ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

PAOLO MAREMONTI
REMIGIO RUSSO

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 21, nº 4 (1994), p. 629-643
<http://www.numdam.org/item?id=ASNSP_1994_4_21_4_629_0>

On the Maximum Modulus Theorem for the Stokes System

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To Professor V.A. Solonnikov on his 60th *birthday*

1. - Introduction

As is well known, the maximum principle in partial differential equations states that a solution to an elliptic equation of the second order with regular coefficients takes its maximum value on the boundary of a smooth regular domain (cf. e.g., [23]). This is no longer true for elliptic systems and equations of higher order. Indeed, in 1930 Pólya [22] gave an example of solution to the system of homogeneous isotropic linear elastostatics whose modulus takes its maximum value in the interior of a ball. However, for systems and equations of higher order in bounded domains, Miranda [19-20], Agmon [1], Fichera [12] and Canfora [5] proved the so-called maximum modulus theorem: there exists a positive constant C depending only on the domain Ω such that the modulus of any solutions $\mathbf{u}(x)$ in Ω is majorized by C times the maximum of the modulus of $\mathbf{u}(x)$ on the boundary of Ω.

In view of its relevance in the linear theory of incompressible media, it is of some interest to detect wether the above result holds for the Stokes system

$$(1.1)\qquad \begin{aligned} &-\Delta\mathbf{u}(x)+\nabla\pi(x)=0,\\ &\nabla\cdot\mathbf{u}(x)=0,\ \ \forall x\in\Omega\subset\mathbb{R}^n\ \ (n=2,3). \end{aligned}$$

The importance of a maximum modulus theorem for the Stokes system lies essentially in the fact that it permits to obtain the existence of a regular solution to the Dirichlet problem by only requiring the datum $\mathbf{a}(x)$ at the boundary to be continuous. Of course, this hypothesis appears to be the most natural and physically meaninful one when dealing with the Dirichlet problem, whereas up to now the existence theory for Stokes problem has required greater regularity

Pervenuto alla Redazione il 31 Dicembre 1993.

properties for $\mathbf{a}(x)$, which, by the way, imply some integral estimates on the derivatives of $\mathbf{u}(x)$ in the whole of Ω that are not *a priori* necessary to the statement of the problem [6, 9, 13-14, 18, 25, 26].

The purpose of the present paper is two-fold: a) to prove the maximum modulus theorem for solutions to system (1.1) in bounded two-dimensional domains; b) as far as three-dimensional domains are concerned, to improve the results of [9, 18] for bounded domains and also for exterior domains. However we have only partial results in case b), more precisely we prove:

$$\max_{\mathcal{K}} |\mathbf{u}(x)| \leq C(\mathcal{K}) \max_{\partial\Omega} |\mathbf{a}(x)|$$

where $\mathcal{K}$ is any compact set properly contained in Ω;

$$|\mathbf{u}|_p \leq C(p) \max_{\partial\Omega} |\mathbf{a}| \ \ (p > 3);$$

$\forall \varepsilon \in (0,1) \exists C(\varepsilon) > 0$:

$$\max_{\bar{\Omega}} |\mathbf{u}(x)| \leq C(\varepsilon)\left\{ \max_{\partial\Omega} |\mathbf{a}(x)| + \langle \mathbf{a} \rangle_p^{\varepsilon} \left(\max_{\partial\Omega} |\mathbf{a}(x)| \right)^{1-\varepsilon} \right\}, \ \ (p > 3),$$

where $\langle \mathbf{a} \rangle_p$ denotes a suitable trace norm (cf. notation).

Acknowledgements

The paper is performed under the auspices of GNFM of Italian CNR and MURST Contracts at the University of Napoli "Federico II" and University of Basilicata (Potenza).

2. - Notation and statement of the problem

Throughout the paper Ω will denote a domain of $\mathbb{R}^n$ $(n = 2, 3)$ of class C^2, bounded for $n = 2$ and either bounded or exterior to a compact region for $n = 3$.

$L^p(\Omega)$ $(p \geq 1)$ stands for the set of all (scalar, vector or tensor) fields $\varphi(x)$ on Ω such that $|\varphi|_p^p = \int\limits_{\Omega} |\varphi(x)|^p dx < \infty$;

$$L^\infty(\Omega) = \left\{ \varphi(x) \text{ Lebesgue measurable} : \sup_{\Omega} |\varphi(x)| < +\infty \right\};$$

$$W^{m,p}(\Omega) = \left\{ \varphi \in L^p(\Omega) : |\varphi|_{m,p}^p = \sum_{|\alpha|=0}^{m} |D^\alpha \varphi|_p^p < +\infty \right\},$$

where $D^\alpha\varphi(x)$ denotes a weak derivative of $\varphi(x)$ of order $|\alpha|$; $W^{1-1/p,p}(\partial\Omega)$ is the trace space of the fields $\varphi(x) \in W^{1,p}(\Omega)$ with norm

$$\|\varphi\|^p_{W^{1-1/p,p}(\partial\Omega)} = |\varphi|^p_{L^p(\partial\Omega)} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{p+n-2}}\, d\sigma_x d\sigma_y = |\varphi|^p_{L^p(\partial\Omega)} + \langle\varphi\rangle^p_p;$$

$H^{1,p}_0(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the $|\cdot|_{1,p}$ norm. Let $1/p+1/q=1$. Any $f(x) \in C_0^\infty(\Omega)$ defines a bounded, linear functional on $H_0^{1,q}(\Omega)$ by $\mathcal{L}_f(u) = \int_\Omega f(x)u(x)dx$. $H^{-1,q}(\Omega)$ is the completion of these functionals with respect to the norm

$$|f|_{-1,q} = \sup_{0\neq u(x)\in H_0^{1,p}(\Omega)} \frac{|(f,u)|}{|u|_{1,p}}.$$

$H^{-1/p}(\partial\Omega)$ stands for the dual space of $H^{1-1/p,p}(\partial\Omega)$. $\mathcal{C}_0(\Omega)$ is the whole set of C^∞ vector fields $\varphi(x)$ with compact support on Ω such that $\nabla\cdot\varphi(x)=0$; $J^p(\Omega)$ and $J^{1,p}(\Omega)$ denote the completion of $\mathcal{C}_0(\Omega)$ in $L^p(\Omega)$ and $W^{1,p}(\Omega)$ respectively. $\hat{J}^{1,p}(\Omega)$ stands for the completion of $\mathcal{C}_0(\Omega)$ in $|\nabla\cdot|_p$. It is well known that $L^p(\Omega) = J^p(\Omega)\oplus G^p(\Omega)$, where $G^p(\Omega) = \{\psi(x) : \psi(x) = \nabla h(x),\ h(x)\in L^p_{\mathrm{loc}}(\Omega)\}$. As is standard, we set $(f,g) = \int_\Omega f(x)g(x)dx$, $\forall f(x)$, $g(x)$ such that $f(x)g(x)$ is integrable over Ω. If $1/p+1/q=1$ we have:

$$(\mathbf{f},\psi) = (\mathbf{f},\nabla h) = 0 \ \forall \mathbf{f}(x)\in J^p(\Omega),\ \ \forall\psi(x)\in G^q(\Omega).$$

For details and elementary properties of the spaces introduced abode, we refer the reader to [20, 27]. Finally, the symbol C will be denote a numerical constant whose values is inessential for our aims; the numerical value of C may change from line to line and in the same line it may be $2C \leq C$.

Let $\mathbf{a}(x)$ be a vector field on $\partial\Omega$ such that

$$\mathbf{a}(x)\in C(\partial\Omega) \text{ and } \int_{\partial\Omega} \mathbf{a}(x)\cdot\vec{n}d\sigma = 0; \tag{2.1}$$

$(2.1)_2$ can be omitted if Ω is exterior.

The Dirichlet problem associated to system (1.1) is to find a solution $(\mathbf{u}(x),\pi(x))$ to system (1.1) such that $\mathbf{u}(x)\in C^2(\Omega)\cap C(\bar{\Omega})$, $\pi(x)\in C^1(\Omega)$ and

$$\begin{aligned} &\mathbf{u}(x) = \mathbf{a}(x) \text{ on } \partial\Omega,\\ &|\mathbf{u}(x)| \to 0 \text{ as } |x|\to+\infty; \end{aligned} \tag{2.2}$$

of course $(2.2)_2$ is required if Ω is an exterior domain.

By a weak solution to system (1.1)-(2.1)-(2.2) in the class $\hat{J}^{1,p}(\Omega) \times L^p(\Omega)$ we mean a pair $(\mathbf{u}(x), \pi(x))$ which satisfies the identities

$$
\begin{aligned}
&(\nabla \mathbf{u}, \nabla \varphi) = -(\pi, \nabla \cdot \varphi), \ \forall \varphi(x) \in C_0^\infty(\Omega),\\
&\nabla \cdot \mathbf{u}(x) = 0, \ \text{a.e. on } \Omega,\\
&\mathbf{u}(x)|_{\partial\Omega} = \mathbf{a}(x), \mathbf{u}(x) \to 0 \text{ for } |x| \to +\infty.
\end{aligned}
$$

The aim of the present paper is to prove the following theorems.

THEOREM 2.1. *Let Ω be a bounded domain of $\mathbb{R}^2$ of class C^2. Then, the Dirichlet problem associated with system* (1.1) *admits a unique solution* $(\mathbf{u}(x), \pi(x))$ *such that*

$$
\max_{K} |\nabla \pi(x)| + \max_{K} \sum_{|\alpha|=1}^{2} |D^\alpha \mathbf{u}(x)| \leq C(K) \max_{\partial\Omega} |\mathbf{a}(x)|, \tag{2.3}
$$

for any region K properly contained in Ω and for some positive constant $C(K)$. Moreover,

$$
\max_{\bar{\Omega}} |\mathbf{u}(x)| \leq C \max_{\partial\Omega} |\mathbf{a}(x)|. \tag{2.4}
$$

THEOREM 2.2. *Let Ω be either a bounded domain or an exterior domain of $\mathbb{R}^3$ of class C^2. Assume that* (2.1) *holds and $\mathbf{a}(x) \in W^{1-1/p,p}(\partial\Omega)$, for some $p > 3$. Then, the Dirichlet problem associated with system* (1.1) *admits a unique solution* $(\mathbf{u}(x), \pi(x))$ *such that*

$$
|\mathbf{u}|_q \leq C(q) \max_{\partial\Omega} |\mathbf{a}(x)|, \tag{2.5}
$$

$\forall q \in (1, +\infty)$ *and* $q \in (3, +\infty)$ *respectively for bounded and exterior. Moreover,*

$$
\max_{K} |\nabla \pi(x)| + \max_{K} \sum_{|\alpha|=1}^{2} |D^\alpha \mathbf{u}(x)| + \max_{K} |\mathbf{u}(x)| \leq C(K) \max_{\partial\Omega} |\mathbf{a}(x)|, \tag{2.6}
$$

for any region K properly contained in Ω and for some positive constant $C(K)$. Finally, $\forall \varepsilon \in (0,1)$ there exists a constant $C(\varepsilon)$ such that

$$
\max_{\bar{\Omega}} |\mathbf{u}(x)| \leq C(\varepsilon) \left\{ \max_{\partial\Omega} |\mathbf{a}(x)| + \left(\max_{\partial\Omega} |\mathbf{a}(x)| \right)^{1-\varepsilon} \langle \mathbf{a} \rangle_p^\varepsilon \right\}. \tag{2.7}
$$

REMARK 2.1. We confine ourselves to deal with $n = 2, 3$. However, we aim at pointing out that, by reproducing step by step our methods, (2.5)-(2.7) can be proved for any n.

It is worth noting that the constant C in (2.4) cannot be equal to 1. Indeed, let Ω be the unit disk. It is readily seen that the couple

$$\mathbf{u}(x) = (x_2^2 - 1, x_1^2 - 1), \quad \pi(x) = 2(x_2 + x_1), \quad \forall (x_1, x_2) \in \mathbb{R}^2,$$

is a solution to system (1.1). Also, the value of $|\mathbf{u}(x)|$ is $\sqrt{2}$ at the center of the disk and the maximum of $|\mathbf{u}(x)|$ at the boundary $(x_1^2 + x_2^2 = 1)$ is 1.

A problem of some interest is to find a numerical value of the constant C in (2.5). If Ω is a bounded domain, as will appear clear in the proof of the theorem, C depends on the constant C_b of the biharmonic problem, the constant C_s of the Sobolev imbedding theorem and on the measure of Ω. However, if Ω is a half-space, by properly reproducing a technique by P. Villaggio [28] and making use of the representation formula for the solution to system (1.1) by the Green tensor [6], it is possible to prove the maximum modulus theorem and to give a numerical value of the constant.

Finally, we observe that Theorem 2.2 implies the following result. Assume that $\mathbf{a}(x) \in C(\partial\Omega)$ (i.e. the datum only satisfies (2.1)), then from (2.5)-(2.6) and $(1.1)_2$ one deduces the existence of a pair $(\mathbf{u}(x), \pi(x)) \in C^2(\Omega) \times C^1(\Omega)$, which satisfies system (1.1) and $\mathbf{u}(x) \cdot \vec{n}|_{\partial\Omega} = \mathbf{a}(x) \cdot \vec{n}$, where the trace is understood to belong $H^{-1/p}(\partial\Omega)$, $\forall p \in (1, +\infty)$.

3. - Some preliminary results

We collect the main preliminary results we shall need in the sequel.
Consider the system

$$\begin{aligned} &\nabla \cdot \mathbf{w}(x) = \mathbf{f}(x) \text{ in } \Omega, \\ &\mathbf{w}(x)|_{\partial\Omega} = \hat{\mathbf{w}}(x), \end{aligned} \tag{3.1}$$

with the condition

$$\int_\Omega \mathbf{f}(x) dx = \int_{\partial\Omega} \hat{\mathbf{w}}(x) \cdot \vec{n} d\sigma \tag{3.2}$$

if Ω is bounded.

LEMMA 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with compact boundary. If $\mathbf{f}(x) \in L^p(\Omega)$ $(p > 1)$ and $\hat{\mathbf{w}}(x) \in W^{1-1/p,p}(\partial\Omega)$, then system* (3.1) *admits a solution $\mathbf{w}(x)$ such that $\nabla \mathbf{w}(x) \in L^p(\Omega)$. Moreover, if Ω is bounded and $\hat{\mathbf{w}}(x) = 0$, then*

$$\|\mathbf{w}\|_{1,p} \le C(p)|\mathbf{f}|_p. \tag{3.3}$$

PROOF. See [2-3]. □

LEMMA 3.2. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain and $\psi(x)$ a function such that $\nabla\psi(x) \in L^q(\Omega)$, $q \in (1,3)$. Then there exists a constant ψ_0 such that*

$$|\psi - \psi_0|_{3q/(3-q)} \leq C|\nabla\psi|_q,$$

with a constant C independent of ψ.

PROOF. See [9] Lemma 1.3. □

Consider the following Stokes problem

$$\begin{aligned} &\Delta\varphi(x) - \nabla p(x) = \mathbf{f}(x) \text{ in } \Omega, \\ &\nabla \cdot \varphi(x) = 0 \text{ in } \Omega, \\ &\varphi(x)_{\partial\Omega} = 0 \end{aligned} \tag{3.4}$$

and, if Ω is a three-dimensional exterior domain,

$$\varphi(x) \to 0 \text{ as } |x| \to +\infty. \tag{3.5}$$

Let us recall two well-known results. The former furnishes two *a priori* estimates and existence of solutions to system (3.4) and (3.4)-(3.5) of Cattabriga-Solonnikov type [6, 23-25-26]; the latter is concerned with interior regularity of solutions to elliptic systems [4, 11, 15].

LEMMA 3.3. *Let $\mathbf{f}(x) \in \mathcal{C}_0(\Omega)$. Then, system* (3.4) *admits a unique solution $(\varphi(x), p(x))$ such that*

$$\|\varphi\|_{2,q} + |\nabla p|_q \leq C|\mathbf{f}|_q, \ \forall q \in (1, +\infty). \tag{3.6}$$

Moreover, system (3.4)-(3.5) *admits a unique solution $(\varphi(x), p(x))$ such that*

$$\begin{aligned} &\varphi(x) \in \bigcap_{1<q}^{q<3/2} L^{3q/(3-2q)}(\Omega), \\ &\nabla\varphi(x) \in \bigcap_{1<q}^{q<3/2} L^{3q/(3-q)}(\Omega), \\ &D^2\varphi(x),\ \nabla\varphi(x) \in \bigcap_{1<q}^{q<3/2} L^q(\Omega); \end{aligned} \tag{3.7}$$

and

$$|\varphi|_{\frac{3q}{3-2q}} + |\nabla\varphi|_{\frac{3q}{3-q}} + |D^2\varphi|_q + |\nabla p|_q \leq C|\mathbf{f}|_q, \ \forall q \in (1, 3/2). \tag{3.8}$$

with a constant C independent of $\mathbf{f}(x)$.

PROOF. Inequality (3.6) is proved in [6, 25-26] and in [10, 13] for $n=3$ and $n=2$ respectively; inequality (3.8) is proved in [10, 13-15, 27]. Property (3.7) is trivial if Ω is bounded and can be considered for $q \in (1,+\infty)$. While for exterior domain it is necessary a suitable uniqueness theorem.

In order to prove (3.7), denote by $(\varphi_1(x), p_1(x))$ and $(\varphi_2(x), p_2(x))$ two solutions to system (3.4)-(3.5) corresponding to the same function $\mathbf{f}(x)$, where $\mathbf{f}(x) \in L^{q_1}(\Omega)$ and $\mathbf{f}(x) \in L^{q_2}(\Omega)$ respectively, with $q_2 > q_1$. Set $\mathbf{w}(x) = \varphi_1(x) - \varphi_2(x)$, $p(x) = p_1(x) - p_2(x)$. Since $(\varphi_2(x), p_2(x))$ satisfies (3.8), $\varphi_2(x) \in L^{3q_1/(3-2q_1)}_{\text{loc}}(\Omega)$, $\nabla\varphi_2(x) \in L^{3q_1/(3-q_1)}_{\text{loc}}(\Omega)$ and $D^2\varphi_2(x)$, $\nabla p_2(x) \in L^{q_1}_{\text{loc}}(\Omega)$. Let $(\psi(x), q(x))$ be a solution to system (3.4)-(3.5) corresponding to $\mathbf{g}(x) \in \mathcal{C}_0(\Omega)$. Of course, if $\mathbf{g}(x) \in L^{3q_1/(5q_1-3)}$, from (3.8) it follows that $\psi(x) \in L^{q_1/(q_1-1)}(\Omega)$, $\nabla\psi(x) \in L^{3q_1/(4q_1-3)}(\Omega)$ and $D^2\psi(x)$, $\nabla q(x) \in L^{3q_1/(5q_1-3)}(\Omega)$. Then, multiplying system $\Delta\mathbf{w}(x) = \nabla p(x)$ scalarly by $\psi(x)$, integrating by parts over Ω_R (with $R \gg \operatorname{diam}(\mathbb{R}^3 - \Omega)$) we have

$$(3.9)\qquad (\mathbf{w}, \mathbf{g})_{S_R} = \int_{\partial S_R} (p(x)\psi(x) + \nabla\psi(x)\cdot\mathbf{w}(x) - \nabla\mathbf{w}(x)\cdot\psi - \pi(x)\mathbf{w}(x))\cdot\vec{n}d\sigma.$$

Since $(\mathbf{w}(x), \psi(x))$ and $(p(x), q(x))$ are infinitesimal at infinity([1]), and in neighbourhood at infinity they are biharmonic and harmonic, respectively, by classical results (cf. e.g. [22]) we see that the surface integral in (3.9) tends to zero as $R \to +\infty$. Then, from (3.9) it follows that $(\mathbf{w}, \mathbf{g}) = 0$, $\forall\mathbf{g}(x) \in \mathcal{C}_0(\Omega)$. Hence the desired result follows at once. □

LEMMA 3.4. *Let $\mathbf{a}(x) \in W^{1-1/q,q}(\partial\Omega)$ with $q \in (1,+\infty)$ and $q \in (3/2,3)$ for Ω bounded or exterior respectively.*

If Ω is bounded, then system (1.1)-(2.2)$_1$ admits a unique weak solution $(\mathbf{u}(x), \pi(x)) \in J^{1,q}(\Omega) \times L^q(\Omega)$ and

$$(3.10)\qquad \|\mathbf{u}\|_{1,q} + |\pi|_q \le C(q)\|\mathbf{a}\|_{W^{1-1/q,q}(\partial\Omega)}.$$

If Ω is exterior, then system (1.1)-(2.2) has a unique weak solution $(\mathbf{u}(x), \pi(x))$ such that $\nabla\mathbf{u}(x) \in L^q(\Omega)$, $\mathbf{u}(x) \in L^{3q/(3-q)}(\Omega)$ and $\pi(x) \in L^q(\Omega)$.

PROOF. The first part of the lemma is proved in [10, Theorem 2.1]. As far as the second part is concerned, by virtue of Lemma 3.1, there exists a divergence-free field $\mathbf{A}(x) \in W^{1,q}(\Omega)$ such that $\mathbf{A}(x) = \mathbf{a}(x)$ on $\partial\Omega$. Since there exists a regular sequence $\{\mathbf{A}_n(x)\}_{n\in N}$ such that $\nabla\cdot\mathbf{A}_n(x) = 0$ and $\mathbf{A}_n(x)$ is a bounded linear functional on $J^{1,q'}(\Omega)$ $\left(\dfrac{1}{q} + \dfrac{1}{q'} = 1\right)$ for all n, and $\mathbf{A}_n(x) \to \mathbf{A}(x)$ in $W^{1,q}(\Omega)$, by Theorem 5.1 in [10] we can

([1]) Actually, for a function $\chi(x)$ such that $\nabla\chi(x) \in L^r(\Omega)$, with $r \in (1,3)$, from Lemma 3.2, it follows that there exists a constant χ_0 such that $\chi(x) - \chi_0 \to 0$ for $|x| \to \infty$ in a suitable sense. However, since the pressure $((\pi(x), p(x)))$ can be defined up to a constant and satisfies a harmonic equation, then we can understand that the pressure tends to zero at infinity.

find a weak solution $(\mathbf{U}_n(x), \Pi_n(x))$ to system (3.4)- (3.5) corresponding to $\mathbf{f}_n(x) = \Delta\mathbf{A}_n(x) \in H^{-1,q}(\Omega)$ and such that

$$|\nabla\mathbf{U}_n|_q + |\Pi_n|_q \leq C|\Delta\mathbf{A}_n|_{-1,q} \leq C|\nabla\mathbf{A}_n|_q.$$

Hence, by the linearity of the problem, it follows that a pair $(\mathbf{U}(x), \Pi(x))$ with $\mathbf{U}(x) \in J^{3q/(3-q)}(\Omega)$, $\Pi(x)$, $\nabla\mathbf{U}(x) \in L^q(\Omega)$, exists such that

$$\lim_{n\to+\infty}\{|\mathbf{U}_n - \mathbf{U}|_{3q/(3-q)} + |\nabla\mathbf{U}_n - \nabla\mathbf{U}|_q + |\Pi_n - \pi|_q\} = 0.$$

Then, setting $\mathbf{u}(x) = \mathbf{U}(x)+\mathbf{A}(x)$, it is not difficult to see that the pair $(\mathbf{u}(x), \pi(x))$ is a weak solution to system (1.1)-(2.2). As far as uniqueness is concerned, we consider a pair $(\mathbf{W}(x), \Pi(x))$ wich is a weak solution to system (1.1)-(2.2) with homogeneous boundary conditions and $\mathbf{W}(x) \in J^{3q(3-q)}(\Omega) \cap \hat{J}^{1,q}(\Omega)$. A density argument implies that

$$(\nabla\mathbf{W}, \nabla\varphi) = 0, \ \ \forall\varphi(x) \in \hat{J}^{1,q'}(\Omega) \ \ q' = q/(q-1).$$

In particular we have

$$(\mathbf{W}, \Delta\varphi) = 0, \ \ \forall\varphi(x) \in \hat{J}^{1,q'}(\Omega) \text{ with } D^2\varphi(x) \in L^{3q/(4q-3)}(\Omega). \tag{3.11}$$

Let $\varphi(x)$ in (3.11) be the solution to system (3.4)-(3.5), with $\mathbf{f}(x) \in C_0(\Omega)$. Taking into account (3.8), an integration by parts gives

$$0 = (\mathbf{W}, \Delta\varphi) = (\mathbf{W}, \nabla p + \mathbf{f}) = (\mathbf{W}, \mathbf{f}), \ \ \forall\mathbf{f}(x) \in C_0(\Omega),$$

which ensures uniqueness. □

The case of the exterior domain in the above lemma is proved in [13] also; since the proof is very short, we have included it here.

LEMMA 3.5. *Let* $(\mathbf{u}(x), \pi(x))$ *be a weak solution to system* (1.1) *such that* $\mathbf{u}(x) \in W^{1,2}_{\text{loc}}(\Omega)$. *Then,* $\forall R < 2^{-4}\operatorname{dist}(x_0, \partial\Omega)$

$$\max_{S_R(x_0)} \sum_{|\alpha|=0}^{2} |D^\alpha\mathbf{u}(x)| \leq C(R)|\mathbf{u}|_{L^2(S_R(x_0))}, \tag{3.12}$$

for some positive constant $C(R)$ *which diverges as* $\operatorname{dist}(x_0, \partial\Omega) \to 0$.

PROOF. The lemma is proved in [11]. Inequality (3.12) is a suitable coupling of the results of Theorem 1.1, Theorem 1.4 and Remark 1.5 of [11]. □

From Lemma 3.5 it follows that the weak solution to system (1.1)-(2.2), whose existence is given by Lemma 3.4, is regular in Ω and at the boundary for a proper choice of q.

REMARK 3.1. It is possible to deduce inequality (3.12) by taking into account some recent results obtained in [24]. Among other things, in [24] it is proved a mean value formula for solutions to biharmonic equations and Stokes system. This formula implies in particular (3.12).

LEMMA 3.6. *Let $g(x) \in W^{1,p}(\Omega)$, with $p > 3$ and Ω bounded, and let $\bar{p} > 1$. Then*

$$|g|_\infty \le C(|\nabla g|_p^\eta |g|_{\bar{p}}^{1-\eta} + |g|_{\bar{p}}), \tag{3.13}$$

with $\eta = 3p[\bar{p}(p-3)+3p] - 1$.

PROOF. See [8] Lemma 5.V. □

4. - Proof of the theorems

We first prove the theorems by assuming the boundary data $\mathbf{a}(x)$ to be sufficiently smooth, i.e., $\mathbf{a}(x) \in C^1(\partial\Omega)$. Then, starting from the results obtained by means the above regularity assumptions, we are able to prove our theorems under the hypotheses $\mathbf{a}(x) \in C(\partial\Omega)$ and $\mathbf{a}(x) \in W^{1-1/p,p}(\partial\Omega)$ for $n = 2$ and $n = 3$ respectively.

We start by proving the following:

LEMMA 4.1. *Let $\mathbf{a}(x) \in C^1(\partial\Omega)$. Then system* (1.1)-(2.2) *admits a unique weak solution* $(\mathbf{u}(x), \pi(x))$ *such that*

$$|\mathbf{u}|_q \le C(q) \max_{\partial\Omega} |\mathbf{a}(x)|, \tag{4.1}$$

with $q > 1$ and $q > 3$ for Ω bounded and exterior respectively.

PROOF. For the existence and uniqueness of $(\mathbf{u}(x), \pi(x))$ we employ Lemma 3.4. Under our hypotheses on $\mathbf{a}(x)$ we can choose $q = 2$ in Lemma 3.4. Moreover, by density argument we can consider $\mathbf{u}(x)$ satisfying the identity

$$(\nabla\mathbf{u}, \nabla\varphi) = 0, \;\; \forall\varphi(x) \in \hat{J}^{1,2}(\Omega).$$

In particular we have

$$(\mathbf{u}, \Delta\varphi) = \int_{\partial\Omega} \mathbf{a}(x) \cdot (\vec{n} \cdot \nabla\varphi(x)) d\sigma \tag{4.2}$$

$$\forall\varphi(x) \in \hat{J}^{1,2}(\Omega) \text{ with } D^2\varphi(x) \in L^{6/5}(\Omega).$$

Let $\varphi(x)$ in (4.2) be the solution to system (3.4), or (3.4)-(3.5), given by Lemma 3.3, with $\mathbf{f}(x) \in \mathcal{C}_0(\Omega)$. Taking into account (3.8), we have

$$(4.3)\qquad (\mathbf{u},\mathbf{f}) = \int_{\partial\Omega} \mathbf{a}(x)\cdot(\vec{n}\cdot\nabla\varphi(x))d\sigma + \int_{\partial\Omega} \mathbf{a}(x)\cdot\vec{n}(p(x)-\bar{p})d\sigma,\ \ \forall\mathbf{f}(x)\in\mathcal{C}_0(\Omega),$$

where $\bar{p} = [\text{meas}\{\Omega\}]^{-1}\int_{\Omega} p(x)dx$ and $\bar{p}$ is the constant which appears in Lemma 3.2 for Ω bounded and exterior respectively. By the trace theorem [7] and the Hölder inequality we have

$$\left|\int_{\partial\Omega} \mathbf{a}(x)\cdot(\vec{n}\cdot\nabla\varphi(x))d\sigma\right|$$
$$\leq C\max_{\partial\Omega}|\mathbf{a}(x)|\{|\nabla\varphi|_{L^{q'}(\Omega')} + |D^2\varphi|_{L^{q'}(\Omega')}\}$$
$$\leq \max_{\partial\Omega}|\mathbf{a}(x)|\begin{cases} C(\Omega')\{|\nabla\varphi|_{L^{3q'/(3-q')}(\Omega')} + |D^2\varphi|_{L^{q'}(\Omega')}\} \\ \qquad\qquad \forall q'\in(1,3/2)\ \text{if}\ \Omega\ \text{exterior}, \\ C\{|\nabla\varphi|_{L^{q'}(\Omega)} + |D^2\varphi|_{L^{q'}(\Omega)}\} \\ \qquad\qquad \forall q'\in(1,\infty)\ \text{if}\ \Omega\ \text{bounded}, \end{cases}$$

where Ω' is a bounded neighbourhood of $\partial\Omega$, and of course we have majorezed taking (3.7) into account. Moreover, for the latter integral on the right-hand side of (4.3), by employing the trace theorem and Hölder inequality, we have

$$\left|\int_{\partial\Omega} \mathbf{a}(x)\cdot\vec{n}(p(x)-\bar{p})d\sigma\right|$$
$$\leq C\max_{\partial\Omega}|\mathbf{a}(x)|\{|p-\bar{p}|_{L^{q'}(\Omega')} + |\nabla p|_{L^{q'}(\Omega')}\}$$
$$\leq \max_{\partial\Omega}|\mathbf{a}(x)|\begin{cases} C(\Omega')\{|p-\bar{p}|_{L^{3q'/(3-q')}(\Omega')} + |\nabla p|_{L^{q'}(\Omega')}\} \\ \qquad\qquad \forall q'\in(1,3/2)\ \text{if}\ \Omega\ \text{exterior}, \\ C\{|p-\bar{p}|_{L^{q'}(\Omega)} + |\nabla p|_{L^{q'}(\Omega)}\} \\ \qquad\qquad \forall q'\in(1,\infty)\ \text{if}\ \Omega\ \text{bounded}; \end{cases}$$

where Ω' is a bounded neighbourhood of $\partial\Omega$. Finally, applying the Poincaré inequality and Lemma 3.2 we deduce

$$\left|\int_{\partial\Omega} \mathbf{a}(x)\cdot\vec{n}(p(x)-\bar{p})d\sigma\right|$$
$$\leq C\max_{\partial\Omega}|\mathbf{a}(x)||\nabla p|_{L^{q'}(\Omega)}\begin{cases}\forall q'\in(1,3/2) & \text{if}\ \Omega\ \text{exterior}, \\ \forall q'\in(1,\infty) & \text{if}\ \Omega\ \text{bounded}, \end{cases}$$

again we have majorezed taking (3.7) into account. By these last inequalities, (3.6) and (3.8), we can majorize the right-hand side of (4.3) as follows:

$$\left|\int_{\partial\Omega} \mathbf{a}(x)\cdot(\vec{n}\cdot\nabla\varphi(x))d\sigma\right| + \left|\int_{\partial\Omega} \mathbf{a}(x)\cdot\vec{n}(p(x)-\bar{p})d\sigma\right| \le C\max_{\partial\Omega}|\mathbf{a}(x)||\mathbf{f}|_{q'}, \tag{4.4}$$

with $q' \in (1,+\infty)$ and $q' \in (1,3/2)$ respectively for Ω bounded and exterior. Making use of (4.4) in (4.3) we get

$$|(\mathbf{u},\mathbf{f})| \le C\max_{\partial\Omega}|\mathbf{a}(x)||\mathbf{f}|_{q'},$$

which implies (4.1). □

PROOF OF THEOREM 2.1. Since $\mathbf{a}(x)\in C^1(\partial\Omega)$, by virtue of Lemmas 4.1 and 3.5 there exists a unique regular solution to system (1.1)-$(2.2)_1$.

Suppose that $\partial\Omega = \bigcup_{j=1}^{m}\partial\Omega_i$ $(\Omega_i\subset\mathbb{R}^2)$. Let

$$\mathbf{a}^1(x)=\mathbf{a}(x)-\left(\int_{\partial\Omega_i}d\sigma\right)^{-1}\left(\int_{\partial\Omega_i}\mathbf{a}(x)\cdot\vec{n}d\sigma\right)\vec{n},\ \forall x\in\partial\Omega_i,\ i=1,\dots,m; \tag{4.5}$$

$$\mathbf{a}^2(x)=\mathbf{a}(x)-\mathbf{a}^1(x),\ \forall x\in\partial\Omega_i,\ i=1,\dots,m. \tag{4.6}$$

Of course

$$\int_{\partial\Omega_i}\mathbf{a}^1(x)\cdot\vec{n}d\sigma=0,\ \forall i=1,\dots,m, \tag{4.7}$$

$$\int_{\partial\Omega}\mathbf{a}^2(x)\cdot\vec{n}d\sigma=0. \tag{4.8}$$

Denote by $(\mathbf{u}^1(x),\pi^1(x))$ and $(\mathbf{u}^2(x),\pi^2(x))$ the weak solutions to system (1.1) corresponding to $\mathbf{a}^1(x)$ and $\mathbf{a}^2(x)$ respectively. Moreover, from (4.1) and (3.10) we have

$$|\mathbf{u}^1|_p\le C\max_{\partial\Omega}|\mathbf{a}^1(x)|\le C\max_{\partial\Omega}|\mathbf{a}(x)|\ \forall p\in(1,+\infty), \tag{4.9}$$

$$|\mathbf{u}^2|_{1,p}\le C\{|\mathbf{a}^2|_{L^p(\partial\Omega)}+\langle\mathbf{a}^2\rangle_p\}\le C\max_{\partial\Omega}|\mathbf{a}(x)|\ \forall p\in(1,+\infty), \tag{4.10}$$

where the constant C depends on $\partial\Omega$ only. Of course we have

$$\mathbf{u}(x)=\mathbf{u}^1(x)+\mathbf{u}^2(x),\ \forall x\in\Omega. \tag{4.11}$$

As is well known, by virtue of $(1.1)_2$ and (4.7) there exists a stream function $h(x)$ such that

$$u_1^1(x) = \frac{\partial}{\partial x_2} h(x), \;\; u_2^1(x) = -\frac{\partial}{\partial x_1} h(x), \;\; \Delta^2 h(x) = 0, \;\; \forall x \in \Omega,$$

so that applying the maximum modulus theorem by Miranda [18] to the biharmonic equation $\Delta^2 h(x) = 0$, we have that

$$\max_{\bar{\Omega}} |h(x) - H| + \max_{\bar{\Omega}} |\nabla h(x)| \leq C\left\{ \max_{\partial\Omega} |h(x) - H| + \max_{\partial\Omega} |\nabla h(x)| \right\}, \tag{4.12}$$

for any constant H. Now, by Sobolev's imbedding theorem [19], from (4.12) it follows that

$$\max_{\bar{\Omega}} |\nabla h(x)| \leq C\left\{ |h - H|_{p_1} + |\nabla h|_{p_1} + \max_{\partial\Omega} |\nabla h(x)| \right\},$$

for some $p_1 > 2$. Hence, choosing $H = [\text{meas}\{\Omega\}]^{-1} \int_\Omega h(x)dx$ and making use of the Poincaré inequality, we have that

$$\max_{\bar{\Omega}} |\nabla h(x)| \leq C\left\{ |\nabla h|_{p_1} + \max_{\partial\Omega} |\nabla h(x)| \right\}. \tag{4.13}$$

On the other hand, since $|\nabla h(x)| = |\mathbf{u}^1(x)|$ $\forall x \in \bar{\Omega}$, (4.9) and (4.13) imply

$$\max_{\bar{\Omega}} |\mathbf{u}^1(x)| \leq C \max_{\partial\Omega} |\mathbf{a}(x)|.$$

Then, choosing $p > 2$ in (4.10) by appealing to Sobolev's imbedding theorem again, we get

$$\max_{\bar{\Omega}} |\mathbf{u}^2(x)| \leq C \max_{\partial\Omega} |\mathbf{a}(x)|.$$

Taking into account (4.11) we deduce (2.4) with a constant C depending on $\partial\Omega$ only. Moreover, (2.3) is a consequence of (3.12) and (4.1). Finally, the regularity of the pressure $\pi(x)$ can be deduced directly from equation $(1.1)_1$.

Let us prove the theorem under the assumption (2.1). Let $\mathbf{a}(x) \in C(\partial\Omega)$. Then, there exists a sequence $\{\bar{\mathbf{a}}_n(x)\}_{n\in N}$, with $\bar{\mathbf{a}}_n(x) \in C^1(\partial\Omega)$, which converges to $\mathbf{a}(x)$ in $C(\partial\Omega)$.

Let

$$\mathbf{a}_n(x) = \bar{\mathbf{a}}_n(x) - \left(\int_{\partial\Omega} \bar{\mathbf{a}}_n(x) \cdot \vec{n} d\sigma \right) \left(\int_{\partial\Omega} d\sigma \right)^{-1} \vec{n}.$$

It is evident that $\mathbf{a}_n(x) \in C^1(\partial\Omega)$ and

$$\int_{\partial\Omega} \mathbf{a}_n(x) \cdot \vec{n} d\sigma = 0, \;\; \mathbf{a}_n(x) \to \mathbf{a}(x) \text{ in } C(\partial\Omega).$$

By (2.3) (2.4), there exists a sequence of solutions $\{(\mathbf{u}_n(x), \pi(x))\}_{n\in N}$ to system (1.1), such that $\mathbf{u}_n(x) \in C^2(\Omega) \cap C(\bar{\Omega})$, $\pi_n(x) \in C^1(\Omega)$ and

$$\max_K \sum_{|\alpha|=1}^{2} |D^\alpha \mathbf{u}_n(x)| + |\mathbf{u}_n|_p + \max_{\bar{\Omega}} |\mathbf{u}_n(x)| \tag{4.14}$$
$$+ \max_K |\nabla \pi_n| \leq C \max_{\partial\Omega} |\mathbf{a}_n(x)|$$

with C independent of n.

Since system (1.1)-(2.1) is linear, from (4.14) we have that $\{(\mathbf{u}_n(x), \pi_n(x))\}_{n\in N}$ is a Cauchy sequence so that it converges to the desidered solution $(\mathbf{u}(x), \pi(x))$. □

PROOF OF THEOREM 2.2. Inequality (2.5) has been proved in Lemma 4.1; (2.6) is a consequence of (2.5) and Lemma 3.5. In order to prove (2.7) we can reduce only to deal with the case where Ω is a bounded domain. Indeed, if Ω is an exterior domain, by virtue of (2.5)-(2.6), there exist two positive constants $\bar{R} > \text{diam}(\Omega^c)$ and $C(\bar{R})$ such that

$$\max_{|x|\geq \bar{R}} |\mathbf{u}(x)| \leq C(\bar{R}) \max_{\partial\Omega} |\mathbf{a}(x)|.$$

Therefore, the result is achieved if we show that (2.7) holds for the solutions to the following boundary value problem:

$$\begin{aligned} &\Delta \mathbf{V}(x) + \nabla \Pi(x) = 0 && \text{in } \Omega \cap S_{\bar{R}}, \\ &\nabla \cdot \mathbf{V}(x) = 0 && \text{in } \Omega \cap S_{\bar{R}}, \\ &\mathbf{V}(x)|_{\partial\Omega} = \mathbf{a}(x), && \\ &\mathbf{V}(x)|_{|x|=\bar{R}} = \mathbf{u}(x)|_{|x|=\bar{R}}. && \end{aligned}$$

From inequality (3.13) and (3.10) it follows that

$$\begin{aligned} \max_{\bar{\Omega}} |\mathbf{u}(x)| &\leq C(|\nabla \mathbf{u}|_p^\eta |\mathbf{u}|_{\bar{p}}^{1-\eta} + |\mathbf{u}|_{\bar{p}}) \\ &\leq C\{(\langle \mathbf{a} \rangle_p + |\mathbf{a}|_{L^p(\partial\Omega)})^\eta |\mathbf{u}|_{\bar{p}}^{1-\eta} + |\mathbf{u}|_{\bar{p}}\} \end{aligned}$$

with $\eta = 3p[\bar{p}(p-3) + 3p]^{-1}$. Hence, by properly choosing $\bar{p}$ and taking into account (2.5), the desired result follows. Finally, the proof of the theorem under the hypothesis $\mathbf{a}(x) \in W^{1-1/p,p}(\partial\Omega)$ $(p > 3)$ is achieved by repeating the final steps in the proof of Theorem 2.1. □

REFERENCES

[1] S. AGMON, *Maximum theorems for solutions of higher order elliptic equations.* Bull. Amer. Math. Soc. **66** (1960), 77-80.

[2] M.E. BOGOVSKII, *Solution of the the first boundary value problem for the equation on continuity of an incompressible medium.* Soviet Math. Dokl. **20** (1979), 1094-1098.

[3] W. BORCHERS - W. SOHR, *On the equations* rot **v**=**g** *and* div **u**=*f with zero boundary conditions.* Hokkaido Math. J. **19** (1990), 1-67.

[4] S. CAMPANATO, Sistemi ellittici in forma divergenza. Regolarità all'interno. Quaderni Scuola Norm. Sup. Pisa 32, Pisa, 1980.

[5] A. CANFORA, *Teorema del massimo modulo e teorema di esistenza per il problema di Dirichlet relativo ai sistemi fortemente ellittici.* Ricerche Mat. **15** (1966), 249-294.

[6] L. CATTABRIGA, *Su un problema al contorno relativo al sistema di equazioni di Stokes.* Rend. Sem. Mat. Univ. Padova **31** (1961), 308-340.

[7] E. GAGLIARDO, *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili.* Rend. Sem. Mat. Univ. Padova **27** (1957), 284-305.

[8] E. GAGLIARDO, *Proprietà di alcuni classi di funzioni in più variabili.* Ricerche Mat. **7** (1958), 102-137.

[9] G.P. GALDI - P. MAREMONTI, *Monotonic decreasing and asymptotic behaviour of the kinetic energy for weak solutions of the Navier-Stokes equations in exterior domains.* Arch. Rational Mech. Anal. **94** (1986), 253-266.

[10] G.P. GALDI - C.G. SIMADER, *Existence, uniqueness and L^q-estimates for the Stokes problem in an exterior domain.* Arch. Rational Mech. Anal. **112** (1990), 291-318.

[11] M. GIAQUINTA - G. MODICA, *Non linear systems of the type of stationary Navier-Stokes system.* J. Reine Angew. Math. **330** (1982), 157-172.

[12] G. FICHERA, *Il teorema del massimo modulo per l'equazione dell'elastostatica.* Arch. Rat. Mech. Anal. **7** (1961), 373-387.

[13] H. KOZONO - H. SOHR, *New a priori estimates for the Stokes equations inexterior domains.* Indiana Univ. Math. J. **41** (1991), 1-27.

[14] H. KOZONO - H. SOHR, *Or a new class of generalized solutions for the Stokes equations in exterior domains.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **19** (1992), 155-181.

[15] P. MAREMONTI - V.A. SOLONNIKOV, *Su una maggiorazione delle soluzioni del problema di Stokes in domini esterni.* Zap. Nauchn. Sem. Leningrad. Odtel. Mat. Inst. Stiklov. (LOMI) **180** (1990), 105-120.

[16] C.B. MORREY JR., *Second order elliptic systems of differential equations.* Ann. of Math. Stud. **33** (1954), 101-159.

[17] J. NAUMANN, *On a maximum principle for weak solutions of the stationary Stokes System.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **16** (1989), 149-168.

[18] C. MIRANDA, *Formule di maggiorazione e teorema di esistenza per le funzioni biarmoniche di due variabili.* Giorn. Mat. Battaglini **78** (1948), 97-118.

[19] C. MIRANDA, *Teorema del massimo modulo e teorema di esistenza e di unicità per il problema di Dirichlet relativo alle equazioni ellittiche in due variabili.* Ann. Mat. Pura Appl. **46** (1958), 265-312.

[20] C. MIRANDA, Istituzioni di analisi funzionale lineare e applicata I-II. U.M.I., Oderisi da Gubbio Editrice, Gubbio, 1978.

[21] M. PICONE, *Nuovi indirizzi di ricerca nella teoria e nel calcolo delle soluzioni di talune equazioni lineari alle derivate parziali della Fisica Matematica.* Ann. Scuola Norm. Sup. Pisa, **5** (1936), 213-288.

[22] G. PÓLYA, *Liegt die Stelle der gröbten Beansprunchung an der Oberfläche?.* Ztschr. f. angew. Math. und Mech. **10** (1930), 353-360.

[23] M.H. PROTTER - H.F. WEINBERGER, *Maximum principles in differential equations.* Prentice-Hall, Englewood Cliffs, NJ, 1967.

[24] C.G. SIMADER, *Mean value formulas, Weyl's lemma and Liouville theorems for* Δ^2 *and Stokes' system.* Results in Math. **22** (1992), 761-780.

[25] V.A. SOLONNIKOV, *On estimates of Green's tensors for certain boundary problems.* Soviet Math. Dokl. **1** (1960), 128-131.

[26] V.A. SOLONNIKOV, *General boundary value problems for Douglis-Nirenberg elliptic systems.* Proc. Steklov Inst. Math. **92** (1966), 269-339.

[27] V.A. SOLONNIKOV, *Estimates of solutions of nonstationary Navier-Stokes equations,* J. Soviet Math. **8** (1977), 467-529.

[28] P. VILLAGGIO, *Maximum modulus theorems for the elastic half-space,* Riv. Mat. Univ. Parma **5** (1979), 663-672.

Dipartimento di Matematica
Università di Basilicata
Via Nazario Sauro, 85
85100 Potenza

Istituto di Matematica
II Università di Napoli
81100 Caserta