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A Characterization of Integral Elliptic Automorphic Forms

ANDREA MORI

0. - Introduction

(0.1) A basic ingredient of the "geometric" approach [1, 6] to the theory of arithmetic modular forms is the so-called *q-expansion principle*. It says, essentially, that a modular form f of weight k and level N is defined over the $\mathbb{Z}\left[\frac{1}{N}, \varsigma_N\right]$-algebra generated by its Fourier coefficients, ς_N being a primitive n-th root of unity.

The goal of this paper is to prove a similar result, where instead of considering the Fourier expansions, we consider expansions at the points corresponding to elliptic curves with complex multiplications. Let Γ be a congruence subgroup without elliptic elements of $\mathrm{SL}_2(\mathbb{Z})$ acting on the upper half-plane $\mathcal{H}$, let Y_Γ be the affine canonical model [19] of the quotient $\Gamma\backslash\mathcal{H}$, $X_\Gamma = Y_\Gamma \cup \{\text{cusps}\}$ its closure and k_Γ its field of definition. Our main result is the following:

THEOREM 1 (Integrality Criterion). *Let f be a holomorphic Γ-automorphic form of weight k. Let K be a number field containing k_Γ, v a non-archimedean place of K such that X_Γ has good reduction modulo v, E a CM curve defined over K with ordinary good reduction modulo v corresponding to a K-rational point of Y_Γ. Let $\tau \in \mathcal{H}$ such that $E \otimes \mathbb{C} \simeq \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$. Then f is defined over $\mathcal{O}_{(v)} = \mathcal{O}_v \cap K$ if and only if*

$$c_r(f) = \frac{(-4\pi)^r (2\pi i)^{k+2r}}{\Omega_v^{k+2r}} (\delta_k^r f)(\tau) \in \mathcal{O}_{(v)}$$

and

$$v\left(\sum_{i=1}^{r} b_{i,r} c_i(f)\right) \geq v(r!) \tag{1}$$

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for each $r \geq 0$, *where* δ_k^r *is the* r*-th iterate of the Maaß operator* δ_k, Ω_v *is a* v*-adic period of* E *and* $\sum_{i=1}^{r} b_{i,r} X^i = r!\binom{X}{r}$.

The strategy used to prove this result has two distinct phases. First, we find a significant local parameter at a CM point $x \in Y_\Gamma$, which is to play the role of $q = e^{2\pi i z}$ in the Fourier expansion case. Then, once this parameter is chosen, and the form f is expanded around x with respect to it, the second problem is to compute the coefficients of the expansion only in terms of f and of the elliptic curve corresponding to x.

The problem of finding a good local parameter at a CM point is attacked in Sections 1 and 2. We study the natural action of the complex multiplications on the ring $\hat{\mathcal{O}}_x$ (the fiber at x of the jet bundle) and we show that the local parameter eigenvector of this action is in fact defined over a sufficiently large number field (Theorem 3). Moreover, it is shown (with the specified restrictions on the elliptic curve under consideration and on the place of reduction) that this eigenparameter is strongly related to the Serre-Tate parameter classifying formal deformations of ordinary elliptic curves in positive characteristic. This relation with formal geometry allows us to characterize the v-integral jets at x (Theorem 10).

After a brief review of the different aspects of the theory of the Maaß operators (for a more detailed exposition, with proofs, see [3, 4, 8]), the final part of Section 3 contains an explicit computation (based mainly on the results of [9]) of the coefficients. This computation will be used to prove Theorem 1 in the last section.

(0.2) In essence, our proof of Theorem 1 exploits only the fact that the modular curves Y_Γ are naturally the base of algebraic "universal" families of elliptic curves. This seems to suggest that our methods can be generalized in order to prove integrality criteria for much more general automorphic forms; in particular, results of this kind may be of great interest in the case of compact quotients, where Fourier expansions are not available.

In [15] the author extends the result presented here to forms of even weight automorphic with respect to norm 1 subgroups of an indefinite quaternion division algebra over $\mathbb{Q}$ (see [19, § 9.2]). The Shimura curve associated to such an algebra is the moduli space for the family of abelian surfaces (i.e. 2-dimensional abelian varieties) with quaternionic multiplication by the algebra itself (e.g. see [12, 18]).

Also, Theorem 1 can be partially extended to primes dividing the level using the theory [10] of bad reduction of modular curves, see [14].

(0.3) It should also be noted that a possible important consequence of the method of proving Theorem 1 is that the evaluation of the iterates $\delta_k^r f$ at τ offers a way to attach a v-adic power series, i.e. an element of the Iwasawa algebra, to a pair $(f, q : F \hookrightarrow \mathrm{GL}_2(\mathbb{Q}))$, where f is a v-adic modular form, F an imaginary quadratic extension of $\mathbb{Q}$ in which the rational prime under v splits, and q is an embedding normalized in the sense of [19, Ch. 4]. Although the author is presently not able to state any precise result, or even a conjecture, this seems to be relevant for the theory of special values of L-functions.

(0.4) *Acknowledgements.* This paper is a condensed version of the author's Ph.D. thesis written at Brandeis University under the supervision of M. Harris. The author wishes to express his gratitude to Prof. Harris for the invaluable help and guidance.

An earlier version of this paper and the brief proof-less presentation [13] of the main result were written while the author was supported by a research fellowship of the Istituto Nazionale di Alta Matematica in Rome, Italy. The reader should be warned that the expression for the numbers $c_r(f)$ appearing in [13, 14] is correct only up to a constant; the right expression is the one shown here.

(0.5) *Notations and Conventions.* The symbols $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ denote, as usual, the integers, the rational and the complex numbers respectively. By a *number field* we shall always mean a finite extension of $\mathbb{Q}$, which will be thought as a subfield of $\mathbb{C}$ (in other words, let us fix once for all an embedding $\sigma : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$). If v is a non-archimedean place of a number field K, the symbols K_v, $\mathcal{O}_v$ and $\mathcal{O}_v^{\mathrm{nr}}$ denote the v-adic completion of K, the ring of integers of K_v, and the ring of integers of the maximal unramified extension of K_v (which is also, for our purposes, the strict henselization of $\mathcal{O}_v$) respectively.

If X is a scheme over (the spectrum of) a ring R and $\phi : R \to R'$ is a map of rings, we shall denote $X \otimes_\phi R'$ (or simply $X \otimes R'$) the scheme over R' obtained by base extension along the natural map $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$.

If a x is a point of a scheme X we shall denote $J^{(n)}_{x,X}$ (respectively $J^{\infty}_{x,X}$) the stalk at x of the sheaf of the n-jets (respectively the ∞-jets) on X.

If X is a scheme over a DVR R with uniformizer π, and if $x : \mathrm{Spec}(R) \to X$ is an R-rational point of X, set $x_0 = x((0))$ and $x_\pi = (\pi R)$.

1. - Local parameters eigenvectors of complex multiplications

(1.1) Let K be a number field and x a K-rational point on the affine modular curve $Y_\Gamma = \Gamma\backslash\mathcal{H}$ corresponding to an elliptic curve E (endowed with a Γ-structure) defined over K. Assume that E has complex multiplications and let $K_o = \mathrm{End}_o(E) = \mathrm{End}(E) \otimes \mathbb{Q}$. The field K_o is a quadratic imaginary extension of $\mathbb{Q}$ and we shall always assume that $K_o \subset K$. Let

$$\phi_\Gamma : \mathcal{H} \longrightarrow Y_\Gamma \tag{2}$$

be the natural quotient map and pick $\tau = \tau_E \in \mathcal{H}$ such that $\phi_\Gamma(\tau) = x$. The complex torus $E \otimes \mathbb{C}$ is thus isomorphic to $\mathbb{C}/\Lambda_\tau$, where $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$. The action of the complex multiplications on the torsion points of $E \otimes \mathbb{C}$ lifts to an embedding $q_\tau : K_o^\times \hookrightarrow \mathrm{GL}(\Lambda_\tau \otimes \mathbb{Q})$. Explicitly, for any $\mu = \alpha + \beta\tau \in K_o^\times$, with $\alpha, \beta \in \mathbb{Q}$, we have

$$q_\tau(\mu) = \begin{pmatrix} \alpha + \beta \operatorname{Tr}_{K_o/\mathbb{Q}}\tau & -\beta N_{K_o/\mathbb{Q}}\tau \\ \beta & \alpha \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q}). \tag{3}$$

Hence, a complex multiplication $\mu \in K_o^\times$ acts on $\mathcal{H}$ via $q_\tau(\mu)$, fixing τ. Its action does not induce in general an action on the modular curve Y_Γ, because it does not preserve Γ-orbits. Nevertheless we have:

PROPOSITION 2. *The action of $\mu \in K_o^\times$ on $\mathcal{H}$ gives rise to an automorphism ρ_μ of $\hat{\mathcal{O}}_{x,Y_\Gamma}$.*

PROOF. As (2) is a local isomorphism of analytic varieties, it is enough to observe that, by discreteness of Γ, we can find open analytic neighborhoods $U \subset \mathcal{H}$, $\tau \in U$, and $V \subset Y_\Gamma$, $x \in V$, such that $U \simeq V$ via ϕ_Γ and:

(1) for each $z_1, z_2 \in U$, if $z_1 = \gamma z_2$ for some $\gamma \in \Gamma$ then $z_1 = z_2$;

(2) for each $z_1, z_2 \in U$, if $q_\tau(\mu)z_1 = \gamma q_\tau(\mu)z_2$ for some $\gamma \in \Gamma$ then $z_1 = z_2$.

Finally, ρ_μ is non-zero (and in fact invertible) because $\left.\frac{d}{dz}\right|_{z=\tau} q_\tau(\mu)z = \bar{\mu}/\mu \neq 0$, as easily computed from (3). □

(1.2) The rest of this section will be devoted to proving the following statement.

THEOREM 3. *Let K be a number field and x a K-rational point on the modular curve Y_Γ corresponding to the CM curve E. Then there is a local parameter U at x, rational over K, which is an eigenvector for the action of all complex multiplications of E on $\hat{\mathcal{O}}_x$.*

This will be established in three steps. First we will compute explicitly the action in terms of the natural parameter $z - \tau$, obtaining a complex eigenvector. Next, we will establish the K-rationality of the maps ρ_μ (Proposition 5) and finally we will reduce the proof of Theorem 3 to an elementary result of linear algebra (Lemma 6).

(1.3) Working with the analytic varieties $\mathcal{H}$ and Y_Γ (i.e. working "over $\mathbb{C}$") it is natural to choose the local parameter $Z = z - \tau$ to make the identification $\hat{\mathcal{O}}_{\tau,\mathcal{H}}^{\mathrm{hol}} \simeq \mathbb{C}[[Z]]$, which will be used to make the maps ρ_μ of Proposition 2 explicit (at least up to the isomorphism $\hat{\mathcal{O}}_{\tau,\mathcal{H}}^{\mathrm{hol}} \simeq \hat{\mathcal{O}}_{x,Y_\Gamma}$ induced by (2)). For all $n \geq 1$ and $\mu = \alpha + \beta\tau \in K_o^\times$, we have

$$\rho_\mu(Z^n) = \left(\frac{\bar{\mu}}{\mu}\right)^n Z^n \sum_{j=0}^{\infty} \binom{n+j-1}{j} \left(-\frac{\beta}{\mu}\right)^j Z^j. \tag{4}$$

In particular $\rho_\mu(m_\tau^n) \subseteq m_\tau^n$, so that ρ_μ defines maps $\rho_{\mu,n} : \hat{\mathcal{O}}_x/m_x^{n+1} \longrightarrow \hat{\mathcal{O}}_x/m_x^{n+1}$, for $n = 1, 2, \ldots$. For $\mu \notin \mathbb{Q}$ set $\lambda = \lambda_\mu = \dfrac{\bar{\mu}}{\mu}$, $\eta = \eta_\mu = \dfrac{\beta}{\mu}$, $\epsilon = \epsilon_\mu = \dfrac{\eta}{\lambda - 1} = \dfrac{1}{2\,\mathrm{Im}(\tau)}$; then we have:

PROPOSITION 4. *There exists $U_n \in \hat{\mathcal{O}}_x/m_x^{n+1}$ which is a λ-eigenvector for all maps $\rho_{\mu,n}$, with $\mu \in K_o^\times$. After the prescribed identifications*

$$U_n = Z + \epsilon Z^2 + \epsilon^2 Z^3 + \cdots + \epsilon^{n-1} Z^n \pmod{Z^{n+1}}.$$

PROOF. It is enough to check that the given U_n is multiplied by λ mod Z^{n+1}, under (4). As ϵ does not in fact depend on μ, this will also prove the first part of the proposition.

The result can be proven by induction on n, the case $n = 1$ being obvious. Assume that the result is true for $U_{n-1} \in \hat{\mathcal{O}}_x/m_x^n$ and write $U_n = V_n + \epsilon^{n-1} Z^n$ with $V_n \equiv U_{n-1} (\mathrm{mod}\ Z^n)$. Then, by the induction hypothesis, $\rho_{\mu,n}(U_n) = \lambda V_n + cZ^n$ and we have to check that $c = \lambda\epsilon^{n-1}$. Using (4):

$$\begin{aligned} c &= \lambda \sum_{k=0}^{n-1} \binom{n-1}{k} (-\eta)^k \frac{\eta^{n-k-1}}{(\lambda-1)^{n-k-1}} \lambda^{n-k-1} \\ &= \lambda\eta^{n-1} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{\lambda^{n-k-1}}{(\lambda-1)^{n-k-1}} \\ &= \lambda \left(\frac{\eta}{\lambda-1}\right)^{n-1} \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda^{n-k-1}(1-\lambda)^k = \lambda\epsilon^{n-1}. \qquad \square \end{aligned}$$

This explicit description of the eigenvector U_n shows that for $n \to \infty$ the U_n's converge to an element $U \in \hat{\mathcal{O}}_x$ which is a λ-eigenvector for all complex multiplications of $\mathbb{C}/\Lambda_\tau$, and also a local parameter, as $U \in m_x - m_x^2$.

From now on, we will often think of the ring $\hat{\mathcal{O}}_{x,Y_\Gamma}$ as the fiber at x of the jet bundle on Y_Γ [2; § IV.16.4.12].

(1.4) We shall now show that the element $U \in \hat{\mathcal{O}}_x$ constructed in the previous subsection is in fact rational over K. To do this it is enough to establish the rationality over K of at least one of the maps ρ_μ of Proposition 2 with $\mu \in \mathrm{End}(E)$ and $\mu \notin \mathbb{Z}$.

PROPOSITION 5. *Let K be a number field, and x a K-rational point of Y_Γ corresponding to a CM curve E with $K_o = \mathrm{End}_o(E) \subset K$. Then there is a $\mu \in \mathrm{End}(E)$, $\mu \notin \mathbb{Z}$, such that the action ρ_μ on the fiber of the jet bundle at x is rational over K.*

PROOF. Let $\mu \in \mathrm{End}(E)$, $\mu \neq 0$, and let $\tau \in \mathcal{H}$ and $q_\tau(\mu) \in \mathrm{GL}_2^+(\mathbb{Q})$ be defined as in subsection 1.1. Then, as explained in [19, § 7.2], $q_\tau(\mu)$ defines a modular correspondence

$$Y_\mu = \{(\phi_\Gamma(z), \phi_\Gamma(q_\tau(\mu)z)) \mid z \in \mathcal{H}\} \subset Y_\Gamma \times Y_\Gamma.$$

The action ρ_μ can be realized as follows. Look at the maps

$$Y_\Gamma \xleftarrow{\pi_2} Y_\mu \xrightarrow{i} Y_\Gamma \times Y_\Gamma \xrightarrow{\pi_1} Y_\Gamma$$

where i is the inclusion, π_1 is the projection on the first factor and π_2 is the restriction to Y_μ of the projection on the second factor. Then we have a map of sheaves on Y_Γ:

$$\mathcal{O}_{Y_\Gamma} \longrightarrow \pi_{2,*}i^*\pi_1^*\mathcal{O}_{Y_\Gamma},$$

and hence a map of stalks

$$\mathcal{O}_{x,Y_\Gamma} \longrightarrow [\pi_{2,*}i^*\pi_1^*\mathcal{O}_{Y_\Gamma}]_x. \tag{5}$$

Now $[\pi_{2,*}i^*\pi_1^*\mathcal{O}_{Y_\Gamma}]_x = \bigoplus \mathcal{O}_{z,Y_\Gamma}$, the direct sum being extended over the (finite) set $\{z \in Y_\Gamma | (z,x) \in Y_\mu\}$. In particular, $\mathcal{O}_{x,Y_\Gamma}$ is itself a direct summand of the right-hand side of (5). After completing with respect to the m_x-adic topology, the map ρ_μ is exactly the composition of (5) with the projection on the $\mathcal{O}_{x,Y_\Gamma}$ factor.

Thus, it would be enough to know that all the varieties and subvarieties under consideration are defined over K. But in fact Y_Γ has a model over (a subfield of) K, and the same is true for at least a Y_μ, with μ as specified, by [19, §§ 7.2 and 7.3]. □

We can now show that also the eigenvectors U_n and $U = \lim U_n$ are rational over K. To do this, we shall exploit two facts:

(1) the algebras $\hat{\mathcal{O}}_x/m_x^{n+1}$ have a natural filtration which is respected by the action of the complex multiplications;

(2) the eigenvalues are in $K_o \subset K$.

Propositions 4 and 5 reduce the task to the following:

LEMMA 6. *Let $F \subset L$ be two fields. Let V be a finite dimensional vector space over F and $\psi_F \in \mathrm{End}(V)$. Let $W = V \otimes_L F$ and consider $\psi_L = \psi_F \otimes 1 \in \mathrm{End}(W)$. Suppose that:*

(1) V has a filtration $V = V_1 \supset V_2 \supset \cdots \supset V_n \supset V_{n+1} = \{0\}$ with $\dim(V_i/V_{i+1}) = 1$ such that $\psi_F(V_i) \subseteq V_i$, for $i = 1, \ldots, n$;

(2) For each $i = 1, \ldots, n$ there exists $w_i \in W_i = V_i \otimes L$, $w_i \neq 0$, such that $\psi_L(w_i) = \xi^i w_i$ for some $\xi \in F^\times$.

Then, if ξ is not a root of unity, we can find $v_1, \ldots, v_n \in V$ such that $\psi_F(v_i) = \xi^i v_i$.

PROOF. We shall construct $v_1, \ldots, v_n$ inductively, starting with v_n. Since $V_n \otimes L = W_n = Lw_n$, we can find v_n simply by multiplying w_n by a suitable scalar. Suppose that we have already constructed $v_{k+1}, \dots, v_n$ for some $k \geq 1$. Pick $v \in V$ such that $\{v, v_{k+1}, \dots, v_n\}$ is a basis for V_k. Thus

$$w_k = av + \sum_{j=k+1}^{n} a_j v_j \tag{6}$$

where $a_{k+1}, \dots, a_n \in L$ and $a \in L^\times$. If $a_{k+1} = \cdots = a_n = 0$ then v is already a ξ^k-eigenvector, so we may assume that $a_{k+1}, \dots, a_n \in L^\times$ (if only some of these coefficients are non-zero, the following reduction procedure is shorter but not at all different). Apply ψ_L to both sides of (6) to get

$$\xi^k w_k = a\psi_L(v) + \sum_{j=k+1}^{n} a_j \xi^j v_j, \tag{7}$$

and subtract (7) from (6) multiplied by ξ^{k+1}. The result is $w_k = av' + \sum_{j=k+2}^{n} b_j v_j$, where $b_j = a_j(\xi - \xi^{j-k})(\xi - 1)^{-1}$ and $v' = \xi(\xi - 1)^{-1}v - \xi^{-k}(\xi - 1)^{-1}\psi_L(v) \in V$. Iterating this procedure we can eliminate all the coefficients of the v_j's in (6) and finally write $w_k = av_k$ for some $v_k \in V$. □

Let us fix $n > 0$ and apply Lemma 6 to the situation

$$\begin{cases} F = K, \quad L = \mathbb{C}, \\ V = m_{x,Y_\Gamma}/m_{x,Y_\Gamma}^{n+1}, \quad V_i = m_{x,Y_\Gamma}^i V, \\ \psi_F = \rho_\mu|_V \quad \text{(any } \mu \text{ as in Proposition 5)}, \\ \xi = \lambda_\mu = \dfrac{\bar{\mu}}{\mu} \in K^\times. \end{cases}$$

It is now clear that the n-jet U_n has a K-rational multiple. The existence of a K-rational λ_μ-eigenvector in $\hat{\mathcal{O}}_x$ is easily obtained taking the limit. This concludes the proof of Theorem 3.

2. - Integrality properties of the eigenparameter

(2.1) Let K be as above, and let v be a place of K with associated prime $\mathbf{p} = \mathbf{p}_v \subset \mathcal{O}_K$ and residue field k_v of characteristic p, such that the canonical model X_Γ has a smooth v-adic model. If, for instance, Γ is one of the groups $\Gamma(N)$, $\Gamma_0(N)$ or $\Gamma_1(N)$, these places v are exactly those not dividing N. Thus, we will think of Y_Γ as a smooth scheme over (the spectrum of) $\mathcal{O}_v^{\mathrm{nr}}$.

Let x be a K-rational point of Y_Γ corresponding to an elliptic curve E with complex multiplications, and assume that E has *ordinary* good reduction $\tilde{E}$ modulo v. Thus E defines in fact a $\mathcal{O}_v^{\mathrm{nr}}$-rational point of Y_Γ, which we denote by x again. The goal of this section is to characterize numerically the fiber $J^\infty_{x_\pi,Y_\Gamma}$ in terms of the K-rational parameter of Theorem 3. This will be achieved by Theorem 10.

We know from [2, § IV.16.4.2] that there are canonical isomorphisms $J^{(n)}_{x_0} = \mathcal{O}_{x_0}/m^{n+1}_{x_0}$ and $J^\infty_{x_0} = \hat{\mathcal{O}}_{x_0}$. As x is a smooth point, the choice of a $\mathcal{O}_v^{\mathrm{nr}}$-rational local parameter T at x will provide a non-canonical identification

$$J^\infty_{x_\pi} \simeq \mathcal{O}_v^{\mathrm{nr}}[[T]]. \tag{8}$$

For any $n \geq 0$, the map of sheaves $\mathrm{jet}^{(n)} : \mathcal{O}_{Y_\Gamma} \longrightarrow \mathrm{jet}^{(n)}_{Y_\Gamma/\mathcal{O}_v^{\mathrm{nr}}}$ described in [2, § IV.16.3] defines a map of rings

$$H^0(\mathcal{O}_{Y_\Gamma}) \longrightarrow J^{(n)}_x. \tag{9}$$

For $m \geq 0$ set $R^n_m = J^{(n)}_{x_\pi} \otimes (\mathcal{O}_v^{\mathrm{nr}}/\mathbf{p}_v^{m+1})$ and let $\Phi^n_m : \Gamma(\mathcal{O}_{Y_\Gamma}) \longrightarrow R^n_m$ be the composition of (9) with the natural projection. Let $\mathcal{E}_\Gamma$ be the universal elliptic curve defined over the ring $H^0(\mathcal{O}_{Y_\Gamma})$. By extending the scalars via the map Φ^n_m, we can construct elliptic curves $E^n_m = \mathcal{E}_\Gamma \otimes R^n_m$. This construction is clearly functorial with respect to the natural maps $R^n_m \to R^n_{m-1}$ and $R^n_m \to R^{n-1}_m$.

The rings R^n_m are Artin rings with algebraically closed residue field $\mathbf{k} = \bar{k}_v$. In other words, the curves E^n_m are formal deformations of the curve $\tilde{E}\otimes\mathbf{k}$. By the Serre-Tate classification of formal deformations of ordinary abelian varieties in positive characteristic, we can associate to each curve E^n_m a symmetric (because of autoduality) bilinear form

$$q(E^n_m; -, -) : T_p(\tilde{E} \otimes \mathbf{k}) \times T_p(\tilde{E} \otimes \mathbf{k}) \longrightarrow \hat{\mathbb{G}}_m(R^n_m). \tag{10}$$

Let us choose a $\mathbb{Z}_p$-generator P of the Tate module $T_p(\tilde{E}\otimes\mathbf{k})$. Then (10) defines an element $q^n_m = q^n_m(P) = q(E^n_m; P, P) \in \hat{\mathbb{G}}_m(R^n_m)$. As $\lim R^n_m = J^\infty_{x_\pi}$, the elements q^n_m converge to

$$q = q(P) \in \hat{\mathbb{G}}_m(J_x) = 1 + m_x. \tag{11}$$

(2.2) Let us point out that so far the fact that E has complex multiplication has not been used. It is a well-known fact that the reduction map $\mathrm{End}(E) \longrightarrow \mathrm{End}(\tilde{E})$ is injective, and since $\tilde{E}$ is ordinary, $\mathrm{End}(\tilde{E})$ can be embedded in $\mathbb{Z}_p$. Of the two possible ways to embed $\mathrm{End}(E)$ in $\mathbb{Z}_p$, the action on the torsion points of E corresponds to that for which $\mathrm{End}(E)$ acts on the Tate module via the latter's natural $\mathbb{Z}_p$-module structure. Having chosen this embedding, let us denote by $[\mu]$ the complex multiplication corresponding to $\mu \in \mathbb{Z}_p$.

PROPOSITION 7. *If E has complex multiplications, the element $q-1$ of* (11) *is a formal local parameter at x, defined over $\mathcal{O}_v^{\mathrm{nr}}$.*

PROOF. After the identification (8), $q - 1 = a_1T + \cdots \in \mathcal{O}_v^{\mathrm{nr}}[[T]]$ and $R^1_0 = \mathbf{k}[T]/(T^2)$, so that it is enough to prove that $q^1_0 \neq 1$. This is in turn equivalent to asking that the p-divisible group $E_0^1[p^\infty]$ is not isomorphic to the trivial extension $\widetilde{E}(R_0^1)[p^\infty]$. As the latter is characterized by the fact that the short exact sequence

$$0 \to \widetilde{E}(R_0^1)[p^\infty]^o \longrightarrow \widetilde{E}(R_0^1)[p^\infty] \longrightarrow \widetilde{E}(R_0^1)[p^\infty]^{\text{ét}} \to 0$$

splits, the proof reduces to showing that not all endomorphisms of $\widetilde{E} \otimes \mathbf{k}$ lift to endomorphisms of $E_0^1[p^\infty]$.

Let $[\mu]$ be a complex multiplication such that $\bar{\mu}/\mu$ is a unit in $\mathcal{O}_v^{\text{nr}}$. As $[\mu]$ maps points of order p to points of order p, it is enough to check that there is no lifting of $[\mu]$ to $E_0^1[p]$. Indeed, any lifting $\mu_0^1 : E_0^1[p] \longrightarrow E_0^1[p]$ would give rise to an R_0^1- linear map $A(p)_0^1 \longrightarrow A(p)_0^1$ of the corresponding Hopf algebra. This map would have to be the identity on $R_0^1 = \mathbf{k} \oplus (m_x/m_x^2) \otimes \mathbf{k}$, which is impossible because the action of the chosen $[\mu]$ on $(m_x/m_x^2) \otimes \mathbf{k}$ is not trivial. □

COROLLARY 8. *There is a (non-canonical) isomorphism* $J_{x_\pi}^\infty \simeq \mathcal{O}_v^{\text{nr}}[[q-1]]$.

PROOF. It is the identification (8). See also [16, Remarks 2.1, 2.10]. □

(2.3) Now we explain how the complex multiplications act on the parameter $q-1$. Since for CM curves the Rosati involution corresponds to complex conjugation, $\text{End}(E)$ acts on $T_p(\widetilde{E} \otimes \mathbf{k}) \times T_p(\widetilde{E} \otimes \mathbf{k})$ as $[\mu] \cdot (P_1, P_2) = (\mu P_1, \bar{\mu} P_2)$. We have proved in the previous subsection that the complex multiplications do not lift to the deformations E_m^n. This will be still true for a general formal deformation $\mathbf{E}/R$ (R an Artin local ring with residue field $\mathbf{k}$), because the requirement due to the Serre-Tate classification theorem is not met. Indeed

$$q(\mathbf{E}/R; \mu P_1, P_2) = q(\mathbf{E}/R; P_1, P_2)^\mu \neq q(\mathbf{E}/R; P_1, P_2)^{\bar{\mu}} = q(\mathbf{E}/R; P_1, \bar{\mu} P_2)$$

for generic $\mathbf{E}$, $[\mu]$. Nevertheless the following result holds:

LEMMA 9. *Let* $\mathbf{E}$ *be a deformation of* $\widetilde{E} \otimes \mathbf{k}$ *over an Artin ring* R *with residue field* $\mathbf{k}$, *and* $[\mu]$ *a complex multiplication of* E. *If* $\mu \in \mathbb{Z}_p^\times$, *then there exists a deformation* $\mathbf{E}_\mu$ *over* R *and a map* $[\mu]_R : \mathbf{E} \longrightarrow \mathbf{E}_\mu$ *lifting* $[\mu]$.

PROOF. Let $q(\mathbf{E}/R; -, -)$ be the bilinear form associated to $\mathbf{E}$. Define $q_\mu(-, -) = q(\mathbf{E}/R; -, -)^{\bar{\mu}/\mu}$. Clearly, $q_\mu(-, -)$ is a bilinear form on $T_p(\widetilde{E} \otimes \mathbf{k}) \times T_p(\widetilde{E} \otimes \mathbf{k})$. Let $\mathbf{E}_\mu$ be the deformation of $\widetilde{E} \otimes \mathbf{k}$ over R such that $q_\mu(-, -) = q(\mathbf{E}_\mu/R; -, -)$. Then $q(\mathbf{E}/R; P_1, \bar{\mu} P_2) = q(\mathbf{E}_\mu/R; \mu P_1, P_2)$ and the existence of the lifting is guaranteed again by the theorem of Serre and Tate. □

This discussion shows that the action of the complex multiplications corresponding to $\mu \in \mathbb{Z}_p^\times$ sits in the action of $\mathbb{Z}_p^\times$ on $\text{Hom}_{\mathbb{Z}_p}(T_p(\widetilde{E} \otimes \mathbf{k}) \times T_p(\widetilde{E} \otimes \mathbf{k}), \hat{\mathbb{G}}_m(R))$ given by $z \cdot \phi = \phi^{\bar{\mu}/\mu}$. The latter action is functorial with respect to the natural maps $R_m^n \to R_m^{n-1}$ and $R_m^n \to R_{m-1}^n$. By taking the limit, it is therefore clear that also the parameter $q \in 1 + m_x$ is transformed by the complex multiplications according to the law $q \mapsto q^{\bar{\mu}/\mu}$

(2.4) We can now prove the following result.

THEOREM 10. *Let K be a number field, $\mathbf{p} \subset K$ a prime as in (2.1) and x a $K_{\mathbf{p}}$-rational point of the modular curve Y_Γ corresponding to a CM curve E with ordinary good reduction modulo $\mathbf{p}$. Then there is a $K_{\mathbf{p}}$-rational local parameter T at x which is an eigenvector for the action of the complex multiplications of E on the fiber at x of the jet bundle on Y_Γ. Moreover, an element $\sum_{n=0}^{\infty} \frac{\alpha_n}{n!} T^n \in K_{\mathbf{p}}[[T]]$ is $\mathbf{p}$-integral if and only if $\alpha_n \in \mathcal{O}_{\mathbf{p}}$ and*

$$v_{\mathbf{p}}\left(\sum_{j=0}^{n} b_{j,n}\alpha_n\right) \geq v_{\mathbf{p}}(n!), \tag{12}$$

where the coefficients $b_{j,n}$ are defined by the formal identity $\sum_{j=0}^{n} b_{j,n}X^j = n!\binom{X}{n}$.

PROOF. Consider $Q = \log q = (q-1)+\frac{1}{2}(q-1)^2+\frac{1}{3}(q-1)^3+\cdots$. By the results obtained in the previous subsection, Q is a λ_μ-eigenvector for the action of the complex multiplications. Therefore $U = \alpha Q$, where U is the local parameter at x constructed in Section 1, and $\alpha \in K_v^{\mathrm{nr}}$, $\alpha \neq 0$. Up to multiplying U by a scalar in $K_v^\times$ we may always assume that α is a unit in $\mathcal{O}_v^{\mathrm{nr}}$. Let T be any parameter defined over K_v satisfying a relation

$$T = \alpha Q \tag{13}$$

with α a unit in $\mathcal{O}_v^{\mathrm{nr}}$. Then $e^T - 1 = \sum_{n=1}^{\infty} \binom{\alpha}{n}(q-1)^n$ is defined over $\mathcal{O}_v^{\mathrm{nr}}$, see [5, § 5.1]. But also $e^T - 1 = T + \frac{1}{2}T^2 + \frac{1}{6}T^3 + \cdots$ so that $e^T - 1$ is defined over $\mathcal{O}_v = \mathcal{O}_v^{\mathrm{nr}} \cap K_v$.

Finally, the second part of the statement is proven exactly as in [5, Theorem 13]. □

3. - Computing the coefficients via the Maaß operators

(3.1) Maaß [11] introduced the differential operators $M_k = (2iy)^{1-k}\frac{d}{dz}((2iy)^k)$ (and the analogous for the Siegel upper half-space). Write $z = x + iy \in \mathcal{H}$ and let k be a positive integer. Define the operator

$$\delta_k = -\frac{1}{4\pi y} M_k = -\frac{1}{4\pi}\left(2i\frac{d}{dz} + \frac{k}{y}\right). \tag{14}$$

For any subgroup Γ of finite index in $\mathrm{SL}_2(\mathbb{Z})$, let $G_k^\infty(\Gamma)$ denote the space of C^∞-modular forms of weight k with respect to Γ. Then, it is routinely checked that the operators δ_k descend to operators $\delta_k : G_k^\infty(\Gamma) \longrightarrow G_{k+2}^\infty(\Gamma)$. The Maaß operators (14) are subject to different interpretations as automorphic forms are seen from different points of view, as briefly explained in the next two subsections.

(3.2) A (possibly C^∞) modular form f of weight k with respect to a subgroup Γ of finite index in the full modular group $\mathrm{SL}_2(\mathbb{Z})$ can be lifted to a C^∞ function $\phi_{k,f}$ on $G = \mathrm{SL}_2(\mathbb{R})$, defined by the formula

$$\phi_{k,f}(g) = (cz+d)^{-k} f(g\cdot i), \qquad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

The automorphic relation satisfied by f forces upon $\phi_{k,f}$ the relations

$$\begin{cases} \phi_{k,f}(\gamma g) &= \phi_{k,f}(g), \qquad \forall \gamma \in \Gamma, \\ \phi_{k,f}(g\kappa) &= e^{-ik\theta}\phi_{k,f}(g), \qquad \forall \kappa = r(\theta) \in K, \end{cases} \tag{15}$$

where $K = \left\{ r(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}$ is the maximal compact subgroup of G that stabilizes $i \in \mathcal{H}$. Conversely, if ϕ is a C^∞-function on G satisfying the relations (15), then we can define a Γ-automorphic form $f_{k,\phi}$ of weight k by

$$f_{k,\phi}(z) = (ci+d)^k \phi(g), \qquad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \text{ such that } g\cdot i = z.$$

An element $A \in \mathrm{Lie}(G)$ acts on the C^∞-functions on G by $(A \star \phi)(g) = \left.\frac{d}{dt}\right|_{t=0} \phi(ge^{tA})$. The adjoint action of K induces a decomposition $\mathrm{Lie}(G) \otimes \mathbb{C} = \mathbb{C}H \oplus \mathbb{C}X \oplus \mathbb{C}Y$ with

$$H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad X = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad Y = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

and

$$\mathrm{Ad}(r(\theta))X = e^{-2i\theta}X, \qquad \mathrm{Ad}(r(\theta))Y = e^{2i\theta}Y. \tag{16}$$

If f is a Γ-automorphic form of weight k, the formulae (16) imply that the function $X \star \phi_{k,f}$ satisfies the relations (15) with $k+2$ in the place of k. Hence, we can define an operator $D_k : G_k^\infty(\Gamma) \longrightarrow G_{k+2}^\infty(\Gamma)$ as $D_k f = f_{k+2,X\star\phi_{k,f}}$. An explicit computation, see [4], shows that $\delta_k = -\frac{1}{4\pi} D_k$.

(3.3) Consider now the analytic family of elliptic curves $\pi : \mathcal{E}_{\mathcal{H}} \longrightarrow \mathcal{H}$, whose fiber over $\tau \in \mathcal{H}$ is the complex torus $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$, and the algebraic families $\pi : \mathcal{E}_\Gamma \longrightarrow Y_\Gamma$ obtained as the space of the orbits of the action of Γ on $\mathcal{E}_{\mathcal{H}}$. To each of these families, we can attach the relative de Rham cohomology bundle H^1_{DR}, whose fiber at $\tau \in \mathcal{H}$ (or $x \in Y_\Gamma$) is just the first de Rham group of the corresponding elliptic curve. Furthermore, we will denote H^1_∞ the associated C^∞-bundles. The fiber-by-fiber Hodge decomposition induces a splitting

$$H^1_\infty \simeq H^{1,0} \oplus H^{0,1}, \tag{17}$$

with an isomorphism of C^∞-bundles $H^1_\infty \simeq \underline{\omega} = \pi_* \Omega^1_{\mathcal{E}_\Gamma/Y_\Gamma}$. Let

$$\mathrm{Split} : H^1_\infty \longrightarrow \underline{\omega} \tag{18}$$

be the projection defined by (17) and the isomorphism just mentioned. If k is a positive integer, we can now define a C^∞ differential operator $\Theta_k : \underline{\omega}^{\otimes k} \longrightarrow \underline{\omega}^{\otimes k+2}$ through the following steps (where Ω^1 denotes the bundle of 1-forms on the base):

Step 1: Embed $\underline{\omega}^{\otimes k} \hookrightarrow S^k H^1_\infty$.

Step 2: Apply the map $\nabla_k : S^k H^1_\infty \longrightarrow (S^k H^1_\infty) \otimes \Omega^1$ obtained by product rule from the *Gauß-Manin connection* $\nabla : H^1_\infty \longrightarrow H^1_\infty \otimes \Omega^1$.

Step 3: Compose the result with the *Kodaira-Spencer isomorphism* $\Omega^1 \simeq \underline{\omega}^{\otimes 2}$ to land in $(S^k H^1_\infty) \otimes \underline{\omega}^{\otimes 2}$.

Step 4: Use the obvious projection induced by (18) $S^k(\mathrm{Split}) : S^k H^1_\infty \longrightarrow \underline{\omega}^{\otimes k}$ to send the result to $\underline{\omega}^{\otimes k} \otimes \underline{\omega}^{\otimes 2} \simeq \underline{\omega}^{\otimes k+2}$.

It is shown in [4] that, after the identification $f \mapsto f(2\pi i\, du)^{\otimes k}$ between elements of $G^k_\infty(\Gamma)$ and global sections of $\underline{\omega}^{\otimes k}$, the operators Θ_k coincide with the operators D_k defined in the previous subsection.

(3.4) Suppose that E is an elliptic curve with complex multiplications, defined over a number field K. Any complex multiplication μ acts on the group $H^1_{\mathrm{DR}}(E/K)$ inducing an eigenspace decomposition $H^1_{\mathrm{DR}}(E/K) = H^1_\mu \oplus H^1_{\bar\mu}$ which is in fact independent of μ. Furthermore, the decomposition of $H^1_{\mathrm{DR}}(E, \mathbb{C}) = H^1_{\mathrm{DR}}(E/K) \otimes \mathbb{C}$ induced by it coincides with the Hodge decomposition, [7, Lemma 4.0.7]. This is an essential ingredient for a number of results about the algebraicity of the values that modular forms attain at CM points. In particular, it shows that at a point $y \in Y_\Gamma$ corresponding to E, the operator

$$\Theta_k(y) : \underline{\omega}_y^{\otimes k} \longrightarrow \underline{\omega}_y^{\otimes k+2} \tag{19}$$

will be actually defined on the fibers of the *algebraic* bundles.

Let R be a sufficiently large ring over which E is defined and such that the R-module $\Omega^1_{E/R}$ is free. The choice of a single R-rational invariant 1-form ω on E induces an identification of $\underline{\omega}_y^{\otimes k}$ with a copy of R; thus (19) defines isomorphisms $[\mathrm{Split}(y), \omega]_k^{k+2} : \underline{\omega}_y^{\otimes k} \longrightarrow R$, and, considering the r-th iterate of Θ_k, isomorphisms $[\mathrm{Split}(y), \omega]_k^{k+2r} : \underline{\omega}_y^{\otimes k} \longrightarrow R$. Katz proves in [8] the following result (where, in view of our application, the restrictions on R are automatically satisfied as E has *ordinary* good reduction):

THEOREM 11. *Let E be a CM curve defined over a subring R of $\mathbb{C}$, ω a R-rational invariant 1-form on E, and f a modular form of weight k defined over R. Then:*

(1) $\Theta_k^{(r)} f(E \otimes \mathbb{C}, \omega) \in R$;

(2) $\Theta_k^{(r)} f(E \otimes \mathbb{C}, \omega) = [\mathrm{Split}(y), \omega]_k^{k+2r}(f(y))$;

where f is identified to a section of $\underline{\omega}_y^{\otimes k}$ and y is the point corresponding to $E \otimes \mathbb{C}$.

(3.5) One of the advantages of the algebraic theory of the Maaß operators as described in subsection 3.3 is that it is well suited for generalizations. In particular, the entire theory can be carried over to the p-adic case, when one uses, instead of (17), the unit root space decomposition of the p-adic de Rham bundle provided by the Frobenius map.

It turns out that for CM curves with ordinary good reduction, also the unit root space decomposition coincides with the Hodge decomposition, [7, Lemma 8.0.13]). Therefore, regarding a modular form defined over a p-adic ring R as a p-adic modular form (in the sense of [8, §§ 1.9-10]), the values at a CM curve with ordinary good reduction of $\Theta_k(f)$ and its p-adic counterpart coincide. We will exploit this fact in the next subsection.

(3.6) Let us go back to the situation and notation of Section 2. Consider the universal formal deformation $\mathcal{E}$ of $\tilde{E} \otimes \mathbf{k}$. The elliptic curve $\mathcal{E}$ is defined over the ring $\mathcal{R}$, an algebra over the Witt vectors $W(\mathbf{k})$. Since the elliptic curves E_m^n are deformations of $\tilde{E} \otimes \mathbf{k}$ to Artin rings, for each of them there is a "classifying map" $\psi_m^n : \mathcal{R} \longrightarrow R_m^n$ such that $E_m^n = \mathcal{E} \otimes_{\psi_m^n} R_m^n$. Passing to the limit over m and n, we can construct a map

$$\psi : \mathcal{R} \longrightarrow \lim R_m^n = J_{x_\pi}^\infty \simeq \mathcal{O}_v^{\mathrm{nr}}[[q-1]]$$

and, in particular, an elliptic curve E_{jet} defined over the ring of $\mathbf{p}_v$-integral jets, by $E_{\mathrm{jet}} = \mathcal{E} \otimes_\psi J_{x_\pi}^\infty$.

Let f be a holomorphic Γ-automorphic form of weight k defined over (a subring of) $\mathbb{C}$. It is understood that we may extend the scalars where the elliptic curves under consideration are defined, in order to be able to evaluate f at them.

Let us recall that the construction of the parameter $q-1$ involved choosing a $\mathbb{Z}_p$-generator P of the Tate module $T_p(\tilde{E}\otimes\mathbf{k})$. Using once again the self-duality of E, P determines, as in [9, § 3.3], a non-zero invariant 1-form $\omega(P)$ on $\mathcal{E}$. This form allows us to identify the fiber of $\underline{\omega}$ (as well as its powers) over the $\mathcal{R}$-rational point corresponding to $\mathcal{E}$, with $\mathcal{R}$ itself. Thus we can write $f(\mathcal{E}) = \tilde{f}\otimes\omega(P)^{\otimes k}$ where $\tilde{f}\in\mathcal{R}$ and $f(E_{\text{jet}}) = (\text{jet}\, f)(x) = \psi(\tilde{f})\otimes(\psi^*\omega(P))^{\otimes k}$. Set $f_{\text{jet}} = \psi(\tilde{f}) \in J_x^{\infty}$ and $\omega_{\text{jet}} = \psi^*\omega(P)$. Let us now use the parameter $Q = \log q$ to identify f_{jet} with a power series, i. e. write

$$f_{\text{jet}} = \sum_{n=0}^{\infty} \frac{b_n(f)}{n!} Q^n. \tag{20}$$

Observe that the coefficients $b_n(f)$ depend in fact also on P. If $P' = \nu P$ with $\nu \in \mathbb{Z}_p^{\times}$ is another $\mathbb{Z}_p$-generator of $T_p(\tilde{E}\otimes\mathbf{k})$, then

$$b_n(f, P') = b_n(f, P)\nu^{-k-2n}, \tag{21}$$

as easily seen.

We shall now use the identification (20) to compute the value at x of $\Theta_k^{(r)}(f)$ for $r = 1, 2, \ldots$. We are going to follow the "instructions" outlined in subsection 3.3, and we will make frequent use of the results and notations of [9]. In fact, we will compute the value at x of the transformed of f by the p-adic Maaß operators. As remarked in the previous subsection, this will not change the final result.

Step 1: Computing the Gauß-Manin connection.

Let P^* be the dual generator of $\text{Hom}_{\mathbb{Z}_p}(T_p(\tilde{E}\otimes\mathbf{k}), \mathbb{Z}_p) \subset \text{Lie}(\mathcal{E}/\mathcal{R})$ and let $\text{Fix}(P^*)$ be the lifting of P^* to the unit root subspace. Then, by [9, Theorem 4.3.1], $\nabla(\omega(P)) = \text{Fix}(P^*)\otimes dQ$, and $\nabla(\text{Fix}(P^*)) = 0$. Therefore, we have

$$\begin{aligned}
\nabla_k(f_{\text{jet}}\otimes\omega_{\text{jet}}^{\otimes k}) &= df_{\text{jet}}\otimes\omega_{\text{jet}}^{\otimes k} + f_{\text{jet}}\nabla_k(\omega_{\text{jet}}^{\otimes k})\\
&= df_{\text{jet}}\otimes\omega_{\text{jet}}^{\otimes k} + k f_{\text{jet}}\otimes\omega_{\text{jet}}^{\otimes k-1}\otimes\nabla(\omega_{\text{jet}})\\
&= df_{\text{jet}}\otimes\omega_{\text{jet}}^{\otimes k} + k f_{\text{jet}}\otimes\omega_{\text{jet}}^{\otimes k-1}\otimes\text{Fix}(P^*)\otimes dQ\\
&= \left[\frac{df_{\text{jet}}}{dQ}\otimes\omega_{\text{jet}}^{\otimes k} + k f_{\text{jet}}\otimes\omega_{\text{jet}}^{\otimes k-1}\otimes\text{Fix}(P^*)\right]\otimes dQ.
\end{aligned}$$

Step 2: Composing with the Kodaira-Spencer map.

We need to compute the image of the differential dQ under the Kodaira-Spencer map $\mathrm{KS} : \Omega \to \underline{\omega}^{\otimes 2}$. In [9] Katz constructs the Kodaira-Spencer map as a map $\mathrm{Kod} : \underline{\omega} \to \mathrm{Lie} \otimes \Omega$ and proves that under the canonical pairing $\underline{\omega} \otimes \mathrm{Lie} \to \mathcal{R}$ one has $\omega(P) \cdot \mathrm{Kod}(\omega(P)) = dQ$. Therefore, we must have $\mathrm{KS}(dQ) = \omega(P)^{\otimes 2}$, and from the computation made in step 1:

$$(1 \otimes \mathrm{KS})(\nabla_k(f_{\mathrm{jet}} \otimes \omega(P)^{\otimes k})) =$$

$$= \left[\frac{df_{\mathrm{jet}}}{dQ} \otimes \omega_{\mathrm{jet}}^{\otimes k} + k f_{\mathrm{jet}} \otimes \omega_{\mathrm{jet}}^{\otimes k-1} \otimes \mathrm{Fix}(P^*)\right] \otimes \mathrm{KS}(dQ) = \tag{22}$$

$$= \frac{df_{\mathrm{jet}}}{dQ} \otimes \omega_{\mathrm{jet}}^{\otimes k+2} + k f_{\mathrm{jet}} \otimes \omega_{\mathrm{jet}}^{\otimes k+1} \otimes \mathrm{Fix}(P^*).$$

Step 3: Projection.

Applying the p-adic splitting induced by the unit root space decomposition, the term in (22) containing $\mathrm{Fix}(P^*)$ vanishes and we get

$$\mathrm{jet}[(p\,\mathrm{Split} \circ (1 \otimes \mathrm{KS}) \circ \nabla_k)(f)](x) = \frac{df_{\mathrm{jet}}}{dQ} \otimes \omega_{\mathrm{jet}}^{\otimes k+2}. \tag{23}$$

The original curve $E \otimes \mathcal{O}_v^{\mathrm{nr}}$ can be recovered from E_{jet} just setting formally $Q = 0$ once the isomorphism $J_x^\infty \simeq \mathcal{O}_v^{\mathrm{nr}}[[q-1]]$ is fixed. Therefore from the identification (20) and formula (23) we obtain

$$\Theta_k(f)(x) = b_1(f) \otimes \omega_{\mathrm{jet}}(x)^{\otimes k+2},$$

whose right hand side is really independent on P (as it must be) as easily confirmed by the relations (21). It is also clear that the computation can be iterated. Since the unit root subspace is horizontal for the Gauß-Manin connection, we have

$$\Theta_k^{(r)}(f)(x) = b_r(f) \otimes \omega_{\mathrm{jet}}(x)^{\otimes k+2r} \qquad \text{for all } r \geq 0. \tag{24}$$

4. - Proof of the main result

(4.1) We shall now define the *period* Ω_v of E entering in the definition of the numbers $c_r(f)$. Since E has complex multiplications, we may assume that E has a smooth model E_v over $\mathcal{O}_{(v)}$ for each non-archimedean place of K. The $\mathcal{O}_{(v)}$-module $H^0(E, \Omega^1_{E/\mathcal{O}_{(v)}})$ of v-integral invariant 1-forms on E is free. Let ω_v be a generator. By definition, ω_v is defined up to a v- adic unit. Pick $\tau \in \mathcal{H}$

such that there is an isomorphism $\Phi : \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau \tilde{\rightarrow} E \otimes \mathbb{C}$. Then the global differential 1-form $\Phi^*(\omega_v)$ will be a scalar multiple of the 1-form defined by dz, i.e. we may write, by abuse of notation, $\Phi^*(\omega_{\text{int}}) = \Omega_v dz$, for some $\Omega_v \in \mathbb{C}$. This complex number can be seen as a period of E as $\Omega_v = \int_0^1 \Phi^*(\omega_v) = \int_{\Phi([0,1])} \omega_v$. The choice of a different τ to write an isomorphism of complex tori as above, is reflected in a different normalization of the period lattice of E, which alters the number Ω_v by a global unit in $\mathcal{O}_K$. Combining the effects of the different choices, Ω_v remains defined up to a v-adic unit.

Let us remark that if K has class number 1, the choice of a v-adic period for E can be globalized. Indeed, in that case the module $H^0(E, \Omega^1_{E/\mathcal{O}_K})$ is free: any generator ω_0 may serve as ω_v for all v's at the same time.

(4.2) We can now use the computation made in subsection 3.6 to prove our integrality criterion (Theorem 1). To avoid any ambiguity, we shall denote by f_{alg} the algebraic modular form defined over $\mathbb{C}$, obtained from f via the relation $f(\tau) = f_{\text{alg}}(\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau, 2\pi i dz)$.

Let us start with f (i.e. f_{alg}) defined over $\mathcal{O}_{(v)} = \mathcal{O}_v \cap K$. From the discussion preceding Theorem 11 we know that also the form $\Theta_k^{(r)}(f)(x)$ must be v-integral. Now compute:

$$\begin{aligned}
\Theta_k^{(r)}(f)(x) &= (\text{by Theorem 11}) = [\text{Split}(x), \omega_v]_k^{k+2r}(f_{\text{alg}}) \otimes \omega_v^{\otimes k+2r} \\
&= \frac{(2\pi i)^{k+2r}}{\Omega_v^{k+2r}} [\text{Split}(x), 2\pi i du]_k^{k+2r}(f_{\text{alg}}) \otimes \omega_v^{\otimes k+2r} \\
&= \frac{(2\pi i)^{k+2r}}{\Omega_v^{k+2r}} \Theta_k^{(r)}(f_{\text{alg}})(E_\tau, 2\pi i du) \otimes \omega_v^{\otimes k+2r} \\
&= \frac{(2\pi i)^{k+2r}}{\Omega_v^{k+2r}} D_k^r(\phi_{f,k}(x)) \otimes \omega_v^{\otimes k+2r} \\
&= \frac{(2\pi i)^{k+2r}(-4\pi)^r}{\Omega_v^{k+2r}} (\delta_k^{(r)} f)(\tau) \otimes \omega_v^{\otimes k+2r} = c_r(f) \otimes \omega_v^{\otimes k+2r}.
\end{aligned}$$

Thus, $c_r(f) \in \mathcal{O}_v$ for all $r \geq 0$.

Let $\alpha \in \mathbb{C}^\times$ be such that $\omega_{\text{jet}}(x) = \alpha\omega_v$. Comparing the above computation of $\Theta_k^{(r)}(f)(x)$ with (24) and (20) yields

$$\text{jet}(f)(x) = \left(\sum_{n=0}^{\infty} \frac{c_n(f)}{n!} \left(\frac{Q}{\alpha^2} \right)^n \right) \otimes \omega_v^{\otimes k}.$$

As the jet of f in x must be $\mathcal{O}_v$-rational, this last expression shows that the local parameter $\alpha^{-2}Q$ is defined over K and that the coefficients $c_r(f)$ must satisfy the Kummer-Serre congruences (1).

Conversely, suppose that the numbers $c_r(f)$ are v-adic integers satisfying the congruences (1). Unwinding the computations done so far shows that the holomorphic section of $\underline{\omega}^{\otimes k}$ corresponding to f has a v-integral jet at x. Thus, it remains to prove that if such a section has a v-integral jet at a $\mathcal{O}_v$-rational point, then it is in fact rational over $\mathcal{O}_v$. This is a consequence of the following general result, where $R \subset \mathbb{C}$ denotes a DVR with uniformizer π and field of quotients K.

LEMMA 12. *Let X be an irreducible, smooth scheme over R of relative dimension ≥ 1. Let $\mathcal{L}$ be an invertible sheaf on X and f a global section of the pull-back of $\mathcal{L}$ to $X \otimes \mathbb{C}$. If the jet of f at a K-rational point is R-rational, then f lifts to a global section of $\mathcal{L}$ on X.*

PROOF. Let $x : \mathrm{Spec}(R) \to X$ be an R-rational point of X. There are natural embeddings $J^{(n)}_{x_\pi, X} \to J^{(n)}_{x_0, X\otimes K} \to J^{(n)}_{x_\pi, X\otimes\mathbb{C}}$ for all n. Let us first prove that f lifts to a K-rational section. On a sufficiently small open neighborhood of x_o the section f can be identified to a section of $\mathcal{O}_X$. Since the stalk $J^{(n)}_{x_0, X\otimes K}$ is generated, as an $\mathcal{O}_{x_0}$-module, by $\mathrm{jet}^n(\mathcal{O}_{x_0})$, we can find elements $f_1, \ldots, f_t, g_1, \ldots, g_t \in \mathcal{O}_{x_0}$ such that, locally at x_0,

$$\mathrm{jet}^n f = \sum_{i=1}^{t} f_i \cdot \mathrm{jet}^n(g_i). \tag{25}$$

Any $h \in \mathcal{O}_{x_0}$ acts on $J^{(n)}_{x_0, X\otimes K} \simeq \mathcal{O}_{x_0}/m^{n+1}_{x_0}$ simply as multiplication, so that (25) can be read as congruence $f = \Sigma f_i g_i \mod m^{n+1}_{x_0}$. By Krull's intersection theorem, $f \in \mathcal{O}_{x_0}$. Therefore f is the extension of the pull-back of a K-rational section defined over an open dense subscheme of $X \otimes K$, and so is itself K-rational.

To achieve R-rationality, argue as above with $f_1, \ldots, f_t, g_1, \ldots, g_t \in \mathcal{O}_{x_\pi}$ in (25) to extend f to a neighborhood of x_π. In this way, f extends to an open subscheme U of X containing all K-rational points. Then $X - U$ is a finite union of closed points, whose local ideals have depth ≥ 2 (by smoothness). Hence f extends to an element of $H^0(X, \mathcal{L})$. □

Theorem 1 is now completely proved.

(4.3) As already remarked in (0.3) our method to prove Theorem 1 fails, in general, for those groups Γ which have elliptic elements, essentially because the map (2) becomes ramified. Nevertheless, the result extends also to automorphic forms with respect to "bad" Γ if we exclude test elliptic curves with j-invariant equal to 0 or 1728. This follows from the fact that any congruence subgroup Γ contains (by definition!) a $\Gamma(N)$ for some $N \geq 3$ and the natural map $Y_{\Gamma(N)} \to Y_\Gamma$ is étale over the open set $\{x | j(x) \neq 0, 1728\} \subset Y_\Gamma$.

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