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0. - Introduction

Let D be a bounded domain in the complex Euclidean space $\mathbb{C}^n$ and $f \in \mathrm{Hol}(D, D)$ a holomorphic self-mapping of D. Consider the sequence $\{f^k\}$ of the iterates of f, defined inductively by $f^1 = f$ and $f^k = f^{k-1} \circ f$. A natural question is then to study the asymptotic behavior of $\{f^k\}$ as k tends to infinity.

In 1926, Denjoy and Wolff proved the first theorem in this direction. They showed that for a holomorphic self-mapping $f \in \mathrm{Hol}(\Delta, \Delta)$ of the unit disk $\Delta \subset \mathbb{C}^1$, $\{f^k\}$ converges uniformly on compacta to a boundary point if and only if f has no fixed point in Δ. Since this important work, much attention has been paid to extending their iteration theory to domains in $\mathbb{C}^n$ for $n \geq 1$. To name a few of the recent results, we mention here those on strongly convex domains in $\mathbb{C}^n$ ([Ab1]) and on contractible strongly pseudoconvex domains in $\mathbb{C}^2$ ([Ma]). For a detailed account of the history and references in this subject, we refer the reader to [Ab2].

In this paper, we are concerned with iteration theory on strongly pseudoconvex domains in $\mathbb{C}^n$ for *any* $n \geq 1$. Our main result is the following Theorem 1, which gives an exact description of the Denjoy-Wolff phenomenon for a large class of non-convex domains in $\mathbb{C}^n$ with $n \geq 1$ (see also [Ab3] for certain partial results in this regard). Theorem 1 answers a problem raised in [Ab3].

THEOREM 1. *Let D be a (topologically) contractible bounded strongly pseudoconvex domain in any dimension with C^3 boundary, and let $f \in \mathrm{Hol}(D, D)$ be a holomorphic self-mapping of D. Then $\{f^k\}$ converges to a boundary point uniformly on compacta if and only if f has no fixed point in D.*

COROLLARY 1. Let $D \subset\subset \mathbb{C}^n$ be a C^3 bounded strongly pseudoconvex domain that is homeomorphic to $\mathbb{C}^n$, and let $f \in \mathrm{Hol}(D, D)$ be a holomorphic

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self-mapping of D. Suppose that there exists $z_0 \in D$ so that $\{f^k(z_0)\}$ is a relatively compact subset of D. Then f fixes some point in D.

In [AH], Abate and Heinzner constructed a bounded taut contractible (non strongly pseudoconvex) domain D in $\mathbb{C}^8$, for which there is a holomorphic self-mapping f so that for some $z_0 \in D$ and some natural number k, it holds that $f^k(z_0) = z_0$, but f has no interior fixed point in D. So the strong pseudoconvexity of D in Corollary 1 (and thus in Theorem 1) is necessary. It is also worth mentioning that Theorem 1 is obviously false for strongly pseudoconvex domains with non-trivial topology.

The key step toward proving Theorem 1 (see Section 2) is to prove a fixed point theorem (Theorem 4 of Section 2) on lower dimensional holomorphic retracts of D. In case D is strongly convex or strongly pseudoconvex in $\mathbb{C}^2$ with trivial topology, this can be achieved by making use of the property that the Kobayashi ball of a *bounded convex* domain is also convex in the euclidean metric, or by making use of the Riemann mapping theorem and the classical Denjoy-Wolff theorem, respectively. Since we now will deal with a *non-convex domain of any dimension*, it does not seem that the aforementioned approaches can be adapted to our situation. The method presented here is based on a non-degeneracy property for extremal mappings near a strongly pseudoconvex point (Theorem 2), which enables us to prove the smooth extendibility of holomorphic retracts across strongly pseudoconvex points and thus leads to the proof of Theorem 4 (to be stated in Section 2). We next present the main technical result. Its statement requires some preliminary notation.

Let D be a bounded domain in $\mathbb{C}^n$ with p a C^2 smooth boundary point. For any $z \in D$ close enough to p, there is a unique point nearest to z in ∂D, which is denoted by $\pi(z)$. For any complex vector $\xi \in T^{(1,0)}D$, in what follows, we will use ξ_T and ξ_N to denote the complex tangential and complex normal components of ξ at $\pi(z)$, respectively.

THEOREM 2. *Let D be a bounded domain in $\mathbb{C}^n$ and $p \in \partial D$ a C^3 strongly pseudoconvex point. Then there is a small neighborhood U of p and a constant C depending only on U so that for any extremal mapping $\phi \in \mathrm{Hol}(\Delta, D)$ of D with $\phi(\Delta) \subset U \cap D$, it holds that $|(\phi'(\tau))_N| \leq C \cdot \eta(\phi)|(\phi'(\tau))_T|$. Here $|\cdot|$ stands for the Euclidean norm in $\mathbb{C}^n$ and $\eta(\phi) = \max_{\xi \in \overline{\Delta}} |\phi(\xi) - p|$.*

COROLLARY 2. Let D be a bounded domain in $\mathbb{C}^n$ and $p \in \partial D$ a C^3 strongly pseudoconvex point. Let $\{\phi_k\}$ be a sequence of extremal mappings of D and ϵ_0 a positive number so that $\{\phi_k(0)\}$ converges to p and $|(\phi_k'(0))_N| \geq \epsilon_0 |(\phi_k'(0))_T|$ for each k. Then the diameter of $\phi_k(\Delta)$ is greater than a fixed positive constant for every k.

The proof of Theorem 2 will be presented in Section 1. However, we remark here that this theorem also has other applications. For example, combining with Proposition 1 of [Hu1], it immediately gives the following useful result, which has been previously obtained in [CHL] by using Lempert's deformation theory in case the boundary is of class C^{14}:

THEOREM 3. *Let D be a bounded C^3 strongly convex domain in $\mathbb{C}^n$. For any given $p \in \partial D$ and complex vector $v \in T^{(1,0)}\mathbb{C}^n$, but not in $T_p^{(1,0)}\partial D$, there exists an extremal mapping ϕ so that $\phi(1) = p$ and $\phi'(1) = \lambda v$ for some real number λ (this ϕ then must be uniquely determined up to an automorphism of Δ according to Lempert* [Lm1]*).*

REMARK. The following simple example shows that Theorem 2 and Corollary 2 fail if ϕ is not extremal.

EXAMPLE. Denote by $\mathbb{B}^2$ the unit two ball. Let σ_t be a conformal mapping from the unit disk Δ to the domain $\{z \in \mathbb{C} : |z - t|^2 < t^2\}$ with $\sigma_t(1) = 0$, where $0 < t < 1$. Define the proper holomorphic embedding ϕ_t of Δ to $\mathbb{B}^2$ by

$$\phi_t(\xi) = (1 - t\sigma_t(\xi), \sqrt{1-t^2}\xi^2\sigma_t(\xi)).$$

Then $\sigma_t(1) = (1, 0) \in \partial\mathbb{B}^2$ and $\mathrm{Diam}(\phi(\Delta)) \to 0$ as $t \to 0$ ($\mathrm{Diam}(E)$ stands for the euclidean diameter of E). But

$$\frac{|(\phi_t'(0))_T|}{|(\phi_t'(0))_N|} \to 0,$$

as t goes to 0.

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1. - A non-degeneracy condition for extremal mappings

The purpose of this section is to prove Theorem 2. The immediate application to the proof of Theorem 3 is also presented.

The point of departure is the characterization of extremal mappings in terms of their Euler-Lagrange equations (see [Lm1] or [P]), which leads to the study of their corresponding meromorphic disks attached to a totally real submanifold. Since we are only interested in the extremes near a boundary point, the poles of the meromorphic disks can be easily controlled. Using the technique of Riemann-Hilbert problems, we then obtain a family of non-linear (but compact) operators, whose fixed points are exactly the boundary values of our meromorphic disks. Finally, a careful analysis of those operators completes the proof of Theorem 2.

Before proceeding, we recall that an extremal mapping ϕ of D is a holomorphic map from the unit disk Δ to D so that for any $\psi \in \mathrm{Hol}(\Delta, D)$ with $\psi(0) = \phi(0)$ and $\psi'(0) = \lambda\phi'(0)$ (where, as usual, λ denotes a real number), it holds that $|\lambda| \leq 1$. A holomorphic mapping from Δ to D is called a complex geodesic in the sense of Vesentini if it realizes the Kobayashi distance between any two points on its image (see [Ve]). For a bounded convex domain, extremal mappings coincide with complex geodesics by a result of Lempert ([Lm1]).

PROOF OF THEOREM 2. We let $D \subset\subset \mathbb{C}^{n+1}$ and $p \in \partial D$ a C^3 strongly pseudoconvex point. We then need to show that for any extremal mapping ϕ of D, when $\phi(\Delta)$ is close enough to p, it holds that $|(\phi'(0))_N| = O(\eta(\phi))|(\phi'(0))_T|$. For this purpose, we start by constructing a C^3 strongly convex domain $\Omega \subset D$ with $\partial\Omega \cap \partial D$ being a piece of hypersurface near p. More precisely, here we should say that Ω is the biholomorphic image of a C^3 strongly convex domain. However, we will not make this distinction in what follows; for all objects involved in this paper are bihilomorphically invariant. Let us assume that $\phi(\Delta) \subset \Omega$. It then follows from the monotonicity of the Kobayashi metric that ϕ is also an extremal mapping of Ω (thus a complex geodesics of Ω). Now we recall a result of Lempert [Lm1], which asserts that ϕ is proper and has a $C^{1+\alpha}$ ($\alpha \in (0,1)$) smooth extension up to $\partial\Delta$. Write $V(q)$ for the unit outward normal vector of Ω at q. The key fact (see [Lm1] of [P]) for our later discussion is that ϕ satisfies the Euler-Lagrange equation in the sense that there exists a $C^{1+\alpha}$ positive function P on $\partial\Delta$ so that $\widetilde{\phi(\xi)} = P\xi\overline{V(\phi(\xi))}$, initially defined on $\partial\Delta$, can be holomorphically extended to Δ (this $\tilde{\phi}$ is called the dual mapping of ϕ).

Since extremal maps are preserved under holomorphic changes of variables, we can assume, without loss of generality, that $p = 0$ and Ω is locally defined by an equation of the form: $\rho(z) = \overline{z}_{n+1} + z_{n+1} + h(z, \overline{z})$ with $h(z, \overline{z}) = \sum_{j=1}^{n} |z_j|^2 + o(|z|^2)$. Moreover, a simple application of the implicit function theorem tells that we can make $h(z, \overline{z})$ depending only on $z' = (z_1, \cdots, z_n)$ and $y_{n+1} = \mathrm{Im}\, z_{n+1}$.

Write $V = (v_1, \ldots, v_{n+1})$ and define

$$W = \left\{ w = (z, \omega) \in \mathbb{C}^{2n+1} : z \in \partial\Omega,\ \ z \approx 0,\ \text{ and } \right.$$
$$\left. \omega = \left(\frac{\overline{v_1(z)}}{\overline{v_{n+1}(z)}}, \ldots, \frac{\overline{v_n(z)}}{\overline{v_{n+1}(z)}} \right) \right\}.$$

Then, by an easy calculation, it can be seen that W is defined near 0 by an equation of the form: $w = (z', iy_{n+1}, \overline{z'}) + O(|z|^2)$. Thus it follows that W is totally real near 0 (this is called the Webster lemma). In fact, the real tangent space of W at 0 is spanned by $\{T_{1,r}, \ldots, T_{n,r}, T_{n+1}, T_{1,i}, \ldots, T_{n,i}\}$, where, for $j \leq n$,

$$T_{j,r} = (0, \ldots, 1, \cdots, 1, \ldots, 0),$$
$$T_{j,i} = (0, \ldots, i, \cdots, -i, \ldots, 0),$$

and

$$T_{n+1} = (0, \ldots, i, \ldots, 0).$$

Write

$$A_0 = \begin{pmatrix} T_{1,r} \\ T_{2,r} \\ \vdots \\ T_{n+1} \\ \vdots \\ T_{n-1,i} \\ T_{n,i} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \ldots & 0 & 1 \\ 0 & 1 & \ldots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ldots & \ldots & i & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & i & \ldots & -i & 0 \\ i & 0 & \ldots & 0 & -i \end{pmatrix}.$$

and let $W^* = WA_0^{-1} = \{wA_0^{-1} : w \in W\}$. Then we have that $T_0W^* = \mathbb{R}^{2n+1} \subset \mathbb{C}^{2n+1}$. From the implicit function theorem, W^* can thus be defined by an equation: $Y = H(X)$ with $X + iY \in \mathbb{C}^{2n+1}$ and $H(0) = \mathrm{d}H(0) = 0$.

We now return to the extremal mapping ϕ (of D and Ω). Assume that $\phi(\Delta)$ is close enough to 0 so that $\Phi(\xi) = (\phi(\xi), \phi^*(\xi))$, defined by

$$\left(\phi(\xi), \frac{\overline{v_1(\phi(\xi))}}{\overline{v_{n+1}(\phi(\xi))}}, \ldots, \frac{\overline{v_n(\phi(\xi))}}{\overline{v_{n+1}(\phi(\xi))}}\right),$$

stays on W for $\xi \in \partial\Delta$. Write $\Phi^*(\xi) = \Phi(\xi)A_0^{-1}$. Then we have that $\Phi^*(\partial\Delta) \subset W^*$.

LEMMA 1. *There exists a* $\sigma \in \mathrm{Aut}(\Delta)$ *so that* $\Phi^* \circ \sigma$ *has a holomorphic extension to* $\Delta \backslash \{0\}$. *Furthermore,* $0 \in \Delta$ *is a simple pole of* $\Phi^* \circ \sigma$.

PROOF OF LEMMA 1. Write $\tilde{\phi}$, the dual mapping of ϕ, as $(\widetilde{\phi_1}, \ldots, \widetilde{\phi_{n+1}})$. We then see that $\widetilde{\phi_{n+1}}(\xi) = \xi P(\xi)\overline{v_{n+1}(\phi(\xi))}$ for $\xi \in \partial D$ and some positive function P. Since $v_{n+1}(\phi(\xi)) \approx 1$, we can conclude that the winding number of $\widetilde{\phi_{n+1}}$ is 1. So it just has a simple zero on Δ, say a. Take $\sigma \in \mathrm{Aut}(\Delta)$ with $\sigma(0) = a$. Then $\widetilde{\phi_{n+1}} \circ \sigma$, has a simple zero at $0 \in \Delta$. Thus $\Phi \circ \sigma$ can be extended to Δ as

$$\left(\phi \circ \sigma, \frac{\widetilde{\phi_1} \circ \sigma}{\widetilde{\phi_{n+1}} \circ \sigma}, \ldots, \frac{\widetilde{\phi_n} \circ \sigma}{\widetilde{\phi_{n+1}} \circ \sigma}\right),$$

which is obviously meromorphic on Δ with a simple pole at 0. Since Φ^* differs from Φ only by a linear transformation, we see that the proof of Lemma 1 is complete. □

For simplicity, let us still write $\Phi^*(\xi) = X(\xi) + iY(\xi)$ for $\Phi^* \circ \sigma$ in what follows. Note that $\Phi^*(\partial\Delta) \subset M^*$. It follows that $Y(\xi) = H(X(\xi))$ $(\xi \in \partial\Delta)$. Let $\xi = e^{i\theta}$ and take the derivative with respect to θ. We then see that $\frac{dY}{d\theta} = \frac{dX}{d\theta}\frac{\partial H}{\partial X}$, where $\frac{\partial H}{\partial X}$ is the Jacobian of H. So,

$$\frac{d\Phi^*}{d\theta} = \frac{dX}{d\theta} + i\frac{dY}{d\theta} = \frac{dX}{d\theta}\left(I_{2n+1} + i\frac{\partial H}{\partial X}\right),$$

or

$$\text{(1.0)} \qquad \operatorname{Im}\left(\frac{dX}{d\theta} + i\frac{dY}{d\theta}\right)\left(I_{2n+1} + i\frac{\partial H}{\partial X}\right)^{-1} = \operatorname{Im}\frac{dX}{d\theta} = 0.$$

Here I_{2n+1} denotes the identical $(2n+1)\times(2n+1)$ matrix and $\|g\| = \max_{\xi\in\partial\Delta}|g(\xi)|$ for each function g in the Banach space $L^\infty(\partial\Delta)$. An easy fact is that $\|X(e^{i\theta})\| \ll 1$ when $\eta(\phi) \approx 0$.

Consider the Riemann-Hilbert problem

$$\text{(1.1)} \qquad \operatorname{Im}\left(Q(X,\xi)\left(I_{2n+1} + i\frac{\partial H}{\partial X}\right)^{-1}\right) = 0, \quad \xi \in \partial\Delta,$$

with $Q(x,\xi)$ holomorphic on $\xi \in \Delta$, L^2 integrable on $\partial\Delta$, and $\operatorname{Re}(Q(X,0)) = I_{2n+1}$.

LEMMA 2. *Then $\|X\| \ll 1$, then* (1.1) *has a unique solution Q. Moreover, $Q^{-1}(X,\xi)$ exists and $\|Q(X,e^{i\theta}) - I_{2n+1}\|_2$, $\|Q^{-1}(X,e^{i\theta}) - I_{2n+1}\|_2 = O(\|X\|)$. Here, we write $\|\circ\|_2$ for the L^2 norm of the Hilbert space $L^2(\partial\Delta)$.*

PROOF OF LEMMA 2. Write $\left(I_{2n+1} + i\frac{\partial H(X)}{\partial X}\right)^{-1} = e_1 + ie_2$ and $Q(X,\xi) = q_1(X,\xi) + iq_2(X,\xi)$. Then we see that $q_1(X,0) = I_{2n+1}$, $\|e_2(X,e^{i\theta})\|_2 = O(\|X\|)$, and (1.1) is equivalent to

$$\text{(1.2)} \qquad q_1e_2 + q_2e_1 = 0.$$

Since $q_1 = -S(q_2) + I_{2n+1}$, where S is the standard Hilbert transform on $\partial\Delta$, (1.2) can therefore be written as

$$-S(q_2)e_2e_1^{-1} + q_2 = -e_2e_1^{-1}.$$

So, when $\|X\| \ll 1$, it follows that $q_2 = (-S(\circ)\times(-e_2e_1^{-1}) + I_{2n+1})^{-1}(-e_2e_1^{-1})$ and

$$\|q_2\|_2 \le \frac{1}{(1 - \|e_2e_1^{-1}\|_2)}\|e_2e_1^{-1}\|_2 = O(\|X\|).$$

Thus Q is uniquely determined and $\|Q(X,\xi) - I_{2n+1}\|_2 \le \|q_2\|_2 + \|S(q_2)\|_2 = 2\|q_2\|_2 = O(\|X\|)$.

We now consider the following equation with respect to Q^*:

$$\mathrm{Im}\left(\left(I_{2n+1}+i\,\frac{\partial H(X)}{\partial X}\right)Q^*(X,\xi)\right)=0,\quad \text{with } \mathrm{Re}(Q^*(X,0))=I_{2n+1}.$$

Similarly, we can obtain a unique solution with $\|Q^*(X,\xi)-I_{2n+1}\|_2=\mathrm{O}(\|X\|)$. Since the holomorphic matrix $Q\times Q^*$ has real value on $\partial\Delta$, it thus follows from the Schwarz reflection principle that $Q(X,\xi)\times Q^*(X,\xi)=C(X)$, some real constant matrix. Here, we remark that, to apply the Schwarz reflection principle, we need obtain $Q(X,\xi)Q^*(X,\xi)\in L^l(\partial\Delta)$ for some $l>1$. But this cas be easily seen by solving the equation (1.1) in the space $L^l(\partial\Delta)$ with $l\gg 1$. We now notice that $|Q(X,0)-I_{2n+1}|\le \frac{1}{2\pi}\int_{\partial\Delta}\left|\frac{Q(X,\xi)-I_{2n+1}}{\xi}\,d\xi\right|=\mathrm{O}(\|X\|)$ and $|Q^*(X,0)-I_{2n+1}|=\mathrm{O}(\|X\|)$ as $\|X\|\to 0$ (by the Hölder inequality). We see, especially, that $C(X)=Q(X,0)Q^*(X,0)=I_{2n+1}+\mathrm{O}(\|X\|)$ as $\|X\|\to 0$. Hence, $C(X)$ is invertible in case $\|X\|\ll 1$. This completes the proof of Lemma 2; for $Q^{-1}(X,\xi)=C^{-1}(X)Q^*(X,\xi)$. □

Now, by making use of Lemma 2, (1.0) becomes

$$\mathrm{Im}\left(\frac{dX}{d\theta}+\frac{dY}{d\theta}\right)Q^{-1}(X,\xi)=0,\quad \text{for } \xi\in\partial\Delta,$$

i.e,

$$\mathrm{Re}\left(\xi\,\frac{d\Phi^*}{d\xi}\,Q^{-1}(X,\xi)\right)=0.$$

Note that $\xi\,\frac{d\Phi^*}{d\xi}\,Q^{-1}(X,\xi)$ is holomorphic on $\Delta\backslash\{0\}$ and has at most a simple pole at 0. We can conclude that

$$\xi\,\frac{d\Phi^*}{d\xi}\,Q^{-1}(X,\xi)=\frac{\alpha}{\xi}-\overline{\alpha}\xi+i\beta,$$

where α is a constant complex vector and β is a constant real vector (depending only on X). In fact, since $\Phi(\xi)=\Phi^*\times A_0=\left(\phi,\frac{\phi^{**}}{\xi}\right)$ with $\phi^{**}=\xi\phi^*$ holomorphic on Δ by Lemma 1, it follows that:

$$\alpha=\lim_{\xi\to 0}\xi^2\,\frac{d\Phi^*}{d\xi}\,Q^{-1}(X,\xi)=(0,-\phi^{**}(0))A_0^{-1}Q^{-1}(X,0). \tag{1.3}$$

Write $R(X,\xi)=Q(X,\xi)\left(I_{2n+1}+i\,\frac{\partial H(X)}{\partial X}\right)^{-1}$ for $\xi\in\partial\Delta$ (we note that R is real). By Lemma 2, it then holds that $\|R-I_{2n+1}\|_2=\mathrm{O}(\|X\|)$. Therefore, the Hölder inequality implies that $\int_0^{2\pi}R(X,\xi)d\theta=2\pi I_{2n+1}+\mathrm{O}(\|X\|)$ is invertible when $\|X\| \ll 1$. On the other hand, we have

$$\frac{dX}{d\theta} = \frac{d\Phi^*}{d\theta}\left(I_{2n+1} + i\,\frac{\partial H(X)}{\partial X}\right)^{-1} = i\xi\,\frac{d\Phi^*}{d\xi}\left(I_{2n+1} + i\,\frac{\partial H(X)}{\partial X}\right)^{-1}$$
$$= i\left(\frac{\alpha}{\xi} - \overline{\alpha}\xi + i\beta\right) Q(X,\xi)\left(I_{2n+1} + i\,\frac{\partial H(X)}{\partial X}\right)^{-1}$$
$$= i\left(\frac{\alpha}{\xi} - \overline{\alpha}\xi + i\beta\right) R(X,\xi).$$

Integrating both sides with respect to θ, we obtain

$$0 = \int_0^{2\pi}\left(\frac{\alpha}{\xi} - \overline{\alpha}\xi + i\beta\right) R(X,\xi)\,d\theta.$$

Thus,

$$\beta = i\left(\int_0^{2\pi}\left(\frac{\alpha}{\xi} - \overline{\alpha}\xi\right) R(X,\xi)\,d\theta\right)\left(\int_0^{2\pi} R(X,\xi)\,d\theta\right)^{-1}. \tag{1.4}$$

Here, as usual, we identify $\xi \in \partial\Delta$ with $e^{i\theta}$. Especially, we easily see that α, $\beta = O(\|X\|)$; for by the Hölder inequality, it holds that $\phi^{**}(0) = O(\|X\|)$.

Consider now the following differential equation with parameters $\gamma \in \mathbb{C}^n$ and $X_0 \in \mathbb{R}^{2n+1}$:

$$\frac{dX(\xi,\gamma,X_0)}{d\theta} = i\left(\frac{\alpha(X,\gamma)}{\xi} - \overline{\alpha(X,\gamma)}\xi + i\beta(X,\gamma)\right) R(X,\xi), \quad \text{with } X(1) = X_0, \tag{1.5}$$

or

$$X(\xi,\gamma,X_0) = i\int_0^{\theta}\left(\frac{\alpha(X,\gamma)}{\xi} - \overline{\alpha(X,\gamma)}\xi + i\beta(X,\gamma)\right) R(X,\xi)\,d\theta + X_0, \tag{1.5$'$}$$

where $\xi = e^{i\theta}$,

$$\alpha(X,\gamma) = (0,\gamma)A_0^{-1}Q^{-1}(X,0), \tag{1.6}$$

and $\beta(X,\gamma)$ is given by (1.4).

LEMMA 3. *For any extremal mapping ϕ with $\phi(\Delta) \approx 0$, there correspond an automorphism σ of Δ, a $\gamma \approx 0$, and an $X_0 \approx 0$ so that the previously* *defined X is a solution of* (1.5). *Conversely, for any γ, $X_0 \approx 0$,* (1.5) *can be uniquely solved, and each of its solutions gives an extremal mapping ϕ of Ω with $\phi(\Delta) \approx 0$ and the last component of its dual mapping having a simple pole at* 0. *Moreover, the solutions of* (1.5) *are uniformly Hölder-$\frac{1}{2}$ continuous with respect to the parameters α and γ. In fact, denoting by $\|\circ\|_{\frac{1}{2}}$ the Hölder-$\frac{1}{2}$ norm in the Banach space $C^{\frac{1}{2}}(\partial\Delta)$, defined by*

$$\|g\|_{\frac{1}{2}} = \|g\| + \sup_{\xi_1,\xi_2} \frac{|g(\xi_1) - g(\xi_2)|}{|\xi_1 - \xi_2|^{\frac{1}{2}}}, \quad \textit{with } g \in C^{\frac{1}{2}}(\partial\Delta),$$

then for each solution X of (1.5), *we have $\|X\|_{\frac{1}{2}} = O(\|X\|)$.*

PROOF OF LEMMA 3. The first part of the lemma follows from the above arguments.

We now present the proof of the last part of the lemma. To this aim, let $X(\xi,\gamma,X_0)$ be a solution of (1.5) with $\|X\| \ll 1$ and let $\Phi^*(\xi,\gamma,X_0) = X(\xi,\gamma,X_0) + iH(X(\xi,\gamma,X_0))$. Then we know from (1.5) that

$$\frac{d\Phi^*(\xi,\gamma,X_0)}{d\xi} = \left(\frac{\alpha(X,\gamma)}{\xi^2} - \overline{\alpha(X,\gamma)} + i\beta(X,\gamma)\frac{1}{3}\right) Q(X,\xi).$$

So (1.3) still holds. Since Φ^* must have a meromorphic extension to 4 (with at most a simple pole at the origin), using the Cauchy formula and the Hölder inequality, we know that α and β are also of $O(\|X\|)$ by (1.3) and (1.4). Now we note that $\|R_2\| = O(1)$ and

$$|X(e^{i\theta_1},\gamma,X_0) - X(e^{i\theta_2},\gamma,X_0)| \le \left| \int_{\theta_1}^{\theta_2} \left(\frac{\alpha(\gamma)}{\xi}| - \overline{\alpha(\gamma)}\xi + i\beta(\gamma)\right) R(X,\xi)d\theta \right|$$

$$\le C(|\alpha| + |\beta|) \int_{\theta_1}^{\theta_2} |R(X,\xi)|d\theta \le C(|\alpha| + |\beta|)\|R\|_2\|\theta_1 - \theta_2\|^{1/2}$$

$$= O(\|X\|)\|\theta_1 - \theta_2\|^{1/2}.$$

It therefore follows that

$$\sup_{\xi_1,\xi_2} \frac{|X(\xi_1,\gamma,X_0) - X(\xi_2,\gamma,X_0)|}{|\xi_1 - \xi_2|^{1/2}} = O(\|X\|).$$

Thus, the Hölder-$\frac{1}{2}$ norm of X

$$\|X\|_{\frac{1}{2}} = \|X\| + \sup_{\xi_1,\xi_2} \frac{|X(\xi_1,\gamma,X_0) - X(\xi_2,\gamma,X_0)|}{|\xi_1 - \xi_2|^{1/2}},$$

is bounded by $C\|X\|$ with some constant C independent of γ and X_0.

It still remains to prove the existence of the solutions of (1.5) and study their behavior. For this purpose, we first notice that, by making use of the just obtained result and by solving (1.1) in the Hölder-$\frac{1}{2}$ space $C^{\frac{1}{2}}$, we see that the holomorphic matrix $Q(X, \xi)$ is also uniformly Hölder-$\frac{1}{2}$ continuous up to the boundary. Moreover it can be similarly seen that $\|Q - I_{2n+1}\|_{\frac{1}{2}}$ and thus $\|R(X, \xi) - I_{2n+1}\|_{\frac{1}{2}} = O(\|X\|)$. Now consider the operator

$$F : C^{\frac{1}{2}}(\partial\Delta) \times \mathbb{C}^n \times \mathbb{R}^{2n+1} \to C^{\frac{1}{2}}(\partial\Delta);$$

$$F(X, \gamma, X_0) = i \int_0^\theta \left(\frac{\alpha(X, \gamma)}{\xi} - \overline{\alpha(X, \gamma)}\xi + i\beta(X, \gamma) \right) R(X, \xi) d\theta + X_0.$$

From the above discussions, it follows that in case γ, $\|X\|_{\frac{1}{2}}$, and $X_0 \approx 0$, we then have $d_X F(0) \approx 0$. Hence, by the implicit function theorem in the Banach space, (1.5) and thus (1.5)$'$ can be uniquely solved for small γ and X_0. Now, for each solution $X(\xi, \gamma, X_0)$, let $\Psi^*(\xi) = X(\xi, \gamma, X_0) + iH(X(\xi, \gamma, X_0))$. Then

$$\frac{d\Psi^*}{d\xi} = \left(\frac{\alpha}{\xi^2} - \overline{\alpha} + \frac{i\beta}{\xi} \right) Q(X, \xi).$$

Denote by $(\psi, \psi^*) = \Psi^* A_0$, where ψ maps $\partial\Delta$ to $\mathbb{C}^{n+1}$. It follows easily that

$$(1.6)' \qquad \psi'_\xi = \left(\frac{\alpha(X, \gamma)}{\xi^2} - \overline{\alpha(X, \gamma)} + i\beta(X, \gamma)\frac{1}{\xi} \right) Q(X, \xi) B.$$

Here we write B for the $(2n + 1) \times (n + 1)$ matrix, formed by the first $(n + 1)$ columns of A_0. Noting that $\alpha(X, \gamma) Q(X, 0) B = (0, \gamma) A_0^{-1} B = 0$, we see that 0 can be at most a simple pole of ψ'. Since ψ is well-defined on $\partial\Delta$, we can conclude that ψ has a holomorphic extension to Δ. Meanwhile, it can be verified that $\psi(\partial\Delta) \subset \partial D$ and ψ satisfies the Euler-Lagrange equation. We thus conclude that ψ is an extremal map of Ω (and of D, in fact) ([Lm1], [Hu1]) with the property described in the lemma. The proof of Lemma 3 is complete. □

We now are in a position to finish the proof of Theorem 2. For the sake of brevity, we retain the above notation and assume that σ in Lemma 1 is the identity.

Let ϕ be an extremal map of D with $\phi(\Delta)$ close to 0. First, we notice that both sides in (1.6)$'$, with ψ being replaced by ϕ, are holomorphic on $\Delta - \{0\}$. We therefore have

$$\phi'(\xi) = \left(\frac{\alpha(\gamma)}{\xi^2} - \overline{\alpha(\gamma)} + i\beta(\gamma)\frac{1}{\xi} \right) Q(X, \xi) B,$$

for $\xi \in \Delta - \{0\}$. Writing $Q_1(X,\xi) = Q(X,\xi) - Q(X,0)$ and $Q_2(X,\xi) = Q_1(X,\xi) - Q'_\xi(X,0)\xi$, we then obtain

$$\phi'_\xi = \left(\frac{\alpha Q_2(X,\xi)}{\xi^2} - \overline{\alpha(\gamma)}Q(X,\xi) + i\beta Q_1(X,\xi)\frac{1}{\xi}\right)B; \tag{1.7}$$

for ϕ is holomorphic on Δ.

LEMMA 4. $\dfrac{Q_1(X,\xi)}{\xi}$ *and* $\dfrac{Q_2(X,\xi)}{\xi^2} = O(\|X\|)$ *as* $\|X\| \to 0$.

PROOF OF LEMMA 4. From the definition, we see that $\dfrac{Q_1(X,\xi)}{\xi}$ and $\dfrac{Q_2(X,\xi)}{\xi^2}$ are holomorphic on Δ. So, by the maximal principle, we have only to show that they converge uniformly to the 0-matrix with the rate of $\|X\|$, when $\xi \in \partial\Delta$ and $\|X\| \to 0$. But this follows obviously from the facts that $Q(X,\xi) = I_{2n+1} + O(\|X\|)$ and $Q'_\xi(X,0) = O(\|X\|)$ (by the Cauchy formula and Hölder inequality). □

Note that $\alpha = (0,\gamma)A_0^{-1}Q(X,0) = (0,\gamma)A_0^{-1} + O(|\gamma|\|X\|)$ and $\beta = O(|\gamma|)$ by (1.4). It can be verified that (1.7) may be written as

$$\frac{1}{|\gamma|}\phi'(\xi) = -\overline{\frac{(0,\gamma)}{|\gamma|}A_0^{-1}} \times B + o(\|X\|),$$

as $\|X\| \to 0$. Now a direct computation shows that

$$A_0^{-1} = 1/2\begin{pmatrix} 1 & 0 & \dots & 0 & -i \\ 0 & 1 & \dots & -i & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & -2i & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & i & 0 \\ 1 & 0 & \dots & 0 & i \end{pmatrix}.$$

So, writing $\overline{\dfrac{\gamma}{|\gamma|}} = (a_1,\dots,a_n)$, we then have

$$\frac{1}{|\gamma|}\phi'(\xi) = -\overline{\frac{(0,\gamma)}{|\gamma|}A_0^{-1}} \times B + O(\|X\|) = -(a_n, a_{n-1},\dots,a_1,0) + O(\|X\|),$$

when $\|X\| \to 0$.

Hence, we obtain

$$\frac{|\phi'_{n+1}(\xi)|}{|\phi'(\xi)|} = O(\|X\|),$$

and

$$\frac{|(\phi_1'(\xi),\dots,\phi_n'(\xi))|}{|\phi'(\xi)|} = 1 + O(\|X\|).$$

Since

$$\frac{|(\phi'(\xi))_N|}{|(\phi'(\xi))|} = \frac{|\phi_{n+1}'(\xi)|}{|\phi'(\xi)|} + O(\|X\|)$$

and

$$\frac{|(\phi'(\xi))_T|}{|\phi'(\xi)|} = \frac{|(\phi_1'(\xi),\dots,\phi_n'(\xi))|}{|\phi'(\xi)|} + O(\|X\|) = 1 + O(\|X\|),$$

we finally conclude that

$$|(\phi'(\xi))_N| = O(\|X\|)|(\phi'(\xi))_T|, \quad \text{for } \xi \in \overline{\Delta},$$

as $\|X\| \to 0$. This completes the proof of Theorem 2; for $\|X\| \approx \eta(\phi)$. □

We conclude this section by proving Theorem 3.

We first fix some notation. For a bounded domain D in $\mathbb{C}^n$, we will use $F(D)$ to denote the collection of all 'normalized' extremal mappings of D in the sense that an extremal map $\phi \in F(D)$ if and only if $\delta(\phi(0)) = \max_{\xi\in\Delta} \delta(\phi(\xi))$. Here $\delta(z)$ stands for the distance between z and ∂D.

PROOF OF THEOREM 3. Let $D \subset\subset \mathbb{C}^n$ be a C^3 strongly convex domain and $p \in \partial D$. For any complex vector v, which is not contained in $T_p^{(1,0)}\partial D$, we then need to find an extremal mapping of D so that $\phi(1) = p$ and $\phi'(1)$ is different from v by a complex number. To this aim, we choose a sequence $\{z_j\} \subset D$ converging to p and choose a sequence of normalized extremal mappings $\{\phi_j\} \subset F(D)$ so that for each j, it holds that $\phi_j(\tau_j) = z_j$ with some $\tau_j \in (0,1)$ and $\phi_j'(\tau_j) = \lambda_j v$ with $\lambda_j \in \mathbb{C}$. Since v is independent of j and is not contained in the complex tangent space of ∂D at p, it follows from Corollary 2, that $\inf_j \phi_j(\Delta) > 0$. In light of Proposition 1 of [Hu1], we therefore see that there is a subsequence of $\{\phi_j\}$, which converges to an extremal mapping ϕ in the topology of $C^1(\overline{\Delta})$. Noting that $\tau_j \to 1$, we can thus conclude that $\phi'(1) = \lambda v$ for some $\lambda \in \mathbb{C}$. The proof is complete. □

2. - Regularity of holomorphic retracts and iterates of holomorphic mappings

In this section, we will focus on the proof of Theorem 1. We will make decisive use of Theorem 2. The key step, as mentioned in Section 0, is to prove the following fixed point theorem:

THEOREM 4. *Let $D \subset\subset \mathbb{C}^n$ be a contractible strongly pseudoconvex domain with C^3 boundary and let M be a holomorphic retract of D. Suppose that*

$f \in \mathrm{Hol}(M, M)$ is elliptic, i.e, no subsequence of $\{f^k\}$ diverges to the boundary of M. Then f has a fixed point in M.

The main idea of the proof of this theorem is to obtain certain regularity results concerning holomorphic retracts so that the Lefschetz fixed point theorem can be applied. The argument will be carried out through several propositions, which are of interest in their own right.

We first recall that a subset M of a bounded domain D is called a holomorphic retract if there is a holomorphic self mapping h of D so that $h^2 = h$ and $h(D) = M$. An obvious fact is that the Kobayashi metric and the Kobayashi distance of M are the same as those inherited from D. Another useful result regarding holomorphic retracts is a theorem of Rossi (see [Ab2] for example), which states that all holomorphic restract of D are closed complex sub-manifolds of D. In what follows, we will also use the notation C^{k-} to denote the function spece $\bigcap_{\alpha<1} C^{k-1+\alpha}$ in case k is an integer, and the space C^k otherwise.

We now start with Proposition 1, which will play a crucial role in the whole discussion.

PROPOSITION 1. *Let $D \subset\subset \mathbb{C}^n$ be either a smooth pseudoconvex domain or a taut domain with a Stein neighborhood basis. Let $p \in \partial D$ be a strongly pseudoconvex point with at least C^3 smoothness. Suppose that $M \subset D$ is a holomorphic retract with complex dimension greater than* 1 *and suppose that $p \in \partial M$. Then for any neighborhood U of p, there is a C^{2-} complex geodesic ϕ of D with $\phi(\Delta) \subset U \cap M$ and $\phi(1) = p$.*

PROOF OF PROPOSITION 1. Choose a sequence $\{z_j\} \subset M$ converging to p and define

$$t_j = \sup_{v \in T^{(1,0)}_{z_j} M} \frac{|v_N|}{|v_T|}.$$

Then we first claim that $\inf_j(t_j) > 0$, i.e, M intersects ∂D transversally at p. If that is not the case, we may just assume that $t_j \to 0$. Then, we let

$$M_j = \cup\{\phi(\Delta) : \phi \text{ is extremal with respect to } D,\ \phi(0) = z_j,$$
$$\text{and } \phi'(0) \in T^{(1,0)}_{z_j} M\}.$$

We first note that M_j is a non-empty set by the tautness of D. In light of a preservation principle of [Hu1] (see Theorem 1 of [Hu1]), we see that, for every C^3 strongly convex domain $\Omega \subset D$ with $\partial\Omega \cap \partial D$ being a piece of hypersurface near p, when $j \gg 1$, it holds that $t_j \ll 1$ and each ϕ in the definition of M_j stays in Ω. Thus $M_j \subset \Omega$. Therefore each ϕ, described in the definition of M_j, is also an extremal mapping of Ω. Now, we notice the tautness of M and the uniqueness property of extremal mappings in Ω. We see, by the fact that each extremal map of M is also extremal with respect to D, that M_j is also a subset of M. We now need use Lempert's spherical representation $\Psi_j(z_j, \circ)$

of Ω with the base point z_j, i.e, we define the map $\Psi_j(z_j, \circ)$ from the closed unit ball $\overline{\mathbb{B}_n}$ to $\overline{\Omega}$ by $\Psi_j(z_j, 0) = z_j$ and $\Psi_j(z_j, b) = \psi_{z_j}(b, |b|)$ for each $b \in \mathbb{B}^n$. Here $\psi_{z_j}(b, \xi)$ stands for the unique extremal mapping of Ω with $\psi_{z_j}(b, 0) = z_j$ and $\psi'_{z_j}(b, 0) = \lambda b$ for some positive λ. Writing $E = \{v \in T^{(1,0)}_{z_j}M : |v| \leq 1\}$, we then get $M_j = \Psi_j(E)$. Notice that Ψ_j is a homeomorphism (in fact, it is a $C^{1+\alpha}$ differeomorphism on $\mathbb{B}^n - \{0\}$, as showed in [Lm2]) and notice that E is a closed submanifold of $\overline{\mathbb{B}_n}$ with real dimension equal to $2\dim_{\mathbb{C}} M$. We therefore see that M_j is a closed open subset of M. From the connectedness of M (since all domains in this paper are assumed to be connected), it hence follows that $M = M_j$. That is a contradiction; for Ω can be made arbitrarily small.

So, there is an $\epsilon_0 > 0$ such that $t_j > \epsilon_0$ for every $j \gg 1$. Pick up two independent unit vectors v_1 and v_2 in the complex tangent space of M at z_j. By the above claim and a simple linear combination, we may assume that $(v_1)_N = 0$ and $|(v_2)_N| > \epsilon_0 |(v_2)_T|$. Let $v(t) = \dfrac{v_1 + tv_2}{|v_1 + tv_2|}$. Then it is easy to see that

$$\frac{|(v(t))_N|}{|(v(t))_T|} = \frac{|t(v_2)_N|}{|(v_1)_T + t(v_2)_T|},$$

can be made to be any number between 0 and ϵ_0 if varying t.

To finish the proof of the proposition, we let U be a small neighborhood of p and construct a C^3 strongly convex domain $\Omega \subset D \cap U$ with $\partial\Omega \cap \partial D$ being a piece of hypersurfaces near p. Again, by making use of Theorem 1 of [Hu1], for $j \gg 1$ and some $\epsilon \ll 1$, we can find a complex geodesic ϕ_j of D with $\phi(0) = z_j$, $\phi'(0) \in T^{(1,0)}_{z_j}M$, $\phi_j(\Delta) \subset \Omega$, and $|(\phi'_j(0))_T| = \epsilon|(\phi'_j(0))_N|$. As argued in Theorem 3, since ϵ is independent of j, after a normalization, Theorem 2 indicates that a subsequence of $\{\phi_j\}$ will converge to a complex geodesic ϕ of D (and also Ω) in the topology of $C^1(\overline{\Delta})$. Noting that $\phi_j(\Delta) \subset M$ for each j, we thus conclude that $\phi(\Delta) \subset M \cap \Omega$ and $\phi(1) = p$. Finally, the regularity of ϕ follows from the reflection principle [Lm1]. □

We now turn to the regularity result for holomorphic retracts.

PROPOSITION 2. *Let $D \subset\subset \mathbb{C}^n$ be either a smooth pseudoconvex domain or a taut domain with a Stein neighborhood basis. Suppose that $p \in \partial D$ is a strongly pseudoconvex point with C^k smoothness $(k \geq 3)$ and suppose that M is a holomorphic retract of D with complex dimension greater than 1. If $p \in \overline{M}$, then $\overline{M}$ is a complex submanifold with a C^{k-1-} smooth boundary near p.*

PROOF OF PROPOSITION 2. As we did before, we first construct a small C^k strongly convex domain Ω with $\partial D \cap \partial\Omega$ being an open subset of $\partial\Omega$ near p. By Proposition 1, we have a complex geodesic ϕ of D, M, and Ω, staying close to p, and with $\phi(1) = p$. Let $z = \phi(0)$ and $v_0 = \dfrac{\phi'(0)}{|\phi'(0)|}$. By Theorem 2, it holds that $|(\phi'(0))_N| \ll |(\phi'(0))_T|$. Hence, from Theorem 1 of [Hu1], it follows that all extremal mappings of D starting from z and with the initial velocity close to v_0 should also stay in Ω. To be more precise, by shrinking ϕ if necessary, there exists a small $\epsilon > 0$ so that, for each extremal mapping ψ with $\psi(0) = z$ and $\left|\frac{\psi'(0)}{|\psi'(0)|} - v_0\right| < \epsilon$, then $\psi(\Delta) \subset \Omega$. Write $E^* = \left\{v \in \overline{\mathbb{B}_n} : v \in T_z^{(1,0)}M, \left|\frac{v}{|v|} - v_0\right| < \epsilon\right\}$ and still denote by $\Psi(z, \circ)$ the spherical representation of Ω with the base point z. Since E^* is a submanifold of $\overline{\mathbb{B}^n}$ with smooth boundary near v_0, hence, by a theorem of Lempert, $M^* = \Psi(z, E^*)$ is a submanifold with C^{k-1-} boundary near p, whose real dimension is obviously $2\dim_{\mathbb{C}} M$. As we have argued before, by noting the fact that all extremal mappings of M are also extremal with respect to D, we can conclude that $M^* \subset \overline{M}$. Now, to complete the proof of the proposition, we need only show that for some small neighborhood U^* of p, it holds that $U^* \cap \overline{M} = U^* \cap M^*$. For this purpose, we proceed by seeking a contradiction if there is no such a U^*. Then, we can find a sequence $\{z_j\} \subset M - M^*$, which converges to p. Choose U_0, a small neighborhood of p, with $U_0 \cap M^*$ being a simply connected submanifold with smooth boundary, and choose a sequence $\{w_j\} \subset M^*$, converging to p.

From an estimate of the Kobayashi distance $K_D(\circ, \circ)$ of D (see [Ab2], for example), we know that

$$
\begin{aligned}
K_D(z_j, w_j) \leq & -\frac{1}{2} \log \delta(z_j) - \frac{1}{2} \log \delta(w_j) \\
& + \frac{1}{2} \log(|z_j - w_j| + \delta(z_j) + \delta(w_j)) + C,
\end{aligned} \tag{2.1}
$$

with C independent of j. On the other hand, since M is connected, there is a curve $\gamma(t)$ on M, connecting z_j to w_j, so that

$$
K_D(z_j, w_j) = K_M(z_j, w_j) \geq \int_0^1 \kappa_D(\gamma(t), \gamma'(t))dt - 1.
$$

Here $\kappa_D(z, v)$ denotes the Kobayashi metric of D at z and in the direction v. We remark that such a curve must intersect the boundary $U_0 \cap M^*$ if we choose U_0 small enough. Let t_0 be such that $\gamma(t_0) \in \partial U_0 \cap M^*$ but $\gamma(t) \notin U_0 \cap M^*$ for $t < t_0$. Then we see that

$$
\begin{aligned}
\int_0^1 \kappa_D(\gamma(t), \gamma'(t))dt &= \int_0^{t_0} \kappa_D(\gamma(t), \gamma'(t))dt + \int_{t_0}^1 \kappa_D(\gamma(t), \gamma'(t))dt \\
&\geq K(z_j) + K(w_j),
\end{aligned} \tag{2.2}
$$

where $K(z) = \inf_{w \in \partial U_0 \cap M^*} K_D(z, w)$. Now, from the strong pseudoconvexity of D at p, it follows that (see [Ab2], for example) $K(z) \geq -\frac{1}{2} \log \delta(z) + C$. Thus, combining (2.1) with (2.2), we come up with

$$\log(|z_j - w_j| + \delta(z_j) + \delta(w_j)) \geq C.$$

Since C is independent of j and $|z_j - w_j| + \delta(z_j) + \delta(w_j) \to 0$, we obtain a contradiction. Therefore, the proof of Proposition 2 is complete. □

PROPOSITION 3. *Let $D \subset\subset \mathbb{C}^n$ be a C^k strongly pseudoconvex domain with $k \geq 3$. Suppose that M is a holomorphic retract of D with complex dimension greater than 1. Then the following holds:*

(1) *Every automorphism of M has C^{k-1-} smooth extension up to $\overline{M}$.*

(2) *Let $\{f_j\}_j$, $f \subset \mathrm{Aut}(M)$ with $\{f_j\}$ converging to f uniformly on compacta. Then it follows that $f_j \to f$ in the topology of $C^{k-1-}(\overline{M})$.*

PROOF OF PROPOSITION 3. First of all, Proposition 2 tells that M is a complex submanifold with a C^{k-1-} boundary. Thus, it makes sense to talk about the regularity (less than C^{k-1-}) extension up to the boundary for its automorphisms.

Choose $p \in \partial M$. By using Proposition 1, we can find a sequence of complex geodesics $\{\phi_j\}$ of M with $\phi_j(\Delta)$ shrinking to p as $j \to \infty$ and with $\phi_j(1) = p$. Let f be an automorphism of M. Then we claim that the diameter of $f \circ \phi_j(\Delta)$ goes to 0 as $j \to \infty$. If that is not the case, then since $\{f \circ \phi_j\}$ are also complex geodesics, we may assume, without loss of generality, that $f \circ \phi_j \in F_D$ for each j. Thus $f \circ \phi_j$'s can be easily shown to be uniformly Hölder-$\frac{1}{4}$ continuous on $\overline{\Delta}$ (see [CHL], for example). Hence, by passing to a subsequence, we may assume that $f \circ \phi_j$ converges uniformly to certain complex geodesics ϕ of D. This implies that there is a sequence $\{z_j\} \to p$ with $f(z_j) \to z \in M$, and thus contradicts the properness of f.

The rest of the argument for (1) is now similar to that in [Lm1]. For simplicity, we assume that $\phi_j(\overline{\Delta})$ converges to $q \in \partial D$. As we did before, construct two small C^k strongly convex domains Ω_1 and Ω_2 near p and q, respectively. Choose $j \gg 1$ so that ϕ_j and $f \circ \phi_j$ are, respectively, complex geodesics of Ω_1 and Ω_2. Denote by Ψ_1 the spherical representation of Ω_1 based at $z_j = \phi_j(0)$, and by Ψ_2 the spherical representation of Ω_2 based at $z_j^* = f \circ \phi_j(0)$. Then $f(z) = \Psi_2\left(z_j^*, \frac{df(\phi_j(0))\Psi_1^{-1}(z_j, z)}{\|df(\phi_j(0))\Psi_1^{-1}(z_j, z)\|}\,\|\Psi_1^{-1}(z_j, z)\|\right)$ for $z \approx p$. Since Ψ_1 and Ψ_2 give the local coordinates charts of M at p and q, respectively, we see that f has the same regularity at p as M does at p and q. Because p is arbitrary, we obtained the proof for (1).

To prove (2), we still pick up an arbitrary boundary point p of M, and write $q = f(p)$. Define similarly Ω_1, Ω_2, ϕ, Ψ_1 and Ψ_2. Using the fact that f_j converges uniformly to f on a small neighborhood of $z_0 = \phi(0)$, we know, by the preservation principle (Theorem 1 of [Hu1]), that $f_j \circ \phi$ is also a complex geodesic of Ω_2 for $j \gg 1$. Denote by $\Psi_2(z_j, \circ)$ the spherical representation of Ω_2 at $z_j = f_j(z_0)$ when $j \gg 1$. Then we see that $f_j(z) = \Psi_2\left(z_j, \frac{df_j(z_0)\Psi_1^{-1}(\phi(0), z)}{\|df_j(z_0)\Psi_1^{-1}(\phi(0), z)\|}\|\Psi_1^{-1}(\phi(0), z)\|\right)$ for z near $p \in \overline{M}$. Thus we can conclude that f_j converges to f in the topology of $C^{k-1-}(p)$; for the matrix sequence $df_j(z_0)$ converges to $df(z_0)$ and $\Psi(z_j, \circ)$ converges to $\Psi_2(z^*, \cdot)(z^* = \lim_{j\to\infty} z_j)$ in $C^{k-1-}(p)$ by the fact that $\Psi(z, w)$ depends C^{k-1-} on the base point z when $w \approx \partial\mathbb{B}_n$. Let p vary, we then complete the proof of Proposition 3. □

REMARK. In case M has the top dimension (i.e, $M = D$), (2) of Proposition 3 can also be obtained by using the asymptotic expansion of the Bergman kernel functions (see [GK]). However, we don't know whether there is a similar Bergman kernel functions argument if M is a holomorphic retract of lower dimension.

Now with all these Propositions at our disposal, the proof of Theorem 4 can be easily achieved by using an idea in [GK].

PROOF OF THEOREM 4. Since a holomorphic retract of M is also a holomorphic retract of D, by results of Bedford [Be] and Abate [Ab3] we may simply assume that $f \in \mathrm{Aut}(M)$ and $\dim_{\mathbb{C}} M > 0$. In case M is a Riemann surface, then the theorem follows easily from the Riemann mapping theorem and the classical Denjoy-Wolff theorem. So we assume that $\dim_{\mathbb{C}} M \geq 2$. Let ρ be a C^3 defining function of D. Then, when restricted to $\overline{M}$, it also gives a C^{2-} defining function of $\overline{M}$ by using the fact that M intersects ∂D transversally (see the claim in the proof of Proposition 1). Let H be the closed subgroup of $\mathrm{Aut}(M)$, generated by f. Then by the Cartan theorem and the given condition, H is a compact Lie group. It thus possesses a regular Harr measure μ. Define $\rho_f = \int_H \rho \circ g d\mu(g)$. By (2) of Proposition 3 and a lemma in [Hu2], it follows that ρ_f is also C^{2-} up to $\overline{M}$ and moreover it is easy to check that ρ_f serves a new defining function of M (an easy application of Hopf's lemma). We now let $M_\epsilon = \{z \in M : \rho_f \leq -\epsilon\}$, for $\epsilon \ll 1$. Then the Morse theory tells that M_ϵ has the same topology type as $\overline{M}$ does; for ρ_f has no critical values between $-\epsilon$ and 0 (including the end points). Since $f(M_\epsilon) \subset M_\epsilon$, we conclude, by using the hypothesis and the Lefschetz fixed point theorem, that f has a fixed point on M_ϵ, which is obviously an interior point of M. □

We now are ready to complete the proof of Theorem 1.

PROOF OF THEOREM 1. We keep the previous notation and consider the sequence $\{f^k\}$. First, by making use of results of Bedford [Be] and Abate [Ab3], we see that either $\{f^k\}$ diverges to the boundary or there is a holomorphic retract M of D so that $f|_M$ is an elliptic element of $\mathrm{Aut}(M)$. In the latter case, Theorem 4 tells that f has an interior fixed point.

So, it only remains to explain why the sequence $\{f^k\}$ converges on compacta to a boundary point in case it diverges to the boundary. This part has actually been argued in [Ma] and [Ab3]. However, for completeness, we include here a proof which is slightly different but much simpler. First, the strong pseudoconvexity of D indicates that there is no non-trivial complex sub-variety in ∂D. Hence, if a subsequence of $\{f^k\}$ converges on compacta, the limit has to be a boundary point. Pick up $z_0 \in D$, and choose, by induction, a subsequence $\{m_1 < m_2 < \cdots, m_j, \cdots\}$ so that $K_D(z_0, f^j(z_0)) \geq K_D(z_0, f^{m_1}(z_0))$ for each $j \geq 1, \ldots, K_D(z_0, f^j(z_0) \geq K_D(z_0, f^{m_l}(z_0))$ for every $j > m_{l-1}$. By passing to a subsequence, we assume that $\{f^{m_j}\}$ converges on compacta to $p \in \partial D$. We will complete the proof by showing that $\{f^k\}$ converges on compacta to p. In fact, if that is not the case, there would be a subsequence f^{k_i}, which goes to $q(\in \partial D) \neq p$. Since $f^{m_j+k_i}(z_0) = f^{m_j}(f^{k_i}(z_0)) \to p$ as $j \to \infty$), for each fixed k_i, we therefore are able to find a subsequence $\{m_{j_i}\}$ of $\{m_j\}$ so that $f^{m_{j_i}+k_i}(z_0) \to p$ as $i \to \infty$. Noting the length decreasing property of the Kobayashi distance and the way we chose $\{m_j\}$, we have

$$K_D(f^{k_i}(z_0), f^{k_i+m_{j_i}}(z_0)) \leq K_D(z_0, f^{m_{ij}}(z_0)) \leq K_D(z_0, f^{k_i+m_{j_i}}(z_0)). \tag{2.3}$$

On the other hand, by making use of the fact that $f^{m_{j_i}+k_i}(z_0) \to p$ and $f^{k_i}(z_0) \to q(\neq p)$, it follows from the estimates of the Kobayashi distance, that

$$\begin{aligned} &K_D(f^{k_i}(z_0), f^{k_i+m_{j_i}}(z_0)) - K_D(z_0, f^{k_i+m_{j_i}}(z_0)) \\ &\geq -\frac{1}{2}\log\delta(f^{k_i}(z_0)) - \frac{1}{2}\log\delta(f^{k_i+m_{j_i}}(z_0)) + \frac{1}{2}\log\delta(f^{k_i+m_{j_i}}(z_0)) + C \\ &\geq -\frac{1}{2}\log\delta(f^{k_i}(z_0)) + C \to +\infty, \text{ (as } i \to \infty), \end{aligned}$$

where C is a constant independent of i. This contradicts (2.3) and thus finishes the proof of Theorem 1. □

REMARK. The boundary point in Theorem 1 is the so-called Wolff point of f, which is a fixed point of f when understanding the value of f there as the non-tangential boundary limit. It is also worth mentioning that the same argument can be used to show that Theorem 1 actually holds for domains with C^{2+} boundaries (of course, we then have to slightly modify Theorem 2 and Theorem 1 of [Hu1]).

We conclude by presenting two more applications of the results in this paper. The first application is the proof of a boundary version of the classical Cartan uniqueness theorem, while the second one is concerned with the compactness of composition operators on simply connected strongly pseudoconvex domains.

THEOREM 5. *Let $D \subset\subset \mathbb{C}^n$ be either a simply connected smooth pseudoconvex domain or a simply connected taut domain with Stein neighborhood basis. Let $p \in \partial D$ be a strongly pseudoconvex point with at least C^3 smoothness.*

Suppose that $f \in \mathrm{Hol}(D, D)$ is a non-identical holomorphic self mapping of D so that $f(z) = z + \mathrm{o}(|z - p|^k)$ as $z \to p$. Then the following hold:

(1) $k \leq 2$

(2) If $k = 1$, then either f fixes a holomorphic retract with positive dimension or $f^m \to p$. In case D is not biholomorphic to the ball, f cannot be an automorphism.

(3) If $k = 2$, then f can not be an automorphism of D and the sequence $\{f^m\}$ converges to p on compacta.

REMARK. We mention that all statements in Theorem 5 are sharp by examples in [Hu1]. Regarding the proof of this result, the argument for (1) was presented in [BK], while the rest follows from the discussion in Section 3 of [Hu1] if assuming furthermore the following lemma:

LEMMA 5. *Let D, p be as in Theorem 5, and let M be a holomorphic retract of D with complex dimension greater than* 1. *Suppose that $p \in \partial M$ and $f \in \mathrm{Aut}(M)$ is an elliptic element such that $f = z + \mathrm{o}(z - p)$ as $z (\in M) \to p$. Then $f(z) \equiv z$.*

PROOF OF LEMMA 5. By Proposition 1, we can find a complex geodesic ϕ of M with $\phi(1) = p$ and $\phi(\Delta)$ close enough to p. By the hypothesis, it then follows that $\mathrm{Diam}(f \circ \phi(\Delta)) \ll 1$. Since ϕ and $f \circ \phi$ are actually two complex geodesics of a C^3 strongly convex domain (see the proof of Proposition 1) with $|\phi(\xi) - f(\phi(\xi))| = \mathrm{o}(|\xi - 1|)$ and since f is elliptic, it thus follows that $\phi = f \circ \phi$. So f fixes $\phi(\Delta)$. Now, noting that all such $\phi(\Delta)$'s fill in an open subset of M, we see the proof of Lemma 5. □

PROPOSITION 4. *Let $D \subset\subset \mathbb{C}^n$ be a C^{3+} simply connected strongly pseudoconvex domain and let ϕ be a holomorphic self mapping of D. Denote by $H^r(D)$ the standard Hardy space (see* [Kr]*) of D with $r > 1$. Suppose that the composition operator C_ϕ, defined by $C_\phi(g) = g \circ \phi$ for each $g \in H^r(D)$, is a compact self-operator of $H^r(D)$. Then $\{\phi^k\}$ converges uniformly on compacta to a fixed point $z_0 \in D$.*

REMARK When D reduces to the ball or a strongly convex domain, Proposition 4 follows from the work of MacCluer or Mercer, respectively. The argument we will present for the general situation is based on the regularity result in Proposition 2 and the extension theorem of certain Hardy spaces obtained by Cumenge in 1983 [Cu].

PROOF OF PROPOSITION 4. Under the given hypothesis, we first claim that ϕ must be an elliptic element. In fact, if that is not the case, then $\phi^k \to p \in \partial D$ and the angular derivative of ϕ at p is a positive number (see [Ab2]). Thus it follows from a standard argument (see [Me], for example), that C_ϕ cannot be a compact self-operator of $H^r(D)$.

Now, suppose that there is a non-trivial holomorphic retract M of D with $\phi|_M \in \mathrm{Aut}(M)$. Notice that M is a closed complex submanifold of D with

C^2 boundary and intersects ∂D transversally (Proposition 1, Proposition 2 and Proposition 2 of [Hu1]). Let $H^r(M, \mu_{k-1}) = \mathrm{Hol}(M) \cap L^r(\mu_{k-1})$ (where k is the codimension of M in D and the notation μ_{k-1} is explained on Page 59 of [Cu]). Then Theorem 0.1 of [Cu] tells that there exists a bounded linear extension operator $E : H^r(M, \mu_{k-1}) \to H^r(D)$ and moreover the restriction operator $\pi : H^r(D) \to H^r(M, \mu_{k-1})$ is also bounded (see the argument of Corollary 4.1 in [Cu]). Since $C_\phi|_{H^r(M,\mu_{k-1})}$ is an isomorphism of $H^r(M, \mu_{k-1})$ to itself (see Proposition 3), we can easily conclude that C_ϕ is not compact; for C_ϕ cannot map the closed unit ball in $E(H^r(M, \mu_{k-1}))$ to a compact subset of $H^r(D)$.

Applying results in [Be] and [Ab2], we can thus conclude that $\{\phi^k\}$ converges uniformly on compacta to some point $z \in D$. □

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