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Surfaces with Assigned Apparent Contour

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Introduction

In all the paper we will denote by $\mathbf{S}^1$ the unit circle in Euclidean plane $\mathbb{R}^2$, and by $\coprod_{h=1}^{k} \mathbf{S}^1$ the disjoint union of k identical copies of the circle; all surfaces and maps will be supposed smooth (i.e. of class C^∞). Furthermore we will often use "cut and paste" techniques which work fine in C^0 or PL category. Since in low dimension these categories are the same as the C^∞ one, we shall not care to give technical details.

Let S and $\mathbf{N}$ be smooth, compact surfaces, $p \in S$.

DEFINITION. A map $F : S \to \mathbf{N}$ is said excellent at p if its germ at p is equivalent (left-right) to one of the following three normal forms:

$$i)\ (x,y) \mapsto (x,y), \qquad ii)\ (x,y) \mapsto (x,y^2), \qquad iii)\ (x,y) \mapsto (x, y^3 - xy).$$

Clearly, if F is excellent at every p then the set Σ_F of its critical points is a smooth curve in S (i.e. Σ_F is a finite union of disjoint circles).

DEFINITION. F is said excellent if the following two properties hold:

(1) F is excellent at every $p \in S$;

(2) the apparent contour of F, i.e. the set $\Gamma_F = F(\Sigma_F) \subset \mathbf{N}$, is a smooth curve except for a finite number of singularities of the following two local kinds:

$$\textit{Crossing: } \{(x,y) \in \mathbb{R}^2 | xy = 0\}; \qquad \textit{Cusp: } \{(x,y) \in \mathbb{R}^2 | x^3 - y^2 = 0\}.$$

A classical theorem of Whitney [22], asserts that excellent maps are stable and constitute an open, dense subset of the set of C^∞ maps between two surfaces. This theorem shows the reason why excellent maps are sometimes called generic.

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Given a curve $\Gamma \subset \mathbf{N}$ satisfying condition (2) of the previous definition, we ask whether there exist a surface S and a map $F : S \to \mathbf{N}$ such that $\Gamma_F = \Gamma$. Moreover we should like to know how many such maps exist up to right equivalence. This problem can obviously be restated as follows: given a curve, $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$, with only cusps and normal crossings (Definition 1.1) find out all surfaces $S \supset \coprod_{h=1}^{k} \mathbf{S}^1$ and maps $F : S \to \mathbf{N}$ such that $\Sigma_F = \coprod_{h=1}^{k} \mathbf{S}^1$ and $F|_{\coprod_{h=1}^{k} \mathbf{S}^1} = f$.

The basic idea we will use to solve this problem, is not dissimilar from the one used by Francis and Troyer [8], [9] to solve the problem for plane curves, and arise from a very simple remark by Haefliger [11]: let F be an excellent extension of f and U a tubular neighborhood of Σ: then F restricted to $S - U$ is an immersion. Furthermore, one can suppose thet F restricted to ∂U is a curve with only normal crossings. Hence the problem reduces to the following two sub–problems: *i)* find out local, excellent extensions of f to a union of cylinders and Mœbious bands, *ii)* find out immersive extensions of the curves resulting as "boundary of the local extensions". The last problem can be solved using Blank's methods [1], [19].

In §1 we define an extension of Blank's word for curves with cusps and normal crossings, from which, by a purely combinatorical algorithm, we construct a set (the set of minimal assemblages) which is in one-to-one correspondence with the set of excellent extensions of the curve, up to right equivalence (Theorem 1.34).

In §0 we will sketch, without any proof, the methods, firstly introduced by Blank [1], [19] and subsequently developed by other authors ([14], [4], [5], [10], [3], [2]), to find all (up to right equivalence) the immersive extensions of a curve with normal crossings only. We include this section because our notations and statements differ a little from those of the original papers. For a complete survey on this matter see also the first chapter of [13].

Finally, given a line bundle $\pi : \mathbf{E} \to \mathbf{N}$, and a curve with cusps and normal crossings $\Gamma \subset \mathbf{N}$, we ask whether there exists a generic surface (i.e. $\pi|_S$ is excellent) $S \subset \mathbf{E}$ haveing Γ as apparent contour. Since we can construct all excellent extensions of the given curve, the problem reduces to find a factorization $F = \pi \circ F_1$, with $F_1 : S \to \mathbf{E}$ an embedding, of a given excellent map F. In §2 we will use the methods developed in §1 to find combinatorical, necessary and sufficient conditions in order that F possesses such a factorization (Theorem 2.32).

0. - Immersions with assigned boundary

DEFINITION 0.1. Let k be a positive integer; we call generic k-curve an immersion $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$ whose image $\Gamma = \coprod_{h=1}^{k} \Gamma_h$ is smooth up to a finite number of normal crossings.

DEFINITION 0.2. Let S be a surface (not necessarily connected) with boundary and f a generic k-curve, we say that $F : S \to \mathbf{N}$ extends f if and only if F is an immersion, $\partial S = \coprod_{h=1}^{k} \mathbf{S}^1$ and $F|_{\partial S} = f$. We denote by $E(f)$ the set of all extensions of f. Given a point $\infty \in \mathbf{N} - \Gamma$, we denote by $E^{\beta}_{\infty}(f)$ the set of all extensions F of f such that $\#F^{-1}(\infty) = \beta$.

If f is a generic k-curve, then the bundle $f^*T\mathbf{N} \to \coprod_{h=1}^{k} \mathbf{S}^1$ has a canonical trivial sub-bundle, τf, spanned by the never zero cross section f'. Let $\nu f = f^*T\mathbf{N}/_{\tau f}$. These two sub-bundles are called the tangent and normal bundles to f.

DEFINITION 0.3. We say f is sided if ν_f is the trivial bundle. A side for f is a never zero cross section of ν_f, up to multiplication by a positive function.

REMARK 0.4. Since νf can be realized as a sub-bundle of $f^*T\mathbf{N}$, in such a way that $f^*T\mathbf{N} = \tau f \oplus \nu f$, f is sided if and only if there exists a vector field along f which is transversal to f.

A notion of rotation number can be given for a sided curve in a surface, with respect to a fixed vector field X on $\mathbf{N}$, not vanishing on Γ. We denote by $R_X(f)$ the rotation number of the sided curve f with respect to the vector field X. We do not give its definition here (for a definition see [20], [4], [13]), we only remark that this number essentially counts how many times a vector specifyng the side turns with respect to X and that it coincides with the usual rotation number for plane curves, endowing such a curve with the side induced by the orientations of $\mathbb{R}^2$, and taking X to be a constant (never zero) vector field.

Let S be a compact surface with boundary, and $F : S \to \mathbf{N}$ be a generic immersion (i.e. $F|_{\partial S}$ is a generic curve), then $F|_{\partial S}$ has a canonical side defined by the image of an inward-pointing vector field on ∂S. The following fact holds (see [4], [13]):

PROPOSITION 0.5. *Let $F : S \to \mathbf{N}$ be a generic immersion, and let X be a vector field which has at most one isolated zero, at the point $\infty \notin \Gamma$. Then:*

$$R_X(F|_{\partial S}) = \chi(S) - \beta\chi(\mathbf{N}). \tag{0.1}$$

Here χ denotes the Euler characteristic and $\beta = \#F^{-1}(\infty)$.

Let $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$ be a sided curve and $\infty \notin \Gamma$ a fixed point.

DEFINITION 0.6. A set of segments for the curve f is a set $R = \{r_i\} = A \cup \Omega$, $A = \{\alpha_i\}$ and $\Omega = \{\omega_j\}$, such that:

(1) each α_i is an oriented, smooth arc, diffeomorphic to a segment of the real line, starting at a point in some component of $\mathbf{N} - \Gamma$ and ending at ∞;

(2) the ω_j's are smooth (diffeomorphic to $\mathbf{S}^1$) representatives of a minimal system of generators of $\pi_1(\mathbf{N}, \infty)$;

(3) each r_i is in general position with respect to Γ (i.e. misses all crossings and is transversal to Γ);

(4) for all $i \neq j \quad r_i \cap r_j = \infty$.

We say that R is a system of segments if the following holds as well:

(5) if C a component of $\mathbf{N} - \Gamma$ and $\infty \notin C$, then some α_i starts from C.

Given a set of segments for the k-curve f, fix a neighborhood U_∞ of the point ∞ such that $U_\infty \cap \Gamma = \emptyset$. For each $r \in R$ fix an orientable neighborhood U_{r_i} of $r - U_\infty$ and an orientation on it. Label each point in $R \cap \Gamma$ by a letter $x^2_{i,j,\mu}$, where i, j are integers and $e, \mu = \pm 1$, according to the following rules (see Fig. 1):

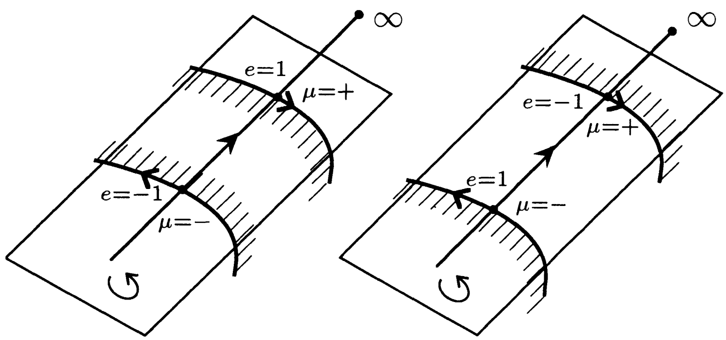


Fig. 1

(1) $x^e_{i,j,\mu} \in r_i$;

(2) $e = 1$ if the segment r_i crosses Γ from left to right (positive point), with respect to the side of f; $e = -1$ otherwise (negative point);

(3) $\mu = 1$ if Γ crosses r_i from left to right, with respect to the orientation of U_{r_i}; $\mu = -1$ otherwise;

(4) for any two points $x^e_{i,j,\mu}, x^e_{i,j',\mu'} \in r_i$, we have $j < j'$ if and only if $x^e_{i,j,\mu} < x^e_{i,j',\mu'}$ with respect to the ordering induced by the orientation on r_i.

Finally, define a set of words $w = \{w_1, \ldots, w_k\}$ by the following construction: fix points $p_h \in \Gamma_h$, which are neither crossings of Γ nor points in $R \cap \Gamma$; for each h, walk along Γ_h, starting from p_h and following its orientation, and write down all the labels you meet, until you come back to p_h. Call w_h the word obtained in this way.

DEFINITION 0.7. The set $w = \{w_1, \ldots, w_k\}$ is called the k-word of the generic k-curve f, with respect to the set of segments R.

REMARK 0.8. Clearly, w depends on the choice of the base points p_h. In fact, if we change the base point p_h, the word w_h changes by the action of a cyclic permutation. We will always consider words up to cyclic permutations.

DEFINITION 0.9. We call 0-assemblage for the k-word w a set $\mathfrak{A}$, whose elements are unordered pairs $\{x^{-1}_{i,j,\mu}, x_{i,j',\mu'}\}$ of letters in w, such that:

(1) if $\{x^{-1}_{i,j,\mu}, x_{i,j',\mu'}\} \in \mathfrak{A}$, then $j < j'$;

(2) each letter corresponding to a point in $\Omega \cap \Gamma$ appears in some pair of $\mathfrak{A}$;

(3) each negative letter (i.e. with $e = -1$) appears in some pair of $\mathfrak{A}$;

(4) each letter appears in at most one pair of $\mathfrak{A}$.

Given f and R as above, fix a neighborhood V_∞ of ∞ and a diffeomorphism $\Phi_\infty : V_\infty \to \mathbf{B}^2$ ($\mathbf{B}^2$ denotes the unit ball in $\mathbb{R}^2$), such that:

(1) $\Gamma \cap V_\infty = \emptyset$;

(2) $\Phi_\infty(R \cap V_\infty)$ is a set of rays starting at the origin;

and for each integer $l \geq 1$ define $g_l : \mathbf{S}^1 \to \mathbf{N}$ as $g_l(t) = \Phi^{-1}_\infty\left(\left(\frac{1}{3} + \frac{1}{3l}\right) e^{it}\right)$, endowed with the side pointing outside V_∞.

DEFINITION 0.10. We call β-expansion of the k-curve f, the $(k+\beta)$-curve f^β, obtained by adding $g_1, \dots, g_\beta$ to f. If w is a word for f, we call β-expansion of the k-word w the $(k+\beta)$-word w^β for f^β defined by the same set of segments as w. Finally, we call β-assemblage for the word w a 0-assemblage for w^β.

REMARK 0.11. $w^\beta = w \cup \{u_1, \dots, u_\beta\}$, u_l being the word for g_l. Observe that the words $u_1, \dots, u_\beta$ are equal up to the shift of the index j of all letters. We define an action of the symmetric group $\mathfrak{S}_\beta$ on the set $\mathcal{A}^\beta(w)$ of β-assemblages of $w\sigma\mathfrak{A}$ being the set obtained from $\mathfrak{A}$ by replacing each letter in u_l with the corresponding one in $u_{\sigma l}$. Clearly $\sigma\mathfrak{A}$ fulfills conditions (1)–(4) in Definition 0.9, hence it is actually a β-assemblage for w.

A β-assemblage $\mathfrak{A}$ defines a graph $G(\mathfrak{A})$ as follows: take $k+\beta$ disjoint, oriented circles $C_1, \dots, C_k$ and $C_{k+1}, \dots, C_{k+\beta}$, on each circle C_h, $h = 1, \dots, k$ [resp. C_{k+l}] choose points corresponding to the letters of the word w_h [resp. u_l], ordered as in w_h [resp. u_l], and join by an extra edge (we call such an edge a proper edge) all pairs of points corresponding to pairs of letters in $\mathfrak{A}$. Finally weight each proper edge by -1 or 1 according as the μ signs of its vertices agree or not.

A graph as just described (i.e. with all vertices standing on k disjoint, oriented circles and all proper edges weighted by ± 1) will be called a weighted k-circular graph. We denoted by $l(G)$ the number of connected components of G.

DEFINITION 0.12. Let G be a connected weighted k-circular graph, and S a surface with k boundary component. We say that G is embeddable in S if there exists a map $\varphi : G \to S$ such that the following three properties hold:

(1) φ is a homeomorphisms onto its image;

(2) $\varphi\left(\bigcup_h C_h\right) = \partial S$;

for each proper edge ℓ of G, let U_ℓ be a neighborhood of $\varphi(\ell)$ homeomorphic to $\ell \times [-1, 1]$. Two semiorientations are naturally defined on $\partial S \cap U_\ell$: the one induced as boundary of U_ℓ and the one induced from the orientation of the circles C_h.

(3) the just described semiorientations on $\partial S \cap U_\ell$ agree if and only if the weight of the edge ℓ is equal to 1.

Obviously, every weighted k-circular graph is embeddable in some surface.

DEFINITION 0.13. If G is a connected, weighted k-circular graph we call genus of G the number $g(G) = \min\{g(S) | G \text{ is embeddable in } S\}$.

The following is easily proved (see [13]):

PROPOSITION 0.14. *Let G' be obtained from G by reversing the orientation of a circle and changing the weight of all proper edges having just one vertex on that circle. Then $g(G') = g(G)$. If G is embeddable in S and $g(G) = g(S)$ then S is orientable if and only if there exists a finite sequence $G = G_0, \ldots, G_n$ of graphs such that for all i, G_{i+1} is obtained from G_i as above, and all proper edges of G_n are positively weighted (in this case we say G is positive).*

The second part of the previous statement allows us to define another noteworthy number associated to the graph:

DEFINITION 0.15. We call weighted genus of the connected k-circular graph G the number $h(G) = 2g(G)$ or $g(G)$ according as G is positive or not. If G is not connected and G_i are its connected components, we call weighted genus of G the number $h(G) = \sum_i h(G_i)$.

We call weighted genus of a β-assemblage the weighted genus of its associated graph, and we denote it by $h(\mathbb{A})$.

The following proposition holds:

PROPOSITION 0.16. *Let $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$ be a generic curve, and w a word for f defined by a system of segments. Then for all $\mathbb{A} \in \mathcal{A}^\beta(w)$*

$$R_X(f) \geq 2l(\mathbb{A}) - k - h(\mathbb{A}) - \beta\chi(\mathbf{N}). \tag{0.2}$$

DEFINITION 0.17. Let w be defined by a system of segments. A β-assemblage $\mathbb{A} \in \mathcal{A}^\beta(w)$ is said minimal if (0.2) is an equality. We denote by $\mathcal{A}^\beta_m(w)$ the set of minimal β-assemblages of w.

It is not hard to see that if $\sigma \in \mathfrak{G}_\beta$, $\mathbb{A} \in \mathcal{A}_m^\beta(w)$ then $\sigma\mathbb{A} \in \mathcal{A}_m^\beta(w)$, hence $\mathfrak{G}_\beta$ acts on $\mathcal{A}_m^\beta(w)$. Denote right equivalence by $\overset{r}{\sim}$, the following theorem holds:

THEOREM 0.18. *Let $f : \coprod_{h=1}^k \mathbf{S}^1 \to \mathbf{N}$ be a generic k-curve and w a k-word for f defined by a system of segments. Then there exists a bijection between $E_\infty^\beta(f)/_{\overset{r}{\sim}}$ and $\mathcal{A}_m^\beta(w)/_{\mathfrak{G}_\beta}$.*

SKETCH OF PROOF. We only show how to define such a bijection. First suppose $\beta = 0$. Let $F \in E_\infty^0(w)$, then $\widetilde{R} = F^{-1}(R)$ is a set of smooth arcs in S, oriented by the immersion. Label each point of $\widetilde{R} \cap \partial S$ by the same label as the corresponding point in Γ, and observe that each positive labeled point (i.e. $e = 1$) is the ending point of some arc, while each negative labeled point (i.e. $e = -1$) is the starting point of some arc. Furthermore, since $F^{-1}(\infty) = \emptyset$, every arc must end on ∂S. Pairing the letters corresponding to the vertices of those arcs in $\widetilde{R}$ which have both vertices on ∂S, we actually get a minimal 0-assemblage, $\mathbb{A}(F)$, and the mapping $F \mapsto \mathbb{A}(F)$ is a bijection $E_\infty^0(f)/_{\overset{r}{\sim}} \to \mathcal{A}_m^0(w)$.

Let $F \in E_\infty^\beta(f)$ and let $D_1, \ldots, D_\beta \subset \mathbf{N}$ be the disks bounded by the curves g_l defining the β-expansion of f (see Definition 0.10), call $\{p_1, \ldots, p_\beta\} = F^{-1}(\infty)$ and U_i^j the interior of the connected component of $F^{-1}(D_i)$ containing p_j. For each $\sigma \in \mathfrak{G}_\beta$, let $S_\sigma = S - \bigcup_i U_i^{\sigma i}$. Clearly $F_\sigma = F|_{S_\sigma} \in E_\infty^0(f^\beta)$, hence, by the case $\beta = 0$, we get an assemblage $\mathbb{A}(F_\sigma) \in \mathcal{A}_m^0(w^\beta) = \mathcal{A}_m^\beta(w)$. Such a construction actually defines a bijection between $E_\infty^\beta(f)/_{\overset{r}{\sim}}$ and $\mathcal{A}_m^\beta(w)/_{\mathfrak{G}_\beta}$. □

The statement of the previous theorem can be slightly improved.

DEFINITION 0.19. A connected component C of $\mathbf{N} - \Gamma$ is said to be positive [resp. negative] if the side of the curve points inward [resp. outward] C at every point in ∂C. We say that a set of segments is a reduced system of segments if:

(5′) at least one α_i starts from each negative component.

REMARK 0.20. Proposition 0.16, Definition 0.17 and finally Theorem 0.18 can be restated, and proved (see [13]), replacing the words "system of segments" by "reduced system of segments", and in this improved form they will be used in §1.

Now we prove a lemma, we will use in §1. Let us give one more definition:

DEFINITION 0.21. We say that a generic k-curve has a curl if there exist $t_0, t_1 \in \mathbf{S}^1$ such that $f(t_0) = f(t_1)$ and $f|_{[t_0,t_1]}$ bounds a disk $D \subset \mathbf{N}$ such that $D \cap \Gamma = \partial D$. We say that the curl is positive according as $D - \partial D$ is a positive component.

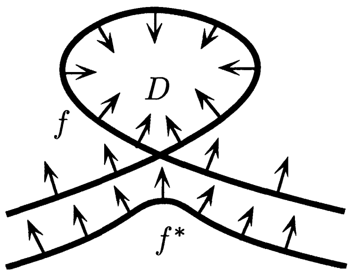


Fig. 2

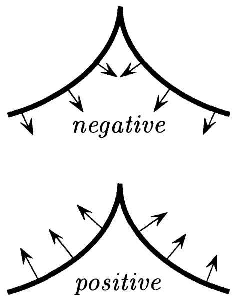


Fig. 3

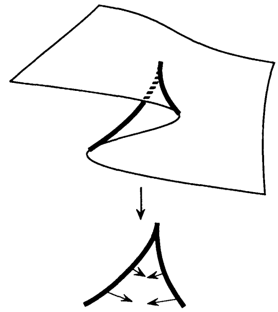


Fig. 4

LEMMA 0.22. *Suppose the k-curve f has a positive curl, then for every word w defined by a reduced system of segments $\mathcal{A}^{\beta}_{m}(w) = \emptyset$.*

PROOF. It is enough to prove that $\mathcal{A}^{\beta}_{m}(w) = \emptyset$ for the word defined by some reduced system of segments. Let D be the disk bounded by the curl and observe that we can construct a reduced system of segment R such that $R \cap D = \emptyset$. In fact, fix $\infty \notin D$. Since the curl is positive, there is no need to draw segments starting from D, and all other segments can be drawn far from D. Let f^* be the curve obtained by removing the curl as suggested in Fig. 2 and let w and w^* be the words defined by R for f and f^*. Clearly, R is a reduced system of segments for f^* too, and since $R \cap D = \emptyset$ we have $w = w^*$. Applying (0.2) to f^* and using $R_X(f) = R_X(f^*) + 1$, we get: $R_X(f) > 2l(\mathfrak{A}) - k - h(\mathfrak{A}) - \beta_X(\mathbf{N})$ for all $\mathfrak{A} \in \mathcal{A}^{\beta}(w)$. □

1. - Extending curves with cusps and normal crossings

Let I be an interval of the real line $\mathbb{R}$, and $f : I \to \mathbf{N}$ be a curve.

DEFINITION 1.1. A point $t_0 \in I$ is called a cusp point if there exist germs of diffeomorphism $\gamma : (I, t_0) \to (\mathbb{R}, 0)$ and $\varphi : (\mathbf{N}, f(t_0)) \to (\mathbb{R}^2, 0)$ such that $\varphi \circ f \circ \gamma^{-1}(s) = (s^2, s^3)$. We say that $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$ is a k-curve with cusp and normal crossing (briefly a CN k-curve) if $f'(t) \neq 0$, except for a finite number of cusp points, and f is injective except for a finite number of normal crossings.

Let C be the set of cusp points of the CN k-curve $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$.

DEFINITION 1.2. A side for $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$ is a side for $f|_{\coprod_{h=1}^{k} \mathbf{S}^1 - C}$ which is directed, in the neighborhood of each cusp, either to the inside or to the outside of the cusp on both branches of the cusp itself. We say that a cusp is positive or negative according as the side is outward or inward pointing (see Fig. 3). Finally, we say that a side for f is coherent if all cusps are negative.

Let $F : S \to \mathbf{N}$ be an excellent mapping, by definition $F|_{\Sigma_F}$ is a CN curve.

PROPOSITION 1.3. *The curve $F|_{\Sigma_F}$ has a coherent side.*

PROOF. Take the folding side of the map (see Fig. 4 and [18]). □

Local extensions. Let $f : \mathbf{S}^1 \to \mathbf{N}$ be a sided CN curve and let $D \subseteq \mathcal{C}$.

DEFINITION 1.4. We call D-deformation of the first kind of f the generic 2-curve $f^*_D = f^*_{D,1} \amalg f^*_{D,2}$ obtained by doubling f and deforming its cusps as suggested in Fig. 5. Fix a point $p \in \Gamma$ which is neither a crossing nor a cusp, and call D-deformation of the second kind the generic 1-curve f^{**} obtained by modifying f^*_D in a neighborhood of p as suggested in Fig. 6.

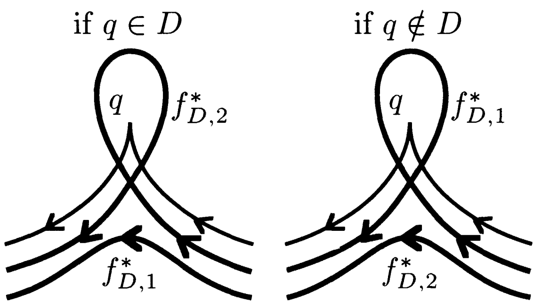


Fig. 5

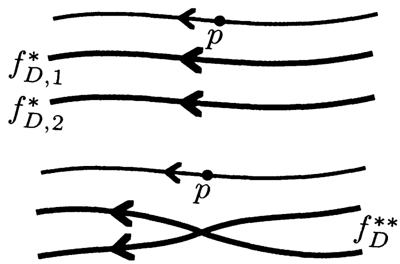


Fig. 6

REMARK 1.5. It is obvious that $f^*_D = f^*_{\mathcal{C}-D}$ and $f^{**}_D = f^{**}_{\mathcal{C}-D}$. By local arguments, it is easily seen that if $F : \mathbf{S}^1 \times [-1,1] \to \mathbf{N}$ is an excellent mapping such that $\Sigma_F = \mathbf{S}^1 = \mathbf{S}^1 \times \{0\}$ and $F|_{\Sigma_F} = f$, then there exists a tubular neighborhood U of $\mathbf{S}^1$ such that $f|_{\partial U} = f^*_D$ for some $D \subseteq \mathcal{C}$. Similarly, if $M = \mathbb{R} \times [-1,1]/_{(x,t)=(x+1,-t)}$ and $F : M \to \mathbf{N}$ is an excellent map with $\Sigma_F = \mathbf{S}^1 = \mathbb{R} \times \{0\}/_{(x,0)=(x+1,0)}$ and $F|_{\Sigma_F} = f$, then there exists a tubular neighborhood U of $\mathbf{S}^1$ such that $F|_{\partial U} = f^{**}_D$. Furthermore this set D is unique up to complementation. We call such a D, up to complementation, the deformation set induced by F.

Let $f : \coprod_{h=1}^k \mathbf{S}^1 \to \mathbf{N}$ be a sided CN k-curve, let D be as above and let $H \subseteq \{1, \ldots, k\}$; we call such a pair (D, H) a deformation pair. Denote by $\bar{H}$ the complement of H.

DEFINITION 1.6. We call (D,H)-deformation of f, the generic k_H-curve $f_{(D,H)} = \{(f_h)^*_D | h \in H\} \amalg \{(f_h)^{**}_D | h \in \bar{H},\}$, where $k_H = 2\#H + \#\bar{H}$.

REMARK 1.7. Let C_h be the set of cusp points in the h^{th} circle and $D_h = D \cap C_h$; if D' is such that either $D'_h = D_h$ or $D'_h = C_h - D_h$ for all h, then $f_{(D,H)} = f_{(D',H)}$. If $F : \coprod_{h=1}^{k} M_h \to \mathbf{N}$ is a local excellent extension of f (i.e. M_h is either a cylinder or a Möbius band and $F|_{M_h}$ is as in Remark 1.5) then there exists a tubular neighborhood U of the k circles such that $F|_{\partial U} = f_{(D,H)}$ form some choice of D and H. Furthermore if (D, H) and (D', H') are two different such choices, then $H = H'$ and D, D' are as above. We call such a pair (D, H), up to this relation, the deformation pair induced by F.

REMARK 1.8. Let M be either a cylinder or a Möbius band. Suppose $F : M \to \mathbf{N}$ is a local generic extension of $f : \mathbf{S}^1 \to \mathbf{N}$, and let t_0 be a cusp point; if $v \in T_{t_0}M - \ker(dF(t_0))$ then $dF(t_0)[v]$ is a vector tangent to the cusp. Identify M with the quotient space of $\mathbb{R} \times [-1, 1]$ by either the relation $(x, t) = (x + 1, t)$ (if S is a cylinder) or the relation $(x, t) = (x + 1, -t)$ (if S is a Mœbious band) in such a way that $\pi^{-1}(p) = \mathbb{Z}$, where p is the fixed point in Γ and π denotes the quotient map. Let $\tilde{F} = F \circ \pi$. It can be easily seen that the deformation set induced by F is given, up to complementation, by $D = \{t \in (0, 1) | (t, 0)$ is a cusp point vand $d\tilde{F}(t, 0)[(0, 1)]$ points inward the cusp$\}$.

Using this characterization, it is not hard to prove the following:

PROPOSITION 1.9. *Let $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$ be a CN k-curve endowed with a coherent side; then for all (D, H) as above, there exists a local excellent extension of f inducing (D, H) as deformation pair.*

Let $F, G : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be two generic, singular germs having the same apparent contour (i.e. $F(\Sigma_F) = G(\Sigma_G) = \Gamma$); then they have the same normal form with respect to left-right equivalence, say *ii)* or *iii)* in Introduction. Standard arguments in singularity theory prove the following:

LEMMA 1.10. *Let F, G be as above and suppose they induce the same side; then there exists a germ of diffeomorphism Φ such that $F = G \circ \Phi$. Moreover if their normal form is iii) such a germ Φ is unique, while if the normal form is ii) there are exactly two such germs, exactly one of which preserves the orientation.*

PROPOSITION 1.11. *Let $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$ be a CN k-curve, and F, G two local, excellent extensions of f. Then F and G are locally (around $\coprod_{h=1}^{k} \mathbf{S}^1$) right equivalent if and only if they induce the same deformation pair.*

PROOF. If $F \overset{r}{\sim} G$ they obviously induce the same deformation pair. Suppose F and G induce the same pair. Clearly it is enough to prove the thesis for a CN 1-curve. Let S be the domain of F and G. By the previous lemma we can take finite open covers, $\{U_i\}_{i=1,\dots,n}$, $\{V_i\}_{i=1,\dots,n}$ of $\mathbf{S}^1$ and diffeomorphisms $\Phi_i : U_i \to V_i$ such that $F|_{U_i} = G|_{V_i} \circ \Phi_i$. Lifting all maps to the universal covering of S, we see that the condition that F and G induce the same deformation set, implies that all uniquely determined diffeomorphisms are orientation-preserving [resp. reversing]. To conclude the proof it is enough to choose all the other ones to be orientation-preserving [resp. reversing] and paste them together. □

Rotation number. Let $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$ be a sided CN k-curve, and let X be a vector field on $\mathbf{N}$ with no zeros on Γ. Denote by f^* the generic k-curve obtained by deforming all cusps in the way suggested in Fig. 7. With the just introduced notations, $f^* = \coprod_{h=1}^{k} (f_h)^*_{\mathcal{C},1}$. Since a deformation of f has a side, canonically induced by the side of f (Fig. 8), we can give the following:

DEFINITION 1.12. We call rotation number of f with respect to X, the number $R_X(f) = R_X(f^*)$, where f^* is endowed with the side induced by f.

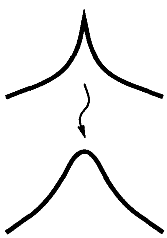


Fig. 7

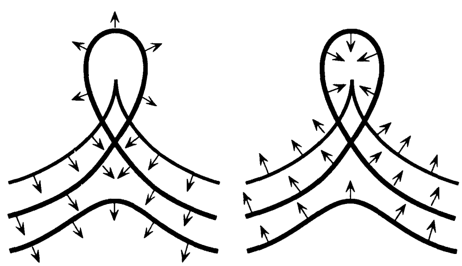


Fig. 8

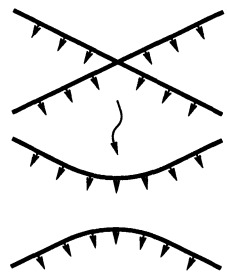


Fig. 9

LEMMA 1.13. *Let $f : \mathbf{S}^1 \to \mathbf{N}$ be a sided, CN 1-curve, and let $D \subseteq \mathcal{C}$; then $R_X(f^{**}_D) = R_X(f^*_D) = 2R_X(f) - c^- + c^+$; where c^+, c^- denote respectively the number of positive and negative cusps.*

PROOF. Since rotation number does not change when modifying a curve as suggested in Fig. 9, the left-hand side equality holds. Denote by $c^{\pm}_D$ the number of positive [resp. negative] cusps in D. Then $f^*_{D,1}$ [resp. $f^*_{D,2}$] is obtained by adding $c^+_{\mathcal{C}-D}$ [resp. c^+_D] positive curls and $c^-_{\mathcal{C}-D}$ [resp. c^-_D] negative curls to f^*, hence $R_X(f^*_{D,1}) = R_X(f^*) + c^+_{\mathcal{C}-D} - c^-_{\mathcal{C}-D}$, and $R_X(f^*_{D,2}) = R_X(f^*) + c^+_D - c^-_D$. Sum up the two equalities to get the thesis. □

LEMMA 1.14. *Let $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$ be a CN k-curve and (D, H) a deformation pair; then $R_X(f_{(D,H)}) = 2R_X(f) - c^- + c^+$.*

PROOF. This follows immediately from the previous Lemma. □

From now on, we will suppose to have fixed a vector field X vanishing at most at a point $\infty \notin \Gamma$.

PROPOSITION 1.15. *Let $F : S \to \mathbf{N}$ be an excellent map; then:*

(1.1) $$2R_X(F|_\Sigma) - c = \chi(S) - \beta\chi(\mathbf{N});$$

here c denotes the number of cusps, and $\beta = \#F^{-1}(\infty)$.

PROOF. Let $f = F|_{\Sigma_F}$. By Proposition 1.3 all cusps are negative and hence, by the previous Lemma, $R_X(f_{(D,H)}) = 2R(f) - c$ for all deformation pairs (D, H). Let U be a tubular neighborhood of Σ_F such that $F|_{\partial U} = f_{(D,H)}$. The sides on $f_{(D,H)}$ induced respectively by the immersion $F|_{S-U}$ and the side of f clearly coincide, and hence, by (0.1), $R_X(f_{(D,H)}) = \chi(S - U) - \beta\chi(\mathbf{N})$. To conclude the proof, observe that $\chi(S) = \chi(S - U)$, since S is obtained pasting a finite number of cylinders and Möbius bands to $S - U$. □

REMARK 1.16. The previous Proposition gives one more condition in order that a curve may be the apparent contour of an excellent map. Nevertheless it is not hard to construct examples of curves fulfilling (1.1), but being the apparent contour of no excellent map $S \to \mathbf{N}$. We remark also that (1.1) implies a generalization of a classical theorem of Thom [21], claiming that $\chi(S) \equiv c \pmod 2$ for all excellent maps $f : S \to \mathbb{R}^2$. More precisely, denoting by $\deg_2$ the modulo two degree:

THEOREM 1.17. *Let $F : S \to \mathbf{N}$ be an excellent map; then $\chi(S) \equiv c \pmod 2$ if and only if either $\chi(\mathbf{N})$ is even or* $\deg_2(F) = 0$.

Words. Let $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$ be a CN k-curve and $\infty \in \mathbf{N} - \Gamma$ a fixed point.

DEFINITION 1.18. A set of segments for f is a set $R = A \cup B \cup \Omega$ of smooth, oriented arcs in $\mathbf{N}$, such that:

(1) each $r \in A \cup B$ is diffeomorphic to a closed interval of the real line. The ending point of these arcs is ∞ and the starting point is either a point in a connected component of $\mathbf{N} - \Gamma$, if $r \in A$, or a cusp, if $r \in B$;

(2) Ω is a set of smooth (diffeomorphic to $\mathbf{S}^1$) representatives of a minimal set of generators of $\pi_1(\mathbf{N}, \infty)$;

(3) $r \cap r' = \infty$ for all $r, r' \in R$, $r \neq r'$;

(4) each $r \in R$ is in general position with respect to Γ, i.e. r contains neither crossings nor cusps (except at most the starting point), is transversal to Γ and if it starts from a cusp, it points to the outside of the cusp.

We say that R is a system of segments for f if the following holds as well:

(5) at least one segment starts from each component of $\mathbf{N} - \Gamma$ not containing ∞ and from each cusp.

DEFINITION 1.19. Suppose f is sided, and let $r \in R$. We say $p \in R \cap \Gamma$ is positive if and only if either p is not a cusp and r crosses Γ from the left to the right or p is a positive cusp; otherwise we say p is negative.

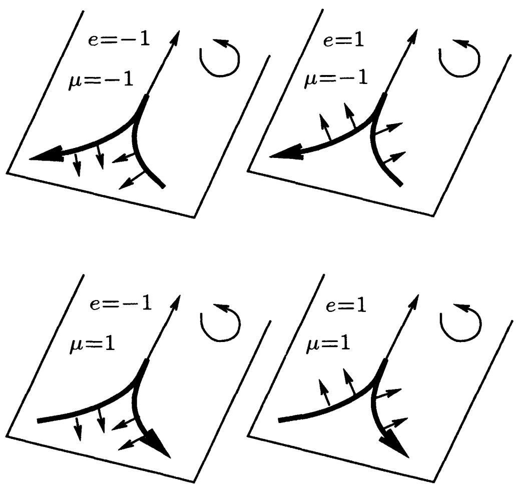


Fig. 10

As in §0, we fix a neighborhood U_∞ of ∞ not intersecting Γ and oriented neighborhoods V_r of $r - U_\infty$ for all $r \in R$. Next we label $x^e_{i,j,\mu}$ each point in $R \cap \Gamma$ by the same rules used in §0, except for cusp points in which the index μ is +1 or -1 according as the curve crosses the segment from left to right or from right to left (see Fig. 10). Associate a set of k words to the CN k-curve f by the same construction described in §0.

DEFINITION 1.20. We call the set $w = \{w_1, \dots, w_k\}$ obtained in this way the k-word of f with respect to the set of segments R.

REMARK 1.21. As for generic curves (Remark 0.8), the word of a CN curve will be considered up to cyclic permutations.

Let $f : \mathbf{S}^1 \to \mathbf{N}$ be a CN curve, and w be a word for f. Let $D \subseteq \mathcal{C}$.

DEFINITION 1.22. We call D-deformations of w respectively of the first and second kind the words $w^*_D = \{w^*_{D,1}, w^*_{D,2}\}$ and $w^{**}_D = w^*_{D,1} w^*_{D,2}$, where $w^*_{D,1}$ is obtained by erasing all letters in w corresponding to cusps in D and $w^*_{D,1}$ by erasing all letters corresponding to cusps in $\mathcal{C} - D$.

Let f be a CN k-curve, w a k-word for f and (D, H) a deformation pair.

DEFINITION 1.23. The (D, H)-deformation of the k-word w is the k_H-word $w_{(D,H)}$ given by $\{(w_h)^*_D | h \in H\} \amalg \{(w_h)^{**}_D | h \in \bar{H}\}$, where $\bar{H}$ and k_H are as in Definition 1.6.

DEFINITION 1.24. A β-assemblage for w is a pair $\mathbf{A} = ((D, H), \mathfrak{A})$, where (D, H) is a deformation pair and $\mathfrak{A}$ is a β-assemblage for the word $w_{(D,H)}$, as defined in Definition 0.10. We denote by $\mathcal{A}^\beta(w)$ the set of β-assemblages of w. Given a β-assemblage $\mathbf{A} = ((D,H),\mathfrak{A})$, we call respectively number of components and weighted genus of $\mathbf{A}$ the numbers $l(\mathbf{A}) = l(\mathfrak{A})$ and $h(\mathbf{A}) = h(\mathfrak{A})$.

REMARK 1.25. All letters, except those corresponding to cusps, appear twice in the deformed word $w_{(D,H)}$. In the previous definition the two copies of the same letter must be considered as different letters.

PROPOSITION 1.26. *Let R be a system of segments for f; then R is a reduced system of segments for $f_{(D,H)}$.*

PROOF. Clearly, R is a set of segment for $f_{(D,H)}$ (the deformation can be done in such a way that R is a set of segments for $f_{(D,H)}$). Each connected components of $\mathbf{N} - \Gamma_{(D,H)}$ either is essentially a component of $\mathbf{N} - \Gamma$ or is generated by the deformation. At least one segment starts from each component of the first kind. The new ones are bounded by either the curl around some cusp or by parallel branches of $\Gamma_{(D,H)}$. In the first case the segment which starts from the corresponding cusp, starts from the new component, in the second case, it is easily seen that the new component is not negative (see Fig. 11), hence condition (5′) holds. □

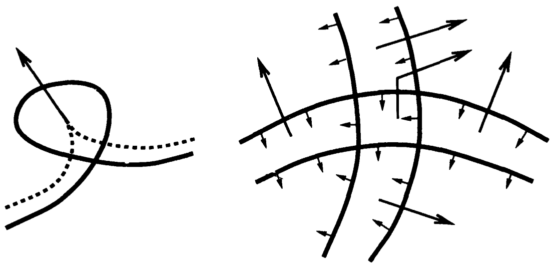


Fig. 11

REMARK 1.27. Let $\tilde{w}_{(D,H)}$ the word for $f_{(D,H)}$ defined by R; clearly $\tilde{w}_{(D,H)}$ is obtained by renumbering the second index of all letters in $w_{(D,H)}$ in such a way to preserve the previous ordering of letters belonging to the same segment; hence, a β-assemblage $\mathbf{A} = ((D,H),\mathfrak{A})$ for the word w obviously defines a β-assemblage $\tilde{\mathfrak{A}}_{\mathbf{A}}$ for $\tilde{w}_{(D,H)}$. Furthermore, since letters having the same indexes in $w_{(D,H)}$ also have the same exponent, then the corresponding letters in $\tilde{w}_{(D,H)}$ cannot be paired by any β-assemblage. This argument proves the following:

PROPOSITION 1.28. *The mapping $\mathbf{A} \mapsto \tilde{\mathfrak{A}}_{\mathbf{A}}$ is a bijection between the set of β-assemblages of w having (D,H) as deformation pair and $\mathcal{A}^\beta(\tilde{w}_{(D,H)})$. Furthermore $\mathbf{A}$ and $\tilde{\mathfrak{A}}_{\mathbf{A}}$ have the same number of components and weighted genus.*

PROPOSITION 1.29. *Let f be a CN k-curve and w be a word defined by a system of segments. Then, for all* $\mathbf{A} \in \mathcal{A}^{\beta}(w)$,

$$(1.2) \qquad 2R_X(f) - c_- + c_+ \geq 2l(\mathbf{A}) - k_H - h(\mathbf{A}) - \beta\chi(\mathbf{N}).$$

PROOF. Use Proposition 1.28, (0.2) and Lemma 1.14. □

DEFINITION 1.30. Let w be a word for the CN k-curve f, defined by a system of segments. A β-assemblage $\mathbf{A}$ of w will be called minimal if in (1.2) equality holds. We denote by $\mathcal{A}^{\beta}_m(w)$ the set of minimal β-assemblages of w.

REMARK 1.31. The mapping $\mathbf{A} \mapsto \tilde{\mathfrak{A}}_{\mathbf{A}}$ is a bijection between the set of minimal β-assemblages of w, having (D, H) as deformation pair, and $\mathcal{A}^{\beta}_m(\widetilde{w}_{(D,H)})$.

Let f be a CN k-curve, and suppose that the h^{th} component of f has no cusps. Let (D, H) be a deformation pair; then $(w_h)^*_{D,1} = (w_h)^*_{D,2} = w_h$. We define an involution ι_h on the set of letters of $w_{(D,H)}$, which maps each letter of $(w_h)^*_{D,1}$ to the corresponding one of $(w_h)^*_{D,2}$ and fixes all other letters. Let $\mathbf{A} = ((D, H), \mathfrak{A})$ be a β-assemblage of w; we denote by $\iota_h\mathbf{A}$ the β-assemblage $((D, H), \iota_h\mathfrak{A})$, where $\iota_h\mathfrak{A} = \{\{\iota_h(x^{-1}_{i,j,\mu}), \iota_h(x_{i,j',\mu'})\} | \{x^{-1}_{i,j,\mu}, x_{i,j',\mu'}\} \in \mathfrak{A}\}$.

DEFINITION 1.32. We say $\mathbf{A}$ is ι-equivalent to $\mathbf{A}'$ (briefly $\mathbf{A} \overset{\iota}{\sim} \mathbf{A}'$) if there exist involutions $\iota_{h_1}, \dots, \iota_{h_n}$ such that $\mathbf{A}' = \iota_{h_n} \dots \iota_{h_1}\mathbf{A}$. We say that $\mathbf{A} = ((D, H), \mathfrak{A})$ and $\mathbf{A}' = ((D', H'), \mathfrak{A}')$ are δ-equivalent (briefly $\mathbf{A} \overset{\delta}{\sim} \mathbf{A}'$) if the two deformation pairs are as in Remark 1.7, and $\mathfrak{A} = \mathfrak{A}'$.

As in §0, we have an action of the symmetric group $\mathfrak{S}_\beta$, on the set $\mathcal{A}^{\beta}(w)$, given by $\sigma\mathbf{A} = ((D, H), \sigma\mathfrak{A})$. We denote by $\sim$ the equivalence relation on $\mathcal{A}^{\beta}(w)$ generated by $\overset{\iota}{\sim}$, $\overset{\delta}{\sim}$ and the action of $\mathfrak{S}_\beta$; we simply call it equivalence.

REMARK 1.33. If $\mathbf{A} \sim \mathbf{A}'$ then $\mathbf{A}$ and $\mathbf{A}'$ have the same number of components and the same weighted genus. Hence $\sim$ can be restricted to the set $\mathcal{A}^{\beta}_m(w)$.

Let $E^{\beta}_{\infty}(f)$ denote the set of all excellent mappings F defined on some surface $S \supset \coprod_{h=1}^{k} \mathbf{S}^1$ such that $\Sigma_F = \coprod_{h=1}^{k} \mathbf{S}^1$, $F|_{\coprod_{h=1}^{k} \mathbf{S}^1} = f$ and $\#F^{-1}(\infty) = \beta$. We will prove the following theorem:

THEOREM 1.34. *Let w a word for f, defined by a system of segments. The set $E^{\beta}_{\infty}(f)/_{\overset{\tau}{\sim}}$ is in one-to-one correspondence with the set $\mathcal{A}^{\beta}_m(w)/_{\sim}$.*

Before proving the Theorem, we give some elementary remarks.

LEMMA 1.35. *Let $\mathbf{A}, \mathbf{A}' \in \mathcal{A}^{\beta}(w)$, $\sigma \in \mathfrak{S}_\beta$. If $\mathbf{A} \overset{\iota}{\sim} \mathbf{A}'$, then $\sigma\mathbf{A} \overset{\iota}{\sim} \sigma\mathbf{A}'$. If $\mathbf{A} \overset{\delta}{\sim} \mathbf{A}'$, then $\sigma\mathbf{A} \overset{\delta}{\sim} \sigma\mathbf{A}'$.*

REMARK 1.36. Denote by $\stackrel{\iota\delta}{\sim}$ the equivalence relation generated by $\stackrel{\iota}{\sim}$ and $\stackrel{\delta}{\sim}$. We can consider the quotient relation over $\mathcal{A}^\beta(w)/\mathfrak{G}_\beta$; we denote it by $\stackrel{\iota\delta}{\sim}$ again. Denote by $[\mathbf{A}]$ the class of $\mathbf{A}$ in $\mathcal{A}^\beta(w)/\mathfrak{G}_\beta$; by Lemma 1.35 we have that $[\mathbf{A}] \stackrel{\iota\delta}{\sim} [\mathbf{A}'] \Leftrightarrow \exists\, \sigma \in \mathfrak{G}_\beta : \sigma\mathbf{A} \stackrel{\iota\delta}{\sim} \mathbf{A}'$; furthermore $\mathcal{A}^\beta_m(w)/_\sim = (\mathcal{A}^\beta_m(w)/\mathfrak{G}_\beta)/_{\stackrel{\iota\delta}{\sim}}$.

REMARK 1.37. Let (D,H) be a deformation pair and $\widetilde{w}_{(D,H)}$ be the word for $f_{(D,H)}$ defined by the same segments as w. Let $\mathbf{A} = ((D,H),\mathfrak{A}) \in \mathcal{A}^\beta(w)$ and let $\widetilde{\mathfrak{A}}_{\mathbf{A}} \in \mathcal{A}^\beta(\widetilde{w}_{(D,H)})$ be the corresponding assemblage (see Proposition 1.28). By the very definitions, we have $\sigma\widetilde{\mathfrak{A}}_A = \widetilde{\mathfrak{A}}_{\sigma\mathbf{A}}$ for all σ and $\mathbf{A}$. It follows that the mapping $\widetilde{\mathfrak{A}}$ defines a map $[\widetilde{\mathfrak{A}}] : [\mathbf{A}] \mapsto [\widetilde{\mathfrak{A}}]_{[\mathbf{A}]} = [\widetilde{\mathfrak{A}}_{\mathbf{A}}]$ which is a bijection between the set of β-assemblages of w having (D,H) as deformation pair, modulo the action of $\mathfrak{G}_\beta$, and the set $\mathcal{A}^\beta(\widetilde{w}_{(D,H)})/\mathfrak{G}_\beta$.

Proof of Theorem 1.34. We start trying to define a map $\mathbf{A}^\beta_1 : E^\beta_\infty(f) \to \mathcal{A}^\beta_m(w)/\mathfrak{G}_\beta$. Let $F \in E^\beta_\infty(f)$, choose a tubular neighborhood U of the set of critical points of F, in such a way that $F|_{\partial(S-U)} = f_{(D,H)}$ for some choice of (D,H). Clearly $F|_{S-U}$ is an immersion which estends $f_{(D,H)}$ and attains exactly β times the value ∞. Let $\mathfrak{A}^\beta(F) \in \mathcal{A}^\beta_m(\widetilde{w}_{(D,H)})/\mathfrak{G}_\beta$ be the class of minimal assemblages associated to $F|_{S-U}$ by Theorem 0.18. Define $\mathbf{A}^\beta_1(F) \in \mathcal{A}^\beta_m(w)/\mathfrak{G}_\beta$ as the unique class such that $[\widetilde{\mathfrak{A}}]_{\mathbf{A}^\beta_1(F)} = \mathfrak{A}^\beta(F)$ (Remark 1.37). Observe that such a construction is not univocal, in fact it depends on:

(a) the choice of the tubular neighborhood;

(b) the choice of the deformation set;

and, if f_h is a component with no cusps and U_h is the correspondent component of U, on:

(c) the choice of the branch of ∂U, used as domain of $f^*_{D,1}$.

As we will see soon, different choices lead to $\stackrel{\iota\delta}{\sim}$-equivalent classes, so that a map $\mathbf{A}^\beta_2 : E^\beta_\infty(f) \to \mathcal{A}^\beta_m(w)/_\sim$ is defined. Let us describe more explicitly the above construction.

Let $\{p_1,\dots,p_\beta\} = F^{-1}(\infty)$ and let $D_1,\dots,D_\beta \subset S$ be disks such that $p_i \in D_i$ and $F(\partial D_i) = g_i(\mathbf{S}_1)$, where $g_1,\dots,g_\beta$ are the curves giving the β-expansion (see Definition 0.10). Denote by Σ^β the set $\Sigma_F \cup \bigcup_{i=1}^{\beta} \partial D_i$. Since the segments of R are in general position with respect to Γ (see condition (4) of Definition 1.18), we see that $(F|_{(S-\bigcup \dot{D}_i)})^{-1}(R)$ is a set of smooth arcs with at least one ending point on Σ^β. Give to all this points the same label as the corresponding point on Γ.

Let U be the tubular neighborhood of Σ, and let γ be an arc of $(F|_{(S-\bigcup \dot{D}_i)})^{-1}(R)$. Clearly, γ intersects ∂U in as many points as $\gamma \cap \Sigma$. Label these points as the corresponding ending point of γ. If we read the letters on ∂U we get the word $w_{(D,H)}$, while if we read the letters on ∂D_i we get the words u_i (for all $i = 1,\dots,k$) of the β-expansion (see §0). Pairing together letters corresponding to ending points of the same arc, we get a β-assemblage for $w_{(D,H)}$, hence a β-assemblage for w, whose class in $\mathcal{A}^{\beta}(w)/_{\mathfrak{G}_\beta}$ is equal to $\mathbf{A}_1(F)$ (compare with the sketch of proof of Theorem 0.18).

From the last construction, it is clear that leaving U and the deformation set D fixed, but exchanging the role of the two branches of ∂U_h, the assemblage changes by the action of the involution ι_h. It is also clear that choosing a different deformation set the assemblage changes by a δ-equivalence. Finally if U, U' are two different tubular neighborhoods, we may define a map from the set of letters in $w_{(D,H)}$ into itself, simply by mapping each letter represented by an intersection of the arc γ with ∂U to the letter represented by the corresponding intersection of γ with $\partial U'$. It is easily seen that the assemblages defined by the two choices differ up to the action of this map and that this map is a composition of ι-involutions, hence the two assemblages are ι-equivalent.

By the very definition, the following is easily proved:

PROPOSITION 1.38. *If $F, F' \in E^{\beta}_{\infty}(f)$ are right equivalent extensions of f, then $\mathbf{A}^{\beta}_2(F) = \mathbf{A}^{\beta}_2(F')$.*

By this Proposition $\mathbf{A}^{\beta}_2$ defines a map: $\mathbf{A}^{\beta} : E^{\beta}_{\infty}(f)/_{\underset{\sim}{r}} \to \mathcal{A}^{\beta}_m(w)/_{\sim}$. To conclude the proof of Theorem 1.34, it is enough to prove the following:

PROPOSITION 1.39. *The just defined map $\mathbf{A}^{\beta}$ is a bijection.*

PROOF. $\mathbf{A}^{\beta}$ is injective. Let $F : S \to \mathbf{N}$ and $F' : S' \to \mathbf{N}$ be such that $\mathbf{A}^{\beta}(F) = \mathbf{A}^{\beta}(F')$, then they induce the same deformation pair. By Proposition 1.11 there exist neighborhoods U and U' of $\coprod_{h=1}^{k} \mathbf{S}^1$ respectively in S and S' and a diffeomorphism $\varphi_1 : U' \to U$ such that $F'|_{U'} = (F|_U) \circ \varphi_1$. On the other hand, $F|_{(S-U)}$ and $F'|_{(S'-U')}$ are two extension of $f_{(D,H)}$ inducing the same class of assemblages in $\mathcal{A}^{\beta}_m(\widetilde{w}_{(D,H)})/_{\mathfrak{G}_\beta}$; hence, by Theorem 0.18, there exists a diffeomorphism $\varphi_2 : S' - U' \to S - U$ such that $F'|_{(S'-U')} = F|_{(}S - U) \circ \varphi_2$. A local analysis shows that the two diffeomorphisms φ_1 and φ_2 paste together, defining a diffeomorphism $\varphi : S' \to S$ such that $F' = F \circ \varphi$.

To prove that $\mathbf{A}^{\beta}$ is surjective we need the following:

LEMMA 1.40. *If w has minimal assemblages then all cusps of f are negative.*

PROOF. By contradiction, suppose that f has a positive cusp; then every deformation of f has a positive curl, and then, by Lemma 0.22, $\mathcal{A}^{\beta}_m(\widetilde{w}_{(D,H)}) = \emptyset$ for all deformation pairs (D, H); by Remark 1.31, $\mathcal{A}^{\beta}_m(w) = \emptyset$. □

Back to the proof of Theorem 1.34, let $\mathbf{A} = ((D, H), \mathfrak{A})$ be a minimal β-assemblage for w. By the previous Lemma f has a coherent side; hence, by Proposition 1.11, there exists a local, excellent extension F_1 of f, defined on a disjoint union of cylinders and Möbius bands M, such that $F_1|_{\partial M} = f_{(D,H)}$. By Proposition 1.28, $\widetilde{\mathfrak{A}}_{\mathbf{A}}$ is a minimal β-assemblage of $\widetilde{w}_{(D,H)}$; hence, by Theorem 0.18, there exist a surface $\widetilde{S}$ and an immersion $F_2 : \widetilde{S} \to \mathbf{N}$ such that $\partial\widetilde{S} = \coprod_{h=1}^{k_H} \mathbf{S}^1$ and $F_2|_{\partial\widetilde{S}} = f_{(D,H)}$. Once again a local analysis shows that

M and $\tilde{S}$ can be pasted along the boundary, so that the map F obtained pasting F_1 and F_2 is an excellent map extending f and, by construction, $\mathbf{A}^{\beta}(F) = \mathbf{A}$. □

2. - Factorization of excellent mappings

Classically, the problem to find out a factorization $F = \pi \circ F_1$ for an excellent map $F : S \to \mathbf{N}$ into a given line bundle $\mathbf{E} \xrightarrow{\pi} \mathbf{N}$, was posed with the requirement that F_1 should be an immersion; this problem was firstly solved by Haefliger [11], in the case $\mathbf{N} = \mathbb{R}^2$ and $\mathbf{E} = \mathbb{R}^3$; successively Millet [15] remarked that Hafliger's proof also worked in the case of an arbitrary surface $\mathbf{N}$ and the trivial bundle ($\mathbf{E} = \mathbf{N} \times \mathbb{R}$). Finally in [12] the author generalized Hafliger's methods to the general case: let C denote a component of Σ_F, and put $c_C = \#\{\text{cusp points in } C\}$, $\nu_C = \pm 1$ according as C has a trivial normal bundle or not and finally $\varepsilon_C = \pm 1$ according as $F|_C^* \mathbf{E} \to C$ is the trivial bundle or not. With these notations the following holds:

THEOREM. *F is factorizable into $\mathbf{E}$ by means of an immersion if and only if, for all components C of Σ_F, $(-1)^{c_C} \nu_C \varepsilon_C = 1$.*

As announced in the introduction, in this section we deal with the problem of finding a factorization $F = \pi \circ F_1$, with F_1 an embedding. Let us begin with some lemmas and propositions wich we will use later.

From now on the set of critical points and the apparent contour of an excellent map will be denoted by Σ and Γ.

LEMMA 2.1. *Let $F : S \to \mathbf{N}$ be an excellent map. Let $D \subset \mathbf{N}$ be an embedded disk such that: ∂D is in general position with respect to Γ; D contains no cusp; $D \cap \Gamma$ is a union of simple arcs. Then F restricted to any component of $F^{-1}(D) - \Sigma$ is 1-1.*

PROOF. Fix a point $\infty \notin D$ and a vector field X on $\mathbf{N}$, vanishing at most at ∞. Let C be a connected component of $F^{-1}(D) - \Sigma$ and let C_ε be obtained by removing a small neighborhood of Σ. The following facts hold: $F|_{C_\varepsilon}$ is an immersion; ∂C_ε consists of branches of $F^{-1}(\partial D)$ and arcs parallel to Σ; if k is the number of boundary components of ∂C_ε, then $F(\partial C_\varepsilon)$ consists of k loops obtained by arcs of ∂D and branches of Γ. Endowing these loops with the side induced by the immersion, it is easy to see that each of these loops has rotation number greater than or equal to 1, hence $R_X(F|_{\partial C_\varepsilon}) \geq k$. On the other hand, by (0.1) of §0, we have $R_X(F|_{\partial C_\varepsilon}) = \chi(C_\varepsilon) \leq 1$; hence $k = 1$, C_ε is a disk and $F|_{\partial C_\varepsilon}$ cannot have any crossing; hence $F|_{C_\varepsilon}$ is 1-1. The Thesis follows by an exhaustion argument. □

Let R be a system of segments for $F|_\Sigma$, the set $F^{-1}(R)$ consists of oriented arcs having at least the ending point on $\Sigma \cup F^{-1}(\infty)$. Denote by $H(F)$ the set of all such arcs starting from $\Sigma \cup F^{-1}(\infty)$.

REMARK 2.2. Since $\omega \in \Omega$ is a closed loop based at ∞, then $F^{-1}(\Omega) \subseteq H(F)$.

LEMMA 2.3. *F restricted to each component of $S - H(F)$ is injective.*

PROOF. Let $p, q \in S - H(F)$ be such that $F(p) = F(q)$ and let $\gamma : [0,1] \to S - H(F)$ be a path joining p and q. It is not hard to see that we can suppose that $p, q \in S - F^{-1}(R)$, $\gamma([0,1]) \subset S - F^{-1}(R)$ and $F \circ \gamma$ is a simple curve in general position with respect to Γ (see [10]). Let $D \subset \mathbf{N}$ be the disk bounded by $F \circ \gamma$. Since $\partial D \cap R = \emptyset$, then $D \cap R = \emptyset$, and since R is a system of segments for $F|_\Sigma$, D contains neither cusps nor curls of Γ. A contradiction follows from Lemma 2.1. □

We now state two topological lemmas, whose proof is an easy exercise.

LEMMA 2.4. *Let X be an arcwise connected topological space, and let U, V be two open subsets such that $X = U \cup V$ and $U \cap V$ is not arcwise connected; then $H_1(X) \neq 0$, where H_1 is the first homology group.*

LEMMA 2.5. *Let $U_1, \dots, U_n$, $n \geq 3$, be open subsets of a topological space X. If $U_i \cap U_j \neq \emptyset \Leftrightarrow |i - j| \equiv 0, 1 \pmod n$, then either $\bigcap_{i=1}^{n} U_i \neq \emptyset$, and in this case $n = 3$, or $\forall i\, U_i \cap \left(\bigcup_{j \neq i} U_j \right)$ is disconnected.*

From the above two lemmas, we have the following:

PROPOSITION 2.6. *Let $U_1, \dots, U_n$ $(n \geq 3)$ arcwise connected, open subsets of the topological space X, satisfying the hypothesis of Lemma 2.5; then either $n = 3$ and $\bigcap_i U_i \neq \emptyset$, or $H_1\left(\bigcup_i U_i\right) \neq 0$.*

From now on, we will denote by $\mathcal{C}$ the set of connected components of $S - H(F)$.

PROPOSITION 2.7. *Every $C \in \mathcal{C}$ is diffeomorphic to an open disk.*

PROOF. Let $C \in \mathcal{C}$. As already remarked, $F^{-1}(\Omega) \subseteq H(F)$, hence $F(C) \subset \mathbf{N} - \Omega$. Since Ω is a minimal system of generators of $\pi_1(\mathbf{N}, \infty)$, $\mathbf{N} - \Omega$ is diffeomorphic to an open disk; hence, by Lemma 2.3, we can say that C is a planar surface, that is to say it is an open disk with some holes. By contradiction, assume $F(C)$ is not an open disk, hence $F(C)$ has at least two boundary components, furthermore the boundary of $F(C)$ consists of branches of Γ and branches of segments of R. Let γ be an interior component of $\partial F(C)$. If γ contains a branch of the segment r, since the segment ends at ∞, the segment itself cuts $F(C)$ from boundary to boundary. Since C does not contain arcs in $F^{-1}(R)$ starting and ending at the boundary, this is a contradiction. If γ consists of branches of Γ only, then it bounds a connected component of $\mathbf{N} - \Gamma$. The segment starting from this connected component leads to a contradiction as in the previous case. □

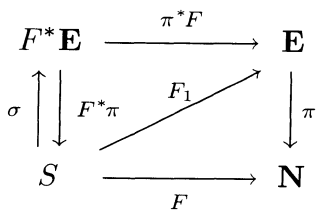
PROPOSITION 2.8. *Let C_1, $C_2 \in \mathcal{C}$, then $F(C_1) \cap F(C_2)$ is connected.*

PROOF. By contradiction, suppose $F(C_1) \cap F(C_2)$ is disconnected. By Lemma 2.4, $F(C_1) \cup F(C_2)$ has at least two boundary components. Contradiction follows as in the previous proposition. □

Factorization. Let S and $\mathbf{N}$ be surfaces and $\mathbf{E} \xrightarrow{\pi} \mathbf{N}$ a line bundle.

DEFINITION 2.9. We say that a map $F : S \to \mathbf{N}$ is factorizable into $\mathbf{E}$ if there exists an embedding $F_1 : S \to \mathbf{E}$ such that $F = \pi \circ F_1$.

REMARK 2.10. Finding a map F_1 such that $F = \pi \circ F_1$ is the same as finding a section σ, of the line bundle $F^*\mathbf{E}$ induced by F (see the diagram below), hence F is factorizable into $\mathbf{E}$ if and only if there exists a section $\sigma : S \to F^*\mathbf{E}$ of the induced bundle, such that $\pi^*F \circ \sigma$ is an embedding. If $\mathbf{E} = \mathbf{N} \times \mathbb{R}$ is the trivial bundle, this condition reduce to the existence of a function $h : S \to \mathbb{R}$ such that $(F, h) : S \to \mathbf{N} \times \mathbb{R}$ is an embedding. We will call such a function a height for F.



REMARK 2.11. Suppose $F : S \to \mathbf{N}$ and $F' : S' \to \mathbf{N}$ are right equivalent maps; it is easily seen that F is factorizable into $\mathbf{E}$ if and only if F' is. For excellent maps this means that, in some sense, factorizability conditions "must" be contained in the assemblage associated to the map.

We now fix our attention on finding factorizability conditions for excellent mappings. Let $F^{-1}(\infty) = \{p_1, \dots, p_\beta\}$ and let $D_1, \dots, D_\beta$ be disks in S such that $p_i \in D_i$ and $F|_{\partial D_i} = g_i$ (g_i as in Definition 0.10), and let $\widetilde{S} = S - \bigcup_{i=1}^{\beta} \dot{D}_i$, $\widetilde{F} = F|_{\widetilde{S}}$.

PROPOSITION 2.12. *F is factorizable into $\mathbf{E}$ if and only if $\widetilde{F}$ is.*

PROOF. Let $\widetilde{F}_1 : \widetilde{S} \to \mathbf{E}$ be an embedding such that $\pi \circ \widetilde{F}_1 = \widetilde{F}$, and U_∞ an open ball around ∞ such that $U_\infty \cap \Gamma = \emptyset$, and U_∞ contains all curves g_i. Let U_i be the connected component of $F^{-1}(U_\infty)$ containing p_i. Clearly, $U_i \cap U_j = \emptyset$ for all $i \neq j$, and F is a diffeomorphism between U_i and U_∞. Take a trivialization $\Phi : \mathbf{E}|_{U_\infty} \to U_\infty \times \mathbb{R}$, then $\Phi \circ \widetilde{F}_1|_{\bigcup_i (U_i - \dot{D}_i)}(p) = (F(p), h(p))$. Since $\widetilde{F}_1$ is injective, we may suppose that h assumes constant value c_i on $U_i - D_i$, such that $c_i \neq c_j$ for all $i \neq j$. Extend $\tilde{F}_1$ to the required embedding F_1, defining $F_1(p) = \Phi^{-1}(F(p), c_i)$ for all $p \in D_i$. □

The trivial bundle. Let us first consider the case $\mathbf{E} = \mathbf{N} \times \mathbb{R}$. In Remark 2.10 we saw that F is factorizable into $\mathbf{E} = \mathbf{N} \times \mathbb{R}$ if and only if there exists a height function h for F.

PROPOSITION 2.13. *Let h be an height function for F; then for all C_1, $C_2 \in \mathcal{C}$ such that $F(C_1) \cap F(C_2) \neq \emptyset$, one (and only one) of the following two holds:*

(1) $\forall x \in C_1,\ y \in C_2 \quad F(x) = F(y) \Rightarrow h(x) < h(y)$;

(2) $\forall x \in C_1,\ y \in C_2 \quad F(x) = F(y) \Rightarrow h(x) > h(y)$.

PROOF. Clearly at most one of the two holds. By contradiction, suppose both are false, then there exist $x_1, x_2 \in C_1$, $y_1, y_2 \in C_2$ such that $F(x_1) = F(y_1) = z_1$, $F(x_2) = F(y_2) = z_2$ and $h(x_1) < h(y_1)$, $h(x_2) > h(y_2)$. By Proposition 2.8 $F(C_1) \cap F(C_2)$ is connected, let $\gamma : [0,1] \to F(C_1) \cap F(C_2)$ be a path joining z_1 and z_2, and let γ_1 and γ_2 be its liftings to C_1 and C_2 respectively (these liftings exist by Lemma 2.3). Let $h_i = h \circ \gamma_i$, then $h_1(0) = h(x_1) < h(y_1) = h_2(0)$, and $h_1(1) = h(x_2) > h(y_2) = h_2(1)$, therefore there exists $t_0 \in (0,1)$ such that $h_1(t_0) = h_2(t_0)$. Call $p_1 = \gamma_1(t_0) \in C_1$, $p_2 = \gamma_2(t_0) \in C_2$, then $p_1 \neq p_2$ and $F(p_1) = F(p_2)$, $h(p_1) = h(p_2)$, that is a contradiction. □

Suppose h is a height for F. We define a structure of oriented graph on the set $\mathcal{C}$, as follows: say that C_1, $C_2 \in \mathcal{C}$ are joined by an edge pointing to C_2 ($C_1 \to C_2$) if and only if C_1 and C_2 verify (1) in Proposition 2.13. We denote by $L(h)$ this oriented graph.

PROPOSITION 2.14. *The graph $L(h)$ has no loops.*

PROOF. By contradiction, let $C_1 \to C_2 \to \ldots \to C_n \to C_1$ be a loop of minimal length. Observe that $F(C_i) \cap F(C_j) \neq \emptyset \Rightarrow |i-j| \equiv 0, 1 \pmod n$, on the contrary we could use Proposition 2.13 to find a shorter loop. By Proposition 2.6, either $n = 3$ and $\bigcap_i F(C_i) \neq \emptyset$ or $H_1\left(\bigcup_i F(C_i)\right) \neq 0$. If the first holds, take points $x_i \in C_i$ such that $F(x_1) = F(x_2) = F(x_3)$, by definition of $\to$, $h(x_1) < h(x_2)$, $h(x_2) < h(x_3)$ and $h(x_3) < h(x_1)$, which is a contradiction; in the other case contradiction follows as in the proof of Proposition 2.7. □

As in the definition of the word of a curve, for all $r \in R$, fix an orientation of a small tubular neighborhood of $r - \infty$.

PROPOSITION 2.15. *Let C_1, C_2, C_3, $C_4 \in \mathcal{C}$ and ℓ, ℓ_1, ℓ_2, be edges of the graph $H(F)$. The following two hold:*

(A) suppose that C_1, C_2 paste along ℓ (i.e. $\bar{C}_1 \cap \bar{C}_2 \supseteq \ell$) and $F(C_3) \cap F(\ell) \neq \emptyset$; then either $C_1 \to C_3$ and $C_2 \to C_3$ or $C_3 \to C_1$ and $C_3 \to C_2$;

(B) suppose that C_1, C_2 paste along ℓ_1, that C_3, C_4 paste along ℓ_2 and $F(\ell_1) \cap F(\ell_2) \neq \emptyset$. Let $r \in R$ be the segment containing $F(\ell_i)$, and suppose $F(C_1)$, $F(C_3)$ be on the left of r and $F(C_2)$, $F(C_4)$ on its right, then: either $C_1 \to C_3$ and $C_2 \to C_4$ or $C_3 \to C_1$ and $C_4 \to C_2$.

PROOF. Let C_1, C_2, C_3 be as in (A); then $F(C_1) \cap F(C_3)$ and $F(C_2) \cap F(C_3)$ are both not empty, hence some edge of $L(h)$ joins C_1, C_3 and C_2, C_3. By contradiction, suppose (A) is false, say $C_1 \to C_3$ and $C_3 \to C_2$. Let $z \in F(C_3) \cap F(\ell)$ and let $p \in \ell$ such that $F(p) = z$. Take a small neighborhood U of p, such that $U \cap C_3 = \emptyset$ and $F(U) \subset F(C_3)$, and a path γ in U joining two points $p_1 \in C_1$, $p_2 \in C_2$. Let γ_3 be the lifting of $F \circ \gamma$ to C_3. Clearly $F \circ \gamma_3 = F \circ \gamma$, but $C_1 \to C_3 \Rightarrow h(\gamma(0)) < h(\gamma_3(0))$ and $C_3 \to C_2 \Rightarrow h(\gamma(1)) > h(\gamma_3(1))$. As in the previous proposition this fact leads to a contradiction. A completely analogous argument proves (B). $\square$

Proposition 2.14 and Proposition 2.15 give necessary conditions in order that an excellent map F may be factorizable. We will prove that these conditions are also sufficient, that is to say the following theorem holds:

THEOREM 2.16. *Let $F : S \to \mathbf{N}$ be an excellent map; then F is factorizable into $\mathbf{N} \times \mathbb{R}$ if and only if the set $\mathcal{C}$ can be given a structure of oriented graph with no loops, satisfying conditions* (A) *and* (B) *of Proposition* 2.15.

PROOF. Let $\tilde{S}$ be as in Proposition 2.12. We proceed in two steps: *Step 1:* definition of a height on $\tilde{S}$ minus a tubular neighborhood of Σ; *Step 2:* extension of such a height to the tubular neighborhood. The thesis will follow by Proposition 2.12.

Let L be the graph in the assumption; since it has no loops, than $\to$ extends to an ordering on $\mathcal{C}$, say $\prec$, and clearly such an ordering fulfills (A) and (B). Let $\mathcal{C} = \{C_1, \dots, C_n\}$, with $C_s \prec C_{s+1}$ for all s.

Step 1. Let U be a tubular neighborhood of Σ such that $F|_{\partial U}$ is a generic curve and denote $S' = \tilde{S} - U$, $\Gamma' = F(\partial S')$ and if $C \in \mathcal{C}$ denote $C' = C \cap S'$. Let V_∞ be a ball around ∞, contained in the interior side of all curves g_i (see Definition 0.10), that is to say $F(\tilde{S}) \subset (\mathbf{N} - V_\infty)$. For all $r \in R$ fix an oriented neighborhood V_r of $r - V_\infty$ in $\mathbf{N} - V_\infty$ and a diffeomorphism $\varphi_r : V_r \to [0, t_r] \times [-1, 1]$, such that:

(1) $V_r \cap V_{r'} = \emptyset \quad \forall r \neq r'$;

(2) $\forall r \in R$, V_r contains no crossings of Γ';

(3) $\forall r \in R$, $\varphi_r(r) \subseteq [0, t_r] \times \{0\}$;

(4) $\forall r \in R$, $(r \cup \Gamma') \cap V_r$ is connected (this fact and (2) say that V_r contains only branches of Γ' intersecting r);

(5) if $\Gamma_{r,j}$ is the component of $\Gamma' \cap V_r$ which intersects r at the j^{th} point, counting according to the orientation of r, then $\varphi_r(\Gamma_{r,j}) = \{j\} \times [-1, 1]$;

(6) $\forall r \in R$, φ_r and $\varphi_r|_r : r \to \mathbb{R}$ are orientation-preserving.

Let $V = \bigcup_{r\in R} V_r$, for any proper edge ℓ of $H(F)$, denote by U_ℓ the connected component of $F^{-1}(V)\cap S'$ containing $\ell\cap S'$, and by $r(\ell)$ the segment containing $F(\ell)$; denote by $s(\ell)$ and $t(\ell)$ the two indexes such that $C_{s(\ell)}$, $C_{t(\ell)} \in \mathcal{C}$ are the two components pasting along ℓ, the former on the left of ℓ, the latter on the right (left and right with respect to the orientations induced by F). For all $C \in \mathcal{C}$ denote $C^* = (C\cap S') - \bigcup_\ell U_\ell$. Finally let $\psi : \mathbb{R}\times[-1,1] \to \mathbb{R}$ be a C^∞ function such that: $\psi(x,y) = 0$, $\forall y \in [-1,-2/3]$; $\psi(x,y) = 1$, $\forall y \in [2/3,1]$ and $\psi(x,y) \in [0,1]$, $\forall x,y$. We define the function $h_1 : S' \to \mathbb{R}$,

$$(2.1) \qquad h_1 : x \longmapsto \begin{cases} s & \text{if } x \in C_s^* \\ s(\ell) + (t(\ell) - s(\ell))\psi(\varphi_{r(\ell)}(F(x))) & \text{if } x \in U_\ell. \end{cases}$$

Clearly such a function is of class C^∞. We now prove that $(F|_{S'}, h_1) : S' \to \mathbf{N}\times\mathbb{R}$ is injective. Let $x \neq y \in S'$, three cases appear:

(I) $x \in C_s^*$ and $y \in C_{s'}^*$;

(II) $x \in C_s^*$ and $y \in U_\ell$;

(III) $x \in U_\ell$ and $y \in U_{\ell'}$.

Case (I). If $s = s'$, by Lemma 2.3, $F(x)\neq F(y)$; if $s\neq s'$, clearly $h_1(x)\neq h_1(y)$.

Case (II). If $F(x)\neq F(y)$ there is nothing to prove; if $F(x) = F(y)$ then $F(C_s^*)\cap F(U_\ell)\neq\emptyset$, therefore, thanks to the choice of the V_r's, $F(C_s)\cap r(\ell)\neq\emptyset$; by (A), this means either $C_s \prec C_{s(\ell)}$ and $C_s \prec C_{t(\ell)}$ or $C_{s(\ell)} \prec C_s$ and $C_{t(\ell)} \prec C_s$, that is to say either $s < s(\ell)$ and $s < t(\ell)$ or $s(\ell) < s$ and $t(\ell) < s$. By definition of h_1, $\min\{s(\ell), t(\ell)\} < h_1(y) < \max\{s(\ell), t(\ell)\}$, hence $s = h_1(x)\neq h_1(y)$.

Case (III). Suppose $F(x) = F(y) = p$; the V_r's are pairwise disjoint, hence $r(\ell) = r(\ell')$, and since $F(U_\ell)\cap F(U_{\ell'})\neq\emptyset$, $F(\ell)\cap F(\ell')\neq\emptyset$. By (B), either $C_{s(\ell)} \prec C_{s(\ell')}$ and $C_{t(\ell)} \prec C_{t(\ell')}$ or $C_{s(\ell')} \prec C_{s(\ell)}$ and $C_{t(\ell')} \prec C_{t(\ell)}$, that is to say either $s(\ell) < s(\ell')$ and $t(\ell) < t(\ell')$ or $s(\ell') < s(\ell)$ and $t(\ell') < t(\ell)$. Using the definition of h_1, an easy computation shows that either $h_1(x) < h_2(y)$ or $h_2(y) < h_1(x)$. This ends the first step.

Step 2. For the sake of simplicity, we suppose Σ is connected, it will be clear that the same proof also works in the general case. Let U as above and put

$$M = \begin{cases} \mathbb{R}\times[-1,1]/_{(x,y)=(x+1,y)} & \text{if } U \text{ is orientable,} \\ \mathbb{R}\times[-1,1]/_{(x,y)=(x+1,-y)} & \text{if } U \text{ is non-orientable.} \end{cases}$$

Let $\varphi : M \to U$ be a diffeomorphism such that $\varphi(\{y = 0\}) = \Sigma$ and the inverse image of $U\cap H(F)$ consists of vertical segments. More precisely, if $p : \mathbb{R}\times[-1,1] \to M$ denotes the quotient map, we suppose that $(\varphi\circ p)^{-1}(H(F))\cap [0,1]\times[-1,1]$ has the form: $\bigcup_i(\{t_i\}\times[0,1])\cup\bigcup_j(\{s_j\}\times[-1,0])\cup\bigcup_l(\{z_l\}\times[-1,1])$. Denote by D_j both the generic component of $M - \varphi^{-1}(H(F))$ and the corresponding component of $U - H(F)$; and by $C_{i(j)} \in \mathcal{C}$ the component of $S - H(F)$ containing D_j. The following two lemmas are an immediate consequence of (A) and (B) respectively.

LEMMA 2.17. *Let $\bar{t} \in [0,1)$ be a cusp point (with the just said notations, either $\bar{t} = t_i$ or $t = s_j$) and let $D_{j_1(\bar{t})}$, $D_{j_2(\bar{t})}$, $D_{j_3(\bar{t})}$ be the three components having $(\bar{t},0)$ in their closure. Let $D_{j_1(\bar{t})}$ be the one opposed to the edge starting at $(\bar{t},0)$ (see Fig. 12); then: either $C_{i(j_2(\bar{t}))} \prec C_{i(j_1(\bar{t}))} \prec C_{i(j_3(\bar{t}))}$ or $C_{i(j_3(\bar{t}))} \prec C_{i(j_1(\bar{t}))} \prec C_{i(j_2(\bar{t}))}$.*

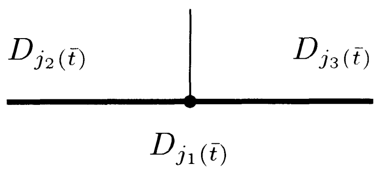


Fig. 12

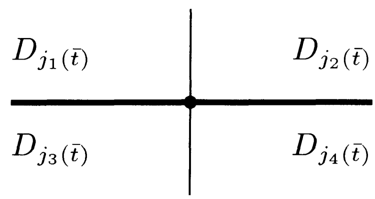


Fig. 13

LEMMA 2.18. *Let $D_{j_1(z_l)}, D_{j_2(z_l)}, D_{j_3(z_l)}, D_{j_4(z_l)}$ the four connected components having $(z_l,0)$ in their closure, numbered as in Fig. 13; then either $C_{i(j_1(z_l))} \prec C_{i(j_3(z_l))}$ and $C_{i(j_2(z_l))} \prec C_{i(j_4(z_l))}$ or $C_{i(j_1(z_l))} \succ C_{i(j_3(z_l))}$ and $C_{i(j_2(z_l))} \succ C_{i(j_4(z_l))}$.*

Denote $h_{1,-1}(t) = h_1 \circ p(t,-1)$, $h_{1,1}(t) = h_1 \circ p(t,1)$. By the previous two lemmas, and the way we constructed h_1 in Step 1, we can suppose $h_{1,1}$ and $h_{1,-1}$ have the following properties:

(1) $h_{1,1}(t+1) = h_{1,1}(t)$ and $h_{1,-1}(t+1) = h_{1,-1}(t)$ if U is orientable, otherwise $h_{1,1}(t+1) = h_{1,-1}(t)$ and $h_{1,-1}(t+1) = h_{1,1}(t)$;

(2) there exists $\varepsilon > 0$ such that:
   a) $h_{1,1}$ is strictly monotonic on $[t_i - \varepsilon, t_i + \varepsilon]$ and $[z_l - \varepsilon, z_l + \varepsilon]$;
   b) $h_{1,-1}$ is strictly monotonic on $[s_j - \varepsilon, s_j + \varepsilon]$ and $[z_l - \varepsilon, z_l + \varepsilon]$;
   c) $h_{1,1}$ e $h_{1,-1}$ assume constant values on the other intervals;

(3) $\forall t \in [0,1)$, $h_{1,1}(t) = h_{1,-1}(t) \Leftrightarrow$ either $t = t_i$ or $t = s_j$ for some i or j.

These conditions imply that we can extend h_1 in a neighborhood $[\bar{t}-\delta, \bar{t}+\delta] \times [-1,1]$ of each cusp point $(\bar{t},0)$, as suggested in Fig. 14, in such a way that $(F,h_1) : [\bar{t}-\delta,\bar{t}+\delta] \times [-1,1] \to \mathbf{N} \times \mathbb{R}$ is an embedding. Moreover, let $D_{j_l(\bar{t})}$, as in Lemma 2.17, be the three components having $(\bar{t},0)$ in their closure and $M(\bar{t}) = \max\{i(j_2(\bar{t})), i(j_3(\bar{t}))\}$, $m(\bar{t}) = \min\{i(j_2(\bar{t})), i(j_3(\bar{t}))\}$, we can suppose that

$$m(\bar{t}) \leq h_1(t,y) \leq M(\bar{t}) \quad \forall (t,y) \in [\bar{t}-\delta, \bar{t}+\delta] \times [-1,1]; \tag{2.2}$$

and, up to taking U and δ small enough, that:

$$F([\bar{t}_1 - \delta, \bar{t}_1 + \delta] \times [-1,1]) \cap F([\bar{t}_2 - \delta, \bar{t}_2 + \delta] \times [-1,1]) = \emptyset. \tag{2.3}$$

for all pairs of cusp points $(\bar{t}_1,0)$, $(\bar{t}_2,0)$.

LEMMA 2.19. *The mapping* $(F, h_1) : S' \cup \bigcup_{\bar t} ([\bar t - \delta, \bar t + \delta] \times [-1, 1]) \to \mathbf{N} \times \mathbb{R}$ *is an embedding.*

PROOF. By the way we extended h_1, the map is clearly an immersion. By contradiction, let x_1, $x_2 \in S' \cup \bigcup_{\bar t} [\bar t - \delta, \bar t + \delta] \times [-1, 1]$ be such that $F(x_1) = F(x_2)$ and $h_1(x_1) = h_1(x_2)$. Using the fact that (F, h_1) restricted to both S' and $[\bar t - \delta, \bar t + \delta] \times [-1, 1]$ is injective and (2.3), we have that $x_1 \in [\bar t - \delta, \bar t + \delta] \times [-1, 1]$ and $x_2 \in S'$. Furthermore h_1 assumes constant value in a neighborhood of x_1; let C_i be the component of $S' - H(F)$ containing it, this means that $F(C_i)$ contains the cusp corresponding to $(\bar t, 0)$, hence $F(C_i) \cap F(D_{j_l(\bar t)}) \neq \emptyset$ for all $l = 1, 2, 3$. Use twice (A) and get either $i < m(\bar t) = \min_l i(j_l(\bar t))$ or $i > M(\bar t) = \max_l i(j_l(\bar t))$. Using (2.2) and the fact that $h_1(x_2) = i$, we get a contradiction. □

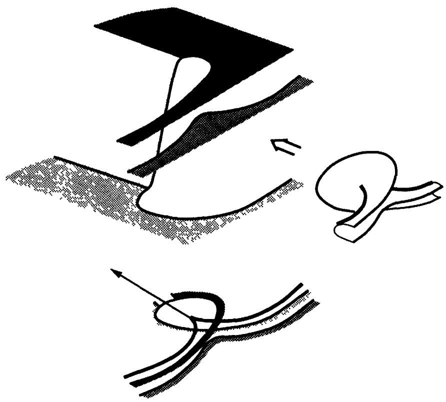


Fig. 14

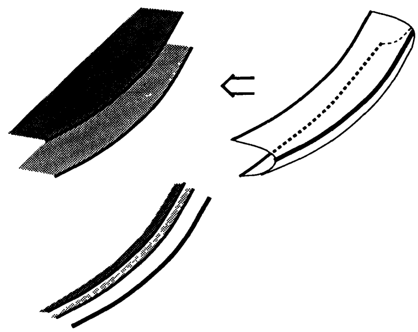


Fig. 15

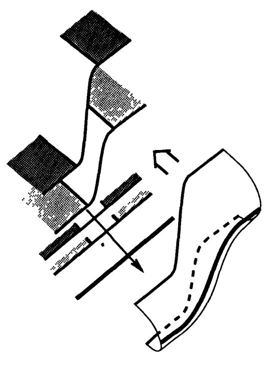


Fig. 16

We are only left to extend h_1 over $U - \bigcup_{\bar{t}}([\bar{t} - \delta, \bar{t} + \delta] \times [-1, 1])$. Where both $h_{1,1}$ and $h_{1,-1}$ assume constant values, an extension is easily found as suggested in Fig. 15. Furthermore, conditions (2) and (3) ensure that we can define local extensions of h_1 over the sets $[z_l - \varepsilon, z_l + \varepsilon] \times [-1, 1]$, in such a way that the value of such an extension at the point (t, y) is between $h_{1,1}(t)$ and $h_{1,-1}(t)$ (see Fig. 16). Clearly all such extensions can be pasted to give a function $h : \widetilde{S} \to \mathbb{R}$. An argument very much like the one used to prove the previous lemma, proves the following too:

LEMMA 2.20. *The map* $(F, h) : S \to \mathbf{N} \times \mathbb{R}$ *is injective.*

Since (F, h) is an immersion, this concludes the proof of Theorem 2.16. □

Now, we give a description of line bundles over surfaces, fitting in with our purpose to find factorability conditions for excellent mappings.

Line bundles. Let $\mathbf{N}_1$ be a connected, compact surface with one boundary component and $\pi_1 : \mathbf{E}_1 \to \mathbf{N}_1$ a line bundle. Suppose $\mathbf{E}_1|_{\partial \mathbf{N}_1}$ is the trivial bundle and let $\Phi : \mathbf{S}^1 \times \mathbb{R} \to \mathbf{E}_1|_{\partial \mathbf{N}_1}$ be a trivialization. Denote by φ the restriction of Φ to the null section. We can consider the line bundle over $\mathbf{N} = \mathbf{N}_1 \cup_\varphi \mathbf{D}^2$ whose total space is $\mathbf{E} = \mathbf{E}_1 \cup_\Phi (\mathbf{D}^2 \times \mathbb{R})$ and whose projection is the obvious one. Here we have denoted by $X \cup_f Y$ the union of X and Y along the mapping $f : A \subset X \to Y$.

DEFINITION 2.21. We call closure of $\mathbf{E}_1$ the bundle constructed this way, and we denote it by $\widehat{\mathbf{E}}_1$.

Let $\mathbf{N}$ be a compact, connected surface without boundary and $D \subset \mathbf{N}$ an embedded disk. Since $\mathbf{E}|_D$ is the trivial bundle, we have the following:

PROPOSITION 2.22. *Every line bundle over* $\mathbf{N}$ *is the closure of some bundle over a compact, connected surface with one boundary component.*

Since the only line bundle over the sphere is the trivial one, from now on we will suppose that $g(\mathbf{N}) > 0$. Call weighted genus of a compact, connected surface $\mathbf{N}$, the number $h(\mathbf{N}) = 2g(\mathbf{N})$ if $\mathbf{N}$ is orientable, $h(\mathbf{N}) = g(\mathbf{N})$ otherwise. It is a classical fact in topology that a surface $\mathbf{N}$ is the quotient of a $2h(\mathbf{N})$-gon, by pairwise identification of its edges, and such a polygon can be found by cutting $\mathbf{N}$ along a minimal system of generators of its fundamental group. Fix the following data: a point $\infty \in \mathbf{N}$; a set $\Omega = \{\omega_i\}$ of smooth curves, as in (2) of Definition 1.18; a small open ball U_∞ around the point ∞. Let $\mathbf{N}_1 = \mathbf{N} - U_\infty$; clearly, cutting $\mathbf{N}_1$ along all the curves $\omega_i \cap \mathbf{N}_1$, we get a $4h$-gon $P_{\mathbf{N}_1}$. Call ℓ_i, ℓ_i' the two edges resulting by the cut along ω_i. For all i, let $\psi_i : \ell_i \to \ell_i'$ be the diffeomorphism giving the identification. Suppose to have a distribution of weights, $\rho_i = \pm 1$, on the set Ω, and define $\mathbf{E}_\rho = P_{\mathbf{N}_1} \times \mathbb{R}/_\sim$ where $(x_1, t_1) \sim (x_2, t_2)$ if and only if either the two pairs are equal or $x_1 \in \ell_i, x_2 = \psi_i(x_1)$ and $t_2 = \rho_i t_1$ for some i. Clearly $\mathbf{E}_\rho$, endowed with the obvious projection, is a line bundle over $\mathbf{N}_1$.

PROPOSITION 2.23. *The line bundle* $\mathbf{E}_\rho|_{\partial \mathbf{N}_1}$ *is trivial.*

PROOF. Such a bundle is obtained by successively pasting $2h$ copies of $[0,1] \times \mathbb{R}$, by means of mappings of the following two kinds: $(1,t) \mapsto (0,t)$, $(1,t) \mapsto (0,-t)$, and both kinds of such identifications are even in number. □

By the previous proposition and Proposition 2.22 we see that every distribution of weights ρ over Ω generates a line bundle over $\mathbf{N}$, simply by taking the closure $\widehat{\mathbf{E}}_\rho$ of the just-constructed bundle $\mathbf{E}_\rho$. Simple technical arguments prove the following:

PROPOSITION 2.24. *If* $\mathbf{E}$ *is a line bundle over* $\mathbf{N}$ *then there exists a distribution of weights* ρ *over* Ω *such that* $\mathbf{E}$ *is isomorphic to* $\widehat{\mathbf{E}}_\rho$.

The general case. Turn back to our problem. Let $F : S \to \mathbf{N}$ be an excellent map and $\mathbf{E} \xrightarrow{\pi} \mathbf{N}$ a line bundle. Let R be a system of segments, U_∞, $\widetilde{S}$ and $\mathbf{N}_1 = \mathbf{N} - U_\infty$ be as before. By Proposition 2.24 we can suppose that $\mathbf{E} = \widehat{\mathbf{E}}_\rho$ for some distribution of weights ρ; hence, by Proposition 2.12 and Remark 2.10, our problem is the same as finding a section σ of $\widetilde{F}^*\mathbf{E}_\rho \to \widetilde{S}$, such that $\pi_\rho^* \widetilde{F} \circ \sigma$ is an embedding. First of all, let us try to understand the meaning of finding a section of the line bundle $\widetilde{F}^*\mathbf{E}_\rho$. Let S' be the surface obtained from $\widetilde{S}$ by cutting along $\widetilde{F}^{-1}(\Omega)$, and let $\xi : S' \to \widetilde{S}$ be the quotient map given by the cut. Let $\varphi : P_{\mathbf{N}_1} \to \mathbf{N}_1$ be the mapping generated by the cut of $\mathbf{N}_1$ along Ω; then there exists a unique map $F' : S' \to P_{\mathbf{N}_1}$ such that $\varphi \circ F' = \widetilde{F} \circ \xi$. By this fact, the way $\mathbf{E}_\rho$ was defined and thanks to the following commutative diagram

$$\begin{array}{ccccccccc}
P_{\mathbf{N}_1} \times \mathbb{R} & \longleftarrow & \xi^* \widetilde{F}^* \mathbf{E}_\rho & \longrightarrow & \widetilde{F}^* \mathbf{E}_\rho & \longrightarrow & \mathbf{E}_\rho & \longleftarrow & P_{\mathbf{N}_1} \times \mathbb{R} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \pi_\rho & & \downarrow \\
P_{\mathbf{N}_1} & \underset{F'}{\longleftarrow} & S' & \underset{\xi}{\longrightarrow} & \widetilde{S} & \underset{\widetilde{F}}{\longrightarrow} & \mathbf{N}_1 & \underset{\varphi}{\longleftarrow} & P_{\mathbf{N}_1}
\end{array}$$

we see that finding a section of $\widetilde{F}^*\mathbf{E}_\rho$ is the same as finding a function $h : S' \to \mathbb{R}$, such that:

$$h(x_1) = \rho_i h(x_2) \text{ if } F(\xi(x_1)) = F(\xi(x_2)) \in \omega_i \quad \forall x_1, x_2 \text{ s.t. } \xi(x_1) = \xi(x_2) \tag{2.4}$$

and satisfying the anologous conditions on the differentials. We will denote by σ_h the section of $\widetilde{F}^*\mathbf{E}_\rho$ defined by h. Clearly the following holds:

PROPOSITION 2.25. *The mapping* $\pi^* \widetilde{F} \circ \sigma_h : \widetilde{S} \to \mathbf{E}_\rho$ *is an embedding if and only if* $(F', h) : S' \to P_{\mathbf{N}_1} \times \mathbb{R}$ *is.*

We will call such a function h a *strange* height for the mapping F.

Let us introduce some more notation. Let $\widetilde{H}(F) = (H(F) \cap \widetilde{S}) \cup \partial \widetilde{S}$, and $H'(F) \subset S'$ be the set $H'(F) = \xi^{-1}(\widetilde{H}(F))$. Denote by $\mathcal{C}$ the set of connected components of $S - H(F)$, by $\widetilde{\mathcal{C}}$ the set of components of $\widetilde{S} - \widetilde{H}(F)$ and finally by $\mathcal{C}'$ the set of components of $S' - H'(F)$. It is immediate that inclusion $\widetilde{S} \hookrightarrow S$

gives a bijection between $\widetilde{\mathcal{C}}$ and $\mathcal{C}$; while $\xi : S' \to \widetilde{S}$ gives a bijection between $\mathcal{C}'$ and $\widetilde{\mathcal{C}}$. For all $C \in \mathcal{C}$ denote by $\widetilde{C} \in \widetilde{\mathcal{C}}$ and by $C' \in \mathcal{C}'$ the corresponding components (i.e. $\widetilde{C}$ is the interior of $C \cap \widetilde{S}$ and C' is the component such that $\xi(C') = \widetilde{C}$).

PROPOSITION 2.26. *Let C_1, $C_2 \in \mathcal{C}$; then $F(C_1) \cap F(C_2) \neq \emptyset$ if and only if $F(\widetilde{C}_1) \cap F(\widetilde{C}_2) \neq \emptyset$, if and only if $F(C_1') \cap F(C_2') \neq \emptyset$.*

PROOF. The second equivalence is immediate, the first follows from the fact that if $F(C_1) \cap F(C_2) \neq \emptyset$ and $F(\widetilde{C}_1) \cap F(\widetilde{C}_2) = \emptyset$ then $F(C_1) \cap F(C_2) \subset U_\infty$ and the fact that $\partial(F(C_1) \cap F(C_2))$ contains branches of Γ, while $U_\infty \cap \Gamma = \emptyset$. □

PROPOSITION 2.27. *Let $C_1, C_2 \in \mathcal{C}$; then $F(\widetilde{C}_1) \cap F(\widetilde{C}_2)$ and $F'(C_1') \cap F'(C_2')$ are connected.*

PROOF. Exactly the same as Proposition 2.8. □

Let h be a *strange* height for the excellent map F; by the same arguments used in the proof of Proposition 2.13, the following can be shown:

PROPOSITION 2.28. *Let C_1', $C_2' \in \mathcal{C}'$ be such that $F'(C_1') \cap F'(C_2') \neq \emptyset$. One of the following two holds:*

(1) $\forall x \in C_1', \ y \in C_2'$ *if* $F(x) = F(y)$ *then* $h(x) < h(y)$;

(2) $\forall x \in C_1', \ y \in C_2'$ *if* $F(x) = F(y)$ *then* $h(x) > h(y)$.

As in the case of the trivial bundle we give a structure of oriented graph to the set $\mathcal{C}$, saying $C_1 \to C_2$ if and only if the corresponding components C_1', $C_2' \in \mathcal{C}'$ satisfy condition (1). We call such a graph the graph of h and denote it by $L(h)$.

REMARK 2.29. By Proposition 2.26 we see that two components C_1, $C_2 \in \mathcal{C}$ are joined by an edge if and only if $F(C_1) \cap F(C_2) \neq \emptyset$, hence the situation is in all similar to the one we had dealing with the trivial bundle.

As in the case of the trivial bundle, the following is proved:

PROPOSITION 2.30. *The graph $L(h)$ has no loop.*

Denote $\Omega^- = \{\omega \in \Omega | \rho(\omega) = -1\}$, an analogue of Proposition 2.15 holds in this case too.

PROPOSITION 2.31. *Let h be a strange height for F, and let C_1, C_2, C_3,$C_4 \in \mathcal{C}$, ℓ, ℓ_1, ℓ_2 be proper edges of $H(F)$; the followings hold:*

(A1) *suppose that C_1, C_2 paste along ℓ (i.e. $\bar{C}_1 \cap \bar{C}_2 \supseteq \ell$) and $F(C_3) \cap F(\ell) \neq \emptyset$, then either $C_1 \to C_3$ and $C_2 \to C_3$ or $C_3 \to C_1$ and $C_3 \to C_2$;*

(B1) *suppose that C_1, C_2 paste along ℓ_1, C_3, C_4 paste along ℓ_2 and $F(\ell_1) \cap F(\ell_2) \neq \emptyset$. Let $r \in R$ be the segment containing $F(\ell_i)$, and suppose that $F(C_1)$, $F(C_3)$ are on the left of r and $F(C_2)$, $F(C_4)$ on its right; then:*

if $r \in R - \Omega^-$ either $C_1 \to C_3$ and $C_2 \to C_4$ or $C_3 \to C_1$ and $C_4 \to C_2$; if $r \in \Omega^-$ either $C_1 \to C_3$ and $C_4 \to C_2$ or $C_3 \to C_1$ and $C_2 \to C_4$.

PROOF. Observe that a situation as in (A1) occurs only if $F(\ell)$ is contained either in Γ or in $R - \Omega$; hence the same argument used to prove (A) of Proposition 2.15 proves (A1) too. We now prove (B1). If $F(\ell_1)$ and $F(\ell_2)$ are contained in some segment $r \in R - \Omega$, as in the previous case, the proof is the same as for Proposition 2.15; hence suppose $F(\ell_1), F(\ell_2) \subset \omega \in \Omega$, and let $x_1 \in \ell_1$, $x_2 \in \ell_2$ be such that $F(x_1) = F(x_2)$. It is easily seen that we can suppose $x_1, x_2 \in \widetilde{S}$. Let ℓ'_1, ℓ''_1 be the edges of $H'(F)$ such that $\xi(\ell'_1) = \xi(\ell''_1) = \ell_1$ and let ℓ'_2, ℓ''_2 be those such that $\xi(\ell'_2) = \xi(\ell''_2) = \ell_2$. The assumption in (B1) on the components C_1, C_2, C_3 and C_4 ensures that C'_1, C'_2, C'_3 and C'_4 verify the following:

$$\partial C'_1 \supset \ell'_1,\ \partial C'_2 \supset \ell''_1,\ \partial C'_3 \supset \ell'_2,\ \partial C'_4 \supset \ell''_2;$$
$$F'(C'_1) \cap F'(C'_3) \neq \emptyset,\ F'(C'_2) \cap F'(C'_4) \neq \emptyset.$$

Let $x'_1 \in \ell'_1$, $x''_1 \in \ell''_1$, $x'_2 \in \ell'_2$, $x''_2 \in \ell''_2$ be such that $\xi(x'_i) = \xi(x''_i) = x_i$. It is not hard to find four path (see Fig. 17) $\gamma'_i : [0,1] \to S'$, $\gamma''_i : [1,2] \to S'$, such that:

(1) $$\gamma'_i(1) = x'_i \text{ and } \gamma''_i(1) = x''_i \quad i = 1,2;$$

(2) $$\forall t \neq 1 \quad \begin{cases} \gamma'_1(t) \in C'_1, & \gamma''_1(t) \in C'_2, \\ \gamma'_2(t) \in C'_3, & \gamma''_2(t) \in C'_4; \end{cases}$$

(3) $$\forall t \quad \begin{cases} F'(\gamma'_1(t)) = F'(\gamma'_2(t)), \\ F'(\gamma''_1(t)) = F'(\gamma''_2(t)). \end{cases}$$

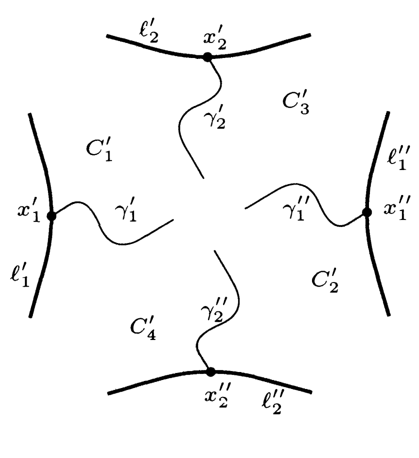


Fig. 17

Define:

$$g_i(t) = \begin{cases} h(\gamma_i'(t)) & \text{if } t \in [0,1] \\ \rho(\omega)h(\gamma_i''(t)) & \text{if } t \in [1,2] \end{cases} \qquad i = 1,2.$$

By definition of a *strange* height and (1), we see that g_1 and g_2 are continuous functions. Consider the two cases $\omega \in \Omega^+ = \Omega - \Omega^-$ and $\omega \in \Omega^-$. In the first case, suppose by contradiction that $C_1' \to C_3'$ and $C_4' \to C_3'$. By definition of $\to$ and (2), (3), $g_1(t) < g_2(t)$ for all $t < 1$ and $g_1(t) > g_2(t)$ for all $t > 1$. This implies that $g_1(1) = g_2(1)$, hence $F'(x_1') = F'(x_2')$ and $h(x_1') = h(x_2')$, contradicting the fact that (F', h) is injective. A similar argument concludes the proof in the second case too. □

As in the case of the trivial bundle, Proposition 2.30 and Proposition 2.31 give necessary and sufficient conditions for the existence of a *strange* height for an excellent map F.

THEOREM 2.32. *Let $F : S \to \mathbf{N}$ be an excellent map, R a system of segments for $F|_\Sigma$ and $\mathbf{E} = \widehat{\mathbf{E}}_\rho$ a line bundle over $\mathbf{N}$. Then F is factorizable into $\mathbf{E}$ if and only if $\mathcal{C}$ can be given a structure of oriented graph with no loops verifying* (A1) *and* (B1) *of Proposition* 2.31.

PROOF. The proof is completely similar to that of Theorem 2.16. □

REMARK 2.33. Repeating almost word by word the statements and proofs in this section, a factorizability theorem for generic immersions can be proved. More precisely, let S be a compact surfece with boundary, $F : S \to \mathbf{N}$ a generic immersion and R a system of segments for $F|_{\partial S}$; denote $G(F) \subset S$ the graph which consists of ∂S and of those arcs in $F^{-1}(R)$ which have both vertices in $\partial S \cup F^{-1}(\infty)$. $G(F)$ is the 1-skeleton of a cell decomposition of S such that F restricted to the interior of each cell is one to one. Denote by $\mathcal{C}$ the set of 2-cells of such a decomposition.

THEOREM 2.34. *A generic immersion F is factorizable into $\mathbf{E} = \widehat{\mathbf{E}}_\rho$ if and only if $\mathcal{C}$ can be given a structure of oriented graph with no loops verifying* (A1) *and* (B1) *of Proposition* 2.31, *seen in the present context.*

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