ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

J. J. L. VELÁZQUEZ

Curvature blow-up in perturbations of minimal cones evolving by mean curvature flow

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 21, nº 4 (1994), p. 595-628

<http://www.numdam.org/item?id=ASNSP_1994_4_21_4_595_0>

Curvature Blow-up in Perturbations of Minimal Cones Evolving by Mean Curvature Flow(*)

J.J.L. VELÁZQUEZ

1. - Introduction

A smooth one-parameter family of hypersurfaces $\{T(t)\} \subset \mathbb{R}^{n+1}$ where $0 < t < T < +\infty$ is said to evolve by its mean curvature if the normal velocity $V_N(P)$ at any point $P \in \Gamma(t)$ coincides with the mean curvature of $\Gamma(t)$ at P, i.e.,

$$V_N = H \text{ on } \Gamma(t), \quad 0 < t < T, \tag{1.1}$$

where H denotes the mean curvature of the hypersurface $\Gamma(t)$. Properties of mean curvature flows (MCF) have been extensively analysed, and in recent years great attention has been paid to developing various theories of generalized solutions for (1.1). For instance, in the seminal work by Brakke [B], a theory of generalized solutions was introduced for varifolds in $\mathbb{R}^{n+1}$ of arbitrary codimension, which satisfy a weak formulation of (1.1). In particular, existence of at least one such global solution for arbitrary initial data was proved there. It was also shown in [B] that such solutions can be nonunique.

In the case where $n = 1$, a theory of generalized solutions for curve shortening on surfaces has been developed in [A1], [A2]. Another method to obtain generalized solutions can be found in [AG]. In the higher dimensional case $n > 1$, an approach has been developed in [CGG] and [ES] which is based on the theory of viscosity solutions for nonlinear elliptic and parabolic equations. In these works, the so-called level set equation in considered. This is a highly degenerate parabolic equation which arises when the level sets of a function evolve by MCF. The theory in [CGG] and [ES] yields existence and uniqueness of generalized solutions for the level set equation and a large class of initial values. Another weak formulation based on a singular limit of a

(*) Partially supported by CICYT Grant PB90-0235 and NATO Grant CRG 920196.
Pervenuto alla Redazione il 2 Agosto 1993.

reaction diffusion equation of the type

$$u_t = \Delta u + \frac{1}{\varepsilon^2} u(1 - u^2) \quad \text{as } \varepsilon \to 0$$

has been proposed by De Giorgi ([DG]). We refer to [ESS] for a discussion of the relations between this approach and those previously discussed. We finally refer to another theory of generalized solutions of MCF that have been suggested in [S].

A question which has deserved much attention is that of blow up, i.e., the description of the possible singularities that smooth hypersurfaces may develop as they evolve by mean curvature flow. One is then led to distinguish between fast and slow blow up. Let $a = (a_{i,j})$ be the second fundamental form of the hypersurface, and let

$$A(t) = \sup_{P \in \Gamma(t)} a(P, t). \tag{1.2}$$

If at a particular time $t = T < \infty$ the evolution of $\Gamma(t)$ cannot be continued smoothly, then $\lim_{t \uparrow T} A(t) = \infty$ (cf. [A4] for details). Two possibilities then arise. Either

$$A(t) \leq C(T - t)^{-1/2} \text{ for } 0 < t < T \text{ and some } C > 0, \tag{1.3}$$

or

$$\lim_{t \uparrow T} \sup (T - t)^{1/2} A(t) = \infty \tag{1.4}$$

when (1.2) holds (resp. when (1.3) is satisfied) we shall say that the solution exhibits fast blow up (resp. slow blow up). The fast blow up case has been considered by Huiskens in [H]. The author proves there the following result. For $x \in \Gamma(t)$ and $P \in \Gamma(t)$, let us write

$$\Phi_\tau^p(x) = e^{\tau/2}(x - P); \quad \tau = -\log(T - t).$$

Then the rescaled hypersurfaces

$$\Sigma^p(\tau) = \Phi_\tau^p(\Gamma(T - e^{-\tau}))$$

converge sequentially (i.e., up to the choice of a suitable subsequence $\{\tau_n\}$ with $\lim_{n \to \infty} \tau_n = \infty$) to a self-similar solution, which is hypersurface of the form

$$\Gamma(t) = (T - t)^{1/2} \Gamma^*,$$

where Γ^* is a hypersurface in $\mathbb{R}^{n+1}$. One often refers to this fact in an abridged way by saying that when fast blow up occurs, the behaviour near a singularity is (sequentially) self-similar.

When the slow blow up case (1.4) holds, one has the following situation (cf. [A4]). For $x \in \Gamma(t)$, $P \in \Gamma(t)$ and $t_0 < T$ with $-t_0 A(t_0)^{-2} < t < (T - t_0)A(t_0)^{-2}$ and $A(t)$ given in (1.2), we define

(1.5a) $$\Gamma^{P,t_0} = \psi^{P,t_0}(\Gamma(t_0 + \varepsilon^2 t))$$

where

(1.5b) $$\varepsilon \equiv \varepsilon(t_0) = (A(t_0))^{-1} \quad \text{(cf. (1.2))}, \quad \psi^{P,t_0}(x) = \frac{x - P}{\varepsilon(t_0)}$$

Then there exist sequences $\{t_n\}$, $\{P_n\}$ with $\lim_{n\to\infty} t_n = T$ and $P_n \in \Gamma(t_n)$, and a subsequence $\{n_j\}$ such that:

(1.6) The family $\Gamma^{n_j}(t) \equiv \Gamma^{p_{n_j}, t_{n_j}}$ converges to a family of hypersurfaces $\{\Gamma^\infty(t)\}$, $-\infty < t < \infty$, whose curvatures are uniformly bounded. Such a family is called an eternal solution in the terminology of [A4]. This eternal solution satisfies
$$|a| \leq 1 \text{ globally}, \ |a| = 1 \text{ at } P = 0, \ t = T$$
where $a = (a_{i,j})$ is the corresponding second fundamental form.

In this paper, we shall describe a slow blow up mechanism for MCF. Consider the Simons cone (cf. [G], [S])

(1.7) $$C_n = \{(x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} : x_1^2 + \ldots + x_n^2 = x_{n+1}^2 + \ldots + x_{2n}^2\}$$

which has dimension $d = 2n - 1$. It has been proved by Bombieri, De Giorgi and Giusti in [BGG] that C_n is a globally minimizing surface for $n \geq 4$. We shall prove here the following:

THEOREM. *Let $n \geq 4$. For $T > 0$ small enough there exists a family of surfaces $\{\Gamma(t)\}$, $0 < t < T$, of dimension $d = 2n - 1$ which evolve by* MCF *and is such that, as $t \uparrow T$, it blows up slowly towards a surface $\Gamma(t)$ which behaves asymptotically as the Simons cone as $|x| \to 0$. Moreover, there exists a smooth minimal surface M and a positive number σ such that $(T - t)^{-\frac{1}{2}\sigma}\Gamma(t)$ approaches to M as $t \uparrow T$, uniformly on compact sets.*

As a matter of fact, precise estimates are obtained on the asymptotics of the family $\{\Gamma(t)\}$ as $t \uparrow T$. The reader is referred to the next Section (cf. in particular Proposition 2.1 and Theorems 2.2 and 2.3 therein) for a detailed statement of such results.

We shall conclude this Introduction by remarking on the relation between this Theorem and other known blow up results for MCF. The structure of singularities is well-known for the case of convex hypersurfaces, a situation in which there is always fast blow up (cf. (1.3)) and the limit surface is a sphere (cf. [Hu], [GH]). For embedded plane curves there is always evolution to a convex curve before collapse to a round point occurs (cf. [Gr]). For immersed curves, both fast and slow blow up can actually occur (cf. for instance [A4] for a recent survey). Finally, for rotationally symmetric surfaces one may again have slow or fast blow up, and near the singularities there is convergence to self-similar spheres or cylinders (cf. [AlAG]).

We should point out here that the structure of singularities which arise in MCF is strikingly similar to those appearing in the semilinear heat equation

$$u_t = \Delta u + u^p; \quad p > 1, \quad x \in \mathbb{R}^N, \ t > 0. \tag{1.8}$$

It is well known that solutions of (1.8) may blow up in a finite time, say $t = T$. If $x = 0$ is a blow up point for (1.8), it was proved in [GK1] that, under the additional hypotheses

$$u(x,t) \le C(T-t)^{-\frac{1}{p-1}} \quad \text{for some } C > 0 \text{ and } t < T, \tag{1.9a}$$

$$N = 1 \quad \text{or} \quad 1 < p < \frac{N+2}{N-2} \text{ when } N \ge 3, \tag{1.9b}$$

one then has that the rescaled function

$$\Phi(y,\tau) = e^{-\frac{1}{p-1}\tau} u(ye^{-\tau/2}, \ T - e^{-\tau}) \tag{1.10}$$

converges uniformly for bounded $|y|$ towards a constant Φ_0, where

$$\Phi_0 = 0 \quad \text{or} \quad \Phi_0 = (p-1)^{-\frac{1}{p-1}}. \tag{1.11}$$

It was proved in [GK2] that (1.9a) holds if (1.9b) holds. The case $\Phi_0 = 0$ was then ruled out in [GK3]; see also [GP] for an independent proof of the one dimensional case $N = 1$. We will say that a solution $v(x,t)$ of (1.8) is self-similar if it has the form

$$v(x,t) = (T-t)^{-\frac{1}{p-1}} \Phi\left(\frac{x}{\sqrt{T-t}}\right);$$

in such a case, Φ satisfies

$$\Delta\Phi - \frac{1}{2} y \nabla\Phi + \Phi^p - \frac{\Phi}{p-1} = 0 \tag{1.12}$$

and from the arguments in [GK1] it readily follows that, if (1.9a) is assumed, function $\Phi(y,\tau)$ in (1.10) converges sequentially to a bounded solution of (1.12). When (1.9b) also holds, the only such solutions of (1.12) are those in (1.11), but for $p > \dfrac{N+2}{N-2}$ there exist nonconstant, radially symmetric solutions of (1.12) (cf. [BQ], [T], [L]). Roughly speaking, when (1.9a) holds, blow up for (1.8) is sequentially self-similar, a case we have already found to happen in MCF when fast blow-up occurs. As a matter of fact, one can distinctly spot deep analogies in the argument in [H] (where fast blow up for MCF was discussed) and those in [GK1], [GK2], [GK3], a series of papers which opened the road towards a detailed analysis of blow up for (1.8). Much progress has been achieved in this direction in recent years, and the subcritical case where (1.9b) holds is by now well understood (cf. [B1], [B2], [FK], [FL], [HV1], [HV2], [HV3], [HV4], [HV5], [V1], [V2], [V3] and the review [V4] for details). In particular, the final profiles of the solutions at blow up time can be computed ([V2]), the $(N-1)$-dimensional Hausdorff measure of the blow up set is bounded in compact sets if $u(x,t) \not\equiv (p-1)(T-t)^{-\frac{1}{p-1}}$ ([V3]), and generic blow up behaviour is shown to correspond to single- point, locally radial blow up patterns ([HV4], [HV5]).

On the other hand, it was implicit in the arguments in [GK2] that, if (1.9a) does not hold, there should be convergence of some subsequence of the form

(1.13)
$$v_{\lambda_n}(x,t) = \lambda_n^{\frac{1}{p-1}} u(\lambda_n^{1/2}x + x_n,\ t_n + \lambda_n t) \quad \text{for some sequences}$$
$$\{x_n\} \subset \mathbb{R}^n \text{ and } \{t_n\} \text{ with } \lim_{n\to\infty} t_n = 0,$$

towards a global, bounded solution $\overline{v}(x,t)$ of (1.8) which is defined for $x \in \mathbb{R}^n$ and $-\infty < t < \infty$ (an eternal solution), which should satisfy $\overline{v}(0,0) = 1$. This is a neat analogue of (1.6) for slow blow up in MCF. It was not clear, however, whether solutions of (1.8) which do not satisfy (1.9a) could possibly exists. This fact has been recently ascertained in [HV6], where the following result was proved:

(1.14) If $N \geq 11$ and $p > \dfrac{N-2(N-1)^{1/2}}{(N-4)-2(N-1)^{1/2}}$, then for any $T > 0$ there exist positive, radial solution $u(|x|,t)$ of (1.8) which blow up at $x = 0$ and $t = T$, and

$$\lim_{t\uparrow T} \sup\, (T-t)^{\frac{1}{p-1}} u(0,t) = +\infty.$$

In [HV6], a precise description of the blow up mechanism in (1.14) is provided. In particular, convergence in suitable scales towards an eternal (actually stationary) solution of (1.8) in an inner layer near $x = 0$ is stated (cf. [HV7] for a sketch of the main arguments of [HV6]).

One can wonder if related analysis can be produced for MCF. We next briefly describe some recent results in this direction which are relevant to our current discussion.

In [AV1], the evolution of rotationally symmetric surfaces under MCF was considered. The authors analysed a situation of collapsing surfaces as depicted in Figures 1 and 2 below.

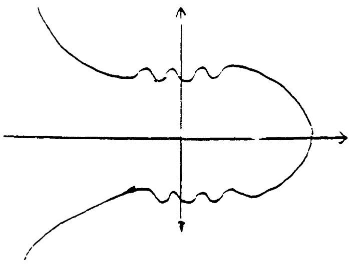

Figure 1
m odd : $m = 3, 5, 7, \ldots$

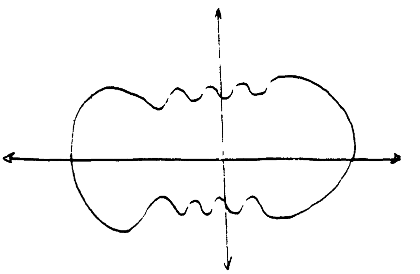

Figure 2
m even : $m = 4, 6, 8, \ldots$

Singularities developed correspond to a slow blow up situation. In particular, the size of the surfaces near collapse, the formation of a travelling wave near the tip and the size of this last were obtained there.

On the other hand, in [AV2] the collapse of a convex, symmetric, immersed loop with one self-intersection was studied (cf. Figure 3).

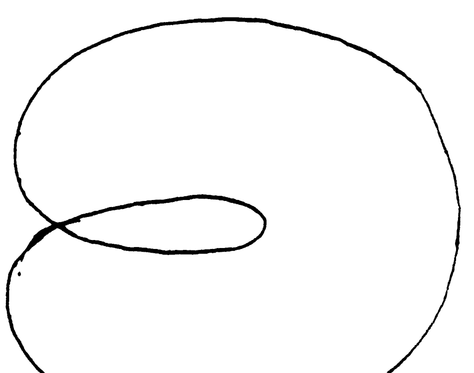

Figure 3

By the results of [A3], it is known that the internal loop must collapse at a rate described in (1.4). Furthermore, it was also proved that, in suitable rescaled variables, a traveling wave will develop near the tip of the internal loop as blow up proceeds.

The detailed asymptotics of the internal loop near blow up (and the precise size of its tip) were then obtained in [AV2]. We point out that, while [AV1], [AV2] describe collapse of type (1.4), the precise blow up mechanisms are different in both cases, even though they are characterized by the generation of a travelling wave near a tip. The result obtained here, and described in the statement of our previous Theorem, corresponds yet to another blow up structure for singularities of type (1.4).

2. - Preliminaries

We shall restrict our attention to hypersurfaces (henceforth referred to as surfaces for short) which are invariant under the action of $O(n) \times O(n)$, where $O(n)$ denotes the orthogonal group in $\mathbb{R}^n$. Let $T > 0$ be given, and for $0 < t < T$ let $\{\Gamma(t)\}$ be any such family of surfaces. We then define a rescaled family $\{\overline{\Gamma}(\tau)\}$ as follows

$$\overline{\Gamma}(\tau) = e^{\tau/2}\Gamma(T - e^{-\tau}), \quad \tau = -\log(T - t) \tag{2.1}$$

A simple calculation reveals that, if $\{\Gamma(t)\}$ evolves by mean curvature flow (MCF), then $\{\overline{\Gamma}(\tau)\}$ changes according to the equation

$$\text{For any } y \in \overline{\Gamma}(\tau),\ V_N(y,\tau) = H(y,\tau) + \frac{1}{2}(y, N) \tag{2.2}$$

where V_N is the normal velocity, H is the mean curvature of the surface, N is a normal vector at $y \in \tilde{\Gamma}(\tau)$ and $(\ ,\ )$ denotes the standard scalar product in $\mathbb{R}^{2n}$.

From now on, we shall specialize to surfaces that can be parametrized in the form

$$y = \tilde{f}_\tau(r, \omega_1, \omega_2); \quad r = |y| \tag{2.3a}$$

under suitable assumptions on the map

$$\tilde{f}_\tau : \mathbb{R}^+ \times S^{n-1}\left(\frac{\sqrt{2}}{2}\right) \times S^{n-1}\left(\frac{\sqrt{2}}{2}\right) \to \mathbb{R}^{2n} \tag{2.3b}$$

which is defined for $\omega_1,\ \omega_2 \in S^{n-1}\left(\frac{\sqrt{2}}{2}\right)$ and $r \in (\delta(\tau), D(\tau))$, where $\delta(\tau)$, $D(\tau)$ will be specified presently (cf. (2.41) below) and $\tau \geq \tau_0$ with τ_0 large enough. Function $\tilde{f}_\tau$ acts as follows

$$\begin{aligned} &\tilde{f}_\tau(r, \omega_1, \omega_2) \equiv \tilde{f}(r, \omega_1, \omega_2, \tau) = (r\omega_1 + \psi(r,\tau)\omega_1,\ r\omega_2 - \psi(r,\tau)\omega_2) \\ &\text{for some function } \psi : (\delta(\tau), D(\tau)) \times (\tau_0, \infty) \to \mathbb{R} \text{ with } |\psi(\sigma,\tau)| < \frac{r}{2}. \end{aligned} \tag{2.3c}$$

For instance, the Simons cone

$$C_n = \{(x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} : x_1^2 + \ldots + x_n^2 = x_{n+1}^2 + \ldots + x_{2n}^2\} \tag{2.4}$$

is indeed invariant under the group $O(n) \times O(n)$, and can be parametrized in the form

$$x = f(r, \omega_1, \omega_2); \quad r = |x|$$

where

$$f(r, \omega_1, \omega_2) = (r\omega_1, r\omega_2)$$

which corresponds to (2.3c) with $\psi \equiv 0$ there.

A routine computation reveals that in the region where $\overline{\Gamma}(\tau)$ can be parametrized in the form (2.3), the normal vector $\overline{N}$ to such surface is given by

$$\overline{N} = \frac{1}{\sqrt{1 + \psi_r^2}} (\omega_1 - \psi_r\omega_1, -\omega_2 - \psi_r\omega_2) \tag{2.5}$$

where $\psi_r = \dfrac{\partial \psi}{\partial r}$, and the normal velocity is

$$V_N = \left(\frac{\partial \tilde{f}_\tau}{\partial \tau}, \overline{N}\right) = \frac{\psi_\tau}{\sqrt{1 + \psi_r^2}} \tag{2.6}$$

so that

$$(y, \overline{N}) = \frac{\psi - r\psi_r}{\sqrt{1 + \psi_r^2}}. \tag{2.7}$$

The metrics at the surface is then given by the matrix

$$(g_{i,j}) = \begin{pmatrix} (1 + \psi_r^2) & 0 & 0 \\ 0 & (r + \psi)^2 \tilde{g}_{\alpha,\beta} & 0 \\ 0 & 0 & (r + \psi)^2 \tilde{g}_{\alpha,\beta} \end{pmatrix},$$

where $(\tilde{g}_{\alpha,\beta})$ is the first fundamental form in the sphere $S^{n-1}\left(\dfrac{\sqrt{2}}{2}\right)$. The inverse matrix $(g^{i,j}) = (g_{i,j})^{-1}$ is then given by

$$(g^{i,j}) = \begin{pmatrix} \dfrac{1}{1 + \psi_r^2} & 0 & 0 \\ 0 & \dfrac{1}{(r + \psi)^2} \tilde{g}^{\alpha,\beta} & 0 \\ 0 & 0 & \dfrac{1}{(r + \psi)^2} \tilde{g}^{\alpha,\beta} \end{pmatrix}, \tag{2.8}$$

where $(\tilde{g}^{\alpha,\beta}) = (\tilde{g}_{\alpha,\beta})^{-1}$. The second fundamental form of $\overline{\Gamma}(\tau)$ in this coordinate system will thus be given by

$$(2.9)\qquad (a_{i,j}) = \\ = (1+\psi_r^2)^{-1/2}\begin{pmatrix} \psi_{rr} & 0 & 0 \\ 0 & 2^{-1/2}(r+\psi)(1-\psi_r)\tilde{a}_{\alpha,\beta} & 0 \\ 0 & 0 & -2^{-1/2}(r+\psi)(1+\psi_r)\tilde{a}_{\alpha,\beta} \end{pmatrix},$$

where $(\tilde{a}_{\alpha,\beta})$ is the second fundamental form of the sphere $S^{n-1}\left(\frac{\sqrt{2}}{2}\right)$. Recalling that the mean curvature H satisfies $H = g^{i,j}a_{i,j}$, it follows from (2.8), (2.9) that

$$(2.10)\qquad H = (1+\psi_r^2)^{-1/2}\left(\frac{\psi_{rr}}{1+\psi_r^2} - (N-1)\left(\frac{1-\psi_r}{r+\psi} - \frac{(1+\psi_r)}{r-\psi}\right)\right)$$

where we have used the fact that $\tilde{g}^{\alpha,\beta}, \tilde{a}_{\alpha,\beta} = -\sqrt{2}(N-1)$. Taking into account (2.2), (2.6), (2.7) and (2.10) we finally arrive at the following equation for $\psi(r,\tau)$ (cf. (2.3c))

$$(2.11)\qquad \psi_\tau = \frac{\psi_{rr}}{1+\psi_r^2} - \frac{1}{2}r\psi_r + (N-1)\left(\frac{1+\psi_r}{r-\psi} - \frac{(1-\psi_r)}{r+\psi}\right) + \frac{\psi}{2}.$$

We shall also make use of a different parametrization for the kind of surfaces that we are considering here. By symmetry, any such surface is determined by its intersection with the plane (x_1, x_{n+1}). For convenience, we shall write $x_{n+1} = v$, $x_1 = u$, and assume that the surfaces $\{\Gamma(t)\}$ may be described by

$$(2.12a)\qquad v = Q(u,t)$$

for some smooth enough function Q such that

$$(2.12b)\qquad Q_u(0,\tau) = 0, \quad Q_u(u,\tau) \geq 0 \text{ for } u \geq 0.$$

Since the arguments to be presented are of a local nature, we shall only need (2.12) to hold for u small enough. In the rest of this Section, however, we will continue to make use of (2.12) as stated above for the sake of simplicity. We then can parametrize the surface $\Gamma(t)$ by means of a function $h(u,\omega_1,\omega_2,t)$ in the form

$$(2.13)\qquad x = h(u,\omega_1,\omega_2,t) = (u\omega_1, Q(u,t)\omega_2).$$

We readily check that one now has the following formulae for normal vector, normal velocity, the inverse of the metrics matrix and the second fundamental form respectively

$$\overline{N} = (1 + D^2)^{-1/2}\left((\omega_1, \omega_2) + D(\omega_1, -\omega_2)\right), \tag{2.14}$$

$$V_N = \left(\frac{\partial h}{\partial \tau}, \overline{N}\right) = -(1 + D^2)^{-1/2}(1 + Q_u(u,t))^{-1}\frac{\partial Q}{\partial \tau}, \tag{2.15}$$

$$(g^{i,j}) = \begin{pmatrix} 2(1 + Q_u(u,t)^2)^{-1} & 0 & 0 \\ 0 & u^{-2}\tilde{g}^{\alpha,\beta} & 0 \\ 0 & 0 & (Q_u(u,t))^{-2}\tilde{g}^{\alpha,\beta} \end{pmatrix} \tag{2.16}$$

$$(a_{i,j}) = \tag{2.17}$$

$$= (1+D^2)^{-1/2}\begin{pmatrix} \frac{Q_u}{2} + \frac{1}{2}DQ_{uu} & 0 & 0 \\ 0 & \sqrt{2}\,uQ_u(1+Q_u)^{-1}\tilde{a}_{\alpha,\beta} & \\ 0 & 0 & \frac{-\sqrt{2}\,h(u)}{1+Q_u}\tilde{a}_{\alpha,\beta} \end{pmatrix}$$

where $(\tilde{a}_{\alpha,\beta})$ is as in (2.9) and

$$D = \frac{Q_u - 1}{Q_u + 1}. \tag{2.18}$$

From (2.16), (2.17) we deduce that

$$\begin{aligned} H = g^{i,j}a_{i,j} = (1 + D^2)^{-1/2}\Bigg(&-\frac{2}{1+Q_u}\cdot\frac{Q_{uu}}{1+Q_u^2} - \\ &-\frac{2(N-1)}{u}\frac{Q_u}{1+Q_u} + \frac{2(N-1)}{Q}\cdot\frac{1}{1+Q_u}\Bigg) \end{aligned} \tag{2.19a}$$

whereas (2.13) and (2.14) yield

$$(X, N) = \frac{1}{2}(1 + D^2)^{-1/2}\left((u - Q) + D(u + Q)\right) \tag{2.19b}$$

We next turn our attention to equation (2.11). If we formally linearize in the right-hand side there, we obtain the differential operator

$$-A\varphi = \varphi_{rr} + \left(\frac{2(N-1)}{r} - \frac{r}{2}\right)\varphi_r + \left(\frac{2(N-1)}{r^2} + \frac{1}{2}\right)\varphi, \tag{2.20}$$

which will play an essential role in what follows. A similar operator has been throughly analysed in [HV5]; we shall therefore refer to that paper for details concerning the properties of A in (2.20) which will be listed below. As in [HV5], we define suitable weighted spaces in the following form

$$L^2_{w,r} \equiv L^2_{w,r} = \left\{ f \in L^2_{\text{loc}}(\mathbb{R}^{2n-1}) : f(y) = f(|y|) \right.$$

$$\left. \text{and} \int_{\mathbb{R}^{2n-1}} |f(y)|^2 e^{-|y|^2/4} dy < \infty \right\}$$

$$H^1_{w,r} \equiv H^1_{w,r}(\mathbb{R}^{2n-1}) = \left\{ f \in H^1_{\text{loc}}(\mathbb{R}^{2n-1}) \cap L^2_{w,r}(\mathbb{R}^{2n-1}) : \right.$$

$$\left. \text{for } 1 \le i \le 2n-1, \frac{\partial f}{\partial x_i} \in L^2_{w,r} \right\}.$$

It is readily seen that $L^2_{w,r}$ (resp. $H^1_{w,r}$) can be endowed with a Hilbert space structure corresponding to the scalar product

$$\langle f, g \rangle = \int_{\mathbb{R}^{2n-1}} f(y) g(y) e^{-|y|^2/4} dy$$

(resp. $\langle f, g \rangle_{(1)} = \langle f, g \rangle + \sum_{i=1}^{2n-1} \left\langle \frac{\partial f}{\partial x_i}, \frac{\partial g}{\partial x_i} \right\rangle$). We are now prepared to describe the basic spectral properties of operator A. These are collected in the following

PROPOSITION 2.1. *Let $n \ge 4$. The operator A in* (2.20) *can then be extended in a unique way to a self-adjoint operator (also denoted by A) which has the following properties*

(2.21a) $D(A) \subset H^1_{r,w}$,

(2.21b) *There exists $C > 0$ such that$\langle \varphi, A\varphi \rangle \ge -C\langle \varphi, \varphi \rangle$ for any $\varphi \in D(A)$.*

Moreover, the spectrum of A consists of a countable sequence of eigenvalues $\{-\lambda\}_{ij=0,1,2,\ldots}$. These and the corresponding eigenfunctions $\{\varphi_j\}$ satisfying $A\varphi_j = \lambda_j \varphi_j$ are given by

(2.22) $$\lambda_j = \frac{\alpha}{2} + j - \frac{1}{2}; \quad j = 0, 1, 2, \ldots$$

(2.23) $$\alpha = \frac{1}{2}(-(2N-3) + ((2N-1)^2 - 16(N-1))^{1/2}) < 0,$$

(2.24) $$\varphi_j(r) = c_j r^{\alpha} M\left(-\lambda_j + \frac{1}{2} + \frac{\alpha}{2}, \alpha + \frac{N}{2}; \frac{r^2}{4}\right),$$

where $M(a, b; z)$ *is the standard Kummer function (cf.* [HV6 *Section* 2] *and* [GaP]*), and for any* $j = 0, 1, 2, \ldots$ c_j *is selected so that* $\langle \varphi_i, \varphi_j \rangle = 1$.

The proof of Proposition 2.1 follows with minor modifications from that of Lemma 2.3 of [HV6]. For the readers convenience, however, we shall merely sketch here one of its main points, which underlines the dimension restriction $n \geq 4$. Let $L > 0$ be any positive integer. Then, as recalled in [HV6], for any $\varphi \in C_0^\infty(\mathbb{R}^L)$ with $L \geq 3$ the following inequalities hold

$$\text{(2.25a)} \qquad \int_{\mathbb{R}^L} \left(\frac{\varphi}{|y|}\right)^2 dy \leq \frac{4}{(L-2)^2} \int_{\mathbb{R}^L} |\nabla \varphi|^2 dy,$$

$$\text{(2.25b)} \qquad \int_{\mathbb{R}^L} (\varphi_r)^2 e^{-|y|^2/4} dy \geq \frac{(L-2)^L}{4} \int_{\mathbb{R}^L} \left(\frac{\varphi}{|y|}\right)^2 e^{-|y|^2/4} dy$$
$$- C \int_{\mathbb{R}^L} \varphi^2 e^{-|y|^2/4} dy \text{ for some } C > 0.$$

As a matter of fact, when $L = 3$ (2.25a) reduces to the classical uncertainty principle Lemma (cf. [RS], vol. II, p. 169), and (2.25b) is an inmediate consequence of (2.25a). From (2.25b) we readily obtain that for any $\varphi \in C_0^\infty(\mathbb{R}^{2n-1})$

$$\langle \varphi, A\varphi \rangle \geq - \left(2(n-1) - \frac{(2n-1-2)^2}{4}\right) \int_{\mathbb{R}^{2n-1}} \left|\frac{\varphi}{y}\right|^2 e^{-|y|^2/4} dy - C\langle \varphi, \varphi \rangle$$

it then follows that A is bounded below whenever $2(n-1) - \dfrac{(2n-3)^2}{4} \leq 0$, which actually holds for $2n \geq 5 + \sqrt{2} \sim 7/8$.

This is the crucial startpoint to show that A has a (unique) Friedrichs extension which satisfies the results described in the Proposition above. □

It will be useful later on to recal here that the Kummer's function $M(a, b; z)$ is an analytic function in the complex plane whenever a, b are complex numbers such that $a \neq -n$ where $n = 0, 1, 2, \ldots$. Furthermore,

$$M(a, b; 0) = 1.$$

Recalling (2.24), one also has that

$$\text{(2.26)} \qquad \varphi_j(r) \sim \frac{\Gamma\left(\alpha + \frac{n}{2}\right)}{\Gamma\left(\alpha + \frac{n}{2} + j\right)} k_j r^{2\lambda_j + 1} \equiv C_j r^{2\lambda_j + 1} \text{ as } r \to \infty,$$

where $k_j = \dfrac{(-1)^j}{4^j} c_j$ $(j = 0, 1, 2, \ldots)$ and $\Gamma(z)$ is the standard Euler's gamma function.

We next proceed to analyse some minimal surfaces which are in a sense tangent to the cone C_n as $|x| \to \infty$. More precisely, the following result holds:

PROPOSITION 2.2. *Let $n \geq 4$. Then for any $a > 0$ there exists a globally defined minimal surface M_a, invariant under the action of $0(n) \times 0(n)$, which may be parametrized in the form*

$$x = g(r, \omega_1, \omega_2)$$

where

$$g : (a, +\infty) \times S^{n-1}\left(\frac{\sqrt{2}}{2}\right) \times S^{n-1}\left(\frac{\sqrt{2}}{2}\right) \to \mathbb{R}^{2n} \tag{2.27a}$$

is given by

$$g(r, \omega_1, \omega_2) = (r\omega_1 + G_a(r)\omega_1, r\omega_2 - G_a(r)\omega_2) \tag{2.27b}$$

for $r \geq a$, and the function $G_a(r)$ is such that $G_a(r) < 0$ for $r \geq a$, $G_a(a) = -a$, $(G_a)_r(a) = 1$ and

$$G_a(r) \sim -k_0 a^{1-\alpha} r^\alpha \text{ as } r \to \infty \text{ in } C^2 \tag{2.28}$$

where $\alpha < 0$ is given in (2.24), and $k_0 > 0$ is a constant which is independent of a. Moreover, if we represent the intersection of M_a with the two dimensional plane (x_1, x_{n+1}) by

$$x_{n+1} \equiv v = B_a(u) \equiv B_a(x_1),$$

we have that:

$$B_a''(u) > 0 \text{ for any } u \geq 0. \tag{2.29}$$

PROOF. Let us write $G(r) \equiv G_a(r)$ for convenience. Recalling (2.10) and (2.27), equation $H = 0$ yields

$$\frac{G_{rr}}{1 + G_r^2} + (n-1)\left(\frac{1 + G_r}{r - G} - \frac{1 - G_r}{r + G}\right) = 0 \tag{2.30}$$

where we impose the boundary conditions

$$G(a) = -a, \quad G_r(a) = 1. \tag{2.31}$$

As $r \gg 1$, we shall assume that

$$G(r) \to \infty \text{ as } r \to \infty. \tag{2.32}$$

We shall obtain a solution of (2.30)-(2.32) by means of classical ODE theory. To this end, we set

$$W(y) = e^{-y}G(e^{y}) \quad y = \log a \tag{2.33}$$

A quick computation reveals now that (2.30) is transformed into

$$\dot{W} = Z \tag{2.34a}$$

$$\dot{Z} = -Z - 2(n-1)(1+(W+Z)^2)\left(\frac{Z+2W}{1-W^2}\right) \tag{2.34b}$$

where $\dot{W} = \dfrac{dW}{dy}$, $\dot{Z} = \dfrac{dZ}{dy}$. Conditions (2.31), (2.32) read now

$$W(\log a) = -1, \quad \dot{W}(\log a) = 2, \tag{2.35a}$$

$$W(y) = 0(e^{-y}) \quad \text{as } y \to \infty. \tag{2.35b}$$

Standard techniques yield at once the following phase diagram for (2.34).

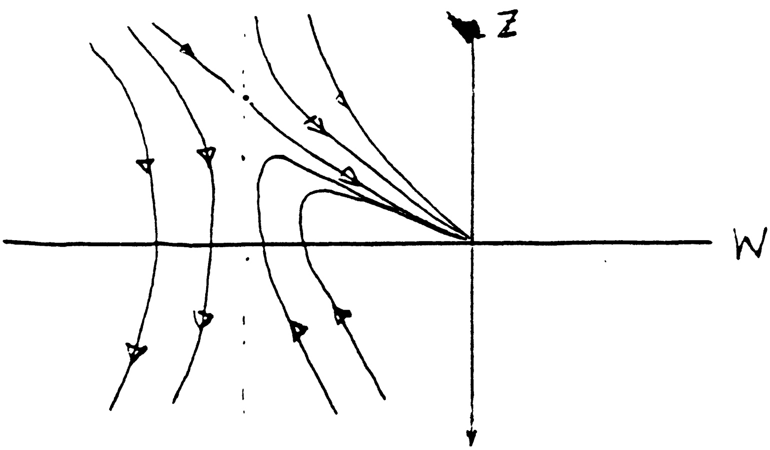

Figure 4: The phase diagram for (2.34) in the region $W < 0$.

The corresponding picture for the semiplane $W > 0$ may now be obtained by reflection with respect to the origin. Notice that no trajectory can cross from the region $W < -1$ to that where $W > -1$, since the line $W = -1$ is singular for (2.34). At the point $(0,0)$, we have two possible asymptotic behaviours, namely

$$W(y) \sim ke^{-\beta_+ y} \text{ as } y \to \infty \tag{2.36a}$$

or

$$W(y) \sim ke^{-\beta_- y} \text{ as } y \to \infty \tag{2.36b}$$

where

$$\beta_\pm = \frac{1}{2}\{-(2n-1) \pm ((2n-1)^2 - 16(n-1))^{1/2})\}$$

We need to show that the trajectory starting at $(-1, 2)$ behaves as indicated in (2.36a). To this end, we use the barrier function $Z = (\beta_- + \varepsilon)W$, where $\varepsilon > 0$ is positive and small. Along such barrier, the slope of the velocity field defined by (2.34) is given by

$$\frac{dZ}{dW} = -1 - 2(n-1)\frac{(1 + (1 + (\beta_- + \varepsilon))^2 W^2}{(\beta_- + \varepsilon)} \cdot \frac{(2 + \beta_- + \varepsilon)}{(1 - W^2)} <$$

$$< -1 - 2(n-1)\left(\frac{2 + \beta_- + \varepsilon}{\beta_- + \varepsilon}\right) < \beta_- + \varepsilon$$

if $\varepsilon > 0$ is small enough. Thus the flow associated to (2.34) points along the line $z = (\beta_- + \varepsilon)W$ towards the region $Z < (\beta_- + \varepsilon)W$.

This show that (2.36a) holds, whence (2.28) in the original set of variables. As a matter of fact, the precise dependence $k_0 a^{-\alpha}$ is obtained by scaling: if $G_a(r)$ is a solution of (2.30)-(2.32), we easily check that $G_b(r) = \frac{b}{a} G_a\left(\frac{ar}{b}\right)$ for any $b > 0$. In particular $G_a(r) = aG_1\left(\frac{r}{a}\right) \sim k_1 a^{1-\alpha} r^\alpha$ as $r \to \infty$. On the other hand, since $W(y)$ is strictly decreasing as y increases, the surfaces M_a never intersect for different values of a. To derive (2.29), we notice that B satisfies

$$\frac{2B''}{(1 + (B')))^2} + \frac{2(n-1)B'}{u} - \frac{2(n-1)}{B} = 0 \tag{2.37}$$

whence $B_a(u) \sim u$ as $u \to \infty$ and $B_a(u) > 0$ since $G > 0$. From (2.27b) we readily see that for $u > 0$, $v > 0$,

$$\frac{\sqrt{2}}{2}(r + G_a(r)) = u,$$

$$\frac{\sqrt{2}}{2}(r - G_a(r)) = B_a(u)$$

hence $B_a(u) > u$ and by (2.28)

$$\frac{dB_a}{du^2} = \frac{1 - G_a'(r)}{1 + G_a'(r)} < 1 \text{ as } r \to \infty$$

and

$$\frac{dB_a}{du^2} = \frac{dr}{du}\frac{d}{dr}\left(\frac{dB_a}{du}\right) = \frac{-2\sqrt{2}}{1 + G_a'(r)} \cdot \frac{G_a''(r)}{(1 + G_a'(r))^2} > 0$$

It follows from (2.37) that $B' \geq 0$. Indeed, if $B' < 0$ in some region, B would have a positive maximum at some value $\overline{u} \in [0, \infty)$ where $B'(\overline{u}) = 0$, and since $B(\overline{u}) \geq \overline{u} \geq 0$, (2.37) would give a contradiction. On the other hand, one clearly has $B''(0) > 0$. Assume that $B''(u^*) = 0$ for some $u^* > 0$. At such a point, we should have

$$B''' = (n-1)(1 + (B')^2)\left(\frac{1}{u^2} - \frac{1}{B^2}\right) B'$$

and by (2.37) $B'(u^*) > 0$. Then $B'''(u^*) > 0$, but this contradicts the existence of the point u^* described above, since B is convex near the origin, as well as when $u \to \infty$. This proves (2.29). Finally, the fact that M_a is a minimal surface follows from standard theory (cf. for instance [F] 5.4.18). $\square$

Let us fix now $T > 0$, $\beta > 0$, $k > 0$, $p > 0$, $\ell = 1, 2, \ldots$ such that $\lambda_\ell > 0$, $\theta \in (0,1)$ and η_1, η_2 so that $0 \leq \eta_2 \leq \eta_1$. We then define the set

$$\mathcal{A}[\eta_1, \eta_2, \theta] \equiv \mathcal{A}[\eta_1, \eta_2, \theta, \beta, k, \rho, \ell, T] \tag{2.38}$$

as the set of families of surfaces $\{\Gamma(t)\}$ with the following properties.

(2.39) For $T - \eta_2 \leq t \leq T - \eta_1$, the families $\Gamma(t)$ are smooth embedded surfaces $g : M \to \mathbb{R}^{2n}$, where M is as in Proposition 3.2, and g conmutes with the action of the group $0(n) \times 0(n)$.

(2.40) For $x \in \Gamma(t)$, $t \in (T - \eta_2, T - \eta_1)$ and $\beta e^{-\tau/2} \leq |x| \leq \rho$, there holds

$$1 - \frac{\theta}{2} \leq \frac{x_1^2 + \ldots + x_n^2}{x_{n+1}^2 + \ldots + x_{2n}^2} < 1 + \theta.$$

(2.41) Let σ_ℓ be $\sigma_\ell = \dfrac{\lambda_\ell}{1 + |\alpha|}$, where α is as in (2.23). In the region $\beta e^{-(\sigma_\ell + \frac{1}{2})\tau} \leq |x| \leq \rho$, the surfaces $(T-t)^{-1/2}\Gamma(t)$ ($t \in (T - \eta_2, T - \eta_1)$) may be parametrized as in (2.3b).
Moreover

$$|\psi| < \theta\beta|k| \left(|y|^{\alpha} + |y|^{2\lambda_\ell + 1}\right) e^{-\lambda_\ell \tau} \text{ for } \beta e^{-\sigma_\ell \tau} \leq |y| \leq \rho e^{\tau/2},$$

$$|\psi_r| < \theta\beta|k| \left(|y|^{\alpha-1} + |y|^{2\lambda_\ell}\right) e^{-\lambda_\ell \tau} \text{ for } \beta e^{-\sigma_\ell \tau} \leq |y| \leq \rho e^{\tau/2},$$

$$|\psi_{rr}| < \theta\beta|k| \left(|y|^{\alpha-2} + |y|^{2\lambda_\ell + 1}\right) e^{-\lambda_\ell \tau} \text{ for } \beta e^{-\sigma_\ell \tau} \leq |y| \leq \rho e^{\tau/2},$$

where $\mu_\ell = \dfrac{1}{2}\left(1 - \dfrac{1}{2\lambda_{\ell+1}}\right)$.

(2.42) The surfaces $\Gamma(t)$ ($t \in (T - \eta_2, T - \eta_1)$) may be parametrized as in (2.12), (2.13). The function $Q(u,t)$ is convex for $|u| \leq \dfrac{1}{\beta} e^{-\tau/2}$ and $|u| \geq \beta e^{-\tau/2}$.

(2.43) Let σ_ℓ be as in (2.41). For $|x| \leq e^{-(\sigma_\ell + \frac{1}{2})\tau}$ the surface $(T-t)^{\frac{\sigma_\ell + 1}{2}}\Gamma(t)$ is contained in the $2n$-dimensional region contained between $M_{k(1-\frac{\theta}{2})}$ and $M_{(1+\theta)k}$.

It is readily seen that for large enough β, small enough ρ and fixed k, the set $\mathcal{A}[\eta_1, \eta_2, \theta]$ is non empty for any $\eta_1 \geq \eta_2 \geq 0$ with η_1 small enough. In an informal manner, (2.39)-(2.43) state that the family of surfaces $\Gamma(t)$ is very close to the Simons cone C_n when $|x| \to 0$, and the rescaled surface $(T-t)^{-\frac{\sigma_\ell+1}{2}}\Gamma(t)$ is close to M_k.

We are now prepared to state in a precise way the main results of this paper.

THEOREM 2.1. *Let us fix $\ell > 0$, $k > 0$, and take β large enoug and ρ small enough. For $\eta > 0$ sufficiently small there exists a family of surfaces $\Gamma(t)$ with $t \in (T-\eta, 0)$ which evolves by mean curvature flow and is such that $\Gamma(t) \in \mathcal{A}(\eta, 0, 1)$.*

Clearly, the family of surfaces $\Gamma(t)$ collapses as $t \uparrow T$ to a surface which satisfies

$$\frac{1}{2} \leq \frac{x_1^2 + \ldots + x_n^2}{x_{n+1}^2 + \ldots + x_{2n}^2} \leq 2.$$

The asymptotic behaviour of the family $\Gamma(t)$ as $t \uparrow T$ is described in detail in the following.

THEOREM 2.2. *For any $\varepsilon > 0$ there exists $\eta > 0$ and $\bar{k} > 0$ with $|k - \bar{k}| < \varepsilon$ such that the following properties hold.*

i) *Let ψ be the function in* (2.3c), (2.41). *Then, as $\tau \to \infty$*

(2.44) $\psi(r,\tau) = \dfrac{\bar{k}}{c_\ell} e^{-\lambda_\ell \tau}\varphi_\ell(r) + o(e^{-\lambda_\ell\tau} r^{2\lambda_\ell+1})$ *in C^2 uniformly on sets $e^{(-\sigma_\ell+\tilde{\varepsilon})\tau} \leq \sigma \leq \mu(\tau)$ for any fixed function $\mu(\tau)$ such that $\lim_{\tau\to\infty} \mu(\tau) = 0$, and $\tilde{\varepsilon} > 0$ small.*

ii) *The surface $(T-t)^{-\frac{1}{2}\sigma_\ell}\Gamma(t)$ approaches to $M_{\bar{k}}$ as $t \uparrow T$, uniformly on compact sets.*

iii) *The surface $\Gamma(T)$ is tangent to the Simons cone C_n at $x = 0$. More precisely, if $\Gamma(T)$ is parametrized as in* (2.12), (2.13), *there holds*

$$Q(u,T) = u + C_\ell \frac{\bar{k}}{c_\ell} u^{2\lambda_\ell+1} + o(u^{2\lambda_\ell+1}) \text{ as } u \to 0,$$

where C_ℓ and c_ℓ are respectively given in (2.26) *and* (2.24).

3. - The topological argument

Let us denote by $G(u, \bar{u}, \tau)$ the Green's function associated to $e^{A\tau}$, where A is the operator defined in (2.20) and Proposition 2.1. It has been proved in

[HV6] that

$$G(y,\overline{y},\tau) = e^{-y^{-2}/4} \sum_{k=0}^{+\infty} e^{-\lambda_k \tau} \varphi_k(y)\varphi_k(\overline{y}) \tag{3.1}$$

Assume that $\Gamma(t) \in \mathcal{A}[\eta, \overline{\eta}, 1]$ where $0 \le \overline{\eta} \le \eta$. Let $\xi : \mathbb{R} \to \mathbb{R}$ be a cutoff function such that $\xi(x) = 0$ for $x \le 0$, $\xi(x) = 1$ for $x \ge 1$, $\xi \in C^\infty(\mathbb{R})$, $\xi' \ge 0$. We introduce the function

$$\Omega(r,\tau) = \xi(re^{\sigma_\ell \tau} - 2\beta)\xi(e^{\mu_\ell \tau} - r)\psi(r,\tau)$$

where $\mu_\ell = \frac{1}{2}\left(1 - \frac{1}{2\lambda_\ell + 1}\right)$ $r = |y|$. We extend Ω as zero for $0 < r < 2\beta e^{-\sigma_\ell \tau}$ $r > e^{\mu_\ell \tau}$.

We then see that Ω satisfies the equation

$$\begin{aligned} \Omega_\tau - \Omega_{rr} - \frac{2(N-1)}{r}\Omega_r + \frac{r\Omega_r}{2} - \frac{2(N-1)}{r^2}\Omega - \frac{1}{2}\Omega &= \\ = f_1(r,\tau) + f_2(r,\tau) + f_3(r,\tau)& \end{aligned} \tag{3.2}$$

for $0 < r < +\infty$, $\tau_0 \equiv -\log(\eta) \le \tau \le -\log(\overline{\eta}) \equiv \tau_1$ where

$$\begin{aligned} f_1(r,\tau) &= \xi(re^{\sigma_\ell \tau} - 2\beta)\xi(e^{\mu_\ell \tau} - r) \\ &\left\{(N-1)\left[\frac{1+\psi_r}{r-\psi} - \frac{1-\psi_r}{r+\psi} - \frac{2}{r}\psi_r - \frac{2}{r^2}\psi\right] - \frac{\psi_r^2\psi_{rr}}{1+\psi^2}\right\}, \end{aligned} \tag{3.3a}$$

$$\begin{aligned} f_2(r,\tau) &= \xi'(re^{\sigma\tau} - 2\beta)\xi(e^{\mu_1 \tau} - r) \\ &\left\{(\sigma_\ell re^{\sigma_\ell \tau})\psi - e^{\sigma_\ell \tau}\psi_r - \frac{2(N-1)}{r}e^{\sigma_\ell \tau}\psi + \frac{r}{2}e^{\sigma_\ell \tau}\psi\right\} \\ &- \xi''(re^{\sigma_\ell \tau} - 2\beta)\xi'(e^{\mu_\ell \tau} - r)e^{2\sigma_\ell \tau}\psi + 2\xi'(re^{\sigma_\ell \tau} - 2\beta)\xi'(e^{\mu_\ell \tau} - r)e^{\sigma_\ell \tau}\psi \end{aligned} \tag{3.3b}$$

$$\begin{aligned} f_3(r,\tau) &= \xi(re^{\sigma\tau} - 2\beta)\xi'(e^{\mu_\ell \tau} - r) \\ &\left\{\mu_\ell e^{\mu_1 \tau}\psi + \psi_r + \frac{2(N-1)}{r}\psi - \frac{r}{2}\psi\right\} - \xi(re^{\sigma_\ell \tau} - 2\beta)\xi''(e^{\mu_\ell \tau} - r)\psi. \end{aligned} \tag{3.3c}$$

We now argue as in [AV1], [HV6]. Let us fix $\varepsilon_0 > 0$ small enough (to be precised). We pick $\alpha = (\alpha_1, \ldots, \alpha_{\ell-1}) \in \mathbb{R}^{\ell-1}$ such that

$$\alpha \in B_{\varepsilon_0 e^{-\lambda_\ell \tau_0}}(0) \subset \mathbb{R}^{\ell-1} \tag{3.4}$$

For each α satisfiying (3.5) we choose a smooth surface $\Gamma_\eta(\alpha) \in \mathcal{A}\left[\eta, \eta, \frac{1}{4}\right]$, where the dependence of $\Gamma_\eta(\alpha)$ is continuous in α, and where

$$\psi(r,\tau_0) = \sum_{j=1}^{\ell-1} \alpha_j \varphi_j(0) + Ke^{-\lambda_\ell \tau_0}\varphi_\ell(r) \text{ for } 2\beta e^{\sigma_\ell \tau_0} \le r \le e^{\mu_\ell \tau_0}, \tag{3.5}$$

where φ_j is as in Proposition 3.1 and k is given in the definition of $\mathcal{A}[\eta,\overline{\eta},\theta]$. A more precise characterization of $\Gamma_\eta(\alpha)$ for $r \le 2\beta e^{\sigma_\ell \tau_0}$, and $r \ge e^{\mu_\ell \tau_0}$ will be given in the following.

We can now solve MCF taking $\Gamma_\eta(\alpha)$ as initial value. Let us denote the family of evolved surfaces as $\Gamma_t(\alpha)$. Given $\overline{\eta} \le \eta$ define the set $U_{\eta,\overline{\eta}} \subset \mathbb{R}^{\ell-1}$ as

$$U_{\eta,\overline{\eta}} = \{\alpha \in \mathbb{R}^{\ell-1} : \text{ (3.5) holds and } \Gamma_t(\alpha) \in \mathcal{A}[\eta,\overline{\eta},1]\}. \tag{3.6}$$

We also define the function $l_{\eta,\overline{\eta}}(\,\cdot\,,\tau_1) : \overline{U}_{\eta,\overline{\eta}} \to \mathbb{R}^{\ell-1}$ given by

$$l_{\eta,\overline{\eta}}(\alpha) = (\langle \varphi_j(\,\cdot\,), \Omega(\,\cdot\,,\tau_1)\rangle)_{j=1\ldots\ell-1} \tag{3.7}$$

By standard continuous dependence results for parabolic differential equations we readily see that $l_{\eta,\overline{\eta}}(\,\cdot\,,\tau_1)$ is a continuous function on $\overline{U}_{\eta,\overline{\eta}}$, and also there is continuity with respect to $\eta \ge \overline{\eta} \ge 0$. Moreover, if $l_{\eta,\overline{\eta}} = (l_{\eta,\overline{\eta},1} \ldots l_{\eta,\overline{\eta},m-1})$ we have that

$$l_{\eta,\eta,i}(\alpha) = \sum_{j=1}^{\ell} Q_{j,i}(\eta,\alpha)\alpha_j \tag{3.8}$$

where $\alpha_\ell = kC^{-\lambda_\ell \tau_0}$ and

$$Q_{j,i}(\eta,\alpha) = \langle \xi(re^{\sigma_\ell \tau_0} - 2\beta)\xi(e^{\mu_\ell \tau} - r)\varphi_j(\,\cdot\,), \varphi_i(\,\cdot\,)\rangle$$

It is easily seen that $Q_{j,i}(\eta,\alpha) \to \delta_{j,i}$ as $\eta \to 0$, uniformly on $B_{\varepsilon_0 e^{-\lambda_\ell \tau_0}}(0)$.

Following the standard notation we shall denote the topological degree of $l_{\eta,\overline{\eta}} : U_{\eta,\overline{\eta}} \to \mathbb{R}^{\ell-1}$ at $\alpha = 0$ as $d(l_{\eta,\overline{\eta}}, 0, U_{\eta,\overline{\eta}})$. A standard homotopy argument proves that $d(l_{\eta,\eta}, 0, U_{\eta,\eta}) = +1$ if η is small enough.

The key result in the proof of Theorem 2.1 is the following a priori estimate

PROPOSITION 3.1. *Assume that $\tilde{\alpha} \in \overline{U}_{\eta,\eta}$ solves the equation $l_{\eta,\overline{\eta}}(\tilde{\alpha}) = 0$ for some $0 \le \overline{\eta} \le \eta$. Then, if η is small enough, the family of surfaces $\Gamma_t(\tilde{\alpha}) \in \mathcal{A}[\eta,\overline{\eta},3/4]$.*

The proof of Proposition 3.1 will be given in Section 4. We will prove now that Proposition 3.1 implies Theorem 2.1.

To this end, we claim that

$$\overline{U}_{\eta,\overline{\eta}} \ne \phi \text{ for any } 0 \le \overline{\eta} \le \eta. \tag{3.9}$$

From (3.9) and Proposition 3.1, Theorem 2.1 readily follows

To prove (3.9) we observe that by Proposition 3.1 and continuous dependence results for MCF, any solution of $l_{\eta,\overline{\eta}}(\tilde{\alpha}) = 0$, $\tilde{\alpha} \in \overline{U}_{\eta,\overline{\eta}}$ is strictly contained at $U_{\eta,\overline{\eta}}$. Then, as $d(l_{\eta,\eta}, 0, U_{\eta,\eta}) = +1$, a standard homotopy argument shows that $d(l_{\eta,\overline{\eta}}, 0, U_{\eta,\overline{\eta}}) = +1$ as far as $\overline{U}_{\eta,\overline{\eta}} \neq \phi$.

Set $\eta^* = \inf\{\overline{\eta} : U_{\eta,\overline{\eta}} \neq \phi\}$.

The compactness of $\overline{U}_{\eta,\eta^*}$ and the continuity of l_{η,η^*} imply the existence of $\alpha^* \in \overline{U}_{\eta,\eta^*}$ such that $l_{\eta,\eta^*}(\alpha^*) = 0$. If $\eta^* = 0$, (3.9) follows. If $\eta^* > 0$, Proposition 3.1 and continuous dependence results for MCF imply that $U_{\eta,\eta^*+\delta} \neq \phi$ for some $\delta > 0$, thus contradicting the definition of η^* whence $\eta^* = 0$. $\square$

4. - The main estimates

In this Section we will prove Proposition 3.1. As a first step we have

LEMMA 4.1. *Assume that $\tilde{\alpha} \in \overline{U}_{\eta,\overline{\eta}}$ is a solution of the equation $l_{\eta,\overline{\eta}}(\tilde{\alpha})$ for some $0 \leq \overline{\eta} \leq \eta$. For any $\mu > 0$ there exists $\eta_0 = \eta_0(\mu)$ such that if $\eta \leq \eta_0$ there holds*

$$|\tilde{\alpha}| \leq \mu\varepsilon_0 e^{-\lambda_\ell \tau_0}. \tag{4.1}$$

PROOF. We can use the variation of constants formula in (3.2) to obtain

$$\Omega(t,\tau) = e^{A(\tau-\tau_0)}\Omega(\,\cdot\,,\tau_0) + \int_{\tau_0}^{\tau} e^{A(\tau-s)}[f_1(\,\cdot\,,s) + f_2(\,\cdot\,,s) + f_3(\,\cdot\,,s)]ds \tag{4.2}$$

where $f_1(\,\cdot\,,\tau)$ $i = 1,2,3$ are given in (3.3). Taking into account (3.5), (3.7), (4.2) we obtain

$$1_{\eta,\overline{\eta},i}(\alpha) = \sum_{j=1}^{\ell} e^{-\lambda_j(\tau_1-\tau_0)}Q_{j,i}(\eta,\alpha)\alpha_j + \int_{\tau_0}^{\tau} e^{-\lambda_\ell(\tau_1-s)}\langle\varphi_1(), f_1(\,\cdot\,,s) +$$
$$+ f_2(\,\cdot\,,s) + f_3(\,\cdot\,,s)\rangle ds \quad i = 1\ldots\ell-1.$$

By (3.3a) and (2.42) we obtain $|f_1(\sigma,\tau)| \leq C_\delta\Big(|\psi_r|^{1+\delta} + \frac{|\psi|^{1+\delta}}{r^{1+\delta}} + \xi(re^{\sigma_\ell\tau} - 2\beta)|\psi_r|^2|\psi_{rr}|\Big)$ where $\delta > 0$ is arbitrarily small. Then, from (2.42) we obtain

$$|f_1(r,\tau)| \leq C_\delta e^{-\lambda_\ell(1+\delta)\tau}\left(\frac{1}{r^{(1+\delta)|\alpha|+2}} + r^{6\lambda_\ell+1}\right)$$

where $\delta > 0$ $\overline{\delta} > 0$ are small.

By Lemma 2.5 in [HV6] we then obtain for $\delta > 0$ sufficiently small

$$\|f_1(\cdot,\tau)\|_{X^{-1/2}} \leq C_\delta e^{-\lambda_\ell(1+\delta)(\tau)} \text{ for } \tau_0 \leq \tau \leq \tau_1 \tag{4.3}$$

where $X = L^2_{w,r}$ and $X^{-1/2}$ is defined in the usual sense of fractional spaces with respect the operator A.

By (2.42) and (3.3) we obtain

$$|f_2(r,\tau)| \leq Ce^{-\lambda_\ell\tau}\frac{e^{\sigma_\ell\tau}}{r^{|\alpha|+1}}\chi_{\{2\beta e^{-\sigma_\ell\tau}\leq r\leq(2\beta+1)e^{-\sigma_\ell\tau}\}} \leq$$
$$\leq C_{\delta,\beta}e^{-\lambda_\ell(1+\delta)\tau}r^{-(|\alpha|+2)(1+\bar{\delta})},$$

for δ, $\bar{\delta} > 0$ small, where χ is the characteristic function of the set

$$\{2\beta e^{-\sigma_\ell\tau} \leq r \leq (2\beta+1)e^{-\sigma_\ell\tau}\}$$
$$\|f_2(\cdot,\tau)\|_{X^{-1/2}} \leq C_\delta e^{-\lambda_\ell(1+\delta)\tau} \text{ for } \tau_0 \leq \tau \leq \tau_1.$$

Finally, $\|f_3(\cdot,\tau)\|$ is readily bounded as $e^{-\Gamma e^{B\tau}}$ for some $\Gamma > 0$, $B > 0$.

On the other hand, since the eigenfunctions $\varphi_i(\cdot)$ $i = 1,\dots,m-1$ are bounded in $H^1_\omega(\mathbb{R}^N)$, we then have

$$\left|\int_{\tau_0}^{\tau_1} e^{-\lambda_\ell(\tau-s)}\langle\varphi_k(\cdot), f_1(\cdot,s)+f_2(\cdot,s)+f_3(\cdot,s)\rangle ds\right| \leq$$
$$\leq Ce^{-\lambda_\ell(\tau_1-\tau_0)}e^{-\lambda_\ell(1+\delta)\tau_1}$$

and taking into account that $Q_{j,k}(\eta,\alpha) \to \delta_{j,k}$ as $\eta \to 0$, uniformly for $|\alpha| \leq \varepsilon_0 e^{-\lambda_\ell\tau_0}$ we obtain (4.1) for τ_0 large enough. □

As a next step we obtain the following

LEMMA 4.2. *For any $\mu > 0$ there exists $A_0 > 0$ such that, if $A > A_0$, there exists $\eta_0 = \eta_0(\mu, A)$ small with the property that, for $\eta < \eta_0$ there holds*

$$\left|\frac{\partial^r\Omega}{\partial\sigma^r} - \frac{K}{C_\ell}e^{-\lambda_\ell\tau}\frac{\partial^r\varphi_\ell}{\partial\sigma^r}\right| \leq \frac{\mu e^{-\lambda_\ell\tau}}{\sigma^{|\alpha|+k}} \text{ for } Ae^{-\sigma_\ell\tau} \leq \sigma \leq A, \tag{4.4}$$
$$\tau_0 \leq \tau \leq \tau_1,\ r = 0,1,2.$$

PROOF. The estimate for $r = 0$ is a simple adaptation of Lemma 4.3, and Lemmata 4.5-4.8 in [HV6], that we will not repeat here. The estimate for the derivatives, may be deduced by rescaling.

Indeed. We define

$$W(\sigma,\tau) = \Omega(\sigma,\tau) - K/C_\ell e^{-\lambda_\ell\tau}\varphi_\ell(\sigma)$$

We easily check that $W(\sigma,\tau)$ satisfies equation (3.2), and $|W(\sigma,\tau)| \le \frac{\mu}{\sigma^{|\alpha|}}$ for $Ae^{-\sigma_\ell \tau} \le \sigma \le A$.

For each fixed $s \ge \tau_0$, and $0 \le \lambda \le 1$ we then set

$$M_{\lambda,s}(u,\sigma) = \lambda^{+\frac{|\alpha|}{2}} W(\lambda^{1/2}u, s+\lambda\sigma). \tag{4.5}$$

Notice that

$$\begin{aligned}(M_{\lambda,s})_\sigma - (M_{\lambda,s})_{uu} - \frac{2(N-1)}{u}(M_{\lambda,s})_u - \frac{2(N-1)}{u^2}M_{\lambda,s}+ \\ +\lambda\left(\frac{u(M_{\lambda,s})_u}{2} - \frac{1}{2}(M_{\lambda,s})\right) = \lambda^{1+\frac{|\alpha|}{2}}[f_1+f_2+f_3](\lambda^{1/2}, s+\lambda\sigma).\end{aligned}$$

Taking into account (2.42) and (2.43), we obtain that for $2\beta C^{-\sigma_\ell \tau} \le \sigma \le 2$

$$\left|\frac{\partial^k f_1}{\partial \sigma^k}(\sigma,\tau)\right| \le C_{\delta,\beta}\left(\frac{e^{-\lambda_\ell \tau}}{r^{|\alpha|+1}}\right)^\delta \frac{e^{-\lambda_\ell\tau}}{r^{|\alpha|+2+k}} \quad k=0,1,2$$

$$\left|\frac{\partial^k f_2}{\partial \sigma^k}(\sigma,\tau)\right| \le Ce^{-\lambda_\ell \tau}\frac{e^{\sigma_\ell \tau}}{r^{|\alpha|+1+k}} \qquad k=0,1,2$$

$f_3(\sigma,\tau)=0$.

If $r \ge Ae^{-\sigma_\ell\tau}$ and A is large enough, and we define $g_i(u,\sigma) = \lambda^{1+\frac{|\alpha|}{2}} f_1(\lambda^{1/2}u, s+\lambda\sigma)$, we easily obtain that $\left(\frac{\partial^k g_i}{\partial u^k}\right)$ is small for $k=0,1,2$, $i=1,2,3$, $\frac{1}{2} \le |u| \le 1$. Then by standard regularizing effects for parabolic equations, we have that $\frac{\partial^k}{\partial u^k}(M_{\lambda,s}(u,\sigma))$ is bounded for $\frac{1}{2} \le |u| \le 1$, $k=0,1$ and $\sigma \sim 1$. If $s=\tau_0$ we obtain similar bounds assuming the estimates on (2.42) as well as for the third derivatives in $\psi(\sigma,\tau_0)$. We finally obtain (4.4) with $k=1,2$ by coming back to the original set of variables. □

As a next step we will obtain the estimates on (2.42) in the region $1 \ll \sigma \le \rho e^{\tau/2}$.

LEMMA 4.3. *Assume that ψ is as in* (2.3c) *and $\tilde\alpha \in \overline{U}_{\eta,\eta}$ is a solution of $l_{\eta,\overline{\eta}}(\tilde\alpha)=0$. Let us fix $\mu>0$ small. If ρ in* (2.42) *is small enough $A>0$ is large enough and η is sufficiently small we have that*

$$\left|\psi(\sigma,\tau) - \frac{K\Gamma_\ell}{C_\ell} r^{2\lambda_\ell+1}e^{-\lambda_\ell \tau}\right| \le \mu r^{2\lambda_\ell+1}e^{-\lambda_\ell\tau}, \tag{4.6}$$

where $\tau_0 \le \tau \le \tau_1$, $A \le r \le \rho e^{\tau/2}$.

PROOF. We can construct sub and supersolutions for (2.11) with the form

$$L(\sigma,\tau) = Be^{\tau/2}\xi^{2\lambda_\ell+1} + C_B e^{-\frac{1}{2}\tau}\xi^{2\lambda_\ell-1},$$

where $\xi = re^{\frac{1}{2}\tau}$, B is an arbitrary constant different from zero and C_B is a large (positive or negative) constant depending on B. It is a straightforward computation to check that $L(r,\tau)$ defines sub and supersolutions for (2.11) for suitable choices of C, $\rho > 0$ small, and η small.

Moreover, if A is large enough $\varphi_\ell(\sigma) \simeq \Gamma_\ell \sigma^{2\lambda_\ell+1}$, and by Lemma 4.2, ψ is arbitrarily clase to $\dfrac{K\Gamma_\ell}{C_\ell} r^{2\lambda_\ell+1} e^{-\lambda_\ell \tau}$ if, $r \geq A$, and A is large enough. Then, we can take in the definition of $L(r,\tau)$ values B_1, B_2, $B_1 < \dfrac{K\Gamma_1}{C_1} < B_2$ and construct the corresponding sub and supersolutions. Taking the initial value $\psi(r,\tau_0)$ between them we finally arrive at (4.6). □

We now proceed to obtain the bounds for the derivatives on (2.42).

LEMMA 4.4. *Assume that ψ, $\tilde{\alpha}$ are as in the previous Lemma. Then, there holds*

(i) *The function $Q(u,t)$ has the convexity indicated in* (2.42) *for $\tau_0 \leq \tau \leq \tau_1$ and $|y| \geq A$ as well as for $|y| \leq \dfrac{1}{A}$ if A is large enough.*

(ii) *We have the estimates*

$$|\psi_r| \leq \mu r^{2\lambda_\ell} e^{-\lambda_\ell \tau} \text{ for } A \leq \sigma \leq \rho e^{\tau/2} \tag{4.7a}$$

$$|\psi_{rr}| \leq \mu r^{2\lambda_\ell+1} e^{-\lambda_\ell \tau} \text{ for } A \leq \sigma \leq \rho e^{\frac{1}{2}\left(1-\frac{1}{2\lambda_\ell+1}\right)\tau} \tag{4.7b}$$

where μ, ρ, η are as in Lemma 4.3.

PROOF. Part i) follows by comparison. Taking into account (2.15) and (2.19) we readily obtain that

$$\frac{\partial Q}{\partial \tau} = 2\frac{Q_{uu}}{1+Q_u^2} + \frac{2(N-1)}{u} Q_u - \frac{2(N-1)}{Q} \tag{4.8}$$

Without loss of generality we can assume that $Q(u,t) \geq u$ for $A \leq |y| \leq \rho e^{\tau/2}$. Moreover by Lemma 4.2 and (2.42) we can assume that Q is convex for $|y| \simeq A$, $\tau \geq \tau_0$ and $A \leq |y| \leq \rho e^{\tau/2}$, $\tau = \tau_0$.

Differentiating (4.8) with respect to u we obtain

$$L_1(Q_u, u, \tau) = 2(N-1)\left(\frac{1}{Q^2} - \frac{1}{u^2}\right) Q_u$$

where L_1 is a parabolic operator satisfying $L_1(c,u,\tau) = 0$ for any $c \in \mathbb{R}$. Selecting the initial values in a suitable way, we have by comparison that

$$0 < Q_u < 1 \text{ for } A \leq |y| \leq \rho e^{\tau/2}, \ \tau \geq \tau_0$$

A new differentiation of (4.8a) yields

$$L_2(Q_{uu}, u, \tau) = 4(N-1)\left(\frac{1}{u^3} - \frac{Q_u}{Q^3}\right) Q_u$$

where L_2 is a parabolic operator satisfying $L_2(0, u, \tau) = 0$. Arguing again by comparison we arrive at $0 < Q_{uu}$ for $A \leq |y| \leq \rho e^{\tau/2}$, $\tau \geq \tau_0$ and part i) of Lemma 4.4 follows. The case $Q(u,t) \leq u$ for $A \leq |y| \leq \rho e^{\tau/2}$, $\tau \geq \tau_0$ is similar.

From the convexity or concavity of the surface we obtain (4.7a) from Lemma 4.3.

To derive (4.7b) we need a more careful argument.

Let us parametrize the surface $\Gamma(t)$ as

$$f_t : ((T-t)^{1/2}, +\infty) \times S^{N-1}\left(\frac{\sqrt{2}}{2}\right) \times S^{N-1}\left(\frac{\sqrt{2}}{2}\right) \to \mathbb{R}^{2N},$$

where $\xi = \sigma e^{-\frac{\tau}{2}}$

$$f_t(\xi, \omega_1, \omega_2) = \begin{pmatrix} \xi\omega_1 + Z(\xi,t)\omega_1 \\ \xi\omega_2 - Z(\xi,t)\omega_2 \end{pmatrix}$$

for some $Z : ((T-t)^{1/2}, +\infty) \to \mathbb{R}$.

Taking into account (2.6), (2.10) we easily obtain that Z satisfies the parabolic equation

$$\frac{\partial z}{\partial t} = \frac{Z_{\xi\xi}}{1 + Z_\xi^2} + (N-1)\left[\frac{1+Z_\xi}{\xi - Z} - \frac{1-Z_\xi}{\xi+Z}\right], \tag{4.9}$$

and taking into account (2.2), (2.3c) we have that

$$Z(\xi,t) = (T-t)^{1/2}\psi\left(\frac{\xi}{(T-t)^{1/2}}, -\log(T-t)\right). \tag{4.10}$$

We define

$$M_s(\gamma,\lambda) = (T-s)^{-\frac{1}{2}} Z(\lambda(s) + (T-s)^{1/2}\gamma, s + \lambda(T-s)), \tag{4.11}$$

where $A(T-s)^{1/2} \leq \lambda(s) \leq ((T-t)^{1/2})^{\frac{1}{2\lambda_\ell+1}}$, $\lambda \in [0,1)$, $|\gamma| \leq A$ and $s \leq \eta$.

It is easily seen that

$$\begin{aligned}(M_s)_\lambda = {} & \frac{(M_s)_{\gamma\gamma}}{1 + ((M_s)_\gamma)^2} + \\ & + (N-1)\left[\frac{1 + (M_s)_\gamma}{\dfrac{\lambda(s)}{(T-s)^{1/2}} + \gamma - M_s} - \frac{1 - (M_s)_\gamma}{\dfrac{\lambda(s)}{(T-s)^{1/2}} + \gamma + M_s}\right].\end{aligned} \tag{4.12}$$

By (4.6) and (4.10) we have that

$$\left|Z(\xi,t) - \frac{K\Gamma_\ell}{C_\ell}\,\xi^{2\lambda_\ell+1}\right| \leq \mu|\xi|^{2\lambda_\ell+1}$$

with μ small if $A(T-t)^{1/2} \leq |\xi| \leq \rho$, and by (4.11) we have

$$\left|M_s(\eta,\lambda) - \frac{K\Gamma_\ell}{C_\ell}\,\frac{(\lambda(s))^{2\lambda_\ell+1}}{(T-s)^{1/2}}\right| \leq \mu\,\frac{(\lambda(s))^{2\lambda_\ell+1}}{(T-s)^{1/2}} \quad . \tag{4.13}$$

for $|\gamma| \leq A$, $\lambda \in (0,1)$.

If $s = \eta$ we can assume that the initial value $\psi(\sigma,\tau_0)$ to be close to $ke^{-\lambda_\ell\tau}r^{2\lambda_\ell+1}$ as well as its first three derivatives. This gives

$$\left|\frac{\partial^j\psi}{\partial r^j} - \frac{K\Gamma_\ell e^{-\lambda_\ell\tau}}{C_\ell}\,(r^{2\lambda_\ell+1})\right| \leq \mu e^{-\lambda_\ell\tau}r^{2\lambda_\ell+1-j},$$

for $j = 0,1,2,3$, $1 \leq r \leq \rho e^{\tau/2}$. On the other hand, taking into account (4.10) and (4.11) we obtain

$$\frac{\partial^j}{\partial\gamma^j}\,(M_\eta(\gamma,0)) = \frac{\partial^j}{\partial\sigma^j}\,(\psi)\left(\frac{\lambda(s)+(T-\eta)^{1/2\gamma}}{(T-\eta)^{1/2}}, -\log(T-\eta)\right),\ \ j = 0,1,2,3$$

whence:

$$\begin{aligned}&\left|\frac{\partial^j}{\partial\gamma^j}\,(M_\eta(\gamma,0)) - \frac{K\Gamma_\ell e^{-\lambda_\ell\tau_0}(T-s)^{-\frac{1}{2}}}{C_\ell}\,\frac{\partial^j}{\partial r^j}\,(r^{2\lambda_\ell+1})\right|_{\sigma=\lambda(s)} \leq \\ &\leq \mu e^{-\lambda_\ell\tau}\,\frac{(\lambda(s))^{2\lambda_\ell+1-j}}{(T-s)^{1/2}},\ \text{ for } j = 0,1,2,3,\ |\gamma| \leq 1,\end{aligned} \tag{4.14}$$

where A is large enough and η small enough. Define

$$R_s(\gamma,\lambda) = M_s(\gamma,\lambda) - \frac{K\Gamma_\ell}{C_\ell}\,\frac{(\lambda(s))^{2\lambda_\ell+1}}{(T-s)^{1/2}}$$

By (4.12) we obtain that $R_s(\gamma,\lambda)$ satisfies the equation

$$(R_s)_\lambda = \frac{(R_s)_{\gamma\gamma}}{1+((R_s)_\gamma)^2} + (N-1)\left[\frac{1+(R_s)_\gamma}{A_1(s)+\gamma-R_s}\,\frac{1-(R_s)_\gamma}{A_2(s)+\gamma+R_s}\right]$$

where

$$A_1(s) = \frac{\lambda(s)}{(T-s)^{1/2}} - \frac{K\Gamma_\ell}{C_\ell}\,\frac{(\lambda(s))^{2\lambda_\ell+1}}{(T-s)^{1/2}},$$

$$A_2(s) = \frac{\lambda(s)}{(T-s)^{1/2}} + \frac{K\Gamma_\ell}{C_\ell}\,\frac{(\lambda(s))^{2\lambda_\ell+1}}{(T-s)^{1/2}}.$$

Clearly $|A_i(s)| \geq \frac{A}{2}$ for $i = 1, 2$, A large enough, η small enough. We now can use (4.13) and interior estimates for quasilinear equations (cf. [LSU], Theorem 5.4, p. 449, as well as (4.14) and Theorem 6.1, p. in [LSU] to prove that $|(R_s)_\gamma|$, $|(R_s)_{\gamma\gamma}|$, $|(R_s)_{\gamma\lambda}|$ are uniformly bounded. Then, we have that R_s satisfies an equation of the form $(R_s)_\lambda = a_s(\gamma,\lambda)(R_s)_{\lambda\lambda} + b_s(\gamma,\lambda)(R_s)_\gamma + c_s(\gamma,\lambda)R_s$, where a_s, b_s, c_s and their derivatives are bounded. By standard Schauder estimates we obtain

$$\left|\frac{\partial^2}{\partial\gamma^2}\left(R_s(\gamma,\lambda)\right)\right| \leq \mu\,\frac{(\lambda(s))^{2\lambda_\ell+1}}{(T-s)^{1/2}}$$

for $s \leq \eta$, $|\gamma| \leq \frac{1}{2}$, $0 \leq \lambda \leq 1$ and taking into account (4.10) (4.11) we can prove that in the original set of variables this estimate implies that

$$\left|\frac{\partial^2\psi}{\partial y^2} - \frac{K\Gamma_\ell}{C_\ell}\,e^{-\lambda_\ell\tau}\,\frac{\partial^2}{\partial y^2}\left(y^{2\lambda_\ell+1}\right)\right| \leq \mu e^{-\lambda_\ell\tau}|y|^{2\lambda_\ell+1}$$

for $\tau_0 \leq \tau \leq \tau_1$, $A \leq |y| \leq e^{\frac{1}{2}\left(1-\frac{1}{2\lambda_\ell+1}\right)\tau}$.

This concludes the proof of Lemma 4.4. □

As a next step we obtain precise estimates for the region $|y| \ll 1$.

LEMMA 4.5. *Let $\mu > 0$ be small enough and let $\tilde{\alpha}$ as in Proposition* 3.1. *The surface $(T-t)^{-\frac{1}{2}-\sigma_\ell}\Gamma_t(\tilde{\alpha})$ is contained between $M_{k-\mu}$, $M_{k+\mu}$ for $T-\eta \leq t \leq T-\eta$, and $|x| \leq C(T-t)^{1/2+\sigma_\ell}$ where M_κ is defined in Proposition* 2.2. *Moreover, if we parametrize $\Gamma_t(\tilde{\alpha})$ as in* (2.13) *we have that the derivatives of the function $L(p,\tau)$ defined by*

$$Q(u,t) = (T-t)^{1/2+\sigma_\ell}L\left(\frac{u}{(T-t)^{1/2+\sigma_\ell}}, -\log(T-t)\right)$$

are uniformly bounded in each compact set of p if η is small enough.

PROOF. The surface defined by the rescaling $(T-t)^{-\frac{1}{2}\sigma}\Gamma(t)$, and the new time variable $s = \frac{1}{2\sigma_\ell}\,e^{2\sigma_\ell\tau}$ evolves according to the equation

$$V_N = H + \left(\frac{1}{2}+\sigma_\ell\right)\frac{2\sigma_\ell}{s}\,\langle\omega, N\rangle,$$

where $\omega = \frac{x}{(T-t)^{1/2+\sigma_\ell}}$. By (2.15), (2.18), (2.19) we have that

$$\begin{aligned}\frac{\partial L}{\partial s} &= \frac{2}{1+L_p^2}\,L_{pp} + \frac{2(N-1)}{p}\,L_p - \frac{2(N-1)}{L} - \\ &\quad - \frac{2\sigma_\kappa}{s}\left(\frac{1}{2}+\sigma_k\right)(pL_p - L).\end{aligned} \tag{4.15}$$

The three first terms on the left-hand side are precisely the minimal surface equation. Notice that by Lemma 4.2.

$$|L(p,\tau) - p - kp^{\alpha}| \le \mu |p|^{\alpha}$$

where $\mu > 0$ is small if $|p| = A$, A is large enough and η is small enough. Given $H_a(u)$, where $H_a(u)$ is as in Proposition 2.2, we can construct sub and supersolutions as follows define $\overline{L}(p) = H_a(p/\theta)$, where $\theta > 0$. Clearly

$$\begin{aligned}
&\overline{L}_{\tau} - \frac{2}{1+(\overline{L}_p)^2}\overline{L}_{pp} - \frac{2(N-1)}{p} + \frac{2(N-1)}{\overline{L}} = \\
&= (\theta^2 - 1)\left[\frac{H_a''(p/\theta)}{(1+(H_a')^2)(\theta^2+(H_a')^2)} + \frac{(N-1)}{p\theta}H_a'(p/\theta)\right].
\end{aligned}$$

By Proposition 2.2 $H_a'(u) \ge 0$, $H_a''(u) > 0$.

Then, for each compact set $|P| \le A$ and η small enough (i.e. s large enough), $\overline{L}$ defines a subsolution for $\theta < 1$ and a supersolution for $\theta > 1$ for the equation (4.15). Taking a close to k, A large enough and η small, and a suitable initial value for $|p| \le A$ we obtain that $\Gamma_t(\tilde{\alpha})$ is contained between the surfaces $M_{k-\mu}$, $M_{k+\mu}$.

In order to bound the derivatives we use an argument rather similar to that already emplayed in the proof of Lemma 4.2. Taking into account that $\Gamma_t(\tilde{\alpha})$ is contained between $M_{k-\mu}$ and $M_{k+\mu}$, as well as the convexity of $Q(u,t)$ for $|u| \le \frac{1}{\beta}e^{-\frac{1}{2}\tau}$, we easily obtain

$$\left|\frac{\partial^j L}{\partial p^j}(p) - \frac{\partial^j H_k}{\partial p^j}(p)\right| \le \mu |p|^{-(|\alpha|+j)},\ j = 0,1,\ \text{for } |p| \le A. \tag{4.16}$$

Define

$$W(p,s) = L(p,s) - H_k(p).$$

By (2.38) and (4.15) we have that

$$\begin{aligned}
\frac{\partial(W)}{\partial s} &= \frac{2}{1+L_p^2}(W)_{pp} + \frac{2(H_p+L_p)H_{pp}}{(1+L_p^2)(1+H_p^2)}W_p + \frac{2(N-1)}{p}(W)_p - \\
&- \frac{2(N-1)}{HL}(W) - \frac{2\sigma_\ell}{s}\left(\frac{1}{2}+\sigma_\ell\right)(pW_p - W) - \\
&- \frac{2\sigma_\ell}{s}\left(\frac{1}{2}+\sigma_\ell\right)(pH_p - p).
\end{aligned}$$

Notice that by (4.16),

$$\left|\frac{\partial^j W}{\partial p^j}\right| \le \frac{\mu}{|p|^{|\alpha|+j}},\ j = 0,1,\ \left|\frac{\partial^j}{\partial p^j}(pH_p - H)\right| \le \frac{C}{|p|^{|\alpha|+j}},\ j = 0,1,2.$$

By standard estimates for quasilinear equations (cf. [LSU]) we have that (W), and its derivatives are bounded for $|p| \le A$, where the bound may be chosen independent of A if η is small enough.

We then define

$$M_{\lambda,s_0}(u,\sigma) = \lambda^{\frac{|\alpha|}{2}} W(\lambda^{1/2}u, s_0 + \lambda\sigma).$$

It is readily seen that $M_{\lambda,s_0}(u,\sigma)$ satisfies the equation

$$\begin{aligned} \frac{\partial}{\partial\sigma}(M_{\lambda,s_0}) &= a_{\lambda,s_0}(u,\sigma)(M_{\lambda,s_0})_{u,u} + b_{\lambda,s_0}(u,\sigma)(M_{\lambda,s_0})_u + \\ &\quad + c_{\lambda,s_0}(u,\sigma)(M_{\lambda,s_0}) - \frac{2\sigma_1\lambda^{\frac{|\alpha|}{2}+1}}{s}\left(\frac{1}{2}+\sigma_1\right)(pH_p - H) \end{aligned} \tag{4.17}$$

where

$$a_{\lambda,s_0}(u,\sigma) = \frac{2}{1 + (L_p(\lambda^{1/2}u, s_0 + \lambda\sigma))^2}$$

$$b_{\lambda,s_0}(u,\sigma) = \left(\frac{2(H_p + L_p)H_{pp}\lambda^{1/2}}{(1 + L_p^2)(1 + H_p^2)} + \frac{2(N-1)}{p} - \frac{2\sigma_\ell}{s\lambda}\left(\frac{1}{2} + \sigma_\ell\right)p\right)\Bigg|_{\substack{p=\lambda^{1/2}u \\ s=s_0+\lambda\sigma}}$$

$$c_{\lambda,s_0}(u,\sigma) = -\frac{2(N-1)}{HL} + \frac{2\sigma_1}{s\lambda}\left(\frac{1}{2} + \sigma_1\right)\Bigg|_{\substack{p=\lambda^{1/2}u \\ s=s_0+\lambda\sigma}}$$

Coefficients $a_{\lambda,s_0}(u,\sigma)$, $b_{\lambda,s_0}(u,\sigma)$, $c_{\lambda,s_0}(u,\sigma)$ depend on M_{λ,s_0}, $(M_{\lambda,s_0})_u$. Taking into account (4.16), we obtain that $|M_{\lambda,s_0}|$, $|(M_{\lambda,s_0})_u|$ are uniformly bounded for $\lambda \ge 1$. Their the functions a_{λ,s_0}, b_{λ,s_0}, c_{λ,s_0} are uniformly bounded as well as their derivatives on u, σ if $\lambda \ge 1$. By standard regularity theory for parabolic equations (cf. [LSU]) and assuming enough regularity for the initial data we obtain that for $\frac{1}{2} \le |u| \le 2$,

$$\left|\frac{\partial^2(M_{\lambda,s_0})}{\partial u^2}\right| \le \mu$$

and in the original variables this implies that

$$\left|\frac{\partial^2 W}{\partial p^2}\right| \le \frac{\mu}{|p|^{|\alpha|+2}} \text{ for } |p| \le A,$$

where A is arbitrarily large, but independent on μ. This concludes the proof of Lemma 4.5. $\square$

End of the Proof of Proposition 3.1.

Assume that $\tilde{\alpha}$ is as the statement of Proposition 3.1. We need to verify that (2.40)-(2.44) hold with $\theta = \frac{1}{2}$. Clearly (2.40) holds by standard regularity theory parabolic equations if (2.42), (2.44) are verified. The estimates on (2.42) follow from Lemmata 4.2, 4.3 and 4.4 part ii) if β is large enough and η sufficiently small; (2.43) follows from Lemma 4.4 part i). (2.44) follows from Lemma 4.5. It only remains to show (2.41) for $|x| \geq \rho$, but this is obtained by taking η small enough. One merely uses continuous dependence results on the initial value for MCF on the region $|x| \geq \rho$. □

5. - The proof of Theorem 2.2

In this Section we prove that the solutions whose existence has been stated in Theorem 2.1 have the asymptotic behaviour prescribed in Theorem 2.1. The proof of part i) of Theorem 2.1 is essentially the same as (1.19b) in [HV6], then. We just will state the main result and refer to that paper for details of the proof. In the rest of this Section we shall prove part ii) and part iii) of Theorem 2.2.

5.1. *The proof of Theorem* 2.2. *Part i)*

The main result is the following

PROPOSITION 5.1. *Let* $\Gamma(t)$ *be the family of surfaces moving by* MCF *that has been obtained in Theorem* 2.1. *Given* $\varepsilon > 0$ *there exists* $k > 0$, $|k - \overline{k}| \leq \varepsilon$ *such that uniformly on sets* $Ae^{-\sigma_1\tau} \leq \sigma \leq B$, *there holds for* η *small enough*

$$\left|\psi(r,\tau) - \frac{\overline{k}}{C_\ell}\varphi_\ell(r)e^{-\lambda_\ell\tau}\right| \leq \frac{\mu(A)}{r^{|\alpha|}}e^{-\lambda_\ell\tau}, \tag{5.1}$$

where $\mu(A) \to 0$ *if* $A \to \infty$. *Moreover, for any given* $B > 0$, *we have that*

$$\psi(r,\tau) - \frac{\overline{K}}{C_\ell}\varphi_1(r)e^{-\lambda_\ell\tau} = o(e^{-\lambda_\ell\tau}) \tag{5.2}$$

as $\tau \to \infty$ *uniformly on sets* $\frac{1}{B} \leq r \leq B$.

The proof of Proposition 5.1 is just a simple modification of Lemma 6.2 in [HV6] and will be omitted.

5.2. *Theorem* 2.2. *Part ii)*

We parametrize the family Γ_t as in (2.13) and define $L(p,s)$ as in Lemma 4.5. We know that function $L(p,s)$ satisfies the equation (4.15). Let us fix $A_1 > 0$ large enough. By (5.1) we have that

$$\left| L(p,s) - p - \frac{\overline{k}}{p^{|\alpha|}} \right| \leq \frac{\tilde{\mu}(A_1)}{p^{|\alpha|}} \text{ as } s \to \infty, \text{ for } |p| \geq A_1, \tag{5.3}$$

where $\tilde{\mu}(A_1) \to 0$ as $A_1 \to \infty$. On the other hand, taking into account the proof of Lemma 4.5 we readily obtain that

$$H_{k_1}\left(\frac{p}{\theta_1}\right) \leq L(p,s) \leq H_{k_2}\left(\frac{p}{\theta_2}\right), \text{ for } s \to \infty \text{ where } |p| \leq A_1 \tag{5.4}$$

$k_1 < \overline{k} < k_2$, $\theta_1 < 1 < \theta_2$. The left hand side of (5.4) is a subsolution, and the right-most hand side is a supersolution.

Assume that $|k_1 - \overline{k}| \leq 2\tilde{\mu}(A_1)$, $i = 1,2$. Then we can take a larger $A_2 > A_1$ such that $|k_1 - \overline{k}| > 2\tilde{\mu}(A_2)$ for some $i = 1,2$. Taking into account (5.3), (5.4) and choosing θ_1, θ_2 closer to 1 if needed we could assume that (5.4) holds for $|p| \leq A_2$.

We then restrict our attention to the case $|k_1 - \overline{k}| > 2\tilde{\mu}(A_1)$, for some $i = 1,2$. Define the sequence of functions

$$L_n(p,s) = L(p, s+n) \quad n = 0,1,2.$$

By standard parabolic theory we have that for some subsequence $n_j \to \infty$, $j \to \infty$, $L_{n_j}(p,s) \to \tilde{L}(p,s)$ as $j \to \infty$, uniformly on $|p| \leq A_1$, $0 \leq s \leq 1$. Moreover, taking into account (5.4) we have

$$H_{k_1}\left(\frac{p}{\theta_1}\right) \leq \tilde{L}(p,0) \leq H_{k_2}\left(\frac{p}{\theta_2}\right) \quad \text{for } |p| \leq A_1$$

and (5.3). The assumption $|k_i - \overline{k}| \leq 2\tilde{\mu}(A_1)$ for some $i = 1,2$ implies then that the least one of the inequalities

$$H_{k_1}\left(\frac{p}{\theta_1}\right) < \tilde{L}(p,0) \text{ or } \tilde{L}(p,s) < H_{k_2}\left(\frac{p}{\theta_2}\right)$$

is satisfied for $0 \leq s \leq 1$ and $|p| = A_1$. Moreover, the function $\tilde{L}(p,s)$ satisfies the equation

$$(\tilde{L})_s = \frac{2}{1 + (\tilde{L}_p)^2} \tilde{L}_{pp} - \frac{2(N-1)}{s} \tilde{L}_p + \frac{2(N-1)}{\tilde{L}}.$$

Then, by the strong maximum principle, at least one of the inequalities

$$H_{k_1}\left(\frac{p}{\theta_1}\right) < \tilde{L}(p,1) \text{ or } \tilde{L}(p,1) < H_{k_2}\left(\frac{p}{\theta_2}\right),$$

is satisfied for $|P| \leq A_1$, whence in turn one of the inequalities

$$H_{k_1}\left(\frac{p}{\theta_1}\right) < L(p,n_j) \text{ or } L(p,n_j) < H_{k_2}\left(\frac{p}{\theta_2}\right)$$

holds for j large enough $|P| \leq A_1$. We can take then some $\overline{k}_1 > k_1$ or some $\overline{k}_2 < k_2$, and suitable $\overline{\theta}_1$, $\overline{\theta}_2$ such that (5.4) holds with $\overline{k}_1$, $\overline{k}_2$, $\overline{\theta}_1$, $\overline{\theta}_2$ instead of k_1, k_2, θ_1, θ_2. This argument can be iterated as long as $|k_1 - \overline{k}| > 2\tilde{\mu}(A_1)$, $i = 1,2$. At this point we take $A_1 > A_1$ reducing the value $\tilde{\mu}(A_2)$. As $\tilde{\mu}(A) \to 0$ as $A \to \infty$, if we repeat the previous argument and take into account the continuity of $H_k(p)$ in k for compact sets of p, we obtain the desired result. □

5.3. *Theorem* 2.2. *Part iii)*

We fix $\overline{\tau} \geq \tau_0$ arbitrary, let h_ε be a smooth function $h_\varepsilon : [0,+\infty) \to \mathbb{R}$, $h_\varepsilon(\xi) = \varepsilon$, $h'_\varepsilon(\xi) \geq 0$, $h_\varepsilon(\infty) = 1$, where $\varepsilon > 0$ arbitrarity small for $\xi \in [0,1]$. We consider the functions

$$W_\pm(\sigma,\tau) = (\overline{k} \pm h_\varepsilon(\sigma e^{-\frac{\tau-\overline{\tau}}{2}}))e^{-\lambda_1\tau}\sigma^{2\lambda_1+1} \mp \overline{C}e^{-\lambda_1\tau}\sigma^{2\lambda_1-1} \tag{5.5}$$

where $\overline{C} \in \mathbb{R}^+$.

It is readily seen that $W_\pm(\sigma,\tau)$ are respectively sub and supersolutions for (2.11) in the region $A \leq \sigma \leq \rho e^{\frac{1}{2}\tau}$ provided that $\overline{C}$ and A are large enough (cf. Lemma 4.3 for related sub and supersolutions). Moreover, taking into account Lemma 4.3 and Proposition 5.1 we have that

$$W_-(\sigma,\overline{\tau}) \leq \psi(\sigma,\overline{\tau}) \leq W_+(\sigma,\overline{\tau}) \text{ if } \overline{\tau} \text{ is large enough.}$$

Then, by comparison there holds

$$W_-(\sigma,\tau) \leq \psi(\sigma,\tau) \leq W_+(\sigma,\overline{\tau}) \text{ for } \tau \geq \overline{\tau}.$$

Taking into account (2.4) we obtain from (5.5) that

$$\left|Q(u,T) - u - \frac{\Gamma_1}{C_1}\overline{k}u^{2\lambda_\ell+1}\right| \leq 2\varepsilon u^{2\lambda_\ell+1}$$

for $u \leq e^{-\frac{\overline{\tau}}{2}}$.

Making $\overline{\tau} \to \infty$ we can take ε arbitrarily small and part iii) of Theorem 2.2 follows. □

REFERENCES

[AlAG] S.J. ALTSCHULER - S.B. ANGENENT - Y. GIGA, *Motion by Mean Curvature through singularities for surfaces of rotation*. Preprint (1991).

[AG] S.J. ALTSCHULER - M.A. GRAYSON, *Shortening space curves and flow through singularities*. IMA Preprint 823 (1991).

[A1] S.B. ANGENENT, *Parabolic equations for curves on surfaces*. Part. I, Ann. Math. **132** (1990), 451-483.

[A2] S.B. ANGENENT, *Parabolic equations for curves on surfaces*. Part. II, Ann. Math. **133** (1991), 171-215.

[A3] S.B. ANGENENT, *On the formation of singularities in the curve shortening flow*. J. Differential Geom. **33** (1991), 601-633.

[A4] S.B. ANGENENT, *Some recent results on Mean Curvature Flow*. In: Recent Advances in PDE's. M.A. Herrero - E. Zuazua eds., Research in Applied Math., Masson & J. Wiley, 1994, 1-18.

[AV1] S.B. ANGENENT - J.J.L. VELÁZQUEZ, *Nonconvex collapse at rotationally symmetric hypersurfaces evolving by mean curvature flow*. In preparation.

[AV2] S.B. ANGENENT - J.J.L. VELÁZQUEZ, *Asymptotic shape of cusp singularities in curve shortening*. Duke Math. J., in press.

[BDGG] E. BOMBIERI - E. DE GIORGI - E. GIUSTI, *Minimal cones and the Bernstein problem*. Inv. Math. **7** (1969), 243-268.

[B] K.A. BRAKKE, The motion of a surface by its mean curvature. Princeton University Press, Math. Notes, Princeton, New Jersey, 1978.

[B1] A. BRESSAN, *On the asymptotic shape of blow-up*. Indiana Univ. Math. J. **39** (1990), 947-960.

[B2] A. BRESSAN, *Stable blow-up patterns*. J. Differential Equations. **98** (1992), 57-75.

[BQ] C.J. BUDD - Y.W. QI, *The existence of bounded solutions of a semilinear heat equation*. J. Differential Equations **82** (1989), 207-218.

[CGG] Y.G. CHEN - Y. GIGA - S. GOTO, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow*. J. Differential Geom. **33** (1991), 749-786.

[DG] E. DE GIORGI, *Some conjectures on flow by mean curvature*. Preprint.

[ESS] L.C. EVANS - H.M. SONER - P.E. SOUGANIDIS, *Phase Transitions and generalized mean curvature flow equations*. To appear in Comm. Pure Appl. Math.

[ES] L.C. EVANS - J. SPRUCK, *Motion of level sets by mean curvature I*. J. Differential Geom. **33** (1991), 635-681.

[F] H. FEDERER, Geometric measure theory. Springer Verlag, New York, 1969.

[FK] S. FILIPPAS - R.V. KOHN, *Refined asymptotics for the blow-up of* $u_t-\Delta u=u^p$. Comm. Pure Appl. Math. **45** (1992), 821-869.

[FL] S. FILIPPAS - W. LIU, *On the blow-up of a multidimensional semilinear heat equation*. Ann. Inst. H. Poincaré Anal. Non Linéaire **10** (1993), 313-344.

[GH] M. GAGE - R.S. HAMILTON, *The heat equation shrinking plane convex curves*. J. Differential Geom. **23** (1986), 69-96.

[GP] V.A. GALAKTIONOV - S.A. POSASHKOV, *The equation* $u_t=u_{xx}+u^B$. *Localization, asymptotic behaviour of unbounded solutions*, Preprint N. 97, M. Keldish, Inst. Appl. Math. (1985), (in russian).

[GaP] A. GALINDO - P. PASCUAL, *Quantum Mechanics*. Vol. I (1990) and vol. II (1991), Springer Verlag.

[G] E. GIUSTI, Minimal surfaces and functions of bounded variation. Birkhäuser, Boston, 1984.

[GK1] Y. GIGA - R.V. KOHN, *Asymptotically self-similar blow-up of semilinear heat equations*. Comm. Pure Appl. Math. **38** (1985), 297-319.

[GK2] Y. GIGA - R.V. KOHN, *Characterizing blow-up using similarity variables*. Indiana Univ. Math. J. **36** (1987), 1-40.

[GK3] Y. GIGA - R.V. KOHN, *Nondegeneracy of blow-up for semilinear heat equations*. Comm. Pure Appl. Math. **42** (1987), 845-884.

[Gr] M. GRAYSON, *The heat equation shrinks embedded plane curves to round points*. J. Differential Geom. **26** (1987), 285-314.

[HV1] M.A. HERRERO - J.J.L. VELÁZQUEZ, *Blow-up behaviour of one-dimensional semilinear parabolic equations*. Ann. Inst. Henri Poincaré **10** (1993), 131-189.

[HV2] M.A. HERRERO - J.J.L. VELÁZQUEZ, *Flat blow-up in one-dimensional semilinear heat equations*. Differential Integral Equations **5** (1992), 973-998.

[HV3] M.A. HERRERO - J.J.L. VELÁZQUEZ, *Blow-up profiles in one-dimensional semilinear parabolic problems*. Comm. Partial Differential Equations **17** (1992), 205-219.

[HV4] M.A. HERRERO - J.J.L. VELÁZQUEZ, *Generic behaviour of one-dimensional blow-up patterns*. Ann. Scuola Norm. Sup. Pisa, Cl. Sci. **19** (1992), 381-450.

[HV5] M.A. HERRERO - J.J.L. VELÁZQUEZ, *Generic behaviour near blow-up points for a N-dimensional semilinear heat equation*. In preparation.

[HV6] M.A. HERRERO - J.J.L. VELÁZQUEZ, *A blow-up result for semilinear heat equations in the supercritical case*. To appear.

[HV7] M.A. HERRERO - J.J.L. VELÁZQUEZ, *Explosion de solutions d'equations paraboliques semilinéaires supercritiques*. C.R. Acad. Sci. Paris **319** (1994), 141-145.

[H] G. HUISKEN, *Asymptotic behaviour for singularities of the mean curvature flow*. J. Differential Geom. **31** (1991), 285-299.

[LSU] O.A. LADYSHENSKAYA - N.N. SOLONNIKOV - V.A. URALCEVA, Linear and quasilinear equations of parabolic type, Translations of Mathematical monographs, A.M.S. vol. 23, 1988.

[L] L.A. LEPIN, *Countable spectrum at eigenfunctions of a nonlinear heat conduction equation with distributed parameters*. Differentsial'nye Uravneiya 24, **7** (1988), 1226-1234.

[RS] M. REED - B. SIMON, Functional analysis, vol. II, Academic Press (1980).

[S] J. SIMONS, *Minimal varieties in riemannian manifolds*. Ann. of Math. **88** (1968), 62-106.

[So] H.M. SONER, *Motion of a set by the curvature of its boundary*. J. Differential Equations **101** (1993), 313-372.

[T] W.C. TROY, *The existence of bounded solutions of a semilinear heat equation*. SIAM J. Math. Anal. vol. 18, **2** (1987), 332-336.

[V1] J.J.L. VELÁZQUEZ, *Classification of singularities for blowing up solutions in higher dimensions*. Trans. Amer. Math. Soc. **338** (1993), 441-464.

[V2] J.J.L. VELÁZQUEZ, *Higher dimensional blow-up for semilinear parabolic equations*. Comm. Partial Differential Equations **17** (1992), 1567-1596.

[V3] J.J.L. VELÁZQUEZ, *Estimates on the* $(N-1)$*-dimensional Hausdorff measure of the blow-up set for a semilinear heat equation.* Indiana Univ. Math. J. **42** (1993), 446-476.

[V4] J.J.L. VELÁZQUEZ, *Blow-up for semilinear parabolic equations.* In Recent Advances in PDE's, M.A. Herrero - E. Zuazua eds., Research in Applied Math., Masson & J. Wiley, 1994, 131-145.

Departamento de Matemática Aplicada
Facultad de Ciencias Matemáticas
Universidad Complutense
Madrid 28040
SPAIN