{-j}}{\sum \overline{\alpha}_{ij} a^{-i} b^{-j} z^i w^j}. \tag{3.4}$$

Take $h'(\varsigma,\omega) := \dfrac{1}{\alpha'}\, g(a'\varsigma, b'\omega)$: by (3.4) we have

$$h'(\varsigma,\omega) = \frac{\alpha}{\alpha'}\, c^h d^k \varsigma^h \omega^k \frac{\sum \alpha_{ij} c^{-i} d^{-j} \varsigma^{-i}\omega^{-j}}{\sum \overline{\alpha}_{ij} c^i d^j \varsigma^i \omega^j}$$

where $c = \dfrac{a'}{a}$ and $d = \dfrac{b'}{b}$. But, by hypothesis, h' too is an inner function for Δ: hence, by Proposition 3.2, the following relations must hold:

$$\frac{\alpha}{\alpha'}\, c^{h-i} d^{k-j} \alpha_{ij} = c^i d^j \alpha_{ij} \tag{3.5}$$

for any i, j. Suppose $c \neq 1$ and solve (3.5), with respect to the integer i: we get that either $\alpha_{ij} \neq 0$ or

$$i = rj + s$$

$$r = -\frac{\log d}{\log c}, \quad s = \frac{h}{2} + \frac{1}{\log c^2}\left[\log \frac{\alpha}{\alpha'} + k \log d\right] =$$
$$= \frac{h}{2} + \frac{1}{2 \log c} \log \frac{\alpha}{\alpha'} - \frac{k}{2} r = \frac{1}{2}[h - \beta - kr].$$

Calling $A_j := \alpha_{rj+s,j}$, the expression (3.4) becomes

$$\begin{aligned} g(z,w) &= \alpha \frac{1}{a^h b^k} z^h w^k \frac{\sum_j A_j a^{rj+s} b^j z^{-rj-s} w^{-j}}{\sum_j \overline{A}_j a^{-rj-s} b^{-j} z^{rj+s} w^j} = \\ &= C z^{h-2s} w^k \frac{\sum_j A_j (a^{-r} b^{-1} z^r w)^{-j}}{\sum_j \overline{A}_j (a^{-r} b^{-1} z^r w)^j} = \\ &= C z^{\beta} (z^r w)^k G(z^r w). \end{aligned}$$

As

$$\beta = 0 \Leftrightarrow \alpha = \alpha'$$

if the function g assumes the same constant value on two different tori, then there exists r such that g is constant on any complex curve $z^r w =$ constant. If $c = 1$ an analogous argument can be performed, arguing with d that must be, by the hypothesis $(\tilde{\Delta} \neq \tilde{\Delta}')$, different from 1. □

4. - Proof of the main theorem

PROOF OF PART a). Let D and D' be bounded pseudoconvex domains in $\mathbb{C}^2$ (with no boundary smoothness hypothesis) and let

$$F : D \to D'$$

be a holomorphic proper mapping. Assume, furthermore, that D (and hence D') satisfies the condition

$\mathcal{A}$: *there is a connected holomorphic variety V and a neighbourhood U of $P \in \partial D$ such that*

$$V \cap U \subset \partial D.$$

As we saw, under these hypotheses, we can construct a maximal foliation $\mathcal{F} \subset \partial D$ and a maximal foliation $\mathcal{F}' \subset \partial D'$ such that

$$F(\partial \mathcal{F}) \subseteq \partial \mathcal{F}'.$$

Furthermore, as the boundary of a foliation may consist of one or two tori (see Section 1), there exists one torus $\mathcal{I}$ on ∂D and one torus $\mathcal{I}'$ on $\partial D'$ such that

$$F(\mathcal{I}) \subseteq \mathcal{I}'.$$

This condition, by the maximum principle for holomorphic functions, in particular says that if Δ and Δ' are the polydiscs (contained respectively in $\overline{D}$ and $\overline{D}'$) with Silov boundaries $\mathcal{I}$ and $\mathcal{I}'$, then

$$F = (F_1, F_2) : \overline{\Delta} \to \overline{\Delta}'. \tag{4.1}$$

CLAIM 4.1$'$. *The mapping F given by* (4.1) *is a holomorphic proper mapping between the polydiscs Δ and Δ', and hence the single components of F depends only on one variable.*

PROOF OF THE CLAIM. We first observe that if $P, Q \in F^{-1}(\mathcal{I}')$ then P, Q belong to the same torus and, hence, to $\mathcal{I}$. In fact, if not, by Proposition 3.6, there would exist two different tori on which $|F_1|$ and $|F_2|$ are constant: Proposition 3.3 then would imply the constancy, for a suitable real r, of F on the family of holomorphic curves of the type $z^r w =$ constant. This is impossible because F is proper.

Thus $F^{-1}(\mathcal{I}')$ must be a single torus and, as $F(\mathcal{I}) \subseteq \mathcal{I}'$, this torus has to be $\mathcal{I}$.

Assume now, by absurd, that $P_0 = (z_0, w_0) \in \Delta$ be such that $F(z_0, w_0) \in \partial\Delta'$. Denoting by F^{-1} the algebroidal corrispondence inverse of the mapping F,

$$F^{-1} = (G_1, G_2) : \Delta' \to D,$$

the existence of P_0 would imply (by the maximum principle for algebroidal correspondences and since $F^{-1}(\mathcal{I}') = \mathcal{S}$) the constancy of G_1 or G_2. Absurd. The claim has been proved. □

By the claim F_1 and F_2 are Blaschke products and each depends only on a single variable.

PROOF OF PART b). Assume, first, that the foliation $\mathcal{F}$ is a disc foliation, say

$$\mathcal{F} = \{|z| \le a,\ |w| = b\}.$$

Then F being proper, $\mathcal{F}'$ too is a disc foliation. We can, without loosing generality, suppose that

$$\mathcal{F}' = \{|z| \le a',\ |w| = b'\}$$

because, otherwise, we can either replace F by $F \circ F$ or rename $\mathcal{F}'$ and $\mathcal{F}'' = F(\mathcal{F}')$ by $\mathcal{F}'$ (with a possible symmetry $z \to w$, $w \to z$). In addition, by a suitable homotety, we can assume

$$a = b = 1.$$

By the maximality of $\mathcal{F}$ and the completeness of D, it follows that

$$a' = b' = 1$$

and thus, denoting by Δ the *unit polydisc*, the mapping F is a holomorphic self-proper mapping of Δ.

If $D \neq \Delta$, again by the maximality of $\mathcal{F}$ and the completeness of D, there exists a boundary point

$$(\varsigma_0, \omega_0) \in \partial D, \ |\varsigma_0|, \ |\omega_0| \neq 1.$$

Let, then, (z_0, w_0) be such that $F(z_0, w_0) = (\varsigma_0, \omega_0)$: by the claim, this implies that the moduli of F_1 and F_2 are constant (different from 1) on the torus generated by (z_0, w_0).

Proposition 3.3 is then applicable and this, jointly with the fact that the components of F depend only on one single variable, implies that, for some positive integers α and β,

$$F_1(z) = cz^\alpha, \ F_2 = kw^\beta \qquad (4.1)$$
$$|h|, \ |k| = 1.$$

The exponent α must be 1: in fact if r is the radius of the disc $\{w = 0\} \cap D$, by (4.1), we have

$$r^\alpha = r$$

with $r > 1$ (otherwise D should be a polydisc). Plugging $\alpha = 1$ in (4.1) we get, in particular, that the disc $D_z(\varsigma_0) = \{z = \varsigma_0\} \cap D$ has to be mapped properly onto itself and thus ($|\omega_0|$ being the radius of this disc)

$$|\omega_0|^\beta = |\omega_0| \neq 1$$

that is $\beta = 1$.

It remains to show the theorem under the assumption that $\mathcal{F}$ and $\mathcal{F}'$ be corona foliations (and that $F(\partial\mathcal{F}) \subseteq \partial\mathcal{F}'$). Let $\mathbb{T}_1$ and $\mathbb{T}_2$ (resp. $\mathbb{T}'_1$ and $\mathbb{T}'_2$) be the two tori such that $\partial\mathcal{F} = \mathbb{T}_1 \cup \mathbb{T}_2$ (resp. $\partial\mathcal{F}' = \mathbb{T}'_1 \cup \mathbb{T}'_2$) and assume $\mathbb{T}_1$ is the Silov boundary of Δ: Proposition 3.3 assures that it cannot happen that $F(\partial\mathcal{F}) \subseteq \mathbb{T}_1$ (or $\mathbb{T}_2$) because in this case F would not be proper; furthermore (Corollary 1.2) the polyradii (a', b') and (c', d') of $\mathbb{T}'_1$ and $\mathbb{T}'_2$ satisfy the conditions

$$a' \neq c', \ b' \neq d'.$$

Then, again by Proposition 3.3 jointly with the fact that the components of F depend each on a single variable, we get

$$F_1(z) = \chi z^\alpha, \ F_2 = \kappa w^\beta. \qquad (4.2)$$

Assume now that F is a proper mapping between Δ and the polydisc whose Silov boundary is $\mathbb{T}'_1$; then by (4.2)

$$|\chi| = a', \ |\kappa| = b'.$$

If, by absurd, $\alpha > 1$ then, denoting by if $r > 1$ the radius of the disc $\{w = 0\} \cap D$ and by ρ the radius of the disc $\{z = 0\} \cap D$, the following must hold:

$$a' = r^{1-\alpha} < 1, \ \ b' = \rho^{1-\beta}$$

and, as b' too cannot be strictly less than 1, it follows that $\beta = 1$ and consequently $b' = 1$. Recalling that D is complete and logarithmically convex, the condition that $(1, 1)$, $(a', 1) \in \partial D (a' < 1)$ means that

$$\{|z| \leq 1, \ \ |w| = 1\} \subset \partial D$$

(see next section) in contrast with $\mathcal{F}$ being a corona foliation. An analogous argument shows that $\beta = 0$. The proof is complete. □

5. - Appendix

This section is devoted to the proof of a property of pseudoconvex Reinhardt domains.

PROPOSITION 5.1. *Let D be a bounded complete Reinhardt domain in $\mathbb{C}^2$ and suppose that (b, a), $(b', a) \in \partial D$ with*

$$|b| > |b'|.$$

Then ∂D contains the complex analytic set

$$\{(z, w) \in \mathbb{C}^2 : |w| = |a|, \ \ |z| \in [|b'|, \ \ |b|]\}.$$

If, in addition, D is pseudoconvex then

$$\{(z, w) \in \mathbb{C}^2 : |w| = |a|, \ \ |z| \leq |b|\} \subset \partial D.$$

PROOF. Consider the open set:

$$A = \{(z, w) \in \mathbb{C}^2 : |z| > |b'|, \ \ |w| > |a|\}$$

then we claim that,

$$A \subseteq \mathbb{C}^2 \backslash D \tag{5.2}$$

In fact, otherwise, there would exist $\varsigma \in A \cap D$ and hence (b', a) would belong to D.

In particular (5.2) reveals that, $\mathbb{C}^2 \backslash D$ being closed,

$$\{(z, w) \in \mathbb{C}^2 : |z| > |b'|, \ \ |w| = |a|\} \subseteq \partial A \subseteq \mathbb{C}^2 \backslash D$$

and this, jointly with the fact that

$$\{(z, w) \in \mathbb{C}^2 : |w| = |a|, \ |z| \in [|b'|, \ |b|\} \subseteq \overline{D}$$

implies the first part of the thesis.

The completeness of D assures, furthermore, that

$$A' = \{|w| = |a|, \ |z| \leq |b|\} \subseteq \overline{D}.$$

To get the second part of the statement it is sufficient to remark that no $(z, w) \in A'$ can be an inner point because the straight line joining $(\log|b|, \log|a|)$ with $(\log|b'|, \log|a|)$ is a supporting line for the log-image of D.

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