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GARY M. LIEBERMAN

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Gradient Estimates for a New Class of Degenerate Elliptic and Parabolic Equations

GARY M. LIEBERMAN

Introduction

It has been known for a long time that the gradient of the solution of a nondegenerate elliptic or parabolic equation can be estimated in terms of the maximum of the solution and certain structure conditions on the equations. For quasilinear non-divergence structure equations, a complete description of these estimates can be found in [19] or [4, Chapter 14]. For divergence structure equations, we refer to [16] and [7]. More precisely, these works consider equations which can be written as

$$u_t = a^{ij}(x,t,u,Du)D_{ij}u + b(x,t,u,Du)$$

(summation convention assumed) under the basic hypothesis that there is a positive constant L such that all eigenvalues of the matrix $(a^{ij}(x,t,u,Du))$ are positive and finite when $|Du| > L$ and Du is finite. Of course, many other hypotheses on a^{ij} and b are important to the gradient estimate, but we wish to focus attention on this one of non-degeneracy.

On the other hand, Mkrtychyan ([17], [18]) has recently considered problems where, for any positive L, there is a choice of Du with $|Du| = L$ which gives (a^{ij}) a zero eigenvalue. Under suitable addition hypotheses, he was able to prove various gradient estimates. Specifically, he consider equations of the form

$$u_t = \operatorname{div} A(u,Du) + f(x,t,u,Du),$$

in two space dimensions, assuming that there are constants $\nu \geq 0$ and $m_i \geq 3$, and nonnegative functions a_i such that

$$A^i(u,p) = \nu p_i + a_i(u)|p_i|^{m_i-2}p_i.$$

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If f satisfies appropriate structure conditions, Mkrtychyan proved two estimates. In [17], the quantity $\sum_{i=1}^{2} a_i(u)^{\alpha_i}|D_iu|^{m_i}$ was estimated for suitable constants α_i provided $\nu = 0$, $|m_1 - m_2|$ is sufficiently small, $a_i(u) = u^{l_i}$ for constants l_i such that $|l_1 - l_2|$ is sufficiently small and $(m_i - 1)^2 - l_i(m_i - l_i - 1)$ is positive for $i = 1, 2$. In [18], $\sum_{i=1}^{2} |D_iu|^m$ is estimated provided $m_1 = m_2 = m$, $\nu > 0$, and $a_1 = a_2$ is a C^2 positive function satisfying a technical condition; the estimate depends on ν. The latter situation provides an approximation scheme for solving the first problem when $m_1 = m_2$ and $l_1 = l_2$.

Our goal in this work is to reproduce Mkrtychyan's results in a more general framework. As the examples in Section 4 demonstrate, we have been only partially successful. If $m_1 = m_2$ and if the a_i's are positive and Lipschitz, we derive a gradient bound, independent of $\nu \in (0, 1)$. If the a_i's are constant, we can allow arbitrary m_i's. In both cases, we need only assume that $m_i \geq 2$. In addition our method applies to many nondegenerate equations.

Our proof follows the general outline of Leon Simon's gradient estimate [20] for nondegenerate elliptic equations, which is based on Moser's iteration scheme; our proofs will therefore be sketchy except when dealing with a new element of these degenerate equations. We start with a suitable version of the Michael-Simon Sobolev-type inequality [16] in Section 1. The gradient estimate for degenerate elliptic equations is proved in Section 2, which includes comments about estimates near the boundary. The modifications needed to handle parabolic equations are given in Section 3, and Section 4 presents examples to illustrate the variety of equations included in our structure conditions.

1. - A Sobolev inequality

An important element of our program is a suitable Sobolev-inequality. The one we use is a consequence of a general result due to Michael and Simon [16].

LEMMA 1.1. *Let m and n be integers with $1 \leq n \leq m$. Let U be an open subset of $\mathbb{R}^m$ and let $M \subset U$. Let μ be a nonnegative measure defined on any set of the form $M \cap B$ for B a Borel subset of $\mathbb{R}^m$ such that $\mu(M \cap C)$ is finite when C is a compact subset of U. Let γ^{ij} $(i = 1, \dots, m)$ and H_i $(i = 1, \dots, m)$ be $L^1_{\text{loc}}(M; d\mu)$ functions. For $h \in C^1(U)$, define $\delta_i h$ $(i = 1, \dots, m)$ by $\delta_i h(x) = \gamma^{ij}(x) D_j h(x)$. Suppose also that*

$$\sum_{i=1}^{m} \gamma^{ii}(x) = m, \quad \gamma^{ij}(x) = \gamma^{ji}(x) \tag{1.1a}$$

$$0 \leq \gamma^{ij}(x)\xi_i\xi_j \leq |\xi|^2 \text{ for all } \xi \in \mathbb{R}^m \tag{1.1b}$$

for μ almost all $x \in M$,

$$\int_M [\delta h + hH] d\mu = 0 \tag{1.2}$$

for each $h \in C^1(U)$ with compact support in U, and

$$\limsup_{\rho \to 0^+} \rho^{-n} \mu(B(\xi, \rho)) \geq \omega_n \tag{1.3}$$

for μ almost all $\xi \in M$, where

$$B(\xi, \rho) = \{x : x \in M \text{ and } |x - \xi| < \rho\} \tag{1.4}$$

and ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Then there is a constant C_0 determined only by n such that

$$\begin{aligned} \left(\int_M h^{2(n+2)/n} d\mu \right)^{n/(n+2)} \leq C_0 & \left(\int_M h^2 \, d\mu \right)^{2/(n+2)} \\ & \left(\int_M (|\delta h|^2 + h^2 |H|^2) d\mu \right)^{n/(n+2)} \end{aligned} \tag{1.5}$$

for any $h \in C^1(U)$ with compact support in U.

PROOF. This inequality was proved from [16, (5) on page 372] as [7, (1.4)]. □

Our application of Lemma 1.1 is based (very loosely) on Example 2 from [16]. Let Ω be an open subset of $\mathbb{R}^n$, let $u \in C^2(\Omega)$, let $b_1, \ldots, b_n$ be positive, increasing, Lipschitz functions on $\mathbb{R}_+$, and define $\overline{B}_i$ by $\overline{B}_i(t) = \int_0^t b_i(s) ds$, $\overline{B} = \sum_{i=1}^n \overline{B}_i$ and $b = \sum_{i=1}^n b_i$. Also we define v to be the positive solution of $\overline{B}(v) = \sum_{i=1}^n \overline{B}_i(|D_i u|)$. (Since $b(t) > 0$ if $t > 0$, this equation uniquely determines v.) Now we choose $m = 2n$, $M = \{(x, 0) \in \mathbb{R}^m : x \in \Omega\}$, and, for a fixed positive τ, we define μ by

$$\mu(B \cap M) = \int_{\pi(B \cap M)} \max\left\{1, \frac{v}{\tau}\right\} dx,$$

where $\pi(x,y)=x$ is the projection of $\mathbb{R}^m$ onto $\mathbb{R}^n$. The matrix (γ^{ij}) is defined by $\gamma^{ij}=0$ if $i\neq j$, $\gamma^{ii}=\overline{B}_i(|D_iu|)/\overline{B}(v)$ if $i\le n$, $\gamma^{ii}=(n-1)/n$ if $i>n$. Finally we define

$$H_i=\frac{1}{\max\left\{1,\frac{v}{\tau}\right\}}D_j\left(\gamma^{ij}\max\left\{1,\frac{v}{\tau}\right\}\right)\ \text{if } i\le n,\ H_i=0 \text{ if } i>n.$$

It is simple to see that the hypotheses of Lemma 1.1 are satisfied, so (1.5) holds in this case. For future reference, we note that

$$H_i=\frac{1}{v}\frac{\overline{B}_i(|D_iu|)}{\overline{B}(v)}D_iv-\frac{b(v)}{\overline{B}(v)^2}\overline{B}_i(|D_iu|)D_iv+\frac{b_i(|D_iu|)}{\overline{B}(v)}D_{ii}u$$

wherever $v>\tau$. Moreover, if we define (g^{ij}) by

$$g^{ij}=0 \text{ if } i\neq j,\ g^{ii}=\frac{vb_i'(|D_iu|)}{b(v)},$$

and if there is a positive constant b_0 such that

$$b_i(t)/b_0\le tb_i'(t)\le b_0b_i(t) \text{ for } t\ge 0,\ i=1,\dots,n,$$

we have

$$v\frac{b(v)\overline{B}_i(|D_iu|)}{\overline{B}(v)^2}\le(1+b_0)\frac{\overline{B}_i(|D_i(u)|)}{\overline{B}(v)}$$

and

$$\left[\frac{b_i(|D_iu|)}{\overline{B}(v)}\right]^2\le b_0(1+b_0)^2\frac{\overline{B}_i(|D_iu|)b_i'(|D_iu|)}{v\overline{B}(v)b(v)}$$

Hence

$$v^2|H|^2\le C(b_0)|\delta v|^2+C(b_0)g^{ij}\gamma^{km}D_{ik}uD_{jm}u.$$

2. - The gradient estimate

To prove our gradient bound, we follow the broad outline of [20] using some ideas from [6], [7], and [9]. In addition we take advantage of some of the special structure of our model equation to adapt some of the structure conditions from these works. Since our model equation typically degenerates on any sphere $|Du|=$ constant, Simon's structure conditions involving the minimum eigenvalue $\underline{\mu}$ must all be recast.

From now on, u denotes a C^2 solution of

$$\operatorname{div}A(x,u,Du)+B(x,u,Du)=0 \text{ in } \Omega. \tag{2.1}$$

and we use β, c, M to denote nonnegative constants with $|u| \le M$ in Ω; β and c will often appear with subscripts. In addition, we assume that β_0 is a positive constant and $b_1, \dots, b_n$ are positive, increasing, Lipschitz functions such that

$$b_i(t)/\beta_0 \le t b_i'(t) \le \beta_0 b_i(t) \text{ for all } t > 0. \tag{2.2}$$

For $v = \overline{B}^{-1}\left(\sum \overline{B}_i(|D_i u|)\right)$, ρ and τ positive constants, and $x_0 \in \Omega$, we define

$$\Omega_\tau = \{x \in \Omega : v > \tau\}, \quad B_\rho = \{x \in \mathbb{R}^n : |x - x_0| < \rho\},$$
$$\Omega_{\tau,\rho} = \Omega_\tau \cap B_\rho.$$

We introduce positive, increasing C^1 structure functions w, λ and Λ, and we assume that

$$w^\beta (\Lambda/\lambda)^{(n+2)/2}/\Lambda \text{ is increasing} \tag{2.3a}$$
$$w^{-\beta} b \text{ is decreasing} \tag{2.3b}$$
$$\xi^{-\beta} w \text{ is decreasing} \tag{2.3c}$$
$$\xi^{-\beta}(\Lambda/\lambda)^{(n+2)/2}/\Lambda \text{ is decreasing,} \tag{2.3d}$$

where ξ is the identity function on $\mathbb{R}$. For some positive constant τ_0 and positive $C^0(\Omega)$ function $\overline{\mu}$ and Λ_0, we assume that the following conditions hold on Ω_{τ_0}:

$$a^{ij}\eta_i\xi_j \le (\overline{\mu}|\eta|^2)^{1/2}(a^{ij}\xi_i\xi_j)^{1/2}, \tag{2.4a}$$
$$\lambda(1 + (v\lambda'/\lambda)^2)\gamma^{ij}\xi_i\xi_j \le a^{ij}\xi_i\xi_j \tag{2.4b}$$

for all η and ξ in $\mathbb{R}^n$ (here and below we omit the argument v from λ, Λ, and w), where γ^{ij} is as in Section 1 and $a^{ij} = v\partial A^i/\partial p_j$. We also assume that

$$\Lambda_0 \le \Lambda, \quad \lambda \le \Lambda, \quad \overline{\mu} \le \Lambda. \tag{2.5}$$

For our next structure conditions, we set $\nu^i = (\operatorname{sgn} D_i u) b_i(|D_i u|)/b(v)$ (so $D_j v = D_{ij} u \nu^i$), we assume there are tensors (C_k^i), (D_k^i) with D_k^i differentiable with respect to (x, z, p) such that

$$D_k u \frac{\partial A^i}{\partial z} + \frac{\partial A^i}{\partial x^k} + B\delta_k^i = C_k^i + D_k^i,$$

and we set

$$D_k^{ij} = \frac{\partial D_k^i}{\partial p_j}, \quad \mathcal{F}_k = p_i \frac{\partial D_k^i}{\partial z} + \frac{\partial D_k^i}{\partial x^i}.$$

With this terminology, we assume that

$$C^i_k g^{jk}\varsigma_{ij} \le \beta_1(\Lambda_0)^{1/2}(a^{ij}g^{km}\varsigma_{ik}\varsigma_{jm})^{1/2}, \tag{2.6a}$$

$$C^i_k \nu^k \xi_i \le \beta_1(\Lambda_0)^{1/2}(a^{ij}\xi_i\xi_j)^{1/2}, \tag{2.6b}$$

$$vD^{ij}_k \nu^k \varsigma_{ij} \le \beta_1(\Lambda_0)^{1/2}(a^{ij}\varsigma_{ik}\varsigma_{jm}g^{km})^{1/2}, \tag{2.6c}$$

$$\mathcal{F}_k \nu^k \le \beta_1^2 \Lambda_0 \tag{2.6d}$$

for all tensors ς and vectors ξ. (The analogs of conditions (2.6a) and (2.6c) in [20] are stated in terms of the minimum eigenvalue $\underline{\mu}$ of (a^{ij}) provided we replace v by $\overline{v} = (1+|Du|^2)^{\frac{1}{2}}$, ν by $\overline{\nu} = Du/\overline{v}$ and g^{km} by $\overline{g}^{km} = \delta^{km} - \overline{\nu}^k\overline{\nu}^m$. Our conditions (2.6a) and (2.6c) with g^{km} replaced by $\delta^{km} = \overline{g}^{km} + \overline{\nu}^k\overline{\nu}^m$ (see our inequality (4.2)) are then consequences of Simon's, and all other conditions are the same for $\Lambda = \Lambda_0$ and the previously indicated replacements.)

We also assume that

$$(\Lambda/\lambda)^{(n+2)/2}v \le \beta_2 w^{\beta_3} Du \cdot A \tag{2.7}$$

$$(w')A \cdot \xi \le \beta_4^{1/2}(v^{-1}a^{ij}\xi_i\xi_j)^{1/2}(Du \cdot A)^{1/2} \text{ for all } \xi \in \mathbb{R}^n, \tag{2.8a}$$

$$w^{\theta/2}|A| \le \beta_4^{1/2} Du \cdot A, \tag{2.8b}$$

$$\overline{\mu} v \le \beta_4 w^{2-\theta} Du \cdot A, \tag{2.8c}$$

$$\Lambda_0 v \le \varepsilon(v) w^2 Du \cdot A, \tag{2.8d}$$

$$|B| \le \beta_5 Du \cdot A, \tag{2.9}$$

for $\theta \in (0, 2]$ a constant and ε a positive decreasing function. (Here, our condition (2.8a) is a consequence of Simon's corresponding hypothesis, while (2.8b,c) are extensions of his conditions; our examples will show the utility of the present form of these conditions.) In fact, we will derive our gradient estimate in the same three stages as in [20]; for some problems, certain structure conditions can be removed from our hypotheses.

To simplify notation, we define

$$\mathcal{C}^2 = v^{-2}a^{ij}g^{km}D_{ik}uD_{jm}u, \quad \mathcal{E} = v^{-2}a^{ij}D_ivD_jv.$$

It follows that

$$\lambda(1+(v\lambda'/\lambda)^2)|\delta v|^2 \le v^2\mathcal{E}, \quad \lambda|H|^2 \le c_1[\mathcal{C}^2 + \mathcal{E}]$$

for some constant $c_1 = c_1(n, \beta_0)$.

Our first step is an energy inequality.

LEMMA 2.1. *Let χ be a nonnegative Lipschitz function on* (τ_0, ∞) *and suppose there are constants* $\tau \geq \tau_0$ *and* $c(\chi) \geq 0$ *such that*

$$(\xi - \tau)\chi'(\xi) \leq c(\chi)\chi(\xi) \text{ for almost all } \xi \geq \tau, \tag{2.10a}$$

$$\chi/b \text{ is increasing on } [\tau, \infty). \tag{2.10b}$$

If conditions (2.2), (2.4a) *and* (2.6) *are satisfied, then there is a constant* $c_2(\beta_0, n)$ *such that*

$$\int\limits_{\Omega_\tau} \left[\left(1 - \frac{\tau}{v}\right) \mathcal{C}^2 + \mathcal{E}\right] \chi\varsigma^2 \, d\mu \leq c_2[1 + c(\chi)] \int\limits_{\Omega_\tau} [\beta_1^2 \Lambda_0 \varsigma^2 + \overline{\mu}|D\varsigma|^2]\chi \, d\mu \tag{2.11}$$

for all nonnegative Lipschitz ς *with compact support in* Ω.

PROOF. We follow the proof of [20, (2.11)] (see also [9, Lemma 3.1)]. Using $D_k(\eta^k)$ as test function in the weak form of (2.1) and integrating by parts gives

$$\int\limits_\Omega [(v^{-1}a^{ij}D_{jk}u + C_k^i)D_i\eta^k - \mathcal{F}_k\eta^k - D_k^{ij}D_{ij}u\eta^k]dx = 0$$

for any Lipschitz vector η with compact support in Ω. Choosing $\eta = \theta\nu$ for some Lipschitz scalar θ with compact support yields

$$\begin{aligned}
&\int\limits_\Omega v^{-2}a^{ij}D_{jk}ug^{km}D_{im}u\theta \, dx \\
&\quad + \int\limits_\Omega v^{-1}a^{ij}D_jvD_i\theta \, dx \\
&\quad - \int\limits_\Omega v^{-1}a^{ij}D_ivD_jv \frac{b'(v)}{b(v)} \theta \, dx \\
&\quad + \int\limits_\Omega \left[C_k^i\nu^kD_i\theta - \frac{b'(v)}{b(v)} C_k^i\nu^kD_iv\theta\right] dx \\
&\quad + \int\limits_\Omega v^{-1}C_k^ig^{km}D_{im}u\theta \, dx \\
&= \int\limits_\Omega \theta\mathcal{F}_k\nu^k \, dx + \int\limits_\Omega \nu^kD_k^{ij}D_{ij}u\theta \, dx.
\end{aligned}$$

Finally we take $\theta = (v-\tau)_+\chi(v)\varsigma^2$ to obtain

$$\int_{\Omega_\tau} v^{-2}a^{ij}D_{jk}ug^{km}D_{im}u\left(1-\frac{\tau}{v}\right)_+\chi\varsigma^2\,d\mu$$

$$+\int_{\Omega_\tau}\mathcal{E}\left[\chi+\chi'(v-\tau)-\frac{b'(v)}{b(v)}\chi(v-\tau)\right]\varsigma^2\,d\mu$$

$$=\int_{\Omega_\tau} v^{-1}C^i_k\nu^kD_iv\left[\chi+\chi'(v-\tau)-\frac{b'(v)}{b(v)}\chi(v-\tau)\right]\varsigma^2\,d\mu$$

$$+\int_{\Omega_\tau} C^i_kg^{mk}D_{im}u\chi\left(1-\frac{\tau}{v}\right)d\mu$$

$$-2\int_{\Omega_\tau} v^{-1}a^{ij}D_jvD_i\varsigma\varsigma\chi\left(1-\frac{\tau}{v}\right)d\mu$$

$$+2\int_{\Omega_\tau} C^i_k\nu^k\chi\left(1-\frac{\tau}{v}\right)D_i\varsigma\varsigma\,d\mu$$

$$+\int_{\Omega_\tau}\mathcal{F}_k\nu^k\left(1-\frac{\tau}{v}\right)d\mu+\int_{\Omega_\tau}\nu^kD^{ij}_kD_{ij}u\chi\left(1-\frac{\tau}{v}\right)d\mu.$$

Now we use (2.10b) to conclude that $b'/b\le\chi'/\chi$, so $\chi+\chi'(v-\tau)-\dfrac{b'(v)}{b(x)}\chi(v-\tau)\ge 0$. Then the terms on the right hand side of this equation are easily estimated via (2.6) and (2.4a) to see that

$$\int_{\Omega_\tau} v^{-2}a^{ij}g^{km}D_{jk}uD_{im}u\left(1-\frac{\tau}{v}\right)\chi\varsigma^2\,d\mu$$

$$+\int_{\Omega_\tau}\mathcal{E}\left[\chi+\chi'(v-\tau)-\frac{b'(v)}{b(v)}\chi(v-\tau)\right]\varsigma^2\,d\mu$$

$$\le c_3(n)\int_{\Omega_\tau}\beta_1^2\Lambda_0\left[\chi+\chi'(v-\tau)-\frac{b'(v)}{b(v)}\chi(v-\tau)\right]\varsigma^2\,d\mu$$

$$+c_3(n)\int_{\Omega_\tau}\overline{\mu}|D\varsigma|^2\chi\,d\mu.$$

The proof is completed by using (2.10) to infer that

$$\chi\le\chi+\chi'(v-\tau)-\frac{b'(v)}{b(v)}\chi(v-\tau)\le(1+c(\chi))\chi. \qquad \square$$

Next, as in [20, Lemma 1], we reduce the gradient estimate to the estimate of an appropriate integral.

LEMMA 2.2. *Let $x_0 \in \Omega$, $\tau \geq \tau_0$ and $\rho > 0$ such that $B_{2\rho} \subset \Omega$. If conditions* (2.2), (2.3), (2.4), (2.5) *and* (2.6) *are satisfied, then*

$$\sup_{\Omega_{\tau,\rho}} \left(1 - \frac{\tau}{v}\right)^{n+2} w^2 \leq c_2(\beta, \beta_0, \beta_1\rho, n)\rho^{-n} \int_{\Omega_{\tau,2\rho}} w^2(\Lambda/\lambda)^{(n+2)/2} d\mu. \tag{2.12}$$

PROOF. We follow the proof of [20, Lemma 1] with the modifications indicated in the proof of [9, Lemma 3.2]. The only change is that we need to verify (2.10b) with

$$\chi = [(\Lambda/\lambda)^{(n+2)/2}/\Lambda] w^{2q} \left(1 - \frac{\tau}{v}\right)^{(n+2)q-n-2}$$

and $q \geq 1 + \beta$. Conditions (2.3a,b) imply that

$$\frac{\chi'}{\chi} \geq (2q - \beta)\frac{w'}{w} \geq \frac{2q-\beta}{\beta}\frac{b'}{b} \geq \frac{b'}{b}. \qquad \square$$

Our third step is to estimate $\int w^2(\Lambda/\lambda)^{(n+2)/2} d\mu$ in terms of $\int Du \cdot A\, dx$. By virtue of (2.7), we need only estimate $\int w^q Du \cdot A\, dx$ for $q = 2 + \beta_3$.

LEMMA 2.3. *With x_0, ρ, τ as in Lemma* 2.2, *if conditions* (2.2), (2.3), (2.4), (2.5), (2.6), (2.8), *and* (2.9) *are satisfied and if there is $\tau_1 \geq \tau$ such that*

$$(\beta_1\sigma)^2 e^{2\beta_5\sigma}\beta_4^2 c_1(2 + \beta q)\varepsilon(\tau_1) \leq \frac{1}{4} \tag{2.13}$$

then there is a constant c_4 $(\beta, \beta_0, \beta_1\rho, n, q, \theta)$ such that

$$\int_{\Omega_{\tau,\rho}} w^q Du \cdot A\, dx \leq c_4 e^{2q\beta_5/\theta}\left(w(\tau_1) + \left(\beta_4\frac{\sigma}{\rho}\right)^{2/\theta}\right)^q \int_{\Omega_{\tau,2\rho}} Du \cdot A\, dx, \tag{2.14}$$

where $\sigma = \operatorname{osc}_{B_{2\rho}} u$.

PROOF. As in [20, Lemma 2], we set $\overline{u} = u - \inf_{B_{2\rho}} u$ (in fact the quantity $u - u(x_0)$ in [20, Lemma 2] must be replaced by $\overline{u}$), $k = 2/\theta$,

$$I = \int_{\Omega_{\tau,2\rho}} w^q \varsigma^{kq} Du \cdot A\, dx,$$

and

$$I' = \int_{B_{2\rho}} \exp(\beta_5\overline{u})(1 + \beta_5\overline{u})\{w^q - w(\tau)^q\}_+ \varsigma^{kq} Du \cdot A\, dx,$$

where ς is the standard cut-off function in $B_{2\rho}$. Integrating by parts, we find that

$$\begin{aligned} I' = &- \int\limits_{\Omega_{\tau,2\rho}} \overline{u} \exp(\beta_5\overline{u}) q w^{q-1} \varsigma^{kq} Dw \cdot A\, dx \\ &- \int\limits_{B_{2\rho}} \overline{u} \exp(\beta_5\overline{u}) \{w^q - w(\tau)^q\}_+ kq \varsigma^{kq-1} D\varsigma \cdot A\, dx \\ &- \int\limits_{B_{2\rho}} \overline{u} \exp(\beta_5\overline{u}) \{w^q - w(\tau)^q\}_+ \varsigma^{kq} \operatorname{div}(A) dx \\ = &\, I_1 + I_2 + I_3. \end{aligned}$$

Now we set $E = \exp(\beta_5\sigma)$ and proceed to estimate I_1, I_2, I_3.

First, using (2.8a), we have

$$\begin{aligned} I_1 &\leq \int\limits_{\Omega_{\tau,2\rho}} \{\beta_4^2 q^2 E^2 \sigma^2 w^{q-2} \varsigma^{kq} \mathcal{E} v\}^{1/2} \{w^q \varsigma^{kq} Du \cdot A\}^{1/2} dx \\ &\leq \frac{1}{4} I + \beta_4 q^2 E^2 \sigma^2 \int\limits_{\Omega_{\tau,2\rho}} w^{q-2} \varsigma^{kq} \mathcal{E}\, d\mu. \end{aligned}$$

Because of (2.3b), we can take $\chi = w^{q-2}$ and $c(\chi) = 1 + \beta q$ in Lemma 2.1 to find that

$$\begin{aligned} I_1 \leq &\frac{1}{4} I + \beta_1^2 \beta_4^2 q^2 E^2 \sigma^2 c_1(2+\beta q) \int\limits_{\Omega_{\tau,2\rho}} \Lambda_0 w^{q-1} \varsigma^{kq} v\, dx \\ &+ \beta_1^2 \beta_4 q^2 E^2 \sigma^2 c_1 (2+\beta q) \int\limits_{\Omega_{\tau,2\rho}} (kq)^2 \overline{\mu} |D\varsigma|^2 \varsigma^{kq-2} w^{q-2} v\, dx \\ \leq &\left\{\frac{1}{4} + (\beta_1\sigma)^2 E^2 \beta_4^2 q^2 c_1 (2+\beta q) \varepsilon(\tau)\right\} I \\ &+ c_5(\beta, \beta_0, \beta_1\rho, n, q, \theta) \beta_4^2 E^2 \left(\frac{\sigma}{\rho}\right)^2 \int\limits_{\Omega_{\tau,2\rho}} \varsigma^{kq-2} w^{q-\theta} Du \cdot A\, dx, \end{aligned}$$

from (2.8c) and (2.8d).

Next, (2.8b) implies that

$$I_2 \leq \int_{\Omega_{\tau,2\rho}} \{\varsigma^{kq} w^q w^{\theta/2}|A|\}^{1/2}\{\sigma^2 E^2 \varsigma^{kq-2} w^{q-\theta/2}|A|\,|D\varsigma|^2\}^{1/2} dx$$

$$\leq \int_{\Omega_{\tau,2\rho}} \{\varsigma^{kq} w^q Du \cdot A\}^{12}\{\beta_4 \sigma^2 E^2 \varsigma^{kq-1} w^{q-\theta} Du \cdot A |D\varsigma|^2\}^{1/2} dx$$

$$\leq \frac{1}{4} I + 4\beta_4^2 \left(\frac{\sigma}{\rho}\right)^2 E^2 \int_{\Omega_{\tau,2\rho}} \varsigma^{kq-2} w^{q-\theta} Du \cdot A\, dx.$$

Finally, using the differential equation to replace $-\operatorname{div} A$ by B and then applying (2.9) yields

$$I_3 \leq \int_{B_{2\rho}} \exp(\beta_5 \overline{u})\beta_5 \overline{u}\{w^q - q(\tau)^q\}_+ \varsigma^{kq} Du \cdot A\, dx.$$

From these estimates, and some rearrangement, we find that

$$I \leq \left\{\frac{1}{2} + (\beta_1\sigma)^2 E^2 \beta_4^2 q^2 c_1 (2 + \beta q)\varepsilon(\tau)\right\} I$$
$$+ E w(\tau)^q \int_{\Omega_{\tau,2\rho}} Du \cdot A\, dx$$
$$+ (4 + c_5)\beta_4^2 E^2 \left(\frac{\sigma}{\rho}\right)^2 \int_{\Omega_{\tau,2\rho}} \varsigma^{kq-2} w^{q-\theta} Du \cdot A\, dx.$$

Now note that $\varsigma^{kq-2} w^{q-\theta} = (w\varsigma^k)^{q-\theta}$. By choosing $\tau = \tau_1$ and using Young's inequality, we have

$$\int_{\Omega_{\tau_1,2\rho}} (\varsigma^k w)^q Du \cdot A\, dx \leq c_6(c_5, q) e^{2q\beta_5\sigma/\theta} \left(w(\tau_1) + \left(\beta_4 \frac{\sigma}{\rho}\right)^{2/\theta}\right)^q \int_{\Omega_{\tau_1,2\rho}} Du \cdot A\, dx.$$

Adding the obvious inequality

$$\int_{\Omega_{\tau,2\rho}\setminus\Omega_{\tau_1,2\rho}} (\varsigma^k w)^q Du \cdot A\, dx \leq w(\tau_1)^q \int_{\Omega_{\tau,2\rho}\setminus\Omega_{\tau_1,2\rho}} Du \cdot A\, dx$$

yields (2.14). □

Of course, a modulus of continuity estimate guarantees (2.13), and hence (2.14), for $\tau_1 = \tau$ and ρ sufficiently small even if ε is constant.

By using the expression $\exp(\beta_9\overline{u})(1+\beta_9\overline{u})$ (present in [20]) rather than $\overline{u}$ the proof of [6, Lemma 4.3], we can replace $\beta_9\varepsilon(v_1)$ by β_9 in [6, (4.8d)] and similarly in [7, (2.6c)], [9, (3.12e)], and [12, (3.10e)]. More significantly, introducing the constant θ allows for consideration of anisotropic growth conditions (see Example 3 in Section 4).

Finally, we estimate $\int Du \cdot A\,dx$ by quoting [20, Lemma 3].

LEMMA 2.4. *With x_0, ρ, τ as in Lemma 2.2, if conditions (2.8b) and (2.9) are satisfied and if there is $\tau_2 \geq \tau$ such that*

$$\beta_4^{1/2} w(\tau_2)^{-\theta/2} \frac{\sigma}{\rho} \leq \frac{1}{16} \tag{2.15}$$

then there is a constant $c_7(n)$ such that

$$\int_{\Omega_{\tau,\rho}} Du \cdot A\,dx \leq c_7 \rho^n \exp(4\beta_5\sigma)\Delta_1(\tau_2), \tag{2.16}$$

where

$$\Delta_1(\tau_2) = \sup_{|p|\leq\tau_2} \left\{ \sigma(B - \beta_6 p \cdot A)_+ + \frac{\sigma}{\rho}|A| \right\}. \tag{2.17}$$

□

Combining Lemmata 2.2, 2.3, and 2.4 with (2.7) gives an estimate on $\sup w$.

THEOREM 2.5. *Let $x_0 \in \Omega$ and suppose $\rho \leq \frac{1}{8}\operatorname{dist}(x_0, \partial\Omega)$. Suppose there are functions w, ε, λ, Λ, b_i, λ_0, and Λ_0 such that conditions (2.2)- (2.9) are satisfied. Let $\tau \geq \tau_0$ and suppose there are constants $\tau_1 \geq \tau$ and $\tau_2 \geq \tau$ such that conditions (2.13), with $q = \beta_3 + 2$, and (2.15) hold. Then there is a constant c_8 $(\beta, \beta_0, \beta_1\rho, \beta_2, \beta_3, \beta_4\sigma, n)$ such that*

$$\left(1 - \frac{\tau}{v}\right)_+^{n+2} w^2(x_0) \leq c_8 \left(w(\tau_1) + \left(\beta_4 \frac{\sigma}{\rho}\right)^{\frac{2}{\theta}} \right)^{2+\beta_3} \Delta_1(\tau_2)/\tau. \tag{2.18}$$

If also $w(\tau) \to \infty$ as $\tau \to \infty$, then

$$v(x_0) \leq w^{-1}\left(\left[2^{n+2} c_8 \left(w(\tau_1) + \left(\beta_4 \frac{\sigma}{\rho}\right)^{\frac{2}{\theta}} \right)^{2+\beta_3} \Delta_1(\tau_2)/\tau \right]^{\frac{1}{2}} \right) + 2\tau. \tag{2.19}$$

□

To convert the estimate on v to an estimate on Du, we note that

$v \geq \overline{B}^{-1}\left(\frac{1}{n}\overline{B}_i(|D_i u|)\right)$ for all i. Since $\overline{B}_i(\tau) \to \infty$ as $\tau \to \infty$ and $\overline{B}$ is strictly increasing, this inequality and (2.19) give a bound for $|Du|$.

It is also possible to derive gradient bounds up to the boundary of the domain. Either by imitating the boundary considerations in [20] or by combining the form of our interior gradient bound with the boundary Lipschitz estimate [4, Theorem 14.1], we find a gradient estimate near the boundary if Ω satisfies a uniform exterior sphere condition and if the quantities

$$\mathcal{E}_1 = a^{ij} p_i p_j, \quad B_1 = \sum_i (C_i^i + D_i^i),$$

and $\overline{\mu}$ are related by

$$|p|\overline{\mu} + v|B_1| \leq \beta_6 \mathcal{E}_1 \text{ for } |p| \geq \tau_0, \tag{2.20}$$

and if the boundary data are C^2.

We also note that boundary gradient estimates can be proved for the conormal derivative in the special case of zero Neumann data, i.e.,

$$A(x,u,Du)\cdot\gamma = 0 \text{ on } \partial\Omega \tag{2.21}$$

with γ the inner normal to $\partial\Omega$. Assuming that there is a scalar function $F(x,z,p)$ with $A = \partial F/\partial p$ and $\partial\Omega \in C^2$, we prove the estimate by imitating the proof in [6] (which allowed $A(x,u,Du)\cdot\gamma = \varphi(x)$ on $\partial\Omega$). If we only assume that $\partial\Omega$ is Lipschitz and satisfies a uniform exterior sphere condition (but still that F exists), we follow [11]. Without the variational structure (i.e. existence of F), the present proof works in the conormal case if the boundary condition implies also that $\nu\cdot\gamma = 0$ on $\partial\Omega$ and $\partial\Omega \in C^2$ (cf. [9]). In [17] the boundary condition (2.21) was assumed with variational structure and $\partial\Omega$ Lipschitz with a uniform exterior sphere condition.

3. - Parabolic estimates

The modifications needed to handle parabolic problems are already present in [7], so we state results here. We consider the problem

$$-u_t + \operatorname{div} A(x,t,u,Du) + B(x,t,u,Du) = 0 \text{ in } Q_T = \Omega \times (0,T).$$

The notation from the previous sections is modified so that the arguments of A, B, and their derivatives also include t and

$$\begin{gathered}\Omega_\tau = \Omega_\tau(t) = \{x \in \Omega : v(x,t) > \tau\},\\ Q(T,\tau) = \{(x,t) \in \Omega \times (0,T) : v(x,t) > \tau\},\\ Q(T,\tau,\rho) = \{(x,t) \in Q(T,\tau) : x \in B_\rho\}.\end{gathered}$$

We also introduce some additional structure conditions (cf. [7, Sect. 2]).

$$\lambda \le \beta_7 v, \tag{3.1}$$

$$v \le \beta_7 \Lambda, \tag{3.2}$$

$$a^{ij}\varsigma_{ij} \le \beta_7\{a^{ij}\varsigma_{ik}\varsigma_{jm}g^{km}\}^{1/2}\{\Lambda_0\}^{1/2}, \tag{3.3}$$

$$v|A_z| + |A_x| \le \beta_5 Du \cdot A \tag{3.4}$$

$$v^2 \le \beta_8 w Du \cdot A. \tag{3.5}$$

Conditions (3.1), (3.2), (3.4) and (3.5) were introduced in [7]. The analog of (3.3) in [7] is a consequence of $\overline{\mu} \le \beta_7^2\Lambda_0$, namely that $a^{ij}\xi_{ij} \le \beta_7\{a^{ij}\xi_{ik}\xi_{jm}(\overline{g}^{km} + \overline{\nu}^k\overline{\nu}^m)\}^{\frac{1}{2}}\{\Lambda_0\}^{\frac{1}{2}}$. As we shall see in Section 4, removing the $\nu^k\nu^m$ term form (3.3) involves no loss of generality.

The analogs of the estimates in Section 2 are as follows.

LEMMA 3.1. *Let χ be a nonnegative Lipschitz function on $[\tau_0, \infty)$ and suppose there are constants $\tau \ge \tau_0$ and $c(\chi) \ge 0$ such that* (2.10a,b) *hold. If conditions* (2.2), (2.4a,c) *and* (2.6) *are satisfied and if*

$$|v| < \tau \ in \ \Omega \in \{0\} \tag{3.6}$$

or if $\varsigma(x,0) = 0$ in Ω, then there is a constant $c_8(\beta_0, n)$ such that

$$\begin{aligned}
&\sup_{0<t<T} \int_{\Omega_\tau} (v-\tau)^2\chi\varsigma^2\,dx + \int_0^T \int_{\Omega_\tau} \left[\left(1 - \frac{\tau}{v}\right)\mathcal{C}^2 + \mathcal{E}\right]\chi\varsigma^2\,d\mu\,dt \\
&\quad \le c_8[1+c(\chi)]^2 \int_0^T \int_{\Omega_\tau} [\beta_1^2\Lambda_0\varsigma^2 + \overline{\mu}|D\varsigma|^2]\chi\varsigma^2\,d\mu\,dt \\
&\quad\quad + c_8[1+c(\chi)]^2 \int_0^T \int_{\Omega_\tau} (v-\tau)^2\chi|\varsigma_t|\varsigma\,dx\,dt
\end{aligned} \tag{3.7}$$

for all nonnegative Lipschitz ς with compact support in $\Omega \times (0,T)$. □

LEMMA 3.2. *Let x_0, τ, and ρ be as in Lemma 2.2. If conditions* (2.2), (2.3), (2.4), (2.5), (2.6), (3.1) *and* (3.6) *are satisfied, then there is a constant c_9 $(\beta, \beta_0, \beta_1\rho, \beta_7, n)$ such that*

$$\sup_{Q(T,\tau,\rho)} \left(1 - \frac{\tau}{v}\right)^{n+2} w^2 \le c_9\rho^{-n-2} \iint_{Q(T,\tau,2\rho)} w^2(\Lambda/\lambda)^{(n+2)/2}\,d\mu\,dt. \tag{3.8}$$

If we replace (3.6) *by* (3.2) *and if* $T \geq 4\rho^2$, *then there is* c_{10} $(\beta, \beta_0, \beta_1\rho, \beta_7, n)$ *such that*

$$\left(1 - \frac{\tau}{v(x_0,T)}\right)^{n+2} w(v(x_0,T))^2 \leq c_{10}\rho^{-n-2}\int_{T-4\rho^2}^{T}\int_{\Omega_{\tau,2\rho}} w^2(\Lambda/\lambda)^{(n+2)/2}d\mu\, dt. \qquad \square$$

LEMMA 3.3. *With* x_0, ρ, τ *as in Lemma* 2.2, *suppose* (2.2), (2.3), (2.4), (2.5), (2.6), (3.1), (3.3), *and* (3.6) *are satisfied. If there is* $\tau_1 \geq \tau$ *such that* (2.13) *holds, then there is* c_{11} $(\beta, \beta_0, \beta_1\rho, \beta_7, n, q, \theta)$ *such that*

$$\begin{aligned}\iint_{Q(T,\tau,\rho)} w^q Du \cdot A\, dx\, dt \leq c_{11}e^{2q\beta_5\sigma/\theta}(w(\tau_1)\\ + (\beta_4\sigma/\rho)^{2/\theta})^q \iint_{Q(T,\tau,2\rho)} Du \cdot A\, dx\, dt,\end{aligned} \tag{3.9}$$

where $\sigma = \sup\{|u(x,t) - u(y,t)| : x,\ y \text{ in } B_\rho,\ 0 < t < T\}$. *If we replace* (3.6) *by* (3.2) *and* (3.5) *and if* $T \geq 4\rho^2$, *then there is* c_{12} $(\beta, \beta_0, \beta_1\rho, \beta_7, n, q, \theta)$ *such that*

$$\begin{aligned}&\int_{T-\rho^2}^{T}\int_{\Omega_{\tau,\rho}} w^q Du \cdot A\, dx\, dt\\ &\leq c_{12}\left(w(\tau_1) + \left(\beta_4\frac{\sigma}{\rho}\right)^{2/\theta} + \beta_8\left(\frac{\sigma}{\rho}\right)^2\right)^q \int_{T-4\rho^2}^{T}\int_{\Omega_{\tau,2\rho}} Du \cdot A\, dx\, dt.\end{aligned} \tag{3.9$'$}$$

PROOF. The important change from the proof of Lemma 2.3 is in the estimate of $\operatorname{div} A$. Using (3.3) and (3.4) gives

$$|\operatorname{div} A| \leq \beta_7\{\mathcal{C}^2 + \mathcal{E}\}^{1/2}\{\Lambda_0\}^{1/2} + \beta_5 Du \cdot A,$$

and then the proof proceeds as before. (See also [9, Lemma 3.3].) $\square$

LEMMA 3.4. *Let* τ, ρ, x_0 *be as in Lemma* 2.2 *and suppose conditions* (2.8b) *and* (2.9) *hold. If* $Q'(T,\tau,\rho) = \{(x,t) \in Q(T,\tau,\rho) : t > T - \rho^2\}$, *if*

$$\operatorname*{osc}_{Q'(T,\tau,2\rho)} u \leq M, \tag{3.10}$$

if $\tau_2 \geq \tau$ *is so large that*

$$\beta_4^{1/2} w(\tau_2)^{-\theta/2} \leq \frac{\rho}{8M}, \tag{3.11}$$

and if

$$\Delta_2 = \sup_{|p|\le\tau_2} \left\{ M(B - \beta_5 Du \cdot A)_+ + |Du \cdot A| + \frac{M|A|}{\rho} \right\}, \tag{3.12}$$

then there is a constant $c_{13}(n)$ *such that*

$$\iint_{Q'(T,\tau,\rho)} Du \cdot A \, dx \, dt \le c_{13} \exp(\beta_5 M)\rho^n [M^2 + \Delta_2 \rho^2]. \tag{3.13}$$ □

For parabolic equations (cf. [8, Theorem 2.2] and [9, p. 47]), a boundary gradient estimate holds if Ω satisfies an exterior sphere condition, if the boundary data have bounded second spatial derivatives and a bounded first time derivative and if, in addition to (2.20), we have

$$\mathcal{E}_1 \ge \beta_9 v \text{ for } |p| \ge \tau_0.$$

4. - Examples

Before presenting our examples, we note some useful inequalities. First,

$$\begin{aligned} vb_i(|p_i|) &\le vb_i(v) + |p_i| b_i(|p_i|) \\ &\le \sum_{k=1}^{n} vb_k(v) + \sum_{k=1}^{k} |p_k| b_k(|p_k|) \\ &\le vb(v) + (1+\beta_0) \sum_{k=1}^{n} \overline{B}_k(|p_k|) \\ &= vb(v) + (1+\beta_0)\overline{B}(v) \le (2+\beta_0) vb(v) \end{aligned} \tag{4.1}$$

by [10, Lemma 1.1(b,e)] and the positivity of b_k. Hence the vector ν has length bounded by $2 + \beta_0$. Furthermore,

$$\begin{aligned} \nu \cdot \xi &= \sum_i \frac{b_i(|D_i u|)}{b(v)} (\operatorname{sgn} D_i u)\xi_i \\ &\le \left\{ \sum_i \frac{b_i(|D_i u|)^2}{vb_i'(|D_i u|)b(v)} \right\}^{1/2} \left\{ \sum_i \frac{vb_i'(|D_i u|)}{b(v)} \xi_i^2 \right\}^{1/2} \\ &\le (\beta_0(1+\beta_0))^{\frac{1}{2}} \left\{ \sum_i \frac{vb_i'(|D_i u|)}{b(v)} \xi_i^2 \right\}^{1/2} \\ &= (\beta_0(1+\beta_0))^{1/2} (g^{ij}\xi_i\xi_j)^{1/2}. \end{aligned} \tag{4.2}$$

Also

$$
\begin{aligned}
g^{ij}\varsigma_{ij} &= \sum_i v\,\frac{b_i'(|D_iu|)}{b(v)}\,\varsigma_{ii} \\
&= \sum_{i,j} v\sqrt{\frac{b_i'(|D_iu|)}{b(v)}}\,\varsigma_{ij}\delta^{ij}\sqrt{\frac{b_j'(|D_ju|)}{b(v)}} \\
&\le v\left(\sum_{i,j}\frac{b_i'(|D_iu|)}{b(v)}\,\frac{b_j'(|D_ju|)}{b(v)}\,\varsigma_{ij}^2\right)^{1/2}\left(\sum_{i,j}\delta^{ij}\right)^{1/2} \\
&= n^{1/2}(g^{ij}g^{km}\varsigma_{ik}\varsigma_{jm})^{1/2}.
\end{aligned}
\tag{4.3}
$$

Hence if $C_k^i = B\delta_k^i$, conditions (2.6a,b) hold with $\beta_1 \ge c(\beta_0, n)\theta_0$ provided there is a function λ_0 such that

$$
\lambda_0 g^{ij}\xi_i\xi_j \le a^{ij}\xi_i\xi_j,\ |B| \le \theta_0(\Lambda_0\lambda_0)^{1/2}. \tag{4.4}
$$

We start with a particularly simple example.

EXAMPLE 1. $A^i = ap_i + |p_i|^{m-1}\operatorname{sgn} p_i$, $B \equiv 0$ for some constant $m > 2$ and $a > 0$. We take $b_1(\tau) = \cdots = b_n(\tau) = \tau^{m-2}$, $\overline{\mu} = \Lambda_0 = \Lambda = mv^{m-1}$, $\lambda = v^{m-1}/(1+m^2)$, $w = v^{(m-1)/2}$. Then conditions (2.2), (2.3), (2.4), (2.5), (2.6) are satisfied with $\beta = m+1$, $\beta_0 = m$, $\beta_1 = 0$, and $\tau_0 > a^{1/(m-2)}$, so Lemma 2.2 gives

$$
\sup_{B(x_0,\rho)} w^2 \le C\rho^{-n}\tau^{-1}\int_{\Omega(\tau,2\rho)} v^m\,dx \le C\rho^{-n}\tau^{-1}\int_{\Omega(\tau,2\rho)} Du\cdot A\,dx
$$

provided $\tau \ge \tau_0$ and $\sup_{B(x_{0,\rho})} v \ge 2\tau$. This last integral is estimated via Lemma 2.4 with $w = v$, $\theta = 2$, $\beta_4 = C(m)$, and $\beta_5 = 0$ to obtain

$$
\int_{\Omega(\tau,2\rho)} Du\cdot A\,dx \le c_7\rho^n C(m,n)\left[a\left(\frac{\sigma}{\rho}\right)^2 + \left(\frac{\sigma}{\rho}\right)^m\right],
$$

and hence

$$
v \le C\left[a\frac{\sigma}{\rho} + \left(\frac{\sigma}{\rho}\right)^{m-1}\right]^{\frac{1}{m-1}} + 2\left(a^{1/(m-2)} + \frac{\sigma}{\rho}\right) \le C\left[a^{1/(m-2)} + \frac{\sigma}{\rho}\right]
$$

by taking $\tau = a^{1/(m-2)} + \dfrac{\sigma}{\rho}$. Since all $\overline{B}_i$'s are equal and convex, it follows that $\max_i |D_iu| \ge v \ge \dfrac{1}{n}\sum |D_iu|$, so we have an easy gradient bound here.

Moreover $\mathcal{E}_1 = (m-1)v^{m+1}$, so (2.20) also holds, and we obtain boundary gradient estimates as well.

This gradient estimate along with classical regularity theory guarantees that $u \in C^2$ as long as $a > 0$. An easy approximation argument and the uniqueness of solutions of the problem

$$\operatorname{div} A(Du) = 0 \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega$$

for $a \geq 0$ show that our gradient holds in the form

$$\sup_{B(x_0,\rho)} |Du| \leq C(m,n)(\operatorname*{osc}_{B(x_0,2\rho)} u)/\rho$$

if $a = 0$.

More generally, we consider a structure which includes the operators of [18]; these operators give rise to the model problem on which the present work is based.

EXAMPLE 2. $A^i = ap_i + a_i(x,u)|p_i|^{m-1} \operatorname{sgn} p_i$ where a and m are constants with $a \in (0,1]$ and $m > 2$, and the a_i's are Lipschitz functions with

$$\theta_1 \leq a_i \leq \theta_2, \quad |a_{i,x}| + |a_{i,z}| \leq \theta_3$$

for θ_1, θ_2, θ_3 positive constants, and

$$|B| \leq \theta_3(v^m + 1).$$

Now our structure functions are

$$b_1(\tau) = \cdots = b_n(\tau) = \tau^{m-2}, \quad \overline{\mu} = m\theta_2 v^{m-1}, \quad \Lambda_0 = \Lambda = \theta_2 v^{m+1},$$
$$\lambda = \theta_1 v^{m-1}, \quad w = v,$$

and conditions (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.8), (2.9) hold with

$$\beta = n + m, \quad \beta_0 = m, \quad \beta_1 = c(n,m,\theta_2/\theta_1)\theta_3/\theta_1, \quad \beta_2 = c(n,m,\theta_2/\theta_1),$$
$$\beta_3 = n+2, \quad \beta_4 = c(m,\theta_2/\theta_1), \quad \beta_5 = 2\theta_3/\theta_1, \quad \varepsilon(v) \equiv \theta_3/\theta_2$$

because (4.3) holds with $\theta_0 = \theta_3/\theta_1$ and $\lambda_0 = \lambda$. According to [5, Theorem 1.1 of Chapter 4], we can estimate the modulus of continuity of u (in terms of θ_1, θ_2, θ_3, n, and m), so Theorem 2.5 provides a gradient estimate in this case. Specifically

$$|Du| \leq c(n,m,\theta_2/\theta_1,\theta_3\sigma/\theta_1,\theta_3\rho/\theta_1)\left(1 + \frac{\sigma}{\rho}\right)^{c(n,m)}. \tag{4.5}$$

(Note that this gradient bound along with classical regularity theory guarantees that $u \in W^{2,2} \cap C^1$, so our derivation is valid here.) Moreover

$$\mathcal{E}_1 = av|p|^2 + (m-1)v \sum a_i(x,u)|p_i|^m,$$

$$B_1 = v\left\{B + \sum_i \frac{\partial a_i}{\partial x^i}|p|^{m-1}\operatorname{sgn} p_i + \sum_i \frac{\partial a_i}{\partial z}|p_i|^m\right\},$$

so a boundary gradient estimate is valid.

For the parabolic version, we can allow a_i and B to depend on t also because the Hölder estimates of Di Benedetto [2, Theorems 2, 3, 4] apply in this case; the only changes are that $\lambda = \theta_1 v/(1+m^2)$ and the constants β and β_3 must be increased approprietely, and condition (3.3) follows by a simple modification of the proof of (4.3).

In fact, our gradient estimate is also valid for $a = 0$. To see this validity, we use an approximation scheme like the one in [21] coupled with an easy variant of the local uniqueness proof from [5, Section 4.2]. For $a \geq 0$, define

$$A^i(x,z,p;a) = ap_i + a_i(x,z)|p_i|^{m-1}\operatorname{sgn} p_i$$

and suppose $u \in W^{1,m} \cap L^\infty$ solves

$$\operatorname{div} A(x,u,Du;0) + B(x,u,Du) = 0 \text{ in } \Omega.$$

Then $b_0(x) = B(x,u,Du)/\left(\sum_{i=1}^n |D_i u|^m + 1\right)$ is a bounded function, so there is a family of uniformly bounded, C^1 functions $(b_a)_{a>0}$ with $b_a \to b_0$ almost everywhere as $a \to 0$. We now define

$$B(x,z,p;a) = b_a(x)\left(\sum_{i=1}^n |p_i|^m + 1\right) + u(x) - z,$$

and for a fixed $x_0 \in \Omega$ and $\rho < \operatorname{dist}(x_0, \partial\Omega)$, we write u_a for the solution of

$$\operatorname{div} A(x,u_a,Du_a;a) + B(x,u_a,Du_a;a) = 0 \text{ in } B_\rho,\ \ u_a = u \text{ on } \partial B_\rho. \tag{4.6}$$

For $a \in (0,1)$, [4, Problem 10.1], the uniform Hölder estimate from [5, Theorem 1.1 of Chapter 4], and our gradient estimate imply the existence of u_a along with positive constants c_1 and α such that $\operatorname{osc}_{B_\rho} u_a \leq c_1\rho^\alpha$ for $a \in (0,1)$. We now show that u_a is also unique provided ρ is small enough.

First (as in [5, Lemma 1.3 of Chapter 4]), for $\xi \in (W_0^{1,m} \cap L^\infty)(B_\rho)$, we use $[u_a - u_a(x_0)]\xi^2$ as a test function to infer that

$$\int_{B_\rho} \left(a|Du_a|^2 + \theta_1 \sum_{i=1}^n |D_i u_a|^m \right) \xi^2 \, dx \le c\rho^\alpha \int_{B_\rho} \xi^2 \, dx$$

$$+ c\rho^{2\alpha} \int_{B_\rho} \left(a|D\xi|^2 + \theta_1 \sum_{i=1}^n |D_i u_a|^{m-2} |D_i \xi|^2 \right) dx.$$

If $\overline{u}_a$ is any other solution, this estimate is also true for $\overline{u}_a$. Now (as in [5, Theorem 2.1 of Chapter 4]), use $\xi = u_a - \overline{u}_a$ as test function in the weak forms of the equations for u_a and $\overline{u}_a$ to infer that

$$\int_{B_\rho} \left[a|D\xi|^2 + \theta_1 \sum_{i=1}^n (|D_i u_a|^{m-2} + |D_i \overline{u}_a|^{m-2}) |D_i \xi|^2 \right] dx$$

$$\le C \int_{B_\rho} \left[(a|Du_a|^2 + |D\overline{u}_a|^2) + \theta_1 \sum_{i=1}^n (|D_i u_a|^m + |D_i \overline{u}_a|^m) \right] \xi^2 \, dx - \int_{B_\rho} \xi^2 \, dx.$$

If ρ is small enough, we have $\int \xi^2 \, dx = 0$ and hence $u_a = \overline{u}_a$. It also follows that $u_a \to u$ uniformly as $a \to 0$ since u is the unique solution of (4.6) with $a = 0$. This uniform convergence, along with the uniform gradient bounds for $a \in (0, 1]$, implies the gradient bound (4.5) also for $a = 0$ provided ρ is sufficiently small.

Example 2 gives a considerable strengthening of the estimate in [18], which was proved only for $n = 2$. In our notation, the additional assumptions are that $a_1(x, z) = a_2(x, z) = \alpha(z)$,

$$|B| + |B_z| + |B_p \cdot p| + \frac{|B_x|}{|p|} = 0(|p|^{m-\varepsilon})$$

for some positive ε, and there are positive constants $\sigma < 1$ and δ_1 such that

$$(m-2)\left(\frac{\alpha'}{\alpha}\right) - \left(\frac{\alpha'}{\alpha}\right)^2 \ge -\sigma(m-1)^2/(\delta_1 + z)^{-2}.$$

Furthermore the estimate in [18] was a global one only.

EXAMPLE 3 (Anisotropic structure conditions I). Now we suppose there are constants M, $\underline{m}$ and $\overline{m}$ with $1 \le \underline{m} \le \overline{m}$, $M \ge 1$ such that

$$\frac{\partial A^i}{\partial p_j} \xi_i \xi_j \ge |p|^{\underline{m}-2} |\xi|^2, \quad |A_p| \le M|p|^{\overline{m}-2} \text{ for } |p| > 1.$$

For simplicity, we also assume that A depends only on p and that $B \equiv 0$. With $b_1(\tau) = \cdots = b_n(\tau) = \tau$, $\overline{\mu} = Mnv^{\overline{m}-1} = \Lambda$, $\Lambda_0 = \lambda = v^{\underline{m}-1}/(1+\underline{m}^2)$, $w = v$, conditions (2.2), (2.3), (2.4), (2.5), (2.6) are satisfied with $\beta = n+2+m$, $\beta_0 = 1$, $\beta_1 = 0$.

In addition (2.7), (2.8), (2.9) hold with $w = v$, $\theta = 2 - (\overline{m} - \underline{m})$ and β_2, β_3, β_4 chosen suitably depending on M, $\underline{m}$, $\overline{m}$, n if also $A(p) = f(|p|)p$ for some function f (which must satisfy the inequalities $\tau^{\underline{m}-2} \le f(t) \le M\tau^{\overline{m}-2}$, $0 \le f'(\tau) \le M\tau^{\overline{m}-3}$) and $\overline{m} - \underline{m} < 2$. In this way, we reproduce the gradient bound of Choe [1] in the scalar case. (The systems case follows from a corresponding modification of the results in [12].) Alternatively, if $\overline{m} < \dfrac{n+4}{n+2}\,\underline{m}$, we can use $w = v^m$ with $m = \underline{m} - (\overline{m} - \underline{m})\dfrac{n+2}{2}$ to obtain

$$\sup w^2 \le C\,\frac{\rho^{-n}}{\tau}\int Du \cdot A\,dx$$

from Lemma 2.2 and then Lemma 2.4 bounds this integral. In either case, there are constants $C(\underline{m}, \overline{m}, M, n)$ and $k(\underline{m}, \overline{m}, n)$ such that $\sup\limits_{B_\rho} |Du| \le C[\operatorname*{osc}\limits_{B_{8\rho}} u/\rho]^k$.

Example 3 should be compared to the results in [13], [14] and [15].

In Examples 1 and 2, we can replace τ^m by $g(\tau)$ for any C^1 function g satisfying

$$1 \le \frac{\tau g'(\tau)}{g(\tau)} \le g_0 \tag{4.7}$$

for some constant g_0. The only change is that we use [10] to infer a modulus of continuity in the elliptic case of Example 2. For Example 3, we replace $\tau^{\underline{m}}$ and $\tau^{\overline{m}}$ by $g_1(\tau)$ and $g_2(\tau)$ with each g_i satisfying (4.7). The restrictions on $\overline{m}$ and $\underline{m}$ are modified as follows: $\overline{m} < \left(\dfrac{n+4}{n+2}\right)\underline{m}$ becomes $g_i^{-\alpha} g_j$ is increasing for all i and j with $\alpha > \dfrac{n+2}{n+4}$ a constant, and $\overline{m} < \underline{m} + 1$ becomes $\tau^\theta g_i(\tau)/g_j(\tau)$ is increasing for all i and j with $\theta \in [0, 1)$ a constant.

The generalization of Example 2 to anisotropic structure conditions is quite surprising.

EXAMPLE 4 (Anisotropic structure conditions II). Let $g_1, \ldots, g_n$ be $C^1[0, \infty)$ functions satisfying (4.7) for $\tau > 0$ with $g_i(1) = 1$, and define $g = \sum\limits_{i=1}^{n} g_i$, $G_i(\tau) = \displaystyle\int_0^\tau g_i(\sigma)\,d\sigma$, and $G = \sum\limits_{i+1}^{n} G_i$. Let a_i be Lipschitz with

$$\theta_1 \le a_i \le \theta_2, \quad |a_{i,x}| \le \theta_3$$

for θ_1, θ_2, θ_3 positive constants, suppose $A^i(x,z,p) = a_i(x)g_i(|p_i|)\operatorname{sgn} p_i$, and suppose there is a decreasing function ε such that $\varepsilon(\tau)\tau$ is increasing with $\varepsilon(\infty) = 0$ and

$$|B(x,z,p)| \le \theta_3 \varepsilon(G^{-1}(\Sigma G_i(|p_i|)))^{\frac{1}{2}} \Sigma G_i(|p_i|).$$

We assume without loss of generality that $\varepsilon(1) = 1$ and we take $\tau_0 = 1$. Our structure functions are

$$b_i(\tau) = g_i(\tau),\ \ \overline{\mu} = g_0\theta_2 v \max_i\{g_i(|p_i|)/|p_i|\},\ \ \Lambda = \theta_2 g_0 G(v)v,\ \ \lambda = \frac{\theta_1 v}{5},$$

$$\Lambda_0 = \theta_1\varepsilon(v)\overline{B}(v)v.$$

From (4.1) and (4.4) with $\theta_0 = c(n, g_0)$ and the easily checked inequalities

$$\overline{B}_i(|D_i u|) \le c(g_0)|D_i u|^2 g_i'(|D_i u|)$$

and

$$|D_i u| \le G_i^{-1}(G(v)) \le G(v)^{\frac{1}{2}},$$

we see that conditions (2.2), (2.3), (2.4), (2.5), and (2.6) are satisfied with $w = v$, $\beta = c(n, g_0)$, $\beta_0 = g_0$, and $\beta_1 = C(n, g_0)\theta_3/\theta_1$, so Lemma 2.2 implies

$$\sup_{B_\rho} v^2 \le \frac{C}{\tau}\rho^{-n} \int_{\Omega_{\tau,2\rho}} G(v)^{n/2} v^3\, dx \le \frac{C\rho^{-n}}{\tau\theta_1} \int_{\Omega_{\tau,2\rho}} v^{c(n,g_0)} Du \cdot A\, dx$$

(assuming $\sup_{B_\rho} v \ge 2\tau$). In additional (2.2)-(2.6), (2.8), and (2.9) hold with w, β, β_0 and β_1 as before, $\beta_4 = C(g_0)\theta_2/\theta_1$ and $\beta_5 = \theta_3/\theta_1$. It follows from Lemmata 2.3 and 2.4 that

$$\sup_{B_\rho} v^2 \le \frac{C}{\tau} \exp(c(n, g_0)\theta_3\sigma)\left(\tau_1 + \frac{\sigma}{\rho}\right)^{c(n,g_0)} [G(\sigma/\rho) + 1]$$

If $\tau \ge 1 + \dfrac{\sigma}{\rho}$ and τ_1 satisfies (2.13) with $q = c(n, g_0)$, it follows that

$$v \le C\left(\tau_1 + 1 + \frac{\sigma}{\rho}\right)^{c(n,g_0)},$$

where C is determined only by g_0, n, θ_2/θ_1, $\rho\theta_3/\theta_1$ and $\theta_3\sigma$, and hence

$$|D_i u| \le G_i^{-1}\left(\sum_k G_k\left(C\left(\tau_1 + 1 + \frac{\sigma}{\rho}\right)^{(n+4)/2}\right)\right) \le C\left(\tau_1 + 1 + \frac{\sigma}{\rho}\right)^{g_0(n+4)/2}.$$

Because $\mathcal{E}_1 = v\sum_i a_i(x)g'(D_i u)(D_i u)^2$ and $B_1 = B + |p|\sum_i \dfrac{\partial a_i}{\partial x_i} g_i(|p_i|)\operatorname{sgn} p_i$, we also have a boundary gradient estimate. For parabolic equations A and B may depend on t also.

Of course, to apply our results directly, we must assume that $g_i'(0)$ is positive. We remove this additional hypothesis by copying the corresponding argument in Example 2. Now we set

$$b_0 = B(x, u, Du) / \left(\sum_{i=1}^{\infty} g_i(|D_i u|)|D_i u|\varepsilon(v) + 1 \right)$$

and

$$B(x, z, p; 0) = b_0(x) \left(\sum_{i=1}^{n} g_i(|p_i|)|p_i|\varepsilon(v) + 1 \right) + k(u(x) - z)$$

for $k > 1$ a constant to be chosen. If $\overline{u}$ is another solution of

$$\operatorname{div} A(x, Du) + B(x, \overline{u}, D\overline{u}; 0) = 0 \text{ in } B_\rho, \quad \overline{u} = u \text{ on } \partial B_\rho,$$

we use first $[\overline{u} - \min \overline{u}](u - \overline{u})_+^2$ and then $[\overline{u} - \max \overline{u}](\overline{u} - u)_+^2$ as text functions in the weak form of the equation for $\overline{u}$ to see that

$$\int_{B_\rho} \sum_i |D_i\overline{u}| g_i(|D_i\overline{u}|)\xi^2 \, dx \leq C \int_{B_\rho} \left\{ \xi^2 + \sum_i g_i'(|D_i\overline{u}|)|D_i\xi|^2 \right\} dx$$

for $\xi = u - \overline{u}$ and the constant C independent of k, and a similar estimate holds with u replacing $\overline{u}$. Then with ξ as test function in the functions for u and $\overline{u}$, it follows that

$$\begin{aligned} &\int_{B_\rho} \sum_i [g_i'(|D_i u|) + g_i'(|D_i\overline{u}|)]|D_i\xi|^2 dx \\ &\qquad \leq \delta \int_{B_\rho} \sum_i [|D_i u| g_i(|D_i u|) + |D_i\overline{u}| g_i(|D_i\overline{u}|)]\xi^2 \, dx \\ &\qquad + [C(\delta) - k] \int_{B_\rho} \xi^2 \, dx \end{aligned}$$

with $C(\delta)$ independent of k, and $\delta > 0$ arbitrary. Choosing first δ small and then k large gives uniqueness here, and uniqueness for the approximating problems (which also use regularized g_i's) follows from [4, Theorem 10.7].

The parabolic version of this estimate is very close to the estimate in [17] in some sense. In that work, Mkrtychyan studied

$$u_t = \sum_{i=1}^{2} D_i(u^{\ell_i}|D_i u|^{m_i-2} D_i u) + f(x, t)$$

with $\ell_i \geq 1$, $m_i \geq 3$ and some additional restrictions on ℓ_i and m_i. If $\ell_i = 0$ our estimate is better because it only requires $m_i \geq 2$, but it is not clear how to handle the u-dependence.

Note that our gradient estimate is true regardless of the "spread" of the g's, once we have a bound on the solution. On the other hand, some restriction on the spread is needed to obtain a local solution bound. For the case $B \equiv 0$, $A^i = p_i$ for $i < n$, $A^n = (p_n)^3$, if $n \geq 6$, Marcellini [13] and Giaquinta [3] found an unbounded solution $u(x) = \sqrt{\frac{n-4}{24}}\, x_n^2 \left(\sum_{i=1}^{n-1} x_i^2\right)^{-1/2}$; hence weak solutions of the equation can be unbounded, but bounded solutions are Lipschitz.

Our method also applies to a number of nondegenerate equations. If we choose $b_1(\tau) = \cdots = b_n(\tau) = \tau$, our structure functions are the same as in [7], so we merely state the results, and only in the elliptic case.

EXAMPLE 5 (Uniformly elliptic equations). Let ψ be an increasing C^1 function on $[0, \infty)$ with $\tau\psi'(\tau) \leq \psi_0\psi(\tau)$ for some positive constant ψ_0. Suppose there are constant $\mu_1, \ldots, \mu_7$ and a decreasing function ε such that

$$Du \cdot A \geq \mu_1 |Du| \psi(|Du|) - \mu_2, \quad |A| \leq \mu_3 \psi(|Du|) + \mu_4,$$

$$a^{ij}\xi_i\xi_j \geq \mu_5 \psi(|Du|)|\xi|^2, \quad |a^{ij}| \leq \mu_6 \psi(|Du|),$$

$$|Du|\,|A_z| + |A_x| + |B| \leq \varepsilon(|Du|)\psi(|Du|)|Du| + \mu_7.$$

In either $\varepsilon(\infty) = 0$ or $\tau\psi'(\tau)/\psi(\tau)$ is bounded away from zero, then we have a gradient estimate.

EXAMPLE 6. $A^i = \exp(|Du|^2)D_iu$, $|B| = O(\exp(|Du|^2)|Du|^2)$.

So far our examples have been of two categories. On a given sphere $\{|p| = \text{const.}\}$ either there are points $p^1, \ldots, p^n$ such that the matrix $\left(\frac{\partial A}{\partial p}\right)$ has a zero eigenvalue with eigenvector ξ^k, having components $\xi_i^k = \delta_i^k$, at p^k (so the equation degenerates in any direction, loosely speaking) or the matrix $\frac{\partial A}{\partial p}$ never degenerates. Our final example only degenerates in one direction.

EXAMPLE 7. $A^1 = |p|^6 p_1$, $A^2 = |p|^4(p_2)^3$, $B \equiv 0$ and $n = 2$. Then we take $b_1(\tau) = b_2(\tau) = \tau$,

$$\lambda = \frac{1}{200}v^7, \quad \Lambda_0 = \Lambda = \overline{\mu} = 1352v^7, \quad w = v^{7/2}$$

$$\beta = 7, \quad \beta_0 = 1, \quad \beta_1 = 0$$

in order to obtain a gradient bound in the form $\sup_{B_\rho} |Du| \leq C \operatorname{osc}_{B_{2\rho}} u/\rho$.

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Department of Mathematics
Iowa State University
Ames, Iowa 50011