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On the Dimension of the Adjoint Linear System for Threefolds

M.C. BELTRAMETTI - A.J. SOMMESE

Introduction.

Let $L^\wedge$ be a very ample line bundle on a smooth, n-dimensional, projective manifold, $X^\wedge$, i.e., assume that $L^\wedge \approx i^*\mathcal{O}_{\mathbb{P}^N}(1)$ for some embedding $i : X^\wedge \to \mathbb{P}^N$. In [S4] it is shown that for such pairs, $(X^\wedge, L^\wedge)$, the Kodaira dimension of $K_{X^\wedge} + (n-2)L^\wedge$ is nonnegative, i.e., there exists some positive integer t such that $h^0(t(K_{X^\wedge} + (n-2)L^\wedge)) \geq 1$, except for a short and well understood list of degenerate examples. It is moreover shown that except for this short list there is a reduction "1-st red." morphism $r : X^\wedge \to X$ expressing $X^\wedge$ as the blowup of a projective manifold X at a finite set B, and such that:

a) $K_{X^\wedge} + (n-1)L^\wedge \cong r^*(K_X + (n-1)L)$ where $L := (r_*L^\wedge)^{**}$ is an ample line bundle, and $K_X + (n-1)L$ is ample;

b) $K_X + (n-2)L$ is nef, i.e., $(K_X + (n-2)L) \cdot C \geq 0$ for every effective curve $C \subset X$.

The hope that except for a few examples, $K_X + (n-2)L$ is not just nef, but spanned at all points by global sections is supported by a number of results:

1. The analogous result is true for $K_{X^\wedge} + (n-1)L^\wedge$ (see [SV] for the history in this case);
2. In [S5] the pairs $(X^\wedge, L^\wedge)$ with the Kodaira dimension of $K_{X^\wedge} + (n-2)L^\wedge$ negative are characterized by $h^0(K_{X^\wedge} + (n-2)L^\wedge) = 0$, and in particular if $K_X + (n-2)L \cong K_{X^\wedge} + (n-2)L^\wedge - \sum_i E_i$, where the E_i's are the exceptional divisors of r, is nef it has a nontrivial global section;
3. If $K_X + (n-2)L$ is nef then $2(K_X + (n-2)L)$ is spanned by global sections at all points [S5];
4. In [BSS] it is shown that if $X^\wedge$ has no rational curves (e.g., if $X^\wedge$ is hyperbolic in the sense of Kobayashi, or if the cotangent bundle $\mathcal{T}^*_{X^\wedge}$ is nef) and if the degree $c_1(L^\wedge)^n \geq 850$, then $K_{X^\wedge} + (n-2)L^\wedge$ is spanned by global sections at all points.

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There is one known counterexample (see [LPS]) of a Del Pezzo threefold of degree 27 with $K_X + L$ ample but not everywhere spanned. A search for other counterexamples led us to the following surprisingly strong result, which would in fact not be implied by spannedness of $K_{X^\wedge} + (n-2)L^\wedge$. (Note that $h^0(K_{X^\wedge} + (n-2)L^\wedge) = h^0(K_X + (n-2)L)$.)

THEOREM. *Let $L^\wedge$ be a very ample line bundle on an n-dimensional projective manifold $X^\wedge$, with $n \geq 3$. If there exist $n-3$ elements $\{A_1, \dots, A_{n-3}\} \subset |L^\wedge|$ meeting transversally in a 3-fold of nonnegative Kodaira dimension, e.g., if the Kodaira dimension of $K_{X^\wedge} + (n-3)L^\wedge$ is nonnegative, then $h^0(K_{X^\wedge} + (n-2)L^\wedge) \geq 5$ with equality only if $n = 3$, and $(X^\wedge, L^\wedge)$ is a degree 5 hypersurface of $\mathbb{P}^4$.*

We also show in Theorem (1.2) that if the Kodaira dimension of $K_{X^\wedge} + (n-2)L^\wedge$ is at least 3, then $h^0(K_{X^\wedge} + (n-2)L^\wedge) \geq 2$.

The method of proof is to use the doublepoint inequality for 3-folds in projective space, Tsuji's inequality, Miyaoka's bound for the number of -2 curves on a surface of general type, Noether inequality $K_S^2 \geq p_g(S) - 4$ for hyperplane sections S, Lefschetz theory and the major results on the adjunction theory of 3-folds.

We refer to [BBeS] for a study of the dimension of the adjoint linear system in the case of quadric fibrations over surfaces.

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0. - Background material.

We work over the complex numbers $\mathbb{C}$. Through the paper we deal with smooth, projective varieties V. We denote by $\mathcal{O}_V$ the structure sheaf of V and by K_V the canonical bundle. For any coherent sheaf $\mathcal{F}$ on V, $h^i(\mathcal{F})$ denotes the complex dimension of $H^i(V, \mathcal{F})$.

Let L be a line bundle on V. L is said to be *numerically effective* (*nef*, for short) if $L \cdot C \geq 0$ for all effective curves C on V. L is said to be *big* if $\kappa(L) = \dim V$, where $\kappa(L)$ denotes the Kodaira dimension of L. If L is nef then this is equivalent to be $c_1(L)^n > 0$, where $c_1(L)$ is the first Chern class of L and $n = \dim V$.

(0.1) The notation used in this paper are standard from algebraic geometry. Let us only fix the following.

$\approx$ (respectively $\sim$), linear (respectively numerical) equivalence of line bundles;

$\chi(L) = \sum_i (-1)^i h^i(L)$, the Euler characteristic of a line bundle L;

$|L|$, the complete linear system associated with a line bundle L on a variety V,

$\Gamma(L) = H^0(L)$, the space of the global sections of L. We say that L is spanned if it is spanned at all points of V by $\Gamma(L)$;

$e(V) = c_n(V)$, the topological Euler characteristic of V, for V smooth, where $c_n(V)$ is the n-th Chern class of the tangent bundle of V. If V is a surface, $e(V) = 12\chi(\mathcal{O}_V) - K_V \cdot K_V$;

$\kappa(V) := \kappa(K_V)$, the Kodaira dimension, for V smooth.

Line bundles and divisors are used with little (or no) distinction. Hence we shall freely switch from the multiplicative to the additive notation and vice versa.

(0.2) For a line bundle L on a variety V of dimension n the sectional genus, $g(L) = g(V, L)$, of (V, L) is defined by $2g(L) - 2 = (K_V + (n-1)L) \cdot L^{n-1}$. Note that if $|L|$ contains $n-1$ elements $H_1, \ldots, H_{n-1}$ meeting in a reduced irreducible curve C, then $g(L) = g(C) = 1 - \chi(\mathcal{O}_C)$, the arithmetic genus of C.

(0.3) Reduction (see e.g., [S4], (0.5), [BFS], (0.2) and [BS], (3.2), (4.3)). Let $(X^\wedge, L^\wedge)$ be a smooth projective variety of dimension $n \geq 2$ polarized with a very ample line bundle $L^\wedge$. A smooth polarized variety (X, L) is called a *(first) reduction* of $(X^\wedge, L^\wedge)$ if there is a morphism $r : X^\wedge \to X$ expressing $X^\wedge$ as the blowing up of X at a finite set of points, B, such that $L := (r_* L^\wedge)^{**}$ is ample and $L^\wedge \approx r^*L - [r^{-1}(B)]$ or, equivalently, $K_{X^\wedge} + (n-1)L^\wedge \approx r^*(K_X + (n-1)L)$.

Note that there is a one to one correspondence between smooth divisors of $|L|$ which contain the set B and smooth divisors of $|L^\wedge|$.

Except for an explicit list of well understood pairs $(X^\wedge, L^\wedge)$ (see [S4], [SV], [BS]) we can assume:

a) $K_{X^\wedge} + (n-1)L^\wedge$ is spanned and big, and $K_X + (n-1)L$ is very ample. Note that in this case this reduction, (X, L), is unique up to isomorphism. We will refer to it as *the first reduction* of $(X^\wedge, L^\wedge)$.

b) $K_X + (n-2)L$ is nef and big, for $n \geq 3$.

Then from the Kawamata-Shokurov base point free theorem (see [KMM], §3) we know that $|m(K_X + (n-2)L)|$, for $m \gg 0$, gives rise to a morphism $\varphi : X \to X'$, with connected fibers and normal image. Thus there is an ample line bundle $\mathcal{K}'$ on X' such that $K_X + (n-2)L \approx \varphi^*\mathcal{K}'$. The pair $(X', \mathcal{K}')$ is known as the *second reduction* of $(X^\wedge, L^\wedge)$. The morphism φ is very well behaved (see e.g., [BFS], (0.2) for a summary of the results). Furthermore X

has terminal, 2-Gorenstein (i.e., $2K_X$ is a line bundle) isolated singularities and $\mathcal{K}' \approx K_{X'} + (n-2)L'$, where $L' := (\varphi_* L)^{**}$ is a 2-Cartier divisor such that $2L \approx \varphi^*(2L') - \mathcal{D}$ for some effective Cartier divisor $\mathcal{D}$ on X which is φ-exceptional (see [BFS], (0.2.4), [BS], (4.2), (4.4), (4.5)). For definition and properties of terminal singularities and for a few facts from Mori theory we use in the sequel, such as the Mori Cone Theorem and the definitions of extremal ray and contraction of an extremal ray we also refer to [KMM].

(0.3.1) ([S5], (0.3.1)). We will use the fact that $\Gamma(aK_{X^\wedge} + bL^\wedge) \cong \Gamma(aK_X + bL)$ for integers a, b with $b \leq a(n-1)$.

(0.4) Pluridegrees. Let $(X^\wedge, L^\wedge)$, (X, L) be as in (0.3) with $n = 3$. Define the *pluridegrees*, for $j = 0, 1, 2, 3$, by

$$d_j^\wedge := (K_{X^\wedge} + L^\wedge)^j \cdot L^{\wedge 3-j} \text{ and } d_j := (K_X + L)^j \cdot L^{3-j}.$$

If γ denotes the number of points blown up under $r : X^\wedge \to X$, then because $K_{X^\wedge} + L^\wedge \cong K_X + L + \sum_i E_i$, the invariants $d_j^\wedge, d_j$ are related by

$$d_0^\wedge = d_0 - \gamma; \; d_1^\wedge = d_1 + \gamma; \; d_2^\wedge = d_2 - \gamma; \; d_3^\wedge = d_3 + \gamma.$$

We put $d^\wedge := d_0^\wedge$, $d := d_0$. If $K_X + L$ is nef, by the generalized Hodge index theorem (see e.g., [BBS], (0.15), [F], (1.2)) one has

(0.4.1) $$d_1^2 \geq dd_2 \text{ and } d_2^2 \geq d_1 d_3$$

and the parity Lemma (1.4) of [BBS] says that

(0.4.2) $$d \equiv d_1 \operatorname{mod}(2); \; d_2 \equiv d_3 \operatorname{mod}(2).$$

Moreover if $K_X + L$ is nef and big the numbers d_j are positive.

If $(X^\wedge, L^\wedge)$ has a second reduction, $(X', \mathcal{K}')$, with $\mathcal{K}' \approx K_{X'} + L'$, we can also define

$$d_j' := \mathcal{K}'^j \cdot L'^{3-j}, \; j = 0, 1, 2, 3, \; d' := d_0'.$$

We will use the fact that

(0.4.3) $$d_2 = d_2'; \; d_3 = d_3'.$$

To see this, let $\varphi : X \to X'$ be the second reduction morphism, recall that $2L \approx \varphi^*(2L') - \mathcal{D}$ for some effective Cartier divisor $\mathcal{D}$ which is φ-exceptional (see (0.3)) and compute

$$d_3 = (K_X + L)^3 = (\varphi^* \mathcal{K}')^3 = \mathcal{K}'^3 = d_3';$$

$$2d_2 = 2(K_X + L)^2 \cdot L = \varphi^* \mathcal{K}' \cdot \varphi^* \mathcal{K}' \cdot (\varphi^*(2L') - \mathcal{D}) = 2\mathcal{K}' \cdot \mathcal{K}' \cdot L' = 2d_2'.$$

(0.5) Double point formula. We need the following result (see also [BBS], (2.11.4)).

(0.5.1) THEOREM. *Let $(X^\wedge, L^\wedge)$ be a smooth projective 3-fold, polarized with a very ample line bundle $L^\wedge$. Let $N := h^0(L^\wedge) - 1$. Let $d_j^\wedge$, $j = 0,1,2,3$, be the pluridegrees of $(X^\wedge, L^\wedge)$ as in* (0.4). *Let $S^\wedge$ be a smooth element of $|L^\wedge|$. Then*

$$e(X^\wedge) - 48\chi(\mathcal{O}_{X^\wedge}) + 84\chi(\mathcal{O}_{S^\wedge}) - 11d_2^\wedge - 17d_1^\wedge - d_3^\wedge + d^\wedge(d^\wedge - 20) \geq 0,$$

with equality if $N \leq 6$.

PROOF. We can assume that $X^\wedge \subset \mathbb{P}^N$ with $N \geq 6$ by using the natural inclusion $\mathbb{P}^a \subset \mathbb{P}^6$ of a linear $\mathbb{P}^a$ when $a \leq 5$. The formula is simply a particular case of the general formula (I, 37), Section D, p. 313 of [K]. It should be noted that the virtual normal bundle, $\mathcal{V}$, in that formula is defined in our situation by the exact sequence

$$0 \to \mathcal{T}_{X^\wedge} \to p^*\mathcal{T}_{\mathbb{P}^6} \to \mathcal{V} \to 0$$

where $p : X^\wedge \to \mathbb{P}^6$ is the restriction to $X^\wedge$ of the projection from a general $\mathbb{P}^{N-7}$ if $N > 6$ and $\mathcal{V} = \mathcal{N}_{X^\wedge}^{\mathbb{P}^6}$, the usual normal bundle, if $N = 6$. □

The following is a consequence of the double point formula above.

(0.5.2) PROPOSITION. *Let $(X^\wedge, L^\wedge)$ be a smooth projective 3-fold, polarized with a very ample line bundle, $L^\wedge$. Let (X, L) and $r : X^\wedge \to X$ be the first reduction and first reduction map respectively. As in* (0.4), *let $d_j^\wedge$, d_j, $0 \leq j \leq 3$, be the pluridegrees of $(X^\wedge, L^\wedge)$ and (X, L) respectively. Let γ be the number of points blown up by r. Let $S^\wedge$ be a smooth element in $|L^\wedge|$. Then*

$$44h^0(K_{X^\wedge} + L^\wedge) + 58\chi(\mathcal{O}_{S^\wedge}) + 2h^0(K_{X^\wedge}) + 4 \geq 12d_2 + 17d_1 + d_3 + (20 - d^\wedge)d^\wedge + 5\gamma.$$

PROOF. Let $S \in |L|$ be the smooth image of $S^\wedge$. Since $h^0(K_{X^\wedge} + L^\wedge) = h^0(K_X + L)$, $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{S^\wedge})$, $h^0(K_X) = h^0(K_{X^\wedge})$, it suffices to prove the formula with $h^0(K_{X^\wedge} + L^\wedge)$, $\chi(\mathcal{O}_{S^\wedge})$, $h^0(K_{X^\wedge})$ replaced with $h^0(K_X + L)$, $\chi(\mathcal{O}_S)$, $h^0(K_X)$ respectively.

Since $\chi(\mathcal{O}_{X^\wedge}) = \chi(\mathcal{O}_X) = \chi(\mathcal{O}_S) - h^0(K_X + L)$, the double point formula (0.5.1) gives

$$e(X^\wedge) + 48h^0(K_X + L) + 36\chi(\mathcal{O}_S) \geq 11d_2^\wedge + 17d_1^\wedge + d_3^\wedge + (20 - d^\wedge)d^\wedge. \tag{0.5.2.1}$$

Let $q := h^1(\mathcal{O}_X) = h^1(\mathcal{O}_S)$, $p_g := p_g(X) = h^0(K_X)$ and $p_g(S) = h^0(K_S)$. Let $h^{p,q} := h^{p,q}(X^\wedge) = h^q(\Omega^p_{X^\wedge})$ be the Hodge numbers. Recall that $h^{p,q} = h^{q,p}$ and the Serre duality $h^{p,q} = h^{3-p,3-q}$. Let $b_j := b_j(X^\wedge) = \sum_{j=p+q} h^{p,q}$ be the j-th Betti number of $X^\wedge$. Using (0.5.2.1) it is enough to prove

$$\begin{aligned} &- e(X^\wedge) - 4h^0(K_X + L) + 22\chi(\mathcal{O}_S) + 2p_g + 4 \\ &\geq d_2 + 11(d_2 - d_2^\wedge) + 17(d_1 - d_1^\wedge) + (d_3 - d_3^\wedge) + 5\gamma = d_2 - 2\gamma. \end{aligned} \tag{0.5.2.2}$$

Now

(0.5.2.3) $e(X^\wedge) := 1 - b_1 + b_2 - b_3 + b_4 - b_5 + 1 = 2 - 4q + 2h^{1,1} + 4h^{0,2} - 2p_g - 2h^{1,2}$.

Note that the exact sequence

$$0 \to K_X \to K_X \otimes L \to K_S \to 0$$

gives $h^{0,2} = p_g(S) - h^0(K_X + L) + p_g$. Thus (0.5.2.3) yields

(0.5.2.4) $$e(X^\wedge) + 4h^0(K_X + L) = 2 - 4q + 2h^{1,1} + 4p_g(S) + 2p_g - 2h^{1,2}.$$

Now

$$e(S^\wedge) := 1 - b_1(S^\wedge) + b_2(S^\wedge) - b_3(S^\wedge) + 1 = 2 - 4q + 2p_g(S) + h^{1,1}(S^\wedge),$$

and hence

$$2 - 4q + 2p_g(S) + h^{1,1}(S^\wedge) = 12\chi(\mathcal{O}_S) - K_{S^\wedge}^2 = 12\chi(\mathcal{O}_S) - K_S^2 + \gamma.$$

Therefore

$$4p_g(S) = 24\chi(\mathcal{O}_S) - 2K_S^2 + 2\gamma - 4 + 8q - 2h^{1,1}(S^\wedge).$$

Substituting in (0.5.2.4) we find

$$\begin{aligned} &e(X^\wedge) + 4h^0(K_X + L) - 2p_g \\ &= -2 + 4q - 2h^{1,1}(S^\wedge) + 2h^{1,1} - 2h^{1,2} + 24\chi(\mathcal{O}_S) - 2K_S^2 + 2\gamma, \end{aligned}$$

which is equivalent to

(0.5.2.5) $$\begin{aligned} &e(X^\wedge) + 4h^0(K_X + L) - 22\chi(\mathcal{O}_S) - 2p_g - 4 + d_2 - 2\gamma \\ &= -2(h^{1,2} - q) - 2(h^{1,1}(S^\wedge) - h^{1,1}) - (K_S^2 - 2p_g(S) + 4). \end{aligned}$$

By the hard Lefschetz theorem it follows that $q \le h^{1,2}$ (see [ShS], (2.73), p. 47) and by the Lefschetz theorem on hyperplane sections (see [GH], p. 157) one has $h^{1,1} \le h^{1,1}(S^\wedge)$. Furthermore $K_S^2 \ge 2p_g(S) - 4$ by the Noether inequality. Thus (0.5.2.5) gives (0.5.2.2) and we are done. □

The following is another special case of the double point formula.

(0.5.3) LEMMA ([Hr], p. 434, [BBS], (0.11)). *Let $(X^\wedge, L^\wedge)$ be as in* (0.3) *with $n = 3$ and let $S^\wedge$ be a smooth element of $|L^\wedge|$. Assume that $\Gamma(L^\wedge)$ embeds $X^\wedge$ in $\mathbb{P}^N$ with $N \ge 5$. Then*

$$d^{\wedge 2} - 5d^\wedge - 10(g(L^\wedge) - 1) + 12\chi(\mathcal{O}_{S^\wedge}) \ge 2K_{S^\wedge} \cdot K_{S^\wedge}$$

with equality if $N = 5$.

(0.6) Tsuji inequality (see [S5], §1, [T], §5). Let $(X^\wedge, L^\wedge)$, (X, L) be as in (0.3) with $n = 3$ and let S be a smooth element of $|L|$. Then we have

$$(K_X + L)^3 + \frac{8}{3} K_S \cdot L_S \leq 32(2h^0(K_X + L) - \chi(\mathcal{O}_S))$$

or

$$h^0(K_X + L) \geq \frac{d_3}{64} + \frac{d_1}{24} + \frac{\chi(\mathcal{O}_S)}{2}.$$

For reader's convenience we give here the argument to show how the inequalities above follow from Tsuji's inequality, the log version of the usual Yau inequality. From the exact sequence

$$0 \to K_X \to K_X \otimes L \to K_S \to 0$$

we have

$$2h^0(K_X + L) - \chi(\mathcal{O}_S) = 2\chi(K_X + L) - \chi(\mathcal{O}_S) = \chi(\mathcal{O}_S) - 2\chi(\mathcal{O}_X).$$

By Riemann-Roch

$$c_2(X) \cdot K_X = -24\chi(\mathcal{O}_X) \text{ and } e(S) = c_2(X) \cdot L + (K_X + L) \cdot L^2.$$

Thus, since $12\chi(\mathcal{O}_S) = e(S) + d_2$, we find

$$2h^0(K_X + L) - \chi(\mathcal{O}_S) = \frac{1}{12}((K_X + L)^2 \cdot L + (K_X + L) \cdot L^2 + c_2(X) \cdot (K_X + L)).$$

Now, Tsuji's inequality for $\Omega^1_X(\log S)$ gives

$$(K_X + L) \cdot (c_2(X) + K_X \cdot L + L^2) \geq \frac{3}{8}(K_X + L)^3.$$

Therefore

$$2h^0(K_X + L) - \chi(\mathcal{O}_S) \geq \frac{1}{12}\left(\frac{3}{8}(K_X + L)^3 + (K_X + L) \cdot L^2\right)$$

or, equivalently,

$$32(2h^0(K_X + L) - \chi(\mathcal{O}_S)) \geq (K_X + L)^3 + \frac{8}{3} K_S \cdot L_S.$$

(0.7) Castelnuovo's bound. Let $(X^\wedge, L^\wedge)$ be as in (0.3) with $n = 3$. Assume that $|L^\wedge|$ embeds $X^\wedge$ in a projective space $\mathbb{P}^N$, $N \geq 4$, and let $d^\wedge := L^{\wedge 3}$. Let $S^\wedge$ be a smooth element of $|L^\wedge|$ and $C^\wedge$ the smooth curve obtained as the transversal intersection of two general members of $|L^\wedge|$. Then $g(L^\wedge) = g(C^\wedge)$ and Castelnuovo's Lemma (see e.g., [H], Theorem 3.7) reads

$$(0.7.1) \qquad g(C^\wedge) \leq \left[\frac{d^\wedge - 2}{N-3}\right]\left(d^\wedge - N + 2 - \left(\left[\frac{d^\wedge - 2}{N-3}\right] - 1\right)\frac{N-3}{2}\right)$$

where $N = h^0(L^\wedge) - 1$ and $[x]$ means the greatest integer $\leq x$.

(0.8) LEMMA (Lefschetz theorem in the singular case). *Let V be an irreducible, normal variety and D an ample effective Cartier divisor on V such that* $\mathrm{Sing}(V) \subset D$ *and* $\dim V \geq 3$. *Then the restriction map* $\mathrm{Pic}(V) \to \mathrm{Pic}(D)$ *is injective.*

PROOF. From the exponential exact sequences for D, X we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^1(V,\mathbb{Z}) & \to & H^1(\mathcal{O}_V) & \to & H^1(\mathcal{O}_V^*) & \to & H^2(V,\mathbb{Z}) \\
\alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow \\
H^1(D,\mathbb{Z}) & \to & H^1(\mathcal{O}_D) & \to & H^1(\mathcal{O}_D^*) & \to & H^2(D,\mathbb{Z})
\end{array}$$

Note that under the assumption $\mathrm{Sing}(V) \subset D$, the usual Lefschetz theorem holds true to say that α is an isomorphism and δ is injective. Note also that β is an injection since $h^1(-D) = 0$ by Kodaira vanishing. Thus a standard diagram chase shows that γ is injective. So we are done. □

We also need the following technical fact.

(0.9) PROPOSITION. *Let X be an irreducible, normal variety with at most rational singularities, and with* $\dim X \geq 3$ *and* $\mathrm{codSing}(X) \geq 3$. *Let L be an ample line bundle on X. Let $\mathcal{L}$ be a line bundle on X such that there are arbitrarily large integers N with $\mathcal{L}_A \sim \mathcal{O}_A$ for a general A in $|NL|$. Then $\mathcal{L} \sim \mathcal{O}_X$.*

PROOF. Let x be a general point of X and let I_x be the ideal sheaf of x in X. Let J be the ideal sheaf of $\mathrm{Sing}(X)$ in X. We can take N arbitrarily large such that $h^1(NL \otimes J \otimes I_x^{\otimes 2}) = 0$. This shows that $|NL \otimes J|$ gives a map which is an embedding in a neighbourhood of x. Therefore $NL \otimes J$ is big and spanned off $\mathrm{Sing}(X)$. Then it is a general fact (see e.g., [Hr], Chap. II, 7.17.3) that there exists a desingularization $p : \tilde{X} \to X$ with a spanned line bundle $\tilde{L}$ on $\tilde{X}$ such that $\tilde{L} \approx p^*(NL) - Z$ for some effective divisor Z on $\tilde{X}$ and with $p_*(\tilde{L}) \cong NL \otimes J$. Since $NL \otimes J$ is big, $\tilde{L}$ is also big.

Since X has rational singularities, the Kawamata-Viehweg vanishing theorem and the Serre duality apply to give $h^1(-A) = h^2(-A) = 0$. Therefore

$$H^1(\mathcal{O}_X) \cong H^1(\mathcal{O}_A).$$

Similarly, since $\tilde{L}$ is spanned and big, $H^1(\mathcal{O}_{\tilde{X}}) \cong H^1(\mathcal{O}_{\tilde{A}})$. Since X has rational singularities we also have $H^1(\mathcal{O}_{\tilde{X}}) \cong H^1(\mathcal{O}_X)$ and therefore

$$H^1(\mathcal{O}_{\tilde{A}}) \cong H^1(\mathcal{O}_A).$$

Consider the exact commutative diagram, given by the exponential sequences for A, $\tilde{A}$, X,

$$\begin{array}{ccccc}
H^1(\mathcal{O}_{\tilde{A}}) & \xrightarrow{\tilde{\psi}} & \mathrm{Pic}(\tilde{A}) & \xrightarrow{\tilde{\varphi}} & H^2(\tilde{A},\mathbb{Z}) \\
p^*\Big\| & & p^*\uparrow & & p^*\uparrow \\
H^1(\mathcal{O}_A) & \xrightarrow{\psi} & \mathrm{Pic}(A) & \xrightarrow{\varphi} & H^2(A,\mathbb{Z}) \\
i^*\Big\| & & i^*\uparrow & & i^*\uparrow \\
H^1(\mathcal{O}_X) & \xrightarrow{\alpha} & \mathrm{Pic}(X) & \xrightarrow{\beta} & H^2(X,\mathbb{Z})
\end{array}$$

where i denotes the inclusion $i : A \hookrightarrow X$. Since $\mathcal{L}_A \sim \mathcal{O}_A$ one has $p^*\mathcal{L}_A \sim \mathcal{O}_{\tilde{A}}$ on $\tilde{A}$. This implies that $\tilde{\varphi}(mp^*\mathcal{L}_A) = 0$ in $H^2(\tilde{A},\mathbb{Z})$ for some positive integer m. Therefore $mp^*\mathcal{L}_A = \tilde{\psi}(\tilde{b})$ for some $\tilde{b} \in H^1(\mathcal{O}_{\tilde{A}})$. Since $\tilde{b} = p^*b$ for some $b \in H^1(\mathcal{O}_A)$, we conclude that

$$p^*(m\mathcal{L}_A - \psi(b)) = 0 \text{ in } \mathrm{Pic}(\tilde{A}).$$

Since X is Cohen-Macaulay and codSing$(X) \geq 3$, A is also Cohen-Macaulay and codSing$(A) \geq 2$. Then A is normal. Since $\tilde{A}$ is smooth and p is birational it thus follows that

$$m\mathcal{L}_A - \psi(b) = 0 \text{ in } \mathrm{Pic}(A),$$

or, since $\mathcal{L}_A = i^*\mathcal{L}$, $b = i^*b'$ for some $b' \in H^1(\mathcal{O}_X)$,

$$i^*(m\mathcal{L} - \alpha(b')) = 0 \text{ in } \mathrm{Pic}(A).$$

Since i^* is an injection by Lemma (0.8), we conclude that $m\mathcal{L} - \alpha(b') = 0$ in Pic(X) and hence $\beta(m\mathcal{L}) = 0$ in $H^2(X,\mathbb{Z})$. This implies that $m\mathcal{L}$, and hence $\mathcal{L}$, is numerically equivalent to $\mathcal{O}_X$. $\square$

(0.10) Threefolds of log-general type. Let $(X^\wedge, L^\wedge)$, (X, L) be as in (0.3) with $n = 3$ and let $d_j^\wedge$, d_j, $j = 0, 1, 2, 3$, the pluridegrees as in (0.4). We say that $(X^\wedge, L^\wedge)$ is of *log-general type* if $K_X + L$ is nef and big. Hence in particular the second reduction $(X', \mathcal{K}')$, $\varphi : X \to X'$, of $(X^\wedge, L^\wedge)$ exists and the numbers d_j are positive in this case.

Let $S^\wedge$ be a smooth element of $|L^\wedge|$ and S the corresponding smooth surface in $|L|$. Then by the adjunction formula K_S is nef. Furthermore K_S is also big since some multiple of K_S is the pullback of some ample divisor under the restriction of φ to S. Then S is a minimal surface of general type. Hence we have

$$d_2 = K_S \cdot K_S < 9\chi(\mathcal{O}_S). \tag{0.10.1}$$

The Miyaoka inequality yields $d_2 \leq 9\chi(\mathcal{O}_S)$. Note that the equality cannot happen. Otherwise S is a ball quotient and hence a $K(\pi,1)$, which would contradict [S1], (1.3).

Assume that $\kappa(X) \geq 0$. Then from [S2], (1.5) and (3.1) we know that

$$d_3 \geq d_2 \geq d_1 \geq d \tag{0.10.2}$$

and

$$8\chi(\mathcal{O}_X) \leq d_2 - d. \tag{0.10.3}$$

Note that if $\kappa(X) \geq 0$, then $(X^\wedge, L^\wedge)$ is of log-general type. Indeed, if $h^0(tK_X) > 0$ for some positive integer t, then $t(K_X + L)$ gives a birational embedding, given on a Zariski open set by sections of $\Gamma(L)$, and thus $\kappa(K_X + L) = 3$. □

We finally need the following general fact.

(0.11) LEMMA. *Let V be a smooth connected variety, L an ample and spanned line bundle and $\mathcal{L}$ any line bundle on V. Let A be a general member of $|L|$. Then $h^0(\mathcal{L}_A) \geq 2$ if $h^0(\mathcal{L}) \geq 2$.*

PROOF. Since $h^0(\mathcal{L}) \geq 2$ we can take two independent sections s, $t \in H^0(\mathcal{L})$. Let D_s, D_t be the divisors defined by s, t. Note that $A \cap D_s \neq A$ and $A \cap D_t \neq A$, since otherwise all $A \in |L|$ would contain either D_s or D_t, contradicting the spannedness assumption. Note also that $A \cap D_s \neq A \cap D_t$ since otherwise we would have equality for all $A \in |L|$ and hence $D_s = D_t$ since L is spanned. This shows that the restrictions s_A, t_A are independent, so we are done. □

Throughout this paper we work under the assumption $n = 3$. Remark (3.4) shows how to reduce the case when $n \geq 4$ to the case $n = 3$.

For any further background material we refer to [S5] and [BS].

1. - The log-general type case.

Let $(X^\wedge, L^\wedge)$ be a smooth threefold polarized with a very ample line bundle $L^\wedge$. Assume that $(X^\wedge, L^\wedge)$ is of log-general type, which implies that the first reduction (X, L) exists and that $(K_X + L)^3 > 0$. In this section we want to show that $h^0(K_{X^\wedge} + L^\wedge) \geq 2$. Let us fix the following.

(1.0) Assumptions. Let $(X^\wedge, L^\wedge)$ be as above and let (X, L) be the first reduction of $(X^\wedge, L^\wedge)$. Let $S^\wedge$ be a smooth element in $|L^\wedge|$ and let S be the corresponding smooth surface in $|L|$. Note that from Tsuji inequality (0.6) it follows that $h^0(K_X + L) \geq 1$. Thus we may assume that $h^0(K_X + L) = 1$ as well as

(1.0.1) $$\chi(\mathcal{O}_S)(=\chi(\mathcal{O}_{S^\wedge})) = 1, \;\; d_1 \leq 11.$$

We can also assume

(1.0.2) $$h^0(K_X) = 0.$$

Indeed, if not, $h^0(K_X + L) \geq h^0(K_X) + h^0(L) - 1 \geq 4$. Therefore $p_g(S) > 0$.

The exact sequence

$$0 \to K_X \to K_X \otimes L \to K_S \to 0$$

gives $h^0(K_X + L) = \chi(K_X) + \chi(K_S) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_X)$, whence, by (1.0.1) and since we are assuming $h^0(K_X + L) = 1$,

(1.0.3) $$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X^\wedge}) = 0$$

and

(1.0.4) $$q(S) = p_g(S) > 0.$$

Therefore $q(X) > 0$, so we can also assume by the Barth-Lefschetz theorem that $\Gamma(L^\wedge)$ embeds $X^\wedge$ in $\mathbb{P}^N$ with

(1.0.5) $$N \geq 6.$$

Let $d_j^\wedge$, d_j, $j = 0, 1, 2, 3$, be the pluridegrees of $(X^\wedge, L^\wedge)$, (X, L) respectively as in (0.4). We also have

(1.0.6) $$d^\wedge \geq 10; \;\; d_1 \geq 6; \;\; d_2 \geq 3.$$

To see this, first note that we can clearly assume $d^\wedge \geq 9$. Indeed, since $S^\wedge$ is a surface of general type (see (0.10)), and $X^\wedge$ is embedded in $\mathbb{P}^N$ with $N \geq 6$, we have from [LS], (0.6) that $d^\wedge = \deg(S^\wedge) > 2(N-3) + 2 \geq 8$. Hence $d \geq 9$. Furthermore d_1, d_2, d_3 are positive. Use the Hodge index relations (0.4.1). From $d_1^2 \geq d_2 d$ we get $d_1 \geq 3$ and therefore $d_3 d_1 \leq d_2^2$ yields $d_2 \geq 2$. If $d_2 = 2$, $d_1^2 \geq d d_2$ gives $d_1 \geq 5$ and by parity $d_3 \geq 2$. Hence $d_2^2 \geq d_1 d_3 \geq 10$ gives $d_2 \geq 4$: a contradiction. Thus $d_2 \geq 3$.

Now $d \geq 9$ and therefore $d_1^2 \geq d d_2$ yields $d_1 \geq 5$. If $d^\wedge = 9$, Castelnuovo's bound (0.7) gives $g(L^\wedge) \leq 7$ and the genus formula leads to the contradiction $14 \leq d + d_1 \leq 12$. Thus $d \geq d^\wedge \geq 10$ and hence $d_1 \geq 6$ from $d_1^2 \geq d d_2 \geq 30$.

From (0.5) we derive the following useful numerical bound.

(1.1) PROPOSITION. *Let $(X^\wedge, L^\wedge)$ be a smooth threefold polarized with a very ample line bundle $L^\wedge$. Assume that $(X^\wedge, L^\wedge)$ is of log-general type and let (X, L), $r : X^\wedge \to X$, be the first reduction of $(X^\wedge, L^\wedge)$. Let γ be the number of points blown up under r. Let $d^\wedge := L^{\wedge 3}$ be the degree of $(X^\wedge, L^\wedge)$ and let $d^\wedge$, d_1, d_2, d_3 be as in* (0.4). *Let $S^\wedge$ be a smooth element in $|L^\wedge|$. Assume that $\chi(\mathcal{O}_{S^\wedge}) = 1$, $\chi(\mathcal{O}_{X^\wedge}) = h^0(K_{X^\wedge}) = 0$ (see* (1.0)). *Then*

$$106 \geq (20 - d^\wedge)d^\wedge + d_3 + 12d_2 + 17d_1 + 5\gamma.$$

PROOF. Since $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_S) - h^0(K_X + L)$, we get $h^0(K_X + L) = 1$. Moreover $h^0(K_X) = 0$. Then the inequality in (0.5.2) gives the result. □

We can now prove the main result of this section. As above, let (X, L) and $r : X^\wedge \to X$ denote the first reduction of $(X^\wedge, L^\wedge)$. Recall that $h^0(K_{X^\wedge} + L^\wedge) = h^0(K_X + L)$ (see (0.3.1)).

(1.2) THEOREM. *Let $(X^\wedge, L^\wedge)$ be a smooth threefold polarized with a very ample line bundle $L^\wedge$. Assume that $(X^\wedge, L^\wedge)$ is of log-general type. Then $h^0(K_{X^\wedge} + L^\wedge) \geq 2$.*

PROOF. We may assume that all the assumptions as in (1.0), (1.1), and (1.0.6) hold. Then, since $d_3 > 0$, $d_2 \geq 3$, $d_1 \geq 6$, from the inequality of (1.1) we find

$$106 \geq (20 - d^\wedge)d^\wedge + 1 + 36 + 102 = (20 - d^\wedge)d^\wedge + 139.$$

Hence $d^\wedge \leq 20$ is clearly not possible. Let $d^\wedge = 21$. Then $106 \geq -21 + 139$, again a contradiction. Thus $d \geq d^\wedge \geq 22$, so that $d_1^2 \geq dd_2$ gives $d_1^2 \geq 66$ or

$$d_1 \geq 9.$$

Let $d^\wedge = 22$. Then (1.1) yields the contradiction $106 \geq -44 + 37 + 153 = 146$. Thus $d^\wedge \geq 23$ and (1.1) gives again the contradiction $106 \geq -69 + 37 + 153 = 121$. Therefore we can assume $d^\wedge \geq 24$.

Case $d_2 = 3$. One has $d_2^2 = d_3 d_1 = 9$ with $d_3 = 1$, $d_1 = 9$. Let $(X', \mathcal{K}')$, $\mathcal{K}' \approx K_{X'} + L'$, be the second reduction of $(X^\wedge, L^\wedge)$ (see (0.3)). Hence on X' we have, for a positive integer m,

$$((m\mathcal{K}')^2 \cdot L')^2 = (m\mathcal{K}')^3 (m\mathcal{K}' \cdot L'^2).$$

Since $\mathcal{K}'$ is ample we can choose $m \gg 0$ and an irreducible divisor $A \in |m\mathcal{K}'|$ which contains all singularities of X' (recall that X' has isolated singularities). Therefore

$$(A_A \cdot L'_A)^2 = (A^2 \cdot L')^2 = (A^3)(A \cdot L'^2) = (A_A^2)(L'^2_A).$$

Then there exist rational numbers λ, μ such that $\lambda L'_A \sim \mu A_A$. Note that we may take λ even so that $\lambda L'$ is a line bundle (see (0.3)). Hence $(\lambda L' - \mu A)_A \sim \mathcal{O}_A$. Therefore, by (0.9), $\lambda L' \sim \mu A$ on X' and hence

$$\lambda L' \sim \mu m \mathcal{K}' \sim \mu m K_{X'} + \mu m L'.$$

Since $\mu m - \lambda > 0$ for $m \gg 0$, this implies that $-K_{X'}$ is ample, so that by Kodaira vanishing $q(X) = q(X') = 0$. This contradicts the assumption (1.0.4).

Case $d_2 \geq 4$. Let $d_2 = 4$. Then by the parity condition (0.4.2) we have $d_3 \geq 2$ and therefore we find the contradiction $16 = d_2^2 \geq d_1 d_3 \geq 18$.

Thus $d_2 \geq 5$. Note that we can assume $K_{X'} + 3\mathcal{K}' \approx 4\mathcal{K}' - L'$ to be nef and not numerically trivial on X'. Indeed otherwise (see [M], (2.1) and also [BS], (2.1), (1.3)) either $(X', \mathcal{K}') \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ or $(X', \mathcal{K}') \cong (Q, \mathcal{O}_Q(1))$, Q quadric in $\mathbb{P}^4$. This contradicts our present assumption $q(X) = q(X') > 0$. Therefore

$$(4\mathcal{K}' - L') \cdot \mathcal{K}' \cdot \mathcal{K}' = 4d'_3 - d'_2 = 4d_3 - d_2 > 0$$

or $4d_3 > d_2 (\geq 5)$. Thus $d_3 \geq 2$. Since $d_2 \geq 5$ and $d \geq d^\wedge \geq 24$, $d_1^2 \geq dd_2 \geq 120$ yields $d_1 \geq 11$. If $d_2 = 5$ then $d_3 \geq 3$ by the parity condition, so that we have the contradiction $25 = d_2^2 \geq d_3 d_1 \geq 33$. Therefore $d_2 \geq 6$ and $d_1^2 \geq dd_2 \geq 144$ gives $d_1 \geq 12$. By using the Tsuji inequality (0.6) (see also (1.0.1)), this implies $h^0(K_X + L) \geq 2$, so we are done. □

2. - The case of nonnegative Kodaira dimension, I.

(2.0) Let $(X^\wedge, L^\wedge)$ be a smooth threefold polarized with a very ample line bundle $L^\wedge$. Let (X, L) be the first reduction of $(X^\wedge, L^\wedge)$. From now on we further assume that $\kappa(X^\wedge) = \kappa(X) \geq 0$. Hence in particular $(X^\wedge, L^\wedge)$ is of log-general type (see (0.10)). □

The aim of this section is to prove that $h^0(K_{X^\wedge} + L^\wedge) \geq 4$. To this purpose let us fix the following.

(2.1) Assumptions. Let $(X^\wedge, L^\wedge)$ be as in (2.0). Let $S^\wedge$ be a smooth element of $|L^\wedge|$ and let S be the corresponding smooth surface in $|L|$. Let $d_j^\wedge$, d_j, $j = 0, 1, 2, 3$, be the pluridegrees of $(X^\wedge, L^\wedge)$, (X, L) respectively as in (0.4). We can assume

$$d_3 \geq d_2 \geq d_1 \geq d \geq 8. \tag{2.1.1}$$

Indeed, let $X^\wedge \hookrightarrow \mathbb{P}^N$, $N \geq 4$, be the embedding given by $\Gamma(L^\wedge)$. If $N = 4$ the assumption $\kappa(X^\wedge) \geq 0$ implies $d^\wedge \geq 5$, so that $h^0(K_{X^\wedge} + L^\wedge) \geq 5$. Therefore we can assume $N \geq 5$ and hence from [LS], (0.6) we have

$d^\wedge := L^{\wedge^3} \geq 3(N-3)+2 \geq 8$. Thus (2.1.1) follows from (0.10.2). We can also assume

$$(2.1.2) \qquad h^0(K_X) = 0.$$

Indeed, otherwise, a section of $\Gamma(K_X)$ gives an embedding $\Gamma(L) \hookrightarrow \Gamma(K_X + L)$, and hence $h^0(K_X + L) \geq 6$.

Moreover from [S5], (2.2), (2.2′) we know that

$$(2.1.3) \qquad h^0(K_X + L) \geq 3; \ \chi(\mathcal{O}_S) = \chi(\mathcal{O}_{S^\wedge}) \geq 3. \qquad \square$$

First let us show some numerical results we need.

(2.2) LEMMA. *Let* $(X^\wedge, L^\wedge)$, (X, L) *be as in* (2.0) *with the assumptions as in* (2.1). *The either* $h^0(K_X + L) \geq h^0(L^\wedge) \geq 5$ *or*

$$d_3 \geq d_2 + 2; \ d_2 \geq d_1 + 2; \ d_1 \geq d + 2.$$

PROOF. Let S be a smooth element in $|L|$ and L_S the restriction of L to S. Since $\kappa(X) \geq 0$, one has $h^0(mK_X) > 0$ for some positive integer m. Then either $K_X \cdot L \cdot L > 0$ or $K_X \sim \mathcal{O}_X$. In the first case we have $d_1 - d = (K_S - L_S) \cdot L_S = K_X \cdot L \cdot L > 0$ or $d_1 \geq d + 1$. By the parity condition (0.4.2) we conclude $d_1 \geq d + 2$. In the second case $L - K_X$ is ample so that $\chi(L) = h^0(L)$. Since $K_X \sim \mathcal{O}_X$ we also have $\chi(L) = \chi(K_X + L) = h^0(K_X + L)$. Thus $h^0(K_X + L)(= h^0(K_{X^\wedge} + L^\wedge)) = h^0(L) \geq h^0(L^\wedge) \geq 5$.

From (0.10.2) we know that $d_3 \geq d_2$. Assume $d_3 = d_2$. Recalling that $d_2 = d'_2$, $d_3 = d'_3$, we have on the second reduction $(X', \mathcal{K}')$, $\mathcal{K}' \approx K_{X'} + L'$ (see (0.3)), $\mathcal{K}' \cdot \mathcal{K}' \cdot (\mathcal{K}' - L') = 0$ and therefore, since $\mathcal{K}'$ is ample, $\mathcal{K}' \sim L'$ that is $K_{X'} \sim \mathcal{O}_{X'}$ and L' is ample. Thus since L' and $L' - K_{X'} \sim L'$ are ample we have $\chi(L') = h^0(L')$ and $\chi(K_{X'} + L') = h^0(K_{X'} + L')$. We claim that

$$(2.2.1) \qquad \chi(K_{X'} + L') \geq \chi(L').$$

We note — as was helpfully pointed out to us by the referee — that we don't necessarily have equality since $K_{X'}$ and L' aren't necessarily Cartier divisors. For full details and properties of the second reduction map we use in what follows we refer e.g., to [BFS], (0.2). To show (2.2.1), factor the second reduction map $\varphi : X \to X'$ as $s \circ r$ with $r : X \to V$ and $s : V \to X'$, V normal variety, where r agrees with φ away from fibers F of φ of the form $F \cong \mathbb{P}^2$, $\mathcal{N}_F^X \cong \mathcal{O}_{\mathbb{P}^2}(-2)$, and r is an isomorphism in a neighborhood of each fiber F. Let $\{F_i, i \in I\}$ denote the possibly empty union of positive dimensional fibers of s, which by construction are all isomorphic to $\mathbb{P}^2$ with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-2)$.

Let $L_V := (r_* L)^{**}$ and $\mathcal{K}_V := K_V + L_V$. Note that L_V is a Cartier divisor and $\mathcal{K}_V \approx s^* \mathcal{K}'$. Note also that $2L'$, $2K_{X'}$ are Cartier divisors and

$$2L_V \approx s^*(2L') - \sum_{i \in I} F_i; \ 2K_V \approx s^*(2K_{X'}) + \sum_{i \in I} F_i \sim \sum_{i \in I} F_i.$$

Moreover $s_*L_V \approx L'$. To see this note that it follows from the definition of L' that $(s_*L_V)^{**} \approx L'$ and we have a natural inclusion of sheaves $s_*L_V \subset L'$. If $s_*L_V \neq L'$ then choose a local section t of L' around $x \in X'$ with $x = s(F_i)$ for some F_i, $i \in I$, and such that t is not a local section of s_*L_V. Now t gives rise to a section v of L_V in a neighborhood of F_i with $\operatorname{div}(v) = -kF_i + D$, where $k > 0$ and D is an effective divisor. Note that $\operatorname{div}(v)_{|F_i} \cong \mathcal{O}_{\mathbb{P}^2}(2k + \delta)$ where $D \cap F_i = \mathcal{O}_{\mathbb{P}^2}(\delta)$ with $\delta \geq 0$. Thus, since $\operatorname{div}(v)_{|F_i} \cong L_{V|F_i} \cong \mathcal{O}_{\mathbb{P}^2}(1)$, we have the absurdity $2 \leq 2k + \delta = 1$.

Since $L_{V|F_i} \cong \mathcal{O}_{\mathbb{P}^2}(1)$ we have by the formal functions theorem that the higher direct images sheaves $s_{(i)}L_V$ are zero for $i > 0$ and therefore $\chi(L_V) = \chi(L')$. Thus, recalling also that $\chi(K_V + L_V) = \chi(K_{X'} + L')$, to show (2.2.1) it suffices to show that

$$\chi(K_V + L_V) \geq \chi(L_V).$$

Using the fact that $2K_V \sim \sum_{i\in I} F_i$ this easily follows from the Riemann-Roch theorem. Using the corrections terms to the Riemann-Roch theorem as in [R], Chapter 3, we could do all calculations directly on X'.

Since $h^0(L') \geq h^0(L) \geq h^0(L^\wedge) \geq 5$ and $h^0(K_X + L) = h^0(K_{X'} + L')$ we get, from (2.2.1), $h^0(K_X + L) \geq h^0(L^\wedge) \geq 5$ in this case.

Thus we conclude that either $h^0(K_X + L) \geq h^0(L^\wedge) \geq 5$ or $d_3 \geq d_2 + 1$. In the latter case $d_3 \geq d_2 + 2$ by parity condition (0.4.2).

It remains to show that $d_2 \geq d_1 + 2$. By the above $d_3 > d_2$. From this and $d_2^2 \geq d_1 d_3$ we get $d_2 > d_1$. If $d_2 = d_1 + 1$ we find

$$d_2^2 = (d_1 + 1)^2 \geq d_1 d_3 \geq d_1(d_2 + 2) = d_1(d_1 + 3).$$

Then $d_1^2 + 2d_1 + 1 \geq d_1^2 + 3d_1$, whence $d_1 \leq 1$. This contradicts (2.1.1). Therefore $d_2 \geq d_1 + 2$ and we are done. □

(2.3) LEMMA. *Let $(X^\wedge, L^\wedge)$, (X, L) be as in* (2.0) *with the assumptions as in* (2.1). *Let $S^\wedge$ be a smooth element of $|L^\wedge|$ and let S be the corresponding smooth surface in $|L|$. Let $d := L^3$. Then*

$$d(d - 17) + 12\chi(\mathcal{O}_S) \geq 18.$$

PROOF. Let $d^\wedge = {L^\wedge}^3$. Let γ be the number of points blown up under the first reduction map $r : X^\wedge \to X$. Then $d^\wedge = d - \gamma$, $K_{S^\wedge} \cdot K_{S^\wedge} = d_2^\wedge = d_2 - \gamma$ and Lemma (0.5.3) yields

$$(d - \gamma)(d - 5 - \gamma) - 10(g(L) - 1) + 12\chi(\mathcal{O}_S) \geq 2d_2 - 2\gamma$$

or

$$d(d - 5) - 10(g(L) - 1) + 12\chi(\mathcal{O}_S) + \gamma(\gamma + 7 - 2d) \geq 2d_2. \tag{2.3.1}$$

We claim that $\gamma(\gamma+7-2d) \le 0$. Indeed otherwise $\gamma+7 > 2d$, or $2d-\gamma = d+d^\wedge \le 6$. This contradicts (2.1.1). Therefore (2.3.1) reads

$$d(d-5) - 10(g(L)-1) + 12\chi(\mathcal{O}_S) \ge 2d_2.$$

Since $2g(L) - 2 = d + d_1$ this is equivalent to

$$d(d-10) + 12\chi(\mathcal{O}_S) \ge 2d_2 + 5d_1. \tag{2.3.2}$$

By Lemma (2.2) we get $2d_2 + 5d_1 \ge 2d + 8 + 5d + 10 = 7d + 18$. Thus (2.3.2) gives the result. □

We can now prove the main result of this section.

(2.4) THEOREM. *Let $(X^\wedge, L^\wedge)$ be a smooth threefold polarized with a very ample line bundle $L^\wedge$. Assume that $\kappa(X^\wedge) \ge 0$. Let (X, L) be the first reduction of $(X^\wedge, L^\wedge)$. Then $h^0(K_{X^\wedge} + L^\wedge) = h^0(K_X + L) \ge 4$.*

PROOF. We can suppose the assumptions in (2.1) are satisfied. Let S be a smooth element in $|L|$. From (2.1.3) we know that $\chi(\mathcal{O}_S) \ge 3$. Use Tsuji inequality (0.6). If $\chi(\mathcal{O}_S) \ge 6$ we have the result. If $\chi(\mathcal{O}_S) = 5$, we have

$$2h^0(K_X + L) \ge 5 + \frac{d_1}{12} + \frac{d_3}{32}.$$

Recall that $(X^\wedge, L^\wedge)$ is of log-general type since $\kappa(X^\wedge) \ge 0$. Then the same argument as in the proof of Theorem (1.2) implies $d_1 \ge 12$. Therefore $\frac{d_1}{12} + \frac{d_3}{32} > 1$ and hence $2h^0(K_X + L) > 6$, that is $h^0(K_X + L) \ge 4$. Thus it remains to consider the cases $\chi(\mathcal{O}_S) = 3, 4$. Recall that $d_2 = K_S \cdot K_S < 9\chi(\mathcal{O}_S)$ by (0.10.1). Let γ be the number of points blown up under the first reduction map $r : X^\wedge \to X$. By combining Lemma (2.2) and Proposition (0.5.2), with $h^0(K_X) = 0$ in view of (2.1.2), and noting that $2d - 15 - \gamma = d^\wedge + d - 15 > 0$ from (2.1), we find

$$44h^0(K_X + L) + 58\chi(\mathcal{O}_S) \ge (50-d)d + 84 := f(d). \tag{2.4.1}$$

Clearly the function $f(d)$ reaches the maximum for $d = 25$ and it is symmetric with respect to the $d = 25$ axis.

Let $\chi(\mathcal{O}_S) = 3$. Then, by Lemma (2.2), $9\chi(\mathcal{O}_S) = 27 > d_2 \ge d + 4$, so that $d \le 22$. Moreover Lemma (2.3) yields $d(d-17) + 18 \ge 0$ or $d \ge 16$. For $16 \le d \le 22$, $f(d) \ge f(16) = 628$. Thus (2.4.1) gives $44h^0(K_X + L) + 174 \ge 628$, or $h^0(K_X + L) \ge 11$.

Let $\chi(\mathcal{O}_S) = 4$. Lemma (2.2) yields $36 > d_2 \ge d + 4$, whence $d \le 31$ and Lemma (2.3) gives $d(d-17) + 30 \ge 0$, or $d \ge 15$. For $15 \le d \le 31$, $f(d) \ge f(15) = 609$. Thus (2.4.1) reads $44h^0(K_X + L) + 232 \ge 609$, or $h^0(K_X + L) \ge 9$. □

The following remark shows that the stable case when $d \gg 0$ is trivial (e.g., $h^0(K_X + L) \geq 6$ as soon as $d \geq 45$).

(2.5) REMARK (the stable case). Notation and assumptions as in (2.4). We have the following explicit lower bound for $h^0(K_X + L)$ in terms of $d := L^3$,

$$d < 9h^0(K_X + L).$$

To see this, use Tsuji inequality (0.6) and inequalities (0.10.2). One has

$$2h^0(K_X + L) - \chi(\mathcal{O}_S) \geq \frac{d_1}{12} + \frac{d_3}{32} \geq \frac{d}{12} + \frac{d}{32} = \frac{11d}{96}.$$

Therefore

$$11d + 96\chi(\mathcal{O}_S) \leq 192h^0(K_X + L). \tag{2.5.1}$$

Note that since $d_2 \geq d$ and $d_2 < 9\chi(\mathcal{O}_S)$ we find $9\chi(\mathcal{O}_S) > d$. Hence (2.5.1) yields

$$11d + \frac{32}{3}d < 192h^0(K_X + L).$$

This gives the result. $\square$

(2.6) REMARK. Let $(X^\wedge, L^\wedge)$ be a smooth projective variety of dimension $n \geq 3$ polarized with a very ample line bundle $L^\wedge$. Assume the first and the second reductions exist and let (X, L), $(X', \mathcal{K}')$, $\varphi : X \to X'$, $\mathcal{K}' \approx K_{X'}+(n-2)L'$, be the first and the second reduction of $(X^\wedge, L^\wedge)$ as in (0.3). As in Lemma (2.2) factor φ as $s \circ r$ with $r : X \to V$ and $s : V \to X'$, V normal variety, where r agrees with φ away from fibers F of φ of the form $F \cong \mathbb{P}^{n-1}$, $\mathcal{N}_F^X \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-2)$ and r is an isomorphism in a neighborhood of such fibers F. The morphism s is a partial resolution of those points $y \in X'$ such that $\varphi^{-1}(y)$ is a fiber F as above. The argument used in the Lemma shows that $s_*L_V \approx L'$, where $L' := (\varphi_*L)^{**}$ and $L_V := (r_*L)^{**}$.

3. - The case of nonnegative Kodaira dimension, II.

Let $(X^\wedge, L^\wedge)$ be a smooth threefold polarized with a very ample line bundle $L^\wedge$. Assume that $\kappa(X^\wedge) \geq 0$. Let (X, L) be the first reduction of $(X^\wedge, L^\wedge)$. The aim of this section is to prove that $h^0(K_{X^\wedge} + L^\wedge) = h^0(K_X + L) \geq 5$ with equality only if $(X^\wedge, L^\wedge)$ is a smooth quintic hypersurface in $\mathbb{P}^4$.

First, let us show an easy consequence of [Mi], (1.1) that we need.

(3.1) PROPOSITION. *Let V be a smooth threefold and let S be a smooth surface which is an ample divisor on V. Let H be the line bundle associated* *to S. Let $d_j := (K_V + H)^j \cdot H^{3-j}$, $j = 0,1,2$, $d_0 = d$. Assume that K_V is nef. Then*

$$d_2 + \frac{d_1 + d}{4} \leq 9\chi(\mathcal{O}_S)$$

and, if $K_V \sim \mathcal{O}_V$, $d_2 \leq 6\chi(\mathcal{O}_S)$.

PROOF. Let H_S be the restriction of H to S. From [Mi], (1.1) we have $K_V \cdot K_V \cdot H \leq 3c_2(V) \cdot H$. Therefore, since $K_V \cdot K_V \cdot H = (K_S - H_S) \cdot (K_S - H_S)$ by the adjunction formula and $c_2(V) \cdot H = e(S) - K_S \cdot H_S$ by the Chern relation $c(S)c(H_S) = c(V)$, where $c(\cdot)$ stands for the total Chern class $1 + c_1(\cdot) + c_2(\cdot) + \ldots$, we find

$$d_2 - 2d_1 + d \leq 3e(S) - 3d_1$$

or, since $d_2 = K_S \cdot K_S$,

$$d_2 + d_1 + d \leq 3e(S) = 36\chi(\mathcal{O}_S) - 3d_2. \tag{3.1.1}$$

Therefore $\frac{d_1 + d}{4} \leq 9\chi(\mathcal{O}_S) - d_2$.

If $K_V \sim \mathcal{O}_V$, we have $d_1 = d_2 = d$ and hence (3.1.1) gives $d_2 \leq 6\chi(\mathcal{O}_S)$. □

The following further numerical condition is the main technical tool we need to improve the results of §2.

(3.2) PROPOSITION (Key-Lemma). *Let $(X^\wedge, L^\wedge)$ be a smooth threefold polarized with a very ample line bundle $L^\wedge$. Assume that $\kappa(X^\wedge) \geq 0$. Let (X, L), $r : X^\wedge \to X$, be the first reduction of $(X^\wedge, L^\wedge)$. Let S be a general smooth element in $|L|$. Further assume that $d_2 = 9\chi(\mathcal{O}_S) - 1$. Then $d_1 \geq 4 + d$.*

PROOF. First, note that the assumption on d_2 implies that

•) S does not contain (-2)-rational curves.

Indeed, since $e(S) = 12\chi(\mathcal{O}_S) - d_2$, we find $3e(S) - d_2 = 4$. Let k be the number of (-2)-rational curves on S. Then the Miyaoka inequality $k \leq \frac{2}{9}(3e(S) - K_S \cdot K_S)$ (see [BPV], p. 215) gives $k \leq \frac{8}{9}$ that is $k = 0$.

Note also that by the parity condition (0.4.2), $d_1 \neq 3 + d$. Therefore it is enough to show that $d_1 > 2 + d$. Then let us assume $d_1 - d \leq 2$. In view of Proposition (3.1) we can assume K_X not nef. Indeed otherwise we would have $d_1 + d \leq 4$ and hence $2d^\wedge \leq 2d \leq 4$, which is clearly not possible. Since K_X is not nef, the Mori Cone theorem says that there exists an extremal ray R. Let $\rho = \mathrm{cont}_R : X \to Y$ be the contraction of R and let E be the locus of R, that is the locus of curves of X whose numerical classes are in R. According to Mori [Mo] we know that either

i) $E \cong \mathbb{P}^2$, $\mathcal{N}^X_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-a)$, $a = 1, 2$;

ii) $E \cong \mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{N}^X_E \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$;

iii) $E \cong Q$, Q quadric cone in $\mathbb{P}^3$, $\mathcal{N}^X_E \cong \mathcal{O}_Q(-1)$, or

iv) E is isomorphic to a $\mathbb{P}^1$ bundle over $\rho(E)$, $\rho(E)$ nonsingular curve, and $\mathcal{N}^X_{E|f} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ for any fiber f of $E \to \rho(E)$.

Furthermore $\rho(E)$ is a point in the first three cases and ρ is the blowing up along $\rho(E)$ in each case.

Case i). Assume $a = 1$. One has $K_X \approx \rho^* K_Y + 2E$. Then

$$d_1 - d = K_X \cdot L \cdot L = \rho^* K_Y \cdot L \cdot L + 2E \cdot L \cdot L.$$

Since $d_1 - d \leq 2$, $\rho^* K_Y \cdot L \cdot L \geq 0$ by the assumption $\kappa(X) \geq 0$, and $E \cdot L \cdot L \geq 1$ we conclude that $\rho^* K_Y \cdot L \cdot L = 0$, $E \cdot L \cdot L = 1$. Thus $L_E \cong \mathcal{O}_{\mathbb{P}^2}(1)$. Since $K_{X|E} \approx K_E - \det \mathcal{N}^X_E \cong \mathcal{O}_{\mathbb{P}^2}(-2)$ we get $(K_X + L)_E \cong \mathcal{O}_{\mathbb{P}^2}(-1)$. Since $(X^\wedge, L^\wedge)$ is of log-general type, $K_X + L$ is nef, so we find a contradiction.

Assume $a = 2$. In this case $2K_X \approx \rho^*(2K_Y) + E$ and Y has a 2-factorial singularity. As above, $\rho^* K_Y \cdot L \cdot L \geq 0$ and therefore

$$4 \geq 2(d_1 - d) = 2K_X \cdot L \cdot L = \rho^*(2K_Y) \cdot L \cdot L + E \cdot L \cdot L \tag{3.2.1}$$

implies $E \cdot L \cdot L = L_E \cdot L_E \leq 4$ and hence $L_E \cong \mathcal{O}_{\mathbb{P}^2}(m)$, $m = 1, 2$. Since $K_{X|E} \approx K_E - \det \mathcal{N}^X_E \approx \mathcal{O}_{\mathbb{P}^2}(-1)$ we get $(K_X + L)_E \cong \mathcal{O}_{\mathbb{P}^2}(m-1)$.

Let $m = 1$ and let C be a line in $|L_E| = |\mathcal{O}_{\mathbb{P}^2}(1)|$. Then $K_{S|C} \approx (K_X + L)_C \approx \mathcal{O}_C$, that is $K_S \cdot C = 0$. Then $C^2 = -2$. This contradicts •) above.

Let $m = 2$. Then $E \cdot L \cdot L = 4$ so that (3.2.1) gives $\rho^* K_Y \cdot L \cdot L = 0$. Since $\kappa(X) \geq 0$, $h^0(NK_X) > 0$ and hence $h^0(2NK_Y) > 0$ for some $N > 0$. Therefore $2K_Y \sim \mathcal{O}_Y$ and hence $2K_X \sim E$. Thus we find the contradiction

$$2K_X \cdot (K_X + L) \cdot (K_X + L) = (K_X + L)_E \cdot (K_X + L)_E = \mathcal{O}_{\mathbb{P}^2}(1) \cdot \mathcal{O}_{\mathbb{P}^2}(1) = 1.$$

Cases ii), iii). In these cases, Y is factorial and $K_X \approx \rho^* K_Y + E$. Let $E \cong \mathbb{P}^1 \times \mathbb{P}^1$. One has again $\rho^* K_Y \cdot L \cdot L \geq 0$ and hence

$$2 \geq d_1 - d = K_X \cdot L \cdot L = \rho^* K_Y \cdot L \cdot L + E \cdot L \cdot L$$

implies $\rho^* K_Y \cdot L \cdot L = 0$, $E \cdot L \cdot L = L_E \cdot L_E = 2$. Then $L_E \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$. Since $K_{X|Q} \approx \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$ we find $(K_X + L)_E \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$. Let C be a smooth curve in $|L_E| = |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$. Therefore $K_{S|C} \approx (K_X + L)_C \approx \mathcal{O}_C$, that is $K_S \cdot C = 0$. Then $C^2 = -2$. This contradicts again •) above.

The same argument rules out the case when E is a quadric cone.

Case iv). In this case Y is smooth and $K_X \approx \rho^* K_Y + E$. Note that since L is very ample outside of a finite set of points and E is a $\mathbb{P}^1$ bundle one has $E \cdot L \cdot L \geq 2$. Thus the usual argument, by using $d_1 - d \leq 2$, $\rho^* K_Y \cdot L \cdot L \geq 0$, implies that $E \cdot L \cdot L = 2$. Hence $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $L_E \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$. Let

f be a fiber of $E \to \rho(E)$. Since $\mathcal{N}_f^E \cong \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{N}_{E|f}^X \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ we get $\det \mathcal{N}_f^X \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ and therefore $K_{X|f} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$, so that $(K_X+L)_f \cong \mathcal{O}_{\mathbb{P}^1}$. By a consequence of Bertini's theorem (see [S3], (0.6.2)) we can assume that the general element S of $|L|$ contains f and S, E intersect transversely along f. Then $K_{S|f} = (K_X+L)_f \cong \mathcal{O}_{\mathbb{P}^1}$, that is $K_S \cdot f = 0$. Therefore $(f \cdot f)_S = -2$, so we contradict again •) and we are done. □

We can prove now the main result of the paper.

(3.3) THEOREM. *Let $(X^\wedge, L^\wedge)$ be a smooth threefold polarized with a very ample line bundle $L^\wedge$. Let $S^\wedge$ be a smooth surface in $|L^\wedge|$. Assume that $\kappa(X^\wedge) \geq 0$. Then $h^0(K_{X^\wedge}+L^\wedge) \geq 5$ with equality only if $(X^\wedge, L^\wedge)$ is a smooth quintic hypersurface in $\mathbb{P}^4$. Furthermore either $p_g(S^\wedge) \geq 6$ or $S^\wedge$ is a degree $d^\wedge = 5$ surface in $\mathbb{P}^3$ with $p_g(S^\wedge) = 4$.*

PROOF. Let (X, L), $r : X^\wedge \to X$ be the reduction of $(X^\wedge, L^\wedge)$. Let S be a smooth element in $|L|$ corresponding to $S^\wedge$ and γ the number of points blown up under r. Let $d_j^\wedge$, d_j, $j = 0, 1, 2, 3$, be the pluridegrees of $(X^\wedge, L^\wedge)$, (X, L). From (2.4) we know that $h^0(K_{X^\wedge}+L^\wedge) \geq 4$. Thus we can assume $h^0(K_X+L) \leq 5$. If $h^0(K_X) > 0$ then

$$h^0(K_X+L) \geq h^0(K_X) + h^0(L) - 1 \geq h^0(K_X) + h^0(L^\wedge) - 1.$$

From this we see that $h^0(K_X) \geq 1$ implies $h^0(L^\wedge) \leq 5$. Since $\kappa(X^\wedge) \geq 0$ one has $h^0(L^\wedge) = 5$ and $(X^\wedge, L^\wedge)$ is a hypersurface in $\mathbb{P}^4$. Since $h^0(K_{X^\wedge}+L^\wedge) = h^0(K_X+L) \leq 5$ we have $d^\wedge = 5$. Therefore in what follows we can assume $h^0(K_{X^\wedge}) = h^0(K_X) = 0$ (compare with (2.1.2)). We can also assume that the numerical inequalities of (2.2) hold true. Indeed, if not, we would have $h^0(K_{X^\wedge}+L^\wedge) \geq h^0(L^\wedge) \geq 5$ and hence either $h^0(K_{X^\wedge}+L^\wedge) \geq 6$, in which case we are done, or $h^0(L^\wedge) = 5$, and we would fall again in the special case above. By using this and all numerical conditions stated in previous sections, namely (note that not all the following conditions are the best possible):

1) $1 \leq \chi(\mathcal{O}_S) \leq 2h^0(K_X+L)$ (from S being of general type and Tsuji inequality (0.6));
2) $5 \leq d \leq d_2 \leq 9\chi(\mathcal{O}_S)$ (from (2.1.1) and Miyaoka inequality);
3) $2g(L) - 2 = d_1 + d$ (the genus formula (0.2));
4) $d + 1 \leq g(L) \leq 12(2h^0(K_X+L) - \chi(\mathcal{O}_S))$ (genus formula and $d_1 \geq d$ (see (2.1.1)) give the lower bound as well as $g(L) \leq d_1 + 1$. Then the Tsuji inequality (0.6) gives the upper bound);
5) $d_1 + 2 \leq d_2 \leq 9\chi(\mathcal{O}_S)$ (from (2.2), and the Miyaoka inequality);
6) $d_1^2 \geq dd_2$ (Hodge index relation (0.4.1));
7) $d_2 \geq 8(\chi(\mathcal{O}_S) - h^0(K_X+L)) + d$ (from (0.10.3));
8) $d_2 + 2 \leq d_3 \leq d_2^2/d_1$ (from (2.2) and (0.4.1));
9) $d_2 = d_3 \operatorname{mod}(2)$ (parity condition (0.4.2));

10) $0 \leq \gamma \leq d - 5$ (from $\gamma = d - d^\wedge$ and $d \geq d^\wedge \geq 5$ (see proof of (2.1.1));

11) $d^\wedge = d - \gamma$, $d_1^\wedge = d_1 + \gamma$, $d_2^\wedge = d_2 - \gamma$ (from (0.4));

12) $d^\wedge(d^\wedge - 5) - 10(g(L) - 1) + 12\chi(\mathcal{O}_S) \geq 2d_2^\wedge$ (Lemma (0.5.3));

13) $2h^0(K_X + L) - \chi(\mathcal{O}_S) \geq (8d_1 + 3d_3)/96$ (from Tsuji inequality (0.6));

14) $44h^0(K_X + L) + 58\chi(\mathcal{O}_S) + 4 \geq 12d_2 + 17d_1 + d_3 + (20 - d^\wedge)d^\wedge + 5\gamma$ (Proposition (0.5.2)),

we find that the only possible invariants are $h^0(K_X + L) = \chi(\mathcal{O}_S) = 5$, $g(L) = 42$, $d = 40$, $d_1 = 42$, $d_2 = 44$, $d_3 = 46$, and $\gamma = 0$. We carried these computations out by using a simple Pascal program that we include for completeness at the end of the proof. In the remaining case above one has $d_2 = 9\chi(\mathcal{O}_S) - 1$ and $d_1 < d + 4$. Therefore Proposition (3.2) applies to rule it out. Thus, except for smooth quintic hypersurfaces in $\mathbb{P}^4$, $h^0(K_X + L) \geq 6$. This proves the first part of the statement.

To show that $p_g(S^\wedge)(= p_g(S)) \geq 6$, look at the exact sequence

$$0 \to K_{X^\wedge} \to K_{X^\wedge} \otimes L^\wedge \to K_{S^\wedge} \to 0. \tag{3.3.1}$$

By what already proven we can assume $h^0(K_{X^\wedge} + L^\wedge) \geq 6$. Indeed otherwise $(S^\wedge, L_{S^\wedge}^\wedge)$ is a smooth quintic surface in $\mathbb{P}^3$ with $p_g(S^\wedge) = 4$. If $h^0(K_{X^\wedge}) = 0$, then $p_g(S) \geq h^0(K_{X^\wedge} + L^\wedge) \geq 6$, so we are done. Thus we may assume $h^0(K_{X^\wedge}) \geq 1$. We may also assume that $S^\wedge$ lies in $\mathbb{P}^N$ with $N \geq 4$, so that

$$h^0(L_{S^\wedge}^\wedge) \geq 5. \tag{3.3.2}$$

Indeed, if $|L_{S^\wedge}^\wedge|$ embeds $S^\wedge$ in $\mathbb{P}^3$ as a surface of degree $d^\wedge = L^\wedge \cdot L^\wedge \cdot L^\wedge$, we have

$$p_g(S) = h^0(K_{S^\wedge}) = h^0(\mathcal{O}_{S^\wedge}(d^\wedge - 4)) = h^0(\mathcal{O}_{\mathbb{P}^3}(d^\wedge - 4)) \geq 6$$

as soon as $d^\wedge \geq 6$. Since $\kappa(X^\wedge) \geq 0$, we have $d^\wedge \geq 5$ and either we are in the special case where $S^\wedge$ is a degree $d^\wedge = 5$ surface in $\mathbb{P}^3$ or $h^0(L_{S^\wedge}^\wedge) \geq 5$. Note that

$$p_g(S) = h^0(K_{X^\wedge|S^\wedge} + L_{S^\wedge}^\wedge) \geq h^0(K_{X^\wedge|S^\wedge}) + h^0(L_{S^\wedge}^\wedge) - 1. \tag{3.3.3}$$

Assume $h^0(K_{X^\wedge}) \geq 2$. Then by Lemma (0.11) we get $h^0(K_{X^\wedge|S^\wedge}) \geq 2$, so we are done by combining (3.3.2) and (3.3.3). Thus, by the above, we can assume $h^0(K_{X^\wedge}) = 1$, $h^0(L_{S^\wedge}^\wedge) = 5$. Hence in particular $X^\wedge$ lies in $\mathbb{P}^5$ so that $q(X^\wedge) = q(S^\wedge) = 0$. Therefore from the exact cohomology sequence associated to (3.3.1) we conclude that $h^1(K_{X^\wedge}) = 0$, $\chi(\mathcal{O}_{X^\wedge})(= \chi(\mathcal{O}_X)) = 0$, and $\chi(\mathcal{O}_{S^\wedge})(= \chi(\mathcal{O}_S)) = 6$. Now, the same Pascal program used above, running now with the invariants $\chi(\mathcal{O}_X) = 0$, $\chi(\mathcal{O}_S) = 6$, and the double point inequality (0.5.3) as an equality, shows that there are no possible cases. □

Pascal Program listing invariants when $h^0(K_{X^\wedge} + L^\wedge) \leq 5$.

```
var     h0, h0KL, chiS, d, g, d1, d2, d3, gamma, dhat, d1hat, d2hat, d3hat: longint;

begin
 writeln(' ','h0KL',' ','chiS',' ','d',' ','g',' ','gamma',' ','d1',' ','d2',' ','d3');
 for h0KL := 1 to 5 do
  begin
   for chiS := 1 to 2 * h0KL do
    begin
     for d := 5 to 9 * chiS do
      begin
       for g := d + 1 to 12 * (2 * h0KL - chiS) + 1 do
        begin
         d1 := 2 * g - 2 - d;
         for d2 := d1 + 2 to 9 * chiS do
          begin
           if d2 <= d1 * d1 div d then
           if 8 * (chiS - h0KL) <= d2 - d then
           for d3 := d2 + 2 to d2 * d2 div d1 do
            begin
             if 0 = (d3 - d2) mod 2 then
             for gamma := 0 to d - 5 do
              begin
               dhat := d - gamma;
               d1hat := d1 + gamma;
               d2hat := d2 - gamma;
               if dhat * (dhat - 5) - 10 * (g - 1) + 12 * chiS >= 2 * d2hat then
                if 2 * h0KL - chiS >= (32 * d1 + 12 * d3 + 12 * 32 - 1) div (12 * 32)
                 then
                 if 44 * h0KL + 58 * chiS + 4 >= 12 * d2 + 17 * d1 + d3
                              + (20 - dhat) * dhat + 5 * gamma then
                  writeln(h0KL, chiS, d, g, gamma, d1, d2, d3);
             end;
           end;
         end;
       end;
     end;
   end;
 end;
end.
```

Let us point out the following standard consequence of the results above in the higher dimensional case.

(3.4) Remark (the higher dimensional case). Let $L^\wedge$ be a very ample line bundle on an projective manifold, $X^\wedge$, of dimension $n \geq 3$. Let V be the 3-fold obtained as the transversal intersection of $n-3$ general elements $A_1, \dots, A_{n-3}$ of $|L^\wedge|$. Let $\mathcal{L}$ be the restriction of $L^\wedge$ to V. Then we have:

1. If $(V, \mathcal{L})$ is of log-general type, then $h^0(K_{X^\wedge} + (n-2)L^\wedge) \geq 2$;

2. If $\kappa(V) \geq 0$, e.g., if the Kodaira dimension of $K_{X^\wedge} + (n-3)L^\wedge$ is nonnegative, then $h^0(K_{X^\wedge} + (n-2)L^\wedge) \geq 5$ with equality only if $n = 3$ and $(X^\wedge, L^\wedge)$ is a degree 5 hypersurface of $\mathbb{P}^4$.

If $n = 3$ the result is proved in the Theorems (1.2) and (3.3). Therefore we can assume $n \geq 4$. The Kodaira vanishing theorem yields

$$h^0(K_{X^\wedge} + (n-2)L^\wedge) \geq h^0(K_V + \mathcal{L}).$$

Then 1) follows from the corresponding $n = 3$ statement (1.2). By using again the inequality above and (3.3) we have either $h^0(K_{X^\wedge} + (n-2)L^\wedge) \geq 6$ or $h^0(K_V + \mathcal{L}) = 5$ and $(V, \mathcal{L})$ is a quintic hypersurface of $\mathbb{P}^4$. In this case, since $L^\wedge$ is very ample, it is easy to see that $(X^\wedge, L^\wedge)$ is a degree 5 hypersurface in $\mathbb{P}^{n+1}$ and $h^0(K_{X^\wedge} + (n-2)L^\wedge) = n + 2 \geq 6$. This shows 2).

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