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C^k-Regularity for the $\overline{\partial}$-Equation on a Piecewise Smooth Union of Strictly Pseudoconvex Domains in $\mathbb{C}^n$

JOACHIM MICHEL - ALESSANDRO PEROTTI

Introduction

In 1980 Lieb-Range [3] proved $C^{k+1/2}$-estimates for an integral solution operator of the equation $\overline{\partial}u = f$ on a smooth strictly pseudoconvex domain (for $\overline{\partial}$-closed C^k $(0,q)$-forms f). Similar results were obtained by Aĭzenberg-Dautov [1].

In 1973, on strictly pseudoconvex piecewise smooth domains, Range-Siu [11] proved uniform and Hölder estimates with a gain of $1/2-\epsilon$, with ϵ arbitrarily small. In [7], Michel-Perotti proved $C^{k-\epsilon}$-estimates, also with arbitrarily small $\epsilon > 0$, on a piecewise strictly pseudoconvex domain which satisfies a condition of real transversality. Under an additional condition for the boundary, involving the complex tangent spaces, the first author showed in [6] that the same solution operator satisfies $C^{k+1/2-\epsilon}$-estimates, for any $\epsilon > 0$.

In this paper we consider the equation $\overline{\partial}u = f$, $f \in C^k_{0,q}(\overline{D})$, on a piecewise smooth union of strictly pseudoconvex domains of $\mathbb{C}^n$, and for any q sufficiently large we construct a solution operator S_q which has some boundary regularity.

We assume $D = D_1 \cup \ldots \cup D_K$, where the domains D_i are smoothly bounded strictly pseudoconvex domains, whose boundaries intersect real-transversally (see Section 1 for exact definitions). Suppose that no more than L among these domains intersect. On these domains we can solve the $\overline{\partial}$-equation for $\overline{\partial}$-closed forms of type $(0,q)$, with $q \geq L$ (see Section 2).

We prove that there exist, for $L \leq q \leq n$ and $k \geq (L+1)/2$, linear operators $S_q : C^k_{0,q}(\overline{D}) \cap \operatorname{Ker}\overline{\partial} \to C^k_{0,q-1}(D)$ with:

(i) $\overline{\partial}S_q(f) = f$ on D,

(ii) for any $\epsilon > 0$, there exists a constant $C_{k,\epsilon}$, independent of f, such that

$$\|S_q(f)\|_{k-(L-1)/2-\epsilon,D} \leq C_{k,\epsilon}\, \|f\|_{k,D}.$$

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The estimates are optimal with the applied methods. There are no hints, if the loss of regularity is really necessary for domains with non-smooth boundary.

If we drop the transversality assumption, the geometry of the boundary becomes very complicated. Solution operators can be written down in an analogous way, but with big losses of regularity. Compare Peters [14] for non-transversal intersections of strictly pseudoconvex domains.

Another generalization can be done for piecewise smooth unions of weakly pseudoconvex domains that have support functions. In this case the losses of regularity would be very large.

1. - Preliminaries

(1.1) Let D be the union of K bounded domains $D_1, D_2, \ldots, D_K$ of $\mathbb{C}^n$. Let L be the integer defined as

$$L = \max\left\{ m \mid \text{there exists a multi-index } I = (i_1, i_2, \ldots, i_m) \right.$$

$$\left. \text{with } 1 \leq i_1 < \cdots < i_m \leq K \text{ such that } \bigcap_{i \in I} D_i \neq \emptyset \right\}.$$

For any multi-index $I = (i_1, i_2, \ldots, i_m)$, with $1 \leq i_1 < i_2 < \cdots < i_m \leq K$, we define

$$D^I = \bigcap_{i \in I} D_i, \qquad C^I = \mathbb{C}^n \setminus \bigcup_{i \in I} \overline{D}_i.$$

Then $D^I = \emptyset$ for $|I| > L$ (here $|I|$ denotes the length of the multi-index I).

We assume that the domains D_i, $1 \leq i \leq K$, are smooth and strictly pseudoconvex and that they intersect real-transversally. This means that there exist C^∞-functions ρ_i on an open neighbourhood U_i of D_i $(1 \leq i \leq K)$, such that:

(a) ρ_i is a strictly plurisubharmonic defining function for D_i;

(b) for any multi-index $I = (i_1, i_2, \ldots, i_m)$ with $D^I \neq \emptyset$, the 1-forms $d\rho_{i_1}, d\rho_{i_2}, \ldots, d\rho_{i_m}$ are linearly independent over $\mathbb{R}$ at every point of $\bigcap_{\nu=1}^{m} U_{i_\nu}$.

(1.2) We want to construct Cauchy-Fantappiè forms on the domain D^I. On any smooth strictly pseudoconvex domain Ω, there exists a Leray map $w(\varsigma, z) = (w_1(\varsigma, z), \ldots, w_n(\varsigma, z))$ which satisfies the following properties:

(a) $w(\varsigma, z)$ is smooth on $(U \times W) \setminus \Delta$, where U is a neighbourhood of $\partial\Omega$, W is a Stein neighbourhood of $\overline{\Omega}$ and Δ is the diagonal of $\mathbb{C}^n \times \mathbb{C}^n$;

(b) $w(\varsigma, \cdot)$ is holomorphic in z on W;

(c) $\langle w(\varsigma, z), \varsigma - z\rangle = \sum_{j=1}^{n} w_j(\varsigma, z)(\varsigma_j - z_j) = 1$;

(d) $w = P/\Phi$, where $P(\varsigma, \cdot)$ is holomorphic in z and Φ is a barrier function for Ω.

REMARK. There exists a neighbourhood N of $(\mathbb{C}^n\backslash\Omega) \times \overline{\Omega}$ such that the Leray map $w(\varsigma, z)$ verifies properties (a)–(c) on $N\backslash\Delta$. Indeed, we can extend $w(\varsigma, z)$ in the following way. Let V be a Stein neighbourhood of $\overline{\Omega}$, with $\Omega \subset\subset V \subset\subset W$. We can assume that $W\backslash\overline{\Omega} = U\backslash\overline{\Omega}$. For any $\varsigma' \in \mathbb{C}^n\backslash W$, the domain $\varsigma' - V = \{v \mid v = \varsigma' - z,\ z \in V\}$ is a Stein domain not containing the origin of $\mathbb{C}^n$. Let U' be any open neighbourhood of ς' such that the set $A = \{v \mid v = \varsigma - z,\ \varsigma \in U',\ z \in V\}$ is contained in a Stein domain B not containing the origin. It is a well-known fact that on B we can find holomorphic functions $w'_1, \ldots, w'_n$ such that $\sum_{i=1}^{n} w'_i(v)v_i = 1$, where $v = (v_1, \ldots, v_n)$ is the coordinate system of $\mathbb{C}^n$.

Setting $w'_i(\varsigma, z) = w'_i(\varsigma - z) \in \mathcal{O}(U' \times V)$, we get $\langle w'(\varsigma, z), \varsigma - z\rangle = 1$ for $(\varsigma, z) \in U' \times V$. In this way we can construct a locally finite open covering $\mathcal{U} = \{U_\alpha\}_{\alpha\in\mathcal{A}}$ of $\mathbb{C}^n\backslash W$, and functions $w_i^\alpha(\varsigma, z) \in \mathcal{O}(U_\alpha \times V)$ ($\alpha \in \mathcal{A}$, $i = 1, \ldots, n$) such that $\langle w^\alpha(\varsigma, z), \varsigma - z\rangle = 1$ for any $\alpha \in \mathcal{A}$.

Let $U_0 = U$, $w^0 = w$, and let $\{\phi_\alpha\}_{\alpha\in\{0\}\cup\mathcal{A}}$ be a smooth partition of unity subordinate to the covering $\mathcal{U} = \{U_\alpha\}_{\alpha\in\{0\}\cup\mathcal{A}}$ of $U \cup (\mathbb{C}^n\backslash\overline{\Omega})$. We set $\tilde{w}(\varsigma, z) = \sum_\alpha \phi_\alpha(\varsigma)w^\alpha(\varsigma, z)$ on $((U \cup \mathbb{C}^n\backslash\overline{\Omega}) \times V)\backslash\Delta$. Then $\tilde{w}$ is a Leray map for Ω which coincides with w on $((U \cap V) \times V)\backslash\Delta$.

Let $\eta_i(\varsigma, z) = \langle w^i, d\varsigma\rangle = \sum_j w_j^i(\varsigma, z)d\varsigma_j = \Phi_i(\varsigma, z)^{-1} \sum_j P_j^i(\varsigma, z)d\varsigma_j$ be the $(1,0)$-form constructed from the Leray map $w^i(\varsigma, z)$ for D_i when $i > 0$, and from the Bochner-Martinelli barrier $\Phi_0 = |\varsigma - z|^2$, $P_j^0(\varsigma, z) = \overline{\varsigma}_j - \overline{z}_j$ when $i = 0$. Like in [7], we define the differential forms

$$D_{n,q} = (2\pi i)^{-n}(-1)^{1/2q(q-1)} \binom{n-1}{q} \eta_I \wedge \bigwedge^{q} \overline{\partial}_z\eta_I \wedge \bigwedge^{n-q-1} \overline{\partial}_{\varsigma,\lambda}\eta_i,$$

for $0 \leq q \leq n-1$, $D_{n,n}(\eta_I) = 0$, where $\overline{\partial}_{\varsigma,\lambda} = \overline{\partial}_\varsigma + d_\lambda$. The forms $D_{n,q}(\eta_I)$ have degree q in z, and degree $2n-q-1$ in (ς, λ), where (ς, z) are in a neighbourhood of $C^I \times D^I$ and λ belongs to the simplex

$$\Delta_{0I} = \{(\lambda_0, \lambda_1, \ldots, \lambda_K) \in \mathbb{R}^{K+1} \mid \lambda_i \geq 0,\ \lambda_0 + \lambda_1 + \cdots + \lambda_K = 1,$$
$$\lambda_j = 0 \text{ for } j \notin 0I = (0, i_1, \ldots, i_m)\}.$$

These Cauchy-Fantappiè kernels satisfy the following properties:

(1) $d_{\varsigma,\lambda}D_{n,q}(\eta_I) = \overline{\partial}_{\varsigma,\lambda}D_{n,q}(\eta_I) = (-1)^q\overline{\partial}_z D_{n,q-1}(\eta_I)$, where $d_{\varsigma,\lambda} = d_\varsigma + d_\lambda$;

(2) $D_{n,q}(\eta_I) = 0$ when $\lambda \in \Delta_I$ and $q > 0$;

(3) $$\int_{\Delta_0} D_{n,q}(\eta_I) = B_q.$$

We set $A_q^I = (-1)^{|I|+1} \int_{\Delta_{0I}} D_{n,q}(\eta_I)$. The forms $A_q^I(\varsigma, z)$ are C^∞-forms of type $(0,q)$ with respect to z, when $z \in D^I$, and type $(n, n-q-|I|-1)$ with respect to ς, $\varsigma \in C^I$, such that, if we define the coboundary operators

$$\delta A_q^I = \sum_{j=1}^{|I|} (-1)^{j-1} A_q^{I_j} \quad \text{for } |I| > 1, \quad \delta A_q^i = -B_q$$

(where $I_j = (i_1, i_2, \ldots, i_{j-1}, i_{j+1}, \ldots, i_m)$, and B_q is the Bochner-Martinelli kernel) and we set $A_{n-|I|}^I \equiv 0$, on a neighbourhood of $C^I \times D^I$ we get

$$(*) \qquad \delta A_q^I = (-1)^{|I|} \{\overline{\partial}_z A_{q-1}^I + (-1)^{q-1} \overline{\partial}_\varsigma A_q^I\}$$

for any $1 \leq q \leq n - |I|$. Equality $(*)$ is an immediate consequence of properties (1)–(3). Compare also Range-Siu [11] and Michel [5] for this construction.

2. - The equation $\overline{\partial} u = f$ on D

(2.1) Let the domain D and the integer L be as in (1.1). Then D is $(L-1)$-cohomologically complete (see [13]). This means that for any analytic coherent sheaf $\mathcal{F}$ on D, the cohomology groups $H^q(D, \mathcal{F})$ vanish for $q \geq L$. This fact is a consequence of Leray's Theorem, since the open covering $\mathcal{U} = \{D_1, \ldots, D_K\}$ is a Leray covering of D, and then we have the isomorphism

$$H^q(D, \mathcal{F}) \simeq \check{H}^q(\mathcal{U}, \mathcal{F}) \quad \text{for } q \geq 0,$$

where $\check{H}^q(\mathcal{U}, \mathcal{F})$ are the Čech cohomology groups. Since $\bigcap_{i \in I} D_i = \emptyset$ if $|I| > L$, when $q \geq L$ there are no q-cochain of $\mathcal{U}$ with coefficients in $\mathcal{F}$ different from zero. Therefore $H^q(D, \mathcal{F}) = 0$ when $q \geq L$.

In particular, the vanishing of the groups $H^q(D, \mathcal{O})$, $q \geq L$, and the Dolbeault isomorphism, imply that the $\overline{\partial}$-problem $\overline{\partial} u = f$ can be solved on D for any $\overline{\partial}$-closed form $f \in C^\infty_{0,q}(D)$, $q \geq L$.

REMARK. If the domains $D_1, \ldots, D_K$ are complex-transversal (this means that $\partial \rho_{i_1}, \ldots, \partial \rho_{i_m}$ are independent over $\mathbb{C}$ at every point of $\bigcap_{\nu=1}^{m} U_{i_\nu}$, for any multi-index $I = (i_1, \ldots, i_m)$), then D satisfies a condition of strict Levi $(L-1)$-concavity, as defined by Nacinovich in [9]. It was proved there that the local cohomology for $\overline{\partial}$ in dimension $L-1$ for differential forms that are C^∞ up to the boundary is infinite-dimensional.

Now we consider the equation $\overline{\partial}u = f$ with $f \in C^k_{0,q}(\overline{D})$, $\overline{\partial} f = 0$. From what we have just seen, if $q \geq L$ it is reasonable to look for solutions u having some boundary regularity. By $C^s_{p,r}(\overline{D})$ we denote the space of all continuous (p,r)-forms on $\overline{D}$ which have continuous derivatives up to order $[s]$ on $\overline{D}$, satisfying on Hölder condition of order $s-[s]$. We prove the following result.

THEOREM 1. *Let D be as in* (1.1), *and $k \geq (L-1)/2$. Then there exist for $L \leq q \leq n$ linear operators $S_q : C^k_{0,q}(\overline{D}) \cap \operatorname{Ker}\overline{\partial} \to C^k_{0,q-1}(D)$ with:*

(i) $\overline{\partial} S_q(f) = f$ *on D;*

(ii) *for any $\epsilon > 0$, S_q is a continuous operator from $C^k_{0,q}(\overline{D}) \cap \operatorname{Ker}\overline{\partial}$ into the space $C^{k'}_{0,q-1}(\overline{D})$, with $k' = k - (L-1)/2 - \epsilon$.*

REMARK. If $f \in C^\infty_{0,q}(\overline{D})$, then $S_q(f) \in C^\infty_{0,q-1}(\overline{D})$.

REMARK. Under the additional condition on the complex tangent spaces of the domains D_i assumed in [6], by using cohomological arguments one can find a solution which satisfies $C^{k'}$-estimates, with $k' = k - (L-2)/2 - \epsilon$, $\epsilon > 0$. Under the hypotheses of Theorem 1, the same reasoning gives only $k' = k - (L-1) + 1/2 - \epsilon$.

(2.2) Let D_0 be an open neighbourhood of $\overline{D}$ with piecewise smooth boundary. We can define a continuous linear extension operator $E : C^{k+\lambda}(\overline{D}) \to C^{k+\lambda}_0(D_0)$ $(k \geq 0,\ 0 \leq \lambda \leq 1)$. The domain D is locally diffeomorphic to a set $S = \bigcup_{i=1}^{t} \{x \in B \mid x_i \leq 0\}$ $(1 \leq t \leq 2n)$, where B is the unit cube of $\mathbb{R}^{2n}$ centered at the origin. For any half-cube $S_i = \{x \in B \mid x_i \leq 0\}$, there exists a continuous extension operator $E_i : C^{k+\lambda}(S_i) \to C^{k+\lambda}(B)$, which satisfies the following property:

$$(**) \qquad E_i(f)|_{S_j} = E_i\left(f|_{S_i \cap S_j}\right)\big|_{S_j} \quad \text{for any } f \in C^{k+\lambda}(S_i) \quad \text{and any } i, j, i \neq j.$$

(see for example [8]-54.XV for the case $k < \infty$, and [12] for $k = \infty$).

Given two extension operators $E_1 : C^{k+\lambda}(A_1) \to C^{k+\lambda}(B)$, $E_2 : C^{k+\lambda}(A_2) \to C^{k+\lambda}(B)$, a new operator $E : C^{k+\lambda}(A_1 \cup A_2) \to C^{k+\lambda}(B)$ can be defined in the following way:

$$Ef = E_1(f|_{A_1}) + E_2(f|_{A_2}) - E_1(E_2(f|_{A_2})|_{A_1}).$$

Applying this operation recursively to the pairs of sets (S_1, S_2), $(S_1 \cup S_2, S_3), \ldots, (S_1 \cup \ldots \cup S_h, S_{h+1})$, and so on, we finally get a continuous linear operator $E : C^{k+\lambda}(S) \to C^{k+\lambda}(B)$, which is an extension operator because of property $(**)$. By a partition of unity argument, we can glue together these local extension operators and obtain the operator $E : C^{k+\lambda}(\overline{D}) \to C^{k+\lambda}_0(D_0)$.

(2.3) Let $f \in C^k_{0,q}(\overline{D})$ be a $\overline{\partial}$-closed form. Applying the Bochner-Martinelli-Koppelman formula to f on D, we get

$$f = \int_{\partial D} f \wedge B_q - \overline{\partial} \int_{D} f \wedge B_{q-1}.$$

If $k > 0$, we can apply the formula to the extension Ef on D_0. Since Ef vanishes on ∂D_0, we have on D

$$f = - \int_{D_0 \setminus \overline{D}} \overline{\partial} Ef \wedge B_q - \overline{\partial} \int_{D_0} Ef \wedge B_{q-1}.$$

Now we want to transform the first integral. For $|I| \leq q \leq n$, we introduce the forms $\beta(I) = \int_{D_0} \overline{\partial} Ef \wedge A^I_{q-|I|}$.

On these forms we define a coboundary operator δ as follows:

$$\delta\beta(I) = \sum_{j=1}^{|I|} (-1)^{j-1} \beta(I_j) \quad \text{if } |I| > 1, \quad \text{and}$$

$$\delta\beta(i) = - \int_{D_0} \overline{\partial} Ef \wedge B_q.$$

LEMMA 1. *Let $l = |I|$, $k \geq 1$ and $\epsilon > 0$. If $f \in C^k_{0,q}(\overline{D})$ is a $\overline{\partial}$-closed form, then $\beta(I) \in C^{k'}_{0,q-l}(\overline{D}^I)$, with $k' = k + (l-1)/2 - \epsilon$. The forms $\beta(I)$ satisfy $\delta\beta(I) = (-1)^l \overline{\partial}\beta(I)$ on D^I. Moreover, there exists a constant $C_{k,\epsilon}$, independent of f, such that*

$$\|\beta(I)\|_{k',D^I} \leq C_{k,\epsilon} \|f\|_{k,D}.$$

PROOF. $\delta\beta(I) = (-1)^l \overline{\partial}\beta(I)$ follows immediately from the properties of the Cauchy-Fantappiè forms (see (1.2)) and the vanishing of $\overline{\partial} Ef$ on the boundary of $D_0 \setminus \overline{D}$. The second statement will be proved in Section 3. □

(2.4) In particular, from the previous lemma, when $q \geq L$ we get on D_i a particular solution $f = \overline{\partial}(\beta(i) - \int_{D_0} Ef \wedge B_{q-1})$. Since $D = \bigcup_{i=1}^{K} D_i$, we must show that we can glue together these solutions and obtain a form of class $C^{k'}(\overline{D})$, with $k' = \max(0, k - (L-1)/2 - \epsilon)$.

PROOF OF THEOREM 1. Here the $C^{k'}$-norm of a q-cochain is defined as the sum of the $C^{k'}$-norms of its components on the sets D^I, $|I| = q + 1$.

If $\mathcal{F}$ is any sheaf on $\overline{D}$, and $\mathcal{U} = \{V_1, \ldots, V_s\}$ is a finite, closed covering of $\overline{D}$ such that the sections of $\mathcal{F}$ on any intersection $V_{i_0} \cap V_{i_1} \ldots \cap V_{i_p}$ ($1 \leq i_m \leq s$ for

$m = 0, 1, \dots, p)$ can be extended to sections on $\overline{D}$, then the Čech cohomology groups $\check{H}^q(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$ with respect to the covering $\mathcal{U}$ vanish for any $q > 0$.

Let $\delta : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$ be the Čech coboundary operator, where $C^q(\mathcal{U}, \mathcal{F})$ is the space of alternating q-cochains of $\mathcal{U}$ with coefficients in $\mathcal{F}$. For any multi-index $I = (i_0, \dots, i_p)$ with $1 \le i_0 < i_1 < \cdots < i_p \le s$, we shall denote by V_I the intersection $V_{i_0} \cap \dots \cap V_{i_p}$. For C^k-estimates we need an explicit construction of the operator inverting δ on cocycles.

Let $c = (c(J))_{|J|=q+1} \in C^q(\mathcal{U}, \mathcal{F})$ be a q-cocycle, and let $|I| = q - 1$. For any i, $1 < i < i_0$, we construct recursively an extension $\tilde{c}(1iI)$ of $c(1iI)$ on V_{iI}. Let $\tilde{c}(12I)$ be any extension of $c(12I)$ on V_{2I}. Since $\delta c = 0$, for any i, j and I, with $1 < i < j < i_0$, we have

$$(\delta c)(1ijI) = c(ijI) - c(1jI) + c(1iI) - (\delta_I c)(1ijI) = 0 \quad \text{on } V_{1ijI},$$

where $\quad (\delta_I c)(1ijI) = \sum_{h=0}^{|I|} (-1)^h c(1ijI_h), \quad$ and $\quad I_h = (i_0, i_1, \dots, i_{h-1}, i_{h+1}, \dots, i_{q-1})$.

Let $j > 2$. Then the equation $(\delta c')(12jI) = 0$ on V_{2jI} defines uniquely an extension $c'(1jI)$ of $c(1jI)$ on V_{2jI}. Let $\tilde{c}(13I)$ be any extension of $c'(13I)$ on V_{3I}. By imposing the condition $(\delta c'')(13jI) = 0$ on $V_{3jI} (j > 3)$, we obtain an extension $c''(1jI)$ of $c'(1jI)$ on V_{3jI}. We extend arbitrarily $c''(14I)$ on V_{4I} by $\tilde{c}(14I)$. Proceeding in this way, we construct the cochains $\tilde{c}(1iI)$ on V_{iI} satisfying the condition $(\delta\tilde{c})(1ijI) = 0$ on V_{ijI} for any i, j and I.

Now we define $b \in C^{q-1}(\mathcal{U}, \mathcal{F})$ by

$$\begin{cases} b(1I) = 0 \\ b(iI) = \tilde{c}(1iI) \quad \text{for any } I \text{ of length } q-1 \text{ and any } i, \ 1 < i < i_0. \end{cases}$$

Then $(\delta b)(ijI) = b(jI) - b(iI) + (\delta_I b)(ijI) = \tilde{c}(1jI) - \tilde{c}(1iI) + (\delta_I \tilde{c})(1ijI) = \tilde{c}(ijI) = c(ijI)$ on V_{ijI}, since $(\delta\tilde{c})(1ijI) = 0$ there. Therefore $\delta b = c$, and the Čech cohomology groups vanish for any $q > 0$.

Let p be an integer, $0 < p \le q - 1$, and let $k' = k + p/2 - \epsilon$, $\epsilon > 0$. Now we apply the above result to the sheafs $\mathcal{C}^{k'}_{0,q-p-1}$ of germs of $(0, q - p - 1)$-forms of class $C^{k'}$, and to the covering $\mathcal{U} = \{\overline{D}_1, \dots, \overline{D}_K\}$ of $\overline{D}$. Given a p-cocycle $c \in C^p(\mathcal{U}, \mathcal{C}^{k'}_{0,q-p-1})$, we can find a $(p-1)$-cochain $b \in C^{p-1}(\mathcal{U}, \mathcal{C}^{k'}_{0,q-p-1})$ such that $\delta b = c$, and

$$\|b\|_{k'} \le C_{k'} \|c\|_{k'},$$

where $C_{k'}$ is independent of c. The constant in such inequalities may change in the course of the proof but we denote it always by the same symbol. There exists a continuous linear extension operator $E^I : C^{k'}(D^I) \to C^{k'}(\overline{D})$ (analogous to E in (2.2)). To get the estimate apply this operator E^I. This so constructed $(p-1)$-cochain b depends linearly on $c = \delta b$.

Now let a^p be the p-cochain defined by

$$a^p(I) = \beta(I) \quad \text{for any multi-index } I \text{ of length } |I| = p+1, \quad p > 0,$$
$$a^0(i) = \beta(i) - \int_{D_0} Ef \wedge B_{q-1} \quad \text{for any } 1 \leq i \leq K,$$

where the forms $\beta(I)$ are those introduced in (2.3).

Then from Lemma 1 we get $(\overline{\partial} a^p)(I) = \overline{\partial}\beta(I) = (-1)^{p+1}\delta\beta(I) = (-1)^{p+1}(\delta a^{p-1})(I)$ for $p > 0$. Therefore $\overline{\partial} a^p = (-1)^{p+1}\delta a^{p-1}$ for any $0 < p \leq q-1$, and

$$||a^p||_{k'} \leq C_{k'}||f||_{k,D}.$$

Here we use the same symbol δ to denote the operator acting on cochains and the coboundary operator defined in (2.3).

Since $\bigcap_{i\in I} D_i = \emptyset$ if $|I| > L$, we have $\delta a^{L-1} = 0$. From the vanishing of the Čech groups, there exists $b^{L-2} \in C^{L-2}(\mathcal{U}, C^{k'}_{0,q-L})$ such that $\delta b^{L-2} = a^{L-1}$, and

$$||b^{L-2}||_{k'} \leq C_{k'}||f||_{k,D},$$

with $k' = k+(L-1)/2-\epsilon$. We set $c^{L-1} := a^{L-1}$ and $c^{L-2} := a^{L-2}-(-1)^L\overline{\partial} b^{L-2}$. Then $\overline{\partial} c^{L-1} = (-1)^L\delta a^{L-2}$, $\overline{\partial} c^{L-2} = (-1)^{L-1}\delta a^{L-3}$, and $\delta c^{L-2} = \delta a^{L-2} - (-1)^L\overline{\partial} c^{L-1} = 0$.

Additionally, we have the estimates

$$||c^{L-1}||_{k'} \leq C_{k'}||f||_{k,D}, \qquad ||c^{L-2}||_{k'-1} \leq C_{k'}||f||_{k,D}.$$

Now we proceed by decreasing induction. We construct $c^{L-1}, c^{L-2}, \ldots, c^0$ with $c^p \in C^p(\mathcal{U}, C^{k'}_{0,q-p-1})$ ($k' = \max(0, k-(L-1)/2+p-\epsilon)$), such that

$$\overline{\partial} c^p = \overline{\partial} a^p \quad \text{for any } 0 \leq p \leq L-1, \ \delta c^p = 0, \quad \text{and } ||c^p||_{k'} \leq C_{k'}||f||_{k,D}.$$

In particular, when $p = 0$ we get a global form $S_q(f) := c^0$ such that $\overline{\partial} S_q(f) = \overline{\partial} a^0 = f$ on D. $S_q(f)$ is linear and has the regularity property given in the statement of the theorem. □

3. - C^k-Estimates

(3.1) We finish the proof of Lemma 1. Let $D \subseteq \mathbb{C}^n$ be the union of K smoothly bounded strictly pseudoconvex domains $D_1, \ldots, D_K$. Let L be defined as in (1.1), and assume that the domains D_i intersect real-transversally. Let

$g \in C^k_{0,q}(\overline{D}_0)$, $\overline{\partial} g = 0$ on D, $l = |I|$, $k \geq 1$ and $l \leq q \leq n-1$. Then the Lemma follows if we can prove for these forms

$$\beta_g(I) := \int_{D_0 \backslash D} \overline{\partial} g \wedge A^I_{q-|I|},$$

the estimate

$$||\beta_g(I)||_{k',D^I} \leq C_{k,\epsilon} ||g||_{k,D}.$$

with $k' = k + (l-1)/2 - \epsilon$, $\epsilon > 0$.

Without restriction of generality, we can assume $I = (1, 2, \dots, l)$. From the definition of $A^I_{q-|I|}$ (see (1.2)), it follows that we must estimate on D^I integrals of type

$$\int_{(D_0 \backslash D)_\varsigma} \overline{\partial} g \wedge \eta_0 \wedge \eta_1 \wedge \dots \wedge \eta_\iota \wedge (\overline{\partial}_\varsigma \eta_0)^{j_0} \wedge \dots \wedge (\overline{\partial}_\varsigma \eta_\iota)^{j_\iota} \wedge (\overline{\partial}_z \eta_0)^{q-\iota},$$

with $j_0 + j_1 + \cdots j_\iota = n - q - 1$.

For the notations see also [7]:

$a(\varsigma)$ is a coefficient of $\overline{\partial} g$; $\quad \eta^*_i(\varsigma, z) = \langle P^i(\varsigma, z), d\varsigma \rangle = \sum_j P^i_j(\varsigma, z) d\varsigma_j$,

$\Theta_L = \eta^*_{i_1} \wedge \dots \wedge \eta^*_{i_r}$ for $L = (l_1, \dots, l_r)$;

$K(a_0, a_1, \dots, a_\iota) = \Phi_0^{\alpha_0} \cdot \Phi_1^{\alpha_1} \cdots \Phi_\iota^{\alpha_\iota}$;

$E_\nu(\varsigma, z)$ is a smooth form on $(\overline{D}_0 \backslash D) \times \overline{D^I}$, independent of g, which vanishes of order v for $\varsigma = z$.

Then we can write the above integral as

$$J(z) = \int_{D_0 \backslash D} \frac{a(\varsigma) \Theta_I(\varsigma, z) \wedge E_1(\varsigma, z)}{K(q - l + 1 + j_0, 1 + j_1, \dots, 1 + j_\iota)}.$$

(3.2) Let D^p be a differentiation in z of order p. Then $D^p J(z)$ is a sum of integrals

$$\int_{D_0 \backslash D} \frac{a(\varsigma) \Theta_L(\varsigma, z) \wedge E_\nu(\varsigma, z)}{K(a_0, a_1, \dots, a_\iota)},$$

with $a_1 \geq j_1 + 1, \dots, a_\iota \geq j_\iota + 1$, $a_1 + \cdots + a_\iota \leq l + n - q - 1 - j_0 + p_1$,

$$q-l+1+j_0 \le a_0 \le q-l+1+j_0+p_3,$$

$$2a_0 - 2(q-l+1+j_0) \le v+p_3-1, \quad L \subseteq I, \ |L| = m = \max(l-p_2, 0),$$

$$p_1+p_2+p_3 = p.$$

REMARK. D^p can be written as $D^{p'} \circ D^{p''}$, where $D^{p'}$ contains only vector fields $\frac{\partial}{\partial z_j}$, and $D^{p''}$ only vector fields $\frac{\partial}{\partial \bar{z}_j}$, then $p = p' + p''$, $p_1 + p_2 \le p'$ and $p_3 \le p''$.

(3.3) Let $z \in D^I$ and $\varsigma_0 \in D_0 \backslash D$ be fixed. On a neighbourhood U of ς_0, with $z \in U$, we can choose

$$x = (x_1, \dots, x_{2n}) = (\rho_1(\varsigma) - \rho_1(z), \rho_2(\varsigma) - \rho_2(z), \dots, \rho_\iota(\varsigma) - \rho_\iota(z), x_{\iota+1}, \dots, x_{2n})$$

as real coordinates. Let $x' = (x_{\iota+1}, \dots, x_{2n})$. The form $\Theta_L(\varsigma, z) \wedge E_\nu(\varsigma, z)$ can be written as a sum of terms

$$E_\nu(\varsigma, z) d\rho_1 \wedge \dots \wedge d\rho_\iota \wedge dx_{\iota+1} \wedge \dots \wedge dx_{2n},$$

where now E_ν denotes a function.

Let $\delta = \delta(z) = \operatorname{dist}(z, \partial D^I)$. Then there exists a constant C such that

$$\delta \le C \cdot \min\{|\rho_1(z)|, \dots, |\rho_\iota(z)|\} \quad \text{for } z \in D^I \cap U,$$

$$|\Phi_j(\varsigma, z)| \ge C(|\operatorname{Im} \Phi_j(\varsigma, z)| + |\rho_j(\varsigma)| + \delta + |\varsigma - z|^2) \quad \text{for } j = 1, 2, \dots, l, \text{ and}$$

$$|\Phi_0(\varsigma, z)| = |\varsigma - z|^2 \ge C(|\rho_1(\varsigma)| + \dots + |\rho_\iota(\varsigma)| + \delta + |x'|)^2$$

for $\varsigma \in (D_0 \backslash \overline{D}) \cap U$ and $z \in D^I \cap U$.

Moreover, since $a(\varsigma)$ is a C^{k-1}-function which vanishes on D, we have

$$|a(\varsigma)| \le C \|g\|_{k, D_0} \rho_1(\varsigma)^{r_1} \cdots \rho_\iota(\varsigma)^{r_\iota} \quad \text{for } \varsigma \in D_0, \text{ with } r_1 + \dots + r_\iota = k-1$$

(see the geometric lemma in [7]). We use the same letter C to denote also different constants, always independent of g.

(3.4) We must estimate the integral

$$I(z) = \int_{D_0 \backslash D} \frac{a(\varsigma) E_\nu(\varsigma, z) d\rho_1 \wedge \dots \wedge d\rho_\iota \wedge dx_{\iota+1} \wedge \dots \wedge dx_{2n}}{|\Phi_1^{a_1} \dots \Phi_\iota^{a_\iota}| \, |\varsigma - z|^{2a_0}}.$$

Assume $p_1 > 0$.

Let l be even, and let $p = p_1 + p_2 + p_3 = k + l/2 - 1 + \Delta$, with $\Delta = 0, 1$.

Since $a_1 + \cdots + a_\iota \leq l + n - q - 1 - j_0 + p_1$ and $n - q - 1 - j_0 \geq 0$, there exists an index i, $1 \leq i \leq l$, such that

$$\frac{1}{|\Phi_1^{a_1} \ldots \Phi_\iota^{a_\iota}|} \leq C \frac{1}{|\Phi_i|\, |\Phi_1 \ldots \Phi_\iota|\, \left|\rho_1(\varsigma)^{b_1} \ldots \rho_\iota(\varsigma)^{b_\iota}\right|\, |\varsigma - z|^{2(n-q-1-j_0)}},$$

with $b_1 + b_2 + \cdots + b_\iota = p_1 - 1$.

We can assume $i = 1$. Then we get

$$|I(z)| \leq C \int\limits_{D_0 \setminus D} \frac{|a(\varsigma)|\, |\varsigma - z|^\nu d\rho_1 \wedge \ldots \wedge d\rho_\iota \wedge dx_{\iota+1} \wedge \ldots \wedge dx_{2n}}{|\Phi_1|^2 |\Phi_2 \ldots \Phi_\iota|\, \left|\rho_1^{b_1} \ldots \rho_\iota^{b_\iota}\right|\, |\varsigma - z|^{2n-2\iota-1+\nu+p_3}}$$

$$\leq C \int\limits_{D_0 \setminus D} \frac{\|g\|_{k,D_0} d\rho_1 \wedge \ldots \wedge d\rho_\iota \wedge dx_{\iota+1} \wedge \ldots \wedge dx_{2n}}{|\Phi_1|^2 |\Phi_2 \ldots \Phi_\iota|\, |\varsigma - z|^{2n-\iota-3+2\Delta}},$$

since $(p_1 - 1) + p_3 \leq (k - 1) + l/2 - 1 + \Delta$.

Set $\sigma^2 = x_{\iota+1}^2 + \cdots + x_{2n}^2$. Let ϵ be a positive number.

(1) $\Delta = 0$. We get

$$|I(z)| \leq C_\epsilon \|g\|_{k,D_0} \int \frac{\sigma^2 d\rho_1 \wedge d\sigma}{(\rho_1 + \delta + \sigma^2)^{2+\epsilon}} \leq C'_\epsilon \|g\|_{k,D_0}.$$

(2) $\Delta = 1$. Analogously, we obtain

$$|I(z)| \leq C_\epsilon \|g\|_{k,D_0} \int \frac{d\rho_1 \wedge d\sigma}{(\rho_1 + \delta + \sigma^2)^{2+\epsilon}} \leq C'_\epsilon \|g\|_{k,D_0} \delta^{-1/2-\epsilon}.$$

Now let l be odd. $p = k + (l-1)/2 - 1 + \Delta$, with $\Delta = 0, 1$. Proceeding in the same way as before, we obtain

$$|I(z)| \leq C \int\limits_{D_0 \setminus D} \frac{\|g\|_{k,D_0} d\rho_1 \wedge \ldots \wedge d\rho_\iota \wedge dx_{\iota+1} \wedge \ldots \wedge dx_{2n}}{|\Phi_1|^2 |\Phi_2 \ldots \Phi_\iota|\, |\varsigma - z|^{2n-\iota-4+2\Delta}}.$$

(1) $\Delta = 0$.

$$|I(z)| \leq C_\epsilon \|g\|_{k,D_0} \int \frac{\sigma^3 d\rho_1 \wedge d\sigma}{(\rho_1 + \delta + \sigma^2)^{2+\epsilon}} \leq C'_\epsilon \|g\|_{k,D_0}.$$

(2) $\Delta = 1$.

$$|I(z)| \leq C_\epsilon \|g\|_{k,D_0} \int \frac{\sigma\, d\rho_1 \wedge d\sigma}{(\rho_1 + \delta + \sigma^2)^{2+\epsilon}} \leq C'_\epsilon \|g\|_{k,D_0} \delta^{-\epsilon}.$$

Now let $p_1 = 0$.

If l is even, since $p_3 \leq p = k + l/2 - 1 + \Delta$, we get

$$|I(z)| \leq C \int\limits_{D_0 \setminus D} \frac{\|g\|_{k,D_0} d\rho_1 \wedge \ldots \wedge d\rho_\iota \wedge dx_{\iota+1} \wedge \ldots \wedge dx_{2n}}{|\Phi_1 \ldots \Phi_\iota| \, |\zeta - z|^{2n-3\iota/2-1+\Delta}}$$

$$\leq C_\epsilon \|g\|_{k,D_0} \int \frac{d\sigma}{(\delta+\sigma)^\epsilon} \leq C'_\epsilon \|g\|_{k,D_0}.$$

If l is odd, $p_3 \leq p = k + (l-1)/2 - 1 + \Delta$, and then

$$|I(z)| \leq C \int\limits_{D_0 \setminus D} \frac{\|g\|_{k,D_0} d\rho_1 \wedge \ldots \wedge d\rho_\iota \wedge dx_{\iota+1} \wedge \ldots \wedge dx_{2n}}{|\Phi_1 \ldots \Phi_\iota| \, |\zeta - z|^{2n-(3\iota+3)/2+\Delta}}$$

$$\leq C_\epsilon \|g\|_{k,D_0} \int \frac{d\sigma}{(\delta+\sigma)^\epsilon} \leq C'_\epsilon \|g\|_{k,D_0}.$$

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