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JEAN-PAUL ZOLÉSIO

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Introduction

Several classical free boundary value problems can be formulated in the following way: given a bounded domain D in $\mathbb{R}^n$, a Hilbert space H of functions defined on D, a closed convex set K in H and a functional J on K, minimize J over K. If y is a minimizer for this problem, then, roughly speaking, the free boundary problem is solved by setting

$$\Omega = \{x \in D | \text{the constraints associated to } K \text{ are not active}\},$$

assuming here, obviously, that K is a pointwise constraint in H. When J is weakly lower semicontinuous on H (w.l.s.c.), the minimization problem can be understood as a variational inequality (v.i., or, more generally, an implicit variational inequality). In other situations, for example in [1], J takes the following form:

$$J(y) = \int_D \{(\operatorname{grad} y)^2 + \chi_{\{y>0\}} G\}\, dx$$

and it is not w.l.s.c. on $H = H^1(D)$; nevertheless some minimization can be performed on an appropriate K (assuming $G \geq 0$) by virtue of Lemma 2.1 below. We shall refer to this situation as weak variational formulation (w.v.f.) of the problem. Using extra regularity results the v.i. leads to the solution of a free boundary problem in strong version. The same kind of conclusion can be derived for the w.v.f., at least in some very specific situations for which we refer to [5].

In order to explain briefly what is the weak shape formulation (w.s.f.), introduced in [11], [12] and [14], we shall refer to the convex subset $K = \{\phi \geq 0$ a.e. in $D\}$ of $H^1(D)$. Both in v.i. and w.v.f. the domain Ω whose boundary is the solution of the free boundary problem is given by $\Omega = \{x \in D | y(x) > 0\}$. The minimization problem is performed on the variable ϕ lying in K while the domain Ω is obtained from the minimizing term y. The fact that the domain Ω does not appear as an explicit variable in the minimization may be an advantage from several viewpoints.

In many examples arising from structural mechanics, fluid dynamics, electrostatics, etc., the free boundary is searched as the boundary (or in fact a part of the boundary) of a domain Ω which is assigned several constraints. The simplest one is the prescription of the measure: $\operatorname{meas}(\Omega) = \alpha$, α given. If the domain Ω is identified to its characteristic function χ_Ω this constraint is linear, but when expressed on ϕ, $\operatorname{meas}\{\phi > 0\} = \alpha$, it is a very severe constraint. The weak shape formulation (w.s.f.) introduces Ω as a variable; Ω belongs to $\{$measurables subsets E of $D | \operatorname{meas}(E) = \alpha\}$, and on this set we minimize the functional

$$J(\Omega) = \min \left\{ \int_\Omega |\operatorname{grad} \phi|^2 + G + \ldots | \phi \in H_0^1(\Omega) \right\}.$$

The positiveness of y will eventually derive from the maximum principle, but the minimization of $J(\Omega)$ will be worked out in Section 2 without any hypothesis on the sign of G. Such a w.s.f. is related to the homogeneous Dirichlet boundary condition and so it will be denoted by $(\mathcal{D}_\alpha)$, with the convention that $\alpha = 0$ means the situation without any constraint on the volume of Ω. The first section deals with an illustrative example with a v.i. and its w.s.f. that we denote by $(\mathcal{P}_\alpha)$.

The problems $(\mathcal{P}_\alpha)$ and $(\mathcal{D}_\alpha)$ of the first two sections are introduced as relaxed formulations respectively of a v.i. and a w.v.f., the v.i. under consideration being not the well-known obstacle problem (for the membrane) to which we referred, but a very simplified version of a free boundary problem arising from plasma physics.

In the next sections we shall investigate other boundary conditions, $\mathcal{N}$ standing for Neumann condition, $\mathcal{T}$ for transmission condition, σ being associated to a constraint on the perimeter (with the same convention, when $\sigma = 0$, as for α).

We shall be concerned with five classical free boundary situations related to scalar elliptic problems and the Bernoulli free boundary condition. Without loss of generality we restrict our study to the Laplace equation and to the basic optimization problems $(\mathcal{D}_0)$, $(\mathcal{D}_\sigma^\alpha)$, $(\mathcal{D}\Sigma\mathcal{N}_\sigma^\alpha)$, $(\mathcal{N}_\sigma^\alpha)$ and $(\mathcal{T}_\alpha^\sigma)$ for which we give existence results. The first two deal with the Dirichlet condition on the boundary of the domain; they correspond for example to the free boundary condition associated to a perfect 2-D fluid (using a stream function representation). The fourth one corresponds to the Neumann condition, which can be related to the 3-D perfect fluid (using a potential formulation). The last one corresponds to transmission conditions through the boundary, while the third one is a mixed situation. The free boundary in these weak shape formulations is the boundary of a measurable subset Ω of D. Following the previous results of [14] we explicit the extra boundary condition in (7.6) and (7.9) with a unified expression for these problems. Finally, in the last section, assuming the optimal solutions smooth enough, we identify (7.6) and (7.9) with a strong boundary condition in each problem. To that purpose we assume that the boundary $\partial\Omega$ has a generalized mean curvature $\mathfrak{H}$: it is by definition the shape gradient of the mapping $\Omega \to P_D(\Omega)$, a vectorial distribution on D (i.e. an element of $\mathcal{D}'(D;\mathbb{R}^n)$) having its support in $\partial\Omega = \mathrm{cl}(\Omega) \cap \mathrm{cl}(\mathrm{cl}(D)\backslash\Omega)$.

The main difficulty is the existence question. The first step is to define the Hilbert Spaces $H_0^1(\Omega), H^1(\Omega), \ldots$ when Ω is just a measurable subset of D. The first two problems $(\mathcal{D}_0)$ and $(\mathcal{D}_\sigma^\alpha)$ are related to the Dirichlet condition and are associated to two different relaxations of the Hilbert space $H_0^1(\Omega)$. In the first one, as we explain in the second section, the idea of this relaxation is close to Caffarelli's work, but here we choose to have Ω explicitly as a control parameter, so even in this first simple problem we face capacity questions in the relaxation. We choose to consider Ω as a control, that is an explicit variable in the problems, to be able to impose some constraints on it, for example on its volume (α refers to the constraint $\mathrm{meas}(\Omega) = \alpha$), or on its perimeter ($\sigma$ refers to the mean curvature H of the boundary $\partial\Omega$). The term σ physically (i.e. in the Bernoulli condition) is the surface tension. In the Dirichlet problems $(\mathcal{D}_0)$ and $(\mathcal{D}_\sigma^\alpha)$, σ can be taken equal to zero, that is to say that the surface tension is not necessary to get existence results for the relaxed problem. Now, it turns out that when $\sigma > 0$ we can relax the Dirichlet condition in a different way, which is more convenient for modelling potential problems, for example hydrodynamical problems. The basic idea is that when $\partial\Omega$ is smooth, but has several connected components, the potential y should be constant on each of these components but equal to zero only on one of them. When $\sigma > 0$ the existence question is helped, for Ω has a bounded perimeter; for any limit of sequence of such measurable sets Ω_n we show (Lemma 4.6) that the Hausdorff limit is easily related to the BPS(D) limit.

The problems we consider have the following form:

$$\inf\{E(\Omega)|\Omega \text{ is a measurable subset in } D\}.$$

In order to derive existence results we use the following three compactness results concerning the family of measurable sets in D, D being smooth enough and bounded:

- $\{\lambda | 0 \leq \lambda \leq 1 \text{ a.e. in } D\}$ is weakly compact in $L^2(D)$.
- $\{C | C \text{ is a compact set in } \mathrm{cl}(D)\}$ is compact for the Hausdorff metric.
- Any bounded family in BPS(D) is compact in $L^2(D)$ (see Section 4 concerning the bounded perimeter sets in D).

Concerning the Neumann condition and the use of the perimeter, Section 5 is an extension of results from Zolésio [1984].

1. - Free interface with continuity condition

We consider a very simple free boundary problem which is not related to the Bernoulli condition but which easily permits to introduce the weak shape formulation for a free boundary problem and to underline that, even when a variational principle exists (say for example a variational inequality) the shape formulation is not equivalent and permits to handle more general free boundary conditions. In fact the problem developed in this section is a simplified version of a free boundary value problem arising in plasma physics.

We first describe a free boundary problem solved by a variational inequality and later we give the associated "shape variational formulation" and then we show the shape extension of that problem as a shape optimization problem which cannot be reformulated as a variational inequality. This extension is obtained by introducing the constraint on the volume of the domain.

1.1. - *Variational inequality*

We consider here the classical solution of the free boundary problem obtained by the minimization of a coercive functional leading to variational inequality. The most famous of such problems is the well-known obstacle problem for the membrane in which the functional to be minimized is quadratic, while the convex set on which the minimization is performed is bounded. In order to avoid that example we choose here an example in which the cost to be minimized has no gradient while the convex set is the whole space. This example is in fact a very simple version of a free boundary problem arising from plasma physics and studied by the author after 1978 (in particular see [11] and [12]). The free boundary appears as a level curve of the solution of the variational inequality and the difficulties are related to the possible existence of level sets with non-zero measure. The main point in the following variational formulation is that the nonlinear term in the variational inequality will force the level set under consideration to have zero measure.

D is a bounded smooth domain in $\mathbb{R}^n$ and the unknown domain is a measurable set Ω in D whose boundary $\Gamma = \text{cl}(\Omega) \cap \text{cl}(D\backslash\Omega)$ is considered as an interface for a BVP posed in the domain D. In this section we are concerned with an interface Γ which is a level curve of the solution u of the boundary value problem.

We consider the following problem: assuming that D is a bounded smooth domain in $\mathbb{R}^n$ and $f \geq 0$ in $L^2(D)$, find a measurable set Ω in D and u in $H_0^1(D) \cap H^2(D)$ such that

$$-\Delta u(x) = \begin{cases} f(x) & \text{a.e. } x \text{ in } \Omega \\ 0 & \text{a.e. } x \text{ in } D\backslash\Omega \end{cases} \tag{1.1}$$

with

$$\text{meas}\ (\{x | u(x) = 1\}) = 0 \tag{1.2}$$

$$u(x) > 1 \text{ in } \Omega,\ \ u(x) < 1 \text{ in } D\backslash\Omega; \tag{1.3}$$

in other words, as $\Omega = \{x \in D | u(x) > 1\}$, if χ_Ω denotes the characteristic function of Ω, we can write the problem (1.1)-(1.3) equivalently as follows:

$$-\Delta u = \chi_\Omega f,\ \ \Omega = \{x | u(x) > 1)\},\ \ \text{meas}(\{x | u(x) = 1\}) = 0. \tag{1.4}$$

We consider the energy functional $W : H_0^1(D) \to \mathbb{R}$ defined by

$$W(\phi) = \int_D (1/2\, |\text{grad}\, \phi|^2 - f(\phi - 1)^+)\, dx. \tag{1.5}$$

LEMMA 1.1. *The Hadamard semi-derivative $W'(\phi; \gamma)$ exists for each ϕ and*

γ in $H_0^1(D)$ and is given by:

$$W'(\phi;\gamma) = \int_D \operatorname{grad}\phi \cdot \operatorname{grad}\gamma \, dx - \int_D f\chi_{\{\phi>1\}}\gamma \, dx - \int_D f\chi_{\{\phi=1\}}\gamma^+ \, dx.$$

PROOF. It can be obtained directly using the Lebesgue dominated convergence theorem. □

Obviously W is weakly lower semi-continuous and coercive on $H_0^1(D)$ so that it attains its minimum on $H_0^1(D)$. Letting ϕ be a local minimum of W, the first-order optimality necessary conditon can be written as follows: for any γ in $H_0^1(D)$ we have

$$\int_D \operatorname{grad}\phi \cdot \operatorname{grad}\gamma \, dx - \int_D f\chi_{\{\phi>1\}}\gamma \, dx \geq \int_D f\chi_{\{\phi=1\}}\gamma^+ \, dx. \tag{1.6}$$

Choosing $\pm\gamma$ in this variational inequality and adding the two equalities we obtain at any local minimum ϕ of W:

$$\int_D f\chi_{\{\phi=1\}}|\gamma| \, dx = 0 \quad \text{for any } \gamma \text{ in } H_0^1(D). \tag{1.7}$$

From (1.7) it easily follows that, if $f > 0$ a.e. in D, $\operatorname{meas}(\{x|\phi(x) = 1\}) = 0$, i.e. (1.2) holds. Then from (1.6) we get that u is solution of the problem (1.1)-(1.3). Thus we can state the following results.

PROPOSITION 1.2. *Let f be given in $L^2(D)$ with $f > 0$ a.e. in D; then W attains its minimum on $H_0^1(D)$. Let u be a local minimum of W; then* $\operatorname{meas}(\{x|u(x) = 1\}) = 0$ *and u is a solution of the problem* (1.1)-(1.3).

PROPOSITION 1.3. *Assume that f is given in $L^p(\Omega)$, with $p > N/2$ and $f > 0$ a.e. in D; then W attains its minimum on $H_0^1(D)$. Let u be a local minimum of W; then* $\operatorname{meas}(\{x|u(x) = 1\}) = 0$ *and u is a solution of the problem* (1.1)-(1.3), *where u belongs to $W^{2,p}(D)$ and the set $\Omega = \{x \in D|u(x) > 1\}$ is open in D.*

REMARK 1.4. In the particular situation $f \geq 0$ a.e. in D, W can be written as follows:

$$W(\phi) = \min\left\{\int_D (1/2\,|\operatorname{grad}\phi|^2 - f\mu(\phi - 1))\,dx \,\middle|\, \mu \in M\right\} \tag{1.8}$$

where

$$M = \{\mu|0 \leq \mu(x) \leq 1 \text{ a.e. } x \text{ in } D\}, \tag{1.9}$$

so that the minimization of W over $H_0^1(D)$ is equivalent to the following problem:

$$\min\left\{\int_D (1/2\,|\mathrm{grad}\,\phi|^2 - f\mu(\phi-1))\,dx \,|\, \phi\in H_0^1(D),\ \ \mu\in M\right\}. \tag{1.10}$$

REMARK 1.5. In problem (1.10) let us assume that f is given in $W^{s,\infty}(D)$ with $0 < s < 1/2$. As $(\phi - 1)$ is an element of $H^s(D)$, the multiplier μ can be taken in $H^{-s}(D)$ with $0 < s < 1/2$; the unit ball of this Hilbert space is strictly convex. It turns out that in fact for each ϕ in $H^1(D)$ there exists a unique minimizer μ_ϕ in the unit ball of $H^{-s}(D)$ enjoying

$$\int_D f\mu_\phi(\phi-1)\,dx = \int_D f(\phi-1)^+\,dx.$$

1.2. - *The weak shape formulation*

The minimization problem

$$(\mathcal{P})\qquad \min\left\{\int_D (1/2\,|\mathrm{grad}\,\phi|^2 - f(\phi-1)^+)\,dx\,|\,\phi\in H_0^1(D)\right\}$$

possesses solutions for any f in $H^{-1}(D)$. This problem can be written as a shape optimization one in the following way.

To any solution u of problem $(\mathcal{P})$, we associate the measurable subset Ω of D defined by

$$\Omega = \{x\in D|u(x) > 1\}.$$

If f is in $L(D)$, u is continuous and then Ω is an open set in D. Its boundary is the level set $u^{-1}(0)$, so that the restrictions v and $w+1$ of u to $D\backslash\Omega$ and Ω (i.e. $w = u_{|\Omega} - 1$) can be considered as independent variables. The minimization is then performed on v, w and Ω, Ω ranging in the family of measurable subsets of D having boundary with zero measure. The elements v and w are respectively in $H_0^1(D)$ and $H_0^1(\Omega)$. The very definition of that space, Ω being just a measurable set, is given in the next sections.

For any measurable set Ω in D let us consider the two functionals defined by:

$$J_1(\Omega) = \min\left\{\int_{D\backslash\Omega} 1/2\,|\mathrm{grad}\,\phi|^2\,dx\,|\,\phi\in H_0^1(D)_+,\ \ \phi = 1 \text{ a.e. in } \Omega\right\}$$

where

$$H_0^1(D)_+ = \{\phi\in H_0^1(D)|\phi\geq 0 \text{ a.e. in } D\},$$

and

$$J_2(\Omega) = \min\left\{ \int_\Omega (1/2\,|\text{grad}\,\phi|^2 - f\phi)\,dx | \phi \in H_0^1(\Omega)_+ \right\};$$

then, given α with $0 < \alpha < \text{meas}(D)$, the shape optimization problem is:

$$(\mathcal{P}_\alpha) \qquad \begin{aligned} &\min\{J(\Omega) = J_1(\Omega) + J_2(\Omega) | \Omega \text{ measurable subset in } D, \\ &\qquad\qquad \text{meas}(\partial\Omega) = 0,\ \text{meas}(\Omega) = \alpha\}. \end{aligned}$$

If v and w are the minimizers of the problems $J_1(\Omega)$ and $J_2(\Omega)$, then we consider $U(\Omega) = \chi_{D\setminus\Omega} v + \chi_\Omega(w+1)$, element of $H_0^1(D)$, so that $J(\Omega) = W(U(\Omega))$.

Let u be the solution of $\mathcal{P}_{\alpha_0}$ with $f > 0$ a.e. (i.e. u solves of (1.1)-(1.3)), and let $\alpha_0 = \text{meas}\{x \in D | u(x) > 1\}$. Then $v = -(u-1)^- + 1$ and $w = (u-1)^+$ are solutions of J_1 and J_2 in $\Omega_0 = \{x \in D | u(x) > 1\}$ and Ω_0 is solution of $\mathcal{P}_{\alpha_0}$. So the problem $\mathcal{P}_{\alpha_0}$ has at least one solution. Conversely if Ω is a smooth solution of $(\mathcal{P}_{\alpha_0})$, $u = U(\Omega)$ is a solution of (1.1)-(1.3); this fact derives from the necessary conditions for optimality of Ω, which will force the normal derivatives of v and w on $\partial\Omega$ to be equal, so that u will belong to $H_0^1(D) \cap H^2(D)$. In general for arbitrary α the problems $(\mathcal{P})$ and $(\mathcal{P}_\alpha)$ are different.

In the next section we shall be concerned with a classical free boundary problem posed in Ω (i.e. without equation in the complement $D\setminus\Omega$). This problem was studied in [1]. After recalling the weak variational formulation in an appropriate setting, we shall introduce the associated weak shape formulation and give existence results for that new problem.

2. - Bernoulli condition associated to homogeneous Dirichlet condition

We turn now to the situation of the free boundary problem of finding Ω in D and a function y on Ω such that, on $\partial\Omega$, we have $y = 0$ and the Neumann condition $\partial_n y = Q^2$, where Q is given over D. Alt and Caffarelli introduced in [1] the following functional:

$$J(\phi) = \int_D (1/2\,|\text{grad}\,\phi|^2 - f\phi)\,dx + \int_D Q^2 \chi_{\{\phi>0\}}\,dx \tag{2.1}$$

to be minimized on

$$K = \{u \in H_0^1(D) | u(x) \geq 0 \text{ for a.e. } x \text{ in } D\}. \tag{2.2}$$

The existence results are based on the following lemma.

LEMMA 2.1. *Let u_n be a sequence in $L^2(D)$ converging to u, and χ_n a sequence of characteristic functions (i.e. $\chi_n(1-\chi_n) = 0$) weakly converging in*

$L^2(D)$ *to* λ. *Assume moreover that* $(1-\chi_n)u_n = 0$ *for all* n; *then*

$$\lambda \geq \chi_{\{u \neq 0\}}. \tag{2.3}$$

PROOF. Since $(1-\chi_n)u_n = 0$ in the limit we get $(1-\lambda)u = 0$ and then on the set $\{x | u(x) \neq 0\}$ we have $\lambda = 1$; on the other hand, as a weak limit of characteristic functions, λ has values between 0 and 1. □

PROPOSITION 2.2. *Let* f *and* Q *be two elements of* $L^2(D)$. *Then there exists* u *in* K *which minimizes the functional* J *over the positive cone* K *of* $H_0^1(D)$.

PROOF. Let u_n be a minimizing sequence of the functional J over the convex set K with u_n in K. We denote by χ_n the characteristic function of the set $\{x \in D | u_n(x) > 0\}$, which is in fact the same as $\{x \in D | u_n(x) \neq 0\}$. It is immediate to verify that the sequence u_n remains bounded in $H_0^1(D)$, so that we can now denote by u_n a subsequence which weakly converges in $H_0^1(D)$ to an element u of K. This convergence holds in $L^2(D)$ so that Lemma 2.1 applies and we get $\lambda \geq \chi_{\{u \neq 0\}}$ for any weak limiting element of the sequence χ_n (which is bounded in $L^2(D)$).

Let j denote the minimum of J over K; then $J(u_n)$ converges to j but in the weak limit we get

$$\int_D (1/2\,|\text{grad}\, u|^2 - fu)\, dx \leq \liminf \int_D (1/2\,|\text{grad}\, u_n|^2 - fu_n)\, dx,$$

$$\int_D \chi_{\{u \neq 0\}} Q^2\, dx \leq \int_D \lambda Q^2\, dx = \lim \int_D \chi_n Q^2\, dx.$$

Finally adding these two inequalities we get $J(u) \leq j$. □

REMARK 2.3. If we assume f to be non-negative a.e. in D and u to be smoothly defined in D we shall see that the set $\Omega = \{x \in D | u(x) > 0\}$ and the element $u_{|\Omega}$ are a weak solution to the free boundary problem

$$-\Delta u = f \text{ in } \Omega, \quad u = 0, \quad \partial_n u = Q^2 \text{ on } \partial\Omega. \tag{2.4}$$

So the idea is now to consider Ω as an independent variable.

The minimization problem (2.1)-(2.2) can be written as a shape optimization problem as follows: for any measurable subset Ω of D define the Sobolev space

$$H_0^1(\Omega) = \{u \in H_0^1(D) | u(x) = 0 \text{ q.e. } x \text{ in } D \backslash \Omega\} \tag{2.5}$$

and the positive cone:

$$H_0^1(\Omega)_+ = \{u \in H_0^1(\Omega) | u(x) \geq 0 \text{ a.e. } x \text{ in } D\};$$

then we consider the shape optimization problem:

$$\inf\{E(\Omega)|\Omega \text{ is a measurable subset of } D\}, \tag{2.6}$$

where the energy functional E is given by:

$$E(\Omega) = \min\left\{\int_\Omega (1/2\,|\text{grad}\,\phi|^2 - f\phi)\,dx + \int_\Omega Q^2\,dx \,|\, \phi \in H_0^1(\Omega)_+\right\}. \tag{2.7}$$

From the definition we have

$$H_0^1(\Omega) = H_0^1(\Omega \cup \mathcal{E}) \quad \text{for any } \mathcal{E} \text{ such that } \text{cap}(\mathcal{E}) = 0.$$

We recall here that the capacity of $\mathcal{E}$ in D is classically defined as

$$\text{cap}(\mathcal{E}) = \min\left\{\left(\int_D |\text{grad}\,\phi|^2\,dx\right)^{1/2} |\phi \in H_0^1(D), \right.$$

$$\left. \phi \geq \chi_{\mathcal{E}} \text{ a.e. in a neighborhood of } D\right\}.$$

We know (see [4] or [6]) that any element u of $H_0^1(D)$ can be defined quasi everywhere and that if u_n is a bounded sequence in $H_0^1(D)$ we can extract a subsequence which converges quasi everywhere to an element u of $H_0^1(D)$. We say that $u(x) = 0$ q.e. x in $D\backslash\Omega$ if there exists $\mathcal{E}$ in D with zero capacity in D such that the equality $u(x) = 0$ holds for any x in $(D\backslash\Omega)\backslash\mathcal{E}$. Let us also recall that if $\mathcal{E}$ is measurable in D with $\text{cap}(\mathcal{E}) = 0$ then $\text{meas}(\mathcal{E}) = 0$, but the converse is false. When equipped with the norm of $H_0^1(D)$, $H_0^1(\Omega)$ is a Hilbert space, so that for any measurable subset Ω of D problem (2.7) has a unique solution y in the closed convex set $H_0^1(\Omega)_+$, and we have an equivalence between problems (2.6)-(2.7) and the minimization of J over K, as expressed in the following result.

PROPOSITION 2.4. *Let u be a minimizing element of J over K. Then $\Omega := \{x \in D|u(x) > 0\}$ is a solution of problem* (2.6) *while $y = u_{|\Omega}$ is a solution of* (2.7). *Conversely, if Ω and y are solution of* (2.6)-(2.7), *Ω being a measurable set in D and y in $H_0^1(\Omega)_+$, the element u defined by $u(x) = y(x)$ in Ω and $u(x) = 0$ in $D\backslash\Omega$, belongs to K and minimizes J over K.*

The sets $\{x \in D|u(x) > 0\}$, $\{x \in D|u(x) = 0\}$ and $\Omega_u = \{x \in D|u(x) \neq 0\}$ are defined as measurable subsets of D up to a set $\mathcal{E}$ with $\text{cap}(\mathcal{E}) = 0$. This fact derives from the quasi everywhere definition of u, element of $H^1(D)$. It is also interesting to build these sets as follows: we recall that any element u in $H_0^1(D)$ possesses a quasi-continuous representative: for any positive ε there exists a set $\mathcal{E}_\varepsilon$ with capacity less then ε such that u is continuous in $D\backslash\mathcal{E}_\varepsilon$.

LEMMA 2.5. *Any u in $H_0^1(D)$ belongs to $H_0^1(\Omega_u)$.*

PROOF. By construction of Ω_u we have $u = 0$ q.e. in $D\backslash\Omega$. □

We turn now to general situations in which the sign of f is not given and the measure of Ω can be prescribed (a value ora a bound), but the control problem $\min E(\Omega)$ is well-defined and possesses solutions even if it does not correspond to the minimization of a functional $J(\phi)$ as in the first two sections.

3. - Shape existence of weak solutions

3.1. - *Dirichlet problem without constraint*

Problem (2.6)-(2.7) can be relaxed as follows: given any f in $L^2(D)$ and G in $L^1(D)$, find

$$(\mathcal{D}_0) \qquad \inf\{E(\Omega)|\Omega \text{ measurable in } D\}, \tag{3.1}$$

where the energy functional is defined by

$$E(\Omega) = \min\left\{\int_\Omega (1/2\,|\mathrm{grad}\,\phi|^2 - f\phi)\,dx + \int_\Omega G\,dx | \phi \text{ in } H_0^1(\Omega)\right\}, \tag{3.2}$$

the Hilbert space $H_0^1(\Omega)$ being defined in (2.5) for any measurable subset Ω of D. From a classical result of Stampacchia [8] we know that for any element u of $H_0^1(\Omega)$ we have $\mathrm{grad}\,u(x) = 0$ a.e. x in $D\backslash\Omega$, so that the functional E can be re-written as follows:

$$E(\Omega) = \min\left\{E(\Omega, \phi) + \int_\Omega G\,dx | \phi \in H_0^1(\Omega)\right\} \tag{3.3}$$

where

$$E(\Omega, \phi) = \int_\Omega (1/2\,|\mathrm{grad}\,\phi|^2 - f\phi)\,dx = \int_D (1/2\,|\mathrm{grad}\,\phi|^2 - f\phi)\,dx$$

for any ϕ in $H_0^1(\Omega)$. We have the following existence result for problem $(\mathcal{D}_0)$.

THEOREM 3.1. *For any f in $L^2(D)$ and $G = Q^2$ in $L^1(D)$ there exists (at least) a solution of problem $(\mathcal{D}_0)$.*

PROOF. Let Ω_n be a minimizing sequence for $(\mathcal{D}_0)$, and for each n let u_n be the (unique) solution of problem (3.3). If χ_n is the characteristic function of the measurable set Ω_n, we have $u_n \in H_0^1(\Omega_n)$, which implies that $(1-\chi_n)u_n = 0$. On the other hand the sequence u_n remains bounded in $H_0^1(D)$ (taking $\phi = 0$ in (3.3) we get $\int_D (1/2\,|\mathrm{grad}\, u_n|^2 - f u_n)\, dx \leq 0$ and the conclusion derives from the equivalence of $H^1(D)$ and $H_0^1(D)$ norms). We can assume that χ_n weakly converges in $L^2(D)$ to an element λ and that u_n weakly converges in $H_0^1(D)$ to an element u. From Lemma 2.1 we get $\lambda \geq \chi_{\{u \neq 0\}}$ a.e. in D.

Let us define $\Omega = \Omega(u) := \{x \in D | u(x) \neq 0\}$. This measurable set Ω is defined up to a set with zero capacity. Then u belongs to $H_0^1(\Omega(u))$ and we have $\mathrm{meas}(\Omega) = \mathrm{meas}(\{x \in D | \lambda(x) = 1\}) \leq \alpha$ (as we have $\alpha = \mathrm{meas}(\{x | \lambda(x) = 1\}) + \mathrm{meas}(\{x | 0 \leq \lambda(x) < 1\})$). In the limit we get

$$\begin{aligned}\int_\Omega (1/2\,|\mathrm{grad}\, u|^2 - f u)\, dx &= \int_D (1/2\,|\mathrm{grad}\, u|^2 - f u)\, dx \\ &\leq \liminf \int_D (1/2\,|\mathrm{grad}\, u_n|^2 - f u_n)\, dx\end{aligned} \tag{3.4}$$

and

$$\int_\Omega G\, dx \leq \int_D \lambda G\, dx = \lim \int_{\Omega_n} G\, dx. \tag{3.5}$$

Summing (3.4) and (3.5) we obtain that Ω minimizes E and u minimizes $E(\Omega, \cdot)$. □

3.2. - *Dirichlet problem with constraint on the measure of the domain*

Problem (2.6)-(2.7) can also be modified as follows: given any f in $L^2(D)$, G in $L^1(D)$ and a real number α, $0 < \alpha < \mathrm{meas}(D)$, find

$$(\mathcal{D}_0^\alpha) \qquad \inf\{E(\Omega) | \mathrm{meas}(\Omega) = \alpha,\ \Omega \text{ in } D\}. \tag{3.6}$$

We have the following existence result for problem $(\mathcal{D}_0^\alpha)$.

THEOREM 3.2. *For any f in $L^2(D)$, $G = 0$ and any real number α, $0 < \alpha < \mathrm{meas}(D)$, there exists (at least) a solution of problem $(\mathcal{D}_0^\alpha)$.*

PROOF. Let Ω_n be a minimizing sequence for $(\mathcal{D}_0^\alpha)$, and for each n let u_n be the (unique) solution of problem (3.3). If χ_n is the characteristic function of the measurable set Ω_n we have $u_n \in H_0^1(\Omega_n)$ which implies that $(1-\chi_n)u_n = 0$. On the other hand the sequence u_n remains bounded in $H_0^1(D)$ (taking $\phi = 0$ in (3.3) we get $\int_D (1/2\,|\mathrm{grad}\, u_n|^2 - f u_n)\, dx \leq 0$ and the conclusion derives from the equivalence of $H^1(D)$ and $H_0^1(D)$ norms). We can assume that χ_n

weakly converges in $L^2(D)$ to an element λ and that u_n weakly converges in $H_0^1(D)$ to an element u. In the limit we get $\int_D \lambda(x)\,dx = \alpha$. From Lemma 2.1 we get $\lambda \geq \chi_{\{u\neq 0\}}$ a.e. in D. Let us define $\Omega = \Omega(u) := \{x \in D|u(x)\neq 0\}$; then u belongs to $H_0^1(\Omega(u))$ and we have $\text{meas}(\Omega) = \text{meas}(\{x \in D|\lambda(x) = 1\}) \leq \alpha$ (as we have $\alpha = \text{meas}(\{x|\lambda(x) = 1\}) + \text{meas}(\{x|0 \leq \lambda(x) < 1\})$). In the limit we get

$$\int_\Omega (1/2\,|\text{grad}\,u|^2 - fu)\,dx = \int_D (1/2\,|\text{grad}\,u|^2 - fu)\,dx \\ \leq \liminf \int_D (1/2\,|\text{grad}\,u_n|^2 - fu_n)\,dx \tag{3.7}$$

and

$$\int_D \lambda G\,dx = \lim \int_{\Omega_n} G\,dx \tag{3.8}$$

so that

$$\int_\Omega (1/2\,|\text{grad}\,u|^2 - fu)\,dx + \int_D \lambda G\,dx \leq \inf\{E(\Omega)|\text{meas}(\Omega) = \alpha\};$$

since $G \geq 0$ we obtain $E(\Omega) \leq \inf\{E(\Omega)|\text{meas}(\Omega) = \alpha\}$; but Ω does not verify the constraint on the measure.

Let us note that for any measurable set Ω' such that Ω' is between Ω and D we have

$$\int_{\Omega'} (1/2\,|\text{grad}\,u|^2 - fu)\,dx = \int_\Omega (1/2\,|\text{grad}\,u|^2 - fu)\,dx$$

so that in (3.7) Ω can be increased to any such Ω'. The inclusion of Ω in Ω' implies the inclusion of $H_0^1(\Omega)$ in $H_0^1(\Omega')$; moreover $G \leq 0$ a.e. in D, and hence it follows immediately from (3.3) that $E(\Omega') \leq E(\Omega)$. To conclude the proof we just have to select Ω' with $\text{meas}(\Omega') = \alpha$; such a measurable set Ω' is admissible and minimizes the cost in (3.6) and we have

$$E(\Omega') = E(\Omega) = \inf\{E(\Omega)|\text{meas}(\Omega) = \alpha\}. \qquad \square$$

COROLLARY 3.3. *Assume* $f \in L^2(D)$ *and* $G = Q^2$ *in* $L^1(D)$; *then the following problem has an optimal solution:*

$$(\mathcal{D}_0^{\alpha-}) \qquad \inf\{E(\Omega)|\text{meas}(\Omega) \leq \alpha,\ \Omega \subset D\}. \tag{3.9}$$

PROOF. The argument is similar to the proof of Theorem 3.2: the minimizing sequence is chosen in such a way that $\text{meas}(\Omega_n) \leq \alpha$, so that in the weak limit we get

$$\text{meas}(\Omega) \leq \int_D \lambda(x)\, dx \leq \alpha. \qquad \square$$

REMARK 3.4. In these problems f has been supposed to be given in $L^2(D)$. This was necessary in the proofs to get in the limit terms $\int_{\Omega_n} f u_n\, dx$ which reduce to $\int_D f u_n\, dx$. Nevertheless in many applications f turns out to be given in $H^{-1}(D)$ with a compact support in D. We briefly show now that the previous existence results are easily extended to this situation.

3.3. - *The situation in which f is given in $H^{-1}(D)$*

Let G be given in $L^1(D)$ and f in $H^{-1}(D)$ such that

$$\mathcal{C} = \text{support of } f \text{ is compact in } D. \tag{3.10}$$

Then we consider the following problem:

$$(\mathcal{DC}_0) \quad \inf\{E(\Omega) | \Omega \text{ measurable in } D,\ \text{meas}(\Omega) \leq \alpha,\ \mathcal{C} \text{ included in } \Omega\} \tag{3.11}$$

where the energy functional is defined by

$$E(\Omega) = \min\left\{ \int_\Omega 1/2\, |\text{grad}\, \phi|^2\, dx + \langle f, \phi^0 \rangle + \int_\Omega G\, dx | \phi \in H_0^1(\Omega) \right\}. \tag{3.12}$$

Here $\langle\, ,\rangle$ is the pairing between $H^{-1}(D)$ and its dual space $H_0^1(D)$, the Hilbert space $H_0^1(\Omega)$ being defined in Remark 2.3 for any measurable subset Ω in D; ϕ^0 is the extension of ϕ by 0, element of $H_0^1(D)$. For any element u in $H_0^1(\Omega)$ we have $\text{grad}\, u(x) = 0$ a.e. x in $D \backslash \Omega$ so that the functional E can be re-written as follows:

$$E(\Omega) = \min\left\{ E(D, \phi) + \int_\Omega G\, dx | \phi \in H_0^1(\Omega) \right\} \tag{3.13}$$

where

$$E(\Omega, \phi) = \int_D 1/2\, |\text{grad}\, \phi|^2\, dx + \langle f, \phi^0 \rangle \quad \text{for any } \phi \text{ in } H_0^1(\Omega).$$

We have the following existence result for problem $(\mathcal{DC}_0)$.

THEOREM 3.5. *For any f in $H^{-1}(D)$ such that the support $\mathcal{C}$ of f is compact in D, and $G = Q^2$ in $L^1(D)$, there exists (at least) a solution of problem $(\mathcal{DC}_0)$.*

PROOF. Let Ω_n be a minimizing sequence for $(\mathcal{DC}_0)$, and for each n let u_n be the (unique) solution of problem (3.12). If χ_n is the characteristic function of the measurable set Ω_n we have $(1 - \chi_n)u_n = 0$. On the other hand the sequence u_n remains bounded in $H_0^1(D)$ (taking $\phi = 0$ in (3.3) we get $\int_D (1/2\,|\operatorname{grad} u_n|^2\, dx - \langle f, u_n\rangle \leq 0$ and the conclusion derives from the equivalence of $H^1(D)$ and $H_0^1(D)$ norms). We can assume that χ_n weakly converges in $L^2(D)$ to an element λ and u_n weakly converges in $H_0^1(D)$ to an element u. From Lemma 2.1 we get $\lambda \geq \chi_{\{u\neq 0\}}$ a.e. in D.

Let us define $\Omega = \Omega(u) := \{x \in D | u(x) \neq 0\} \cup \mathcal{C}$. From the definition of the minimizing sequence we have $\chi_n \geq \chi_{\mathcal{C}}$. Then $\lambda \geq \chi_{\mathcal{C}}$ a.e. in D. The measurable set Ω is defined up to a set with zero capacity.

Since u belongs to $H_0^1(\{x|u(x)\neq 0\})$ then it also belongs to $H_0^1(\Omega(u))$, and we have

$$\operatorname{meas}(\Omega) \leq \operatorname{meas}(\{x \in D | \lambda(x) = 1\}) \leq \alpha$$

(remark that $\alpha = \operatorname{meas}(\{x|\lambda(x) = 1\}) + \operatorname{meas}(\{x|0 \leq \lambda(x) < 1\})$).

In the limit we get

$$\int_\Omega 1/2\,|\operatorname{grad} u|^2\, dx - \langle f, u\rangle = \int_D (1/2\,|\operatorname{grad} u|^2\, dx - \langle f, u\rangle$$
$$\leq \liminf \int_D (1/2\,|\operatorname{grad} u_n|^2\, dx - \langle f, u_n\rangle$$

and

$$\int_\Omega G\, dx \leq \int_D \lambda G\, dx = \lim \int_{\Omega_n} G\, dx.$$

The inclusion of $\mathcal{C}$ in Ω is easily obtained from the definition of Ω. □

4. - Shape weak existence with bounded perimeter sets. Dirichlet condition

We have proved above that problems $(\mathcal{D}_0)$, $(\mathcal{D}_0^\alpha)$ and $(\mathcal{D}_0^{\alpha-})$ do have optimal solutions. However, since they are associated to the homogeneous Dirichlet condition $u \in H_0^1(\Omega)$, the optimal domain Ω is not allowed in general to possess holes; that is to say, roughly speaking, the topology of Ω is given a priori. In many examples it turn out that the solution u can be physically interpreted as a potential, so that the homogeneous Dirichlet condition appears not adequate. The physical condition is that the potential u should be constant on each connected component of the boundary $\partial\Omega$ in D. When Ω is a simply connected domain then $\partial\Omega$ has a single connected component and the constant value of u on it can be taken as zero; in general the constant value of u can be fixed only on one connected component, and the other constant values on the remaining components become unknowns of the problem. To illustrate the reason for which holes cannot occur in the previous problems let us consider a simple example: D is the square $]0,1[\times]0,1[$, $f=1$ and $G=0$. For any smooth domain Ω in D we have $E(\Omega) = -1/2 \int_\Omega u(x)\,dx$ and it can be easily verified that

$$\inf\{E(\Omega) | \Omega \text{ measurable in } D\}$$

is attained at $\Omega = D$. Let us modify this optimal domain by removing a closed subset E such that $|E|_{\mathcal{H}^{n-1}} = 0$ but $\text{cap}(E) > 0$ (for example take E to be line); then it is possible to construct a sequence Ω_n which converges to D but such that the sequence of optimal solutions $u_n = u(\Omega_n)$ converges not to $u(D)$ but to $u(D\backslash E)$. The homogeneous Dirichlet condition in $H_0^1(D\backslash E)$ implies that u is zero on E. In other words the mapping $\chi_\Omega \mapsto u(\Omega)$ is not continuous from $L^2(D)$ to $H_0^1(D)$; nevertheless the infimum of the problem is attained but, at least when f is positive, no hole is allowed in the optimal solutions. These considerations justify the introduction of the following Hilbert space, defined for any measurable set Ω in D:

$$\tilde{H}_0^1(\Omega) = \{\phi \in H_0^1(D) | (1 - \chi_\Omega)\,\text{grad}\,\phi(x) = 0 \text{ a.e. } x \text{ in } D\}. \tag{4.1}$$

When Ω is an open subset of D such that $D\backslash\text{cl}(\Omega)$ is not simply connected, from classical results of distribution theory we know that f is constant on each connected component of $D\backslash\text{cl}(\Omega)$, this constant being zero on the component whose boundary contains ∂D.

The minimization problems $(\mathcal{D}_0)$, $(\mathcal{D}_0^\alpha)$ and $(\mathcal{D}_0^{\alpha-})$ associated to this Hilbert space fail, in the sense that the techniques previously used to establish existence of an optimal Ω cannot be applied. The main reason is that Lemma 2.1 is false when convergence of u_n in $L^2(D)$ is replaced by weak convergence of $\text{grad}\,u_n$ in $L^2(D)^n$. The point is to recover an equivalent of Lemma 2.1 by imposing the (strong) $L^2(D)$ convergence of the sequence u_n. In practice u_n stands for the sequence of characteristic functions χ_{Ω_n} of a minimizing sequence. To obtain the strong $L^2(D)$ convergence of a subsequence we add a constraint on the perimeters. We consider the family of Bounded Perimeter Sets in D defined as follows:

$$\text{BPS}(D) = \left\{ \Omega \subset D \,\middle|\, \sup \left\{ \int_\Omega \text{div}(g)\,dx \,\middle|\, g \in C^\infty_{\text{comp}}(D;\mathbb{R}^n), \|g(x)\| \le 1,\ x \in D \right\} < \infty \right\}.$$

It is immediate to verify that $\{\chi_\Omega | \Omega \in \mathrm{BPS}(D)\}$ is contained in $\mathrm{BV}(D)$. The norm of χ_Ω is given by

$$\|\chi_\Omega\|_{\mathrm{BV}(D)} = \mathrm{meas}\,\Omega + \|\mathrm{grad}\,\chi_\Omega\|_{M^0(D)},$$

where the norm of $\mathrm{grad}\,\chi_\Omega$ in the Banach space $M^0(D)$ is given by

$$\|\mathrm{grad}\,\chi_\Omega\|_{M^0(D)} = \sup\{\langle \mathrm{grad}\,\chi_\Omega, g\rangle | g \in C_{\mathrm{comp}}(D;\mathbb{R}^n),\ \|g(x)\| \leq 1,\ x \in D\}$$

and $\langle \mathrm{grad}\,\chi_\Omega, g\rangle = -\lim \int_\Omega \mathrm{div}(g_n)\,dx$, g_n being a sequence in $C^\infty_{\mathrm{comp}}(D;\mathbb{R}^n)$ which converges to g in $C_{\mathrm{comp}}(D;\mathbb{R}^n)$; it can be easily verified that that limit is independent of the choice of such a sequence g_n.

The perimeter of Ω in D is given by

$$P_D(\Omega) = \sup\left\{\int_\Omega \mathrm{div}(g)\,dx | g \in C^\infty_{\mathrm{comp}}(D;\mathbb{R}^n),\ \|g(x)\| \leq 1,\ x \in D\right\}.$$

When Ω is a smooth subdomain of D the $(n-1)$-dimensional measure of its boundary is given by $|\partial\Omega|_{\mathcal{H}^{n-1}} = P_D(\Omega) + |\partial(\Omega \cap D)|_{\mathcal{H}^{n-1}}$. The inclusion of $\mathrm{BPS}(D)$ in $\mathrm{BV}(D)$ allows us to obtain the following compactness result concerning the family $\mathrm{BPS}(D)$.

LEMMA 4.1. *Let Ω_n be a sequence in* $\mathrm{BPS}(D)$ *such that $P_D(\Omega) \leq M$. Then there exists Ω in* $\mathrm{BPS}(D)$ *and a subsequence, still denoted by Ω_n, such that the characteristic functions converge in $L^1(D)$ to the characteristic function of Ω; moreover for any g in $C_{\mathrm{comp}}(D;\mathbb{R}^n)$ we have*

$$\langle \mathrm{grad}\,\chi_{\Omega_n}, g\rangle \to \langle \mathrm{grad}\,\chi_\Omega, g\rangle$$

and $P_D(\Omega) \leq \liminf P_D(\Omega_n)$.

PROOF. This is a simple translation of the classical "compact embedding" of $\mathrm{BV}(D)$ in $L^1(D)$, see for example [9]. □

We define the perimeter of a measurable subsets Ω of D, as an element of $\mathbb{R} \cup \{+\infty\}$, by

$$P_D(\Omega) = \sup\left\{\int_\Omega \mathrm{div}(g)\,dx | g \in C^\infty_{\mathrm{comp}}(D;\mathbb{R}^n),\ \|g(x)\| \leq 1,\ x \in D\right\}.$$

We introduce the following problem, for any $\sigma > 0$

$$(\mathcal{D}^\alpha_\sigma) \qquad \inf\{E_\sigma(\Omega) | \mathrm{meas}(\Omega) = \alpha,\ \Omega \subset D\} \qquad (4.2)$$

where

$$E_\sigma(\Omega) = E(\Omega) + \sigma P_D(\Omega), \tag{4.3}$$

$$E(\Omega) = \min \left\{ \int_\Omega (1/2\,|\text{grad}\,\phi|^2 - f\phi)\,dx + \int_\Omega G\,dx \,|\, \phi \in \tilde{H}_0^1(\Omega) \right\}. \tag{4.4}$$

THEOREM 4.2. *Assume* $f \in L^2(D)$, $G \in L^1(D)$, $\sigma > 0$ *and* $0 \leq \alpha <$ meas(D). *Then problem* $(\mathcal{D}_\sigma^\alpha)$ *possesses (at least) one optimal solution* Ω *in* BPS(D).

PROOF. Let Ω_n be a minimizing sequence for $(\mathcal{D}_\sigma^\alpha)$; taking $\phi = 0$ in (4.4) we get

$$P_D(\Omega_n) \leq \sigma^{-1} \left(E(D) + \int_\Omega |G|\,dx \right)$$

and then we can consider the subsequence $\chi_n = \chi_{\Omega_n}$ given by Lemma 4.1. For each n, let $u_n \in \tilde{H}_0^1(\Omega_n)$ be the unique minimizer of (4.4). This sequence remains bounded in $H_0^1(D)$ so that we can assume that it is weakly convergent to an element u of $H_0^1(D)$.

From $(1 - \chi_n)\,\text{grad}\,u_n = 0$ a.e. in D we get in the limit $(1 - \chi_\Omega)\,\text{grad}\,u = 0$ a.e. in D, so that the limiting element u belongs to $\tilde{H}_0^1(\Omega)$. We have:

$$\begin{aligned} \int_\Omega (1/2\,|\text{grad}\,u|^2 - fu)\,dx &= \int_D (1/2\,|\text{grad}\,u|^2)\,dx - \int_\Omega fu\,dx \\ &\leq \liminf \int_D (1/2\,|\text{grad}\,u_n|^2)\,dx - \int_{\Omega_n} fu_n\,dx \end{aligned} \tag{4.5}$$

and

$$\int_\Omega fu\,dx = \lim \int_{\Omega_n} fu_n\,dx, \quad \int_\Omega G\,dx = \lim \int_{\Omega_n} G\,dx \tag{4.6}$$

so that

$$\begin{aligned} &\int_\Omega (1/2\,|\text{grad}\,u|^2 - fu)\,dx + \int_\Omega G\,dx + \sigma P_D(\Omega) \\ &\qquad \leq \inf\{E_\sigma(\Omega) | \text{meas}(\Omega) = \alpha\}. \end{aligned} \tag{4.7}$$

□

As in Section 3 we can consider the situation when f is given in $H^{-1}(D)$ with a compact support $\mathcal{C}$ in D. For any $\sigma > 0$ we introduce the following problem:

$(\mathcal{DC}_\alpha^\sigma)$ $\inf\{E_\sigma(\Omega) | \Omega \subset D,\ \Omega \supset \mathcal{C}\}$

where

$$E_\sigma(\Omega) = E(\Omega) + \sigma P_D(\Omega),$$

$$E(\Omega) = \min\left\{\int_\Omega 1/2\,|\operatorname{grad}\phi|^2\,dx - \langle f,\phi\rangle + \int_\Omega G\,dx \,|\, \phi \text{ in } \tilde{H}_0^1(\Omega)\right\}.$$

THEOREM 4.3. *Assume* $f \in H^{-1}(D)$ *has support contained in the compact subset* $\mathcal{C}$ *of* D, $G \in L^1(D)$, $\sigma > 0$ *and* $\operatorname{meas}(\mathcal{C}) < \alpha < \operatorname{meas}(D)$. *Then problem* $(\mathcal{DC}_\alpha^\sigma)$ *possesses (at least) one optimal solution* Ω *in* BPS(D).

PROOF. It is similar to the proof of Theorem 4.2. □

5. - Shape weak existence with bounded perimeter sets. Neumann condition

5.1. - *Min Min formulation*

We turn to the minimization of the energy functional associated to the Neumann condition. The main question is to relax the definition of the Sobolev space $H^1(\Omega)$ when Ω is a measurable subset of D with finite perimeter, i.e. $\Omega \in \text{BPS}(D)$.

$$\begin{aligned} H^1(\Omega) = \{(f,h) \in L^2(\Omega) \times L^2(\Omega)^n \,|\, & \\ & \exists \text{ polyhedral open sets } \Omega_n \text{ and } f_n \in H^1(\Omega_n) \text{ s.t.} \\ & \chi_{\Omega_n} \to \chi_\Omega \text{ in } L^2(D), \\ & \chi_{\Omega_n} f_n \to \chi_\Omega f \text{ weakly in } L^2(D), \\ & \chi_{\Omega_n} \operatorname{grad} f_n \to \chi_\Omega h \text{ weakly in } L^2(D)^n\}. \end{aligned} \tag{5.1}$$

Note that in this definition the sequence of polyhedral open sets may depend on the element (f,h).

For conciseness in the sequel the situation of (5.1) will be denoted by:

$$(\Omega_n, f_n) \twoheadrightarrow (\Omega, f, h).$$

PROPOSITION 5.1. $H^1(\Omega)$ *is a Hilbert space when equipped with the norm:*

$$\|(f,h)\|_{H^1(\Omega)} = ((\|f\|_{L^2(\Omega)})^2 + (\|h\|_{L^2(\Omega)^n})^2)^{1/2}.$$

PROOF. Let (f^k, h^k) be a Cauchy sequence for this norm. Then there exist f and h in $L^2(\Omega)$ and $L^2(\Omega)^n$ such that $f^k \to f$ and $h^k \to h$ strongly in $L^2(\Omega)$ and $L^2(\Omega)^n$. From (5.1) we have that for any k there exists a sequence Ω_n^k of polyhedral open subsets of D and elements f_n^k in $H^1(\Omega_n^k)$ such that, as $n \to \infty$, $\Omega_n^k \to \Omega^k$ in $L^2(D)$,

$$\chi_{\Omega k_n} f_n^k \to \chi_\Omega f^k \text{ weakly in } L^2(\Omega),$$
$$\chi_{\Omega k_n} \operatorname{grad} f_n^k \to \chi_\Omega h^k \text{ weakly in } L^2(\Omega)^n.$$

If $\Omega_k = \Omega_{n(k)}^k$ and $f_k = f_{n(k)}^k$ we have $(\Omega_k, f_k) \twoheadrightarrow (\Omega, f, h)$. So $(f, h) \in H^1(\Omega)$. □

We consider the following problem:

$$(\mathcal{N}_\sigma^\alpha) \qquad \inf\{\mathrm{En}_\sigma(\Omega) | \operatorname{meas}(\Omega) = \alpha,\ \Omega \subset D\} \tag{5.2}$$

where

$$\mathrm{En}_\sigma(\Omega) = \mathrm{En}(\Omega) + \sigma P_D(\Omega) \tag{5.3}$$

and

$$\mathrm{En}(\Omega) = \min\left\{ \int_\Omega 1/2(|h|^2 + f^2)\,dx + \int_\Omega Ff\,d\mu + \int_\Omega G\,dx | (f,h) \in H^1(\Omega) \right\}. \tag{5.4}$$

LEMMA 5.2. *For any measurable subset Ω of D the minimum in* (5.4) *is attained.*

PROOF. Let (f^k, h^k) be a minimizing sequence in $H^1(\Omega)$. Choosing the first element of this sequence to be zero we get

$$1/2 \int_\Omega (|h^k|^2 + (f^k)^2)\,dx - \int_\Omega Ff^k\,dx \leq 0 \tag{5.5}$$

from which we get

$$\|f^k\|_{L^2(\Omega)} \leq \|F\|_{L^2(\Omega)}, \quad \|h^k\|_{L^2(\Omega)^n} \leq 2\|F\|_{L^2(\Omega)}.$$

We can then extract subsequences, still denoted by f^k and h^k, which weakly converge to elements f and h respectively in $L^2(\Omega)$ and $L^2(\Omega)^n$. One can verify that the element (f, h) belongs to $H^1(\Omega)$. A proof is obtained building a "diagonal" subsequence exactly as in the proof of Lemma 4.6, with weak $L^2(\Omega)$ convergence of the sequence replacing the strong one: since on the ball $\{f \in L^2(\Omega) | \|f\| \leq 2\|F\|\}$ the weak topology of $L^2(\Omega)$ is metrisable, to extract the diagonal subsequence one writes the triangular inequality for this distance and then he chooses first k and then $n = n(k)$ large enough in the other term.

Then the fact that (f, h) is a minimizing element for problem (5.4) derives from weak lower semicontinuity of the functional under consideration. □

PROPOSITION 5.3. *Problem* $(\mathcal{N}_\sigma^\alpha)$ *has (at least) a solution* $\Omega \in \mathrm{BPS}(D)$ *with* $\mathrm{meas}(\Omega) = \alpha$.

PROOF. Let Ω_k be a minimizing sequence for $(\mathcal{N}_\sigma^\alpha)$ and for any k let (f^k, h^k) be a minimizing element for (5.4) in $H^1(\Omega_k)$. We have

$$\int_{\Omega_k} 1/2(|h^k|^2 + (f^k)^2)\, dx - \int_{\Omega_k} F f^k\, dx + \sigma P_D(\Omega_k) \leq E_\sigma(\Omega_0) := E(\Omega_0) + \sigma P_D(\Omega_0)$$

for any admissible set Ω_0. But for any Ω_0 (5.5) implies that $E(\Omega_0) \leq 0$, so that:

$$\int_{\Omega_k} 1/2(|h^k|^2 + (f^k)^2)\, dx - \int_{\Omega_k} F f^k\, dx + \sigma P_D(\Omega_k) \leq \sigma\alpha. \tag{5.6}$$

In particular it follows that

$$\begin{aligned}
&\|f^k\|_{L^2(\Omega_k)} \leq \|F\|_{L^2(D)} + (\|F\|_{L^2(D)})^2 + 2\sigma)^{1/2} = b,\\
&\|h^k\|_{L^2(\Omega_k)^n} \leq 2(\sigma\alpha + b\|F\|_{L^2(D)}),\\
&\sigma P_D(\Omega_k) \leq \sigma\alpha + b\|F\|_{L^2(D)}.
\end{aligned}$$

Then, after extracting a subsequence there exist two elements λ and μ in $L^2(D)$ and $L^2(D)^n$ respectively such that the following weak convergences hold: $\chi_{\Omega_k} f^k \to \lambda$, $\chi_{\Omega_k} h^k \to \mu$ and $\Omega_k \to \Omega$, where $\Omega \in \mathrm{BPS}(D)$; now we can prove that (λ, μ) belongs to $H^1(\Omega)$. For each $k, (f^k, h^k)$ belongs to $H^1(\Omega_k)$; then by definition there exists a sequence Ω_k^n of polyhedral open sets in D and f_k^n in $H^1(\Omega_k^n)$ such that

$$\begin{aligned}
&\chi_{\Omega_k^n} \to \chi_{\Omega_k} \text{ in } L^2(D),\\
&\chi_{\Omega_k^n} f_k^n \to \chi_{\Omega_k} f_k \text{ weakly in } L^2(D),\\
&\chi_{\Omega_k^n} \operatorname{grad} f_k^n \to \chi_{\Omega_k} h^k \text{ weakly in } L^2(D)^n
\end{aligned}$$

as $n \to \infty$.

The weak topology of $L^2(D)$ on the ball of radius $2(\sigma\alpha + b\|F\|_{L^2(D)})$ is metrisable; let $d(\cdot,\cdot)$ be the distance. Then we can write

$$d(\chi_{\Omega_k^n} f_k^n, \lambda) \leq d(\chi_{\Omega_k^n} f_k^n, \chi_{\Omega_k} f_k) + d(\chi_{\Omega_k} f_k, \lambda) \tag{5.7}$$

$$d(\chi_{\Omega_k^n} \operatorname{grad} f_k^n, \mu) \leq d(\chi_{\Omega_k^n} \operatorname{grad} f_k^n, \chi_{\Omega_k} h_k) + d(\chi_{\Omega_k} h_k, \mu). \tag{5.8}$$

Given an arbitrary positive ε we choose k large enough, $k = k(\varepsilon)$, so that the last two terms in the right-hand sides of (5.7) and (5.8) are less then ε; now we select n large enough, $n = n(k(\varepsilon))$, so that also the other terms in the right-hand sides of (5.7) and (5.8) are less then ε. Then "diagonal" subsequence defined by

$$(\chi_{\Omega_{k(\varepsilon)}^{n(k(\varepsilon))}} f_{k(\varepsilon)}^{n(k(\varepsilon))}, \chi_{\Omega_{k(\varepsilon)}^{n(k(\varepsilon))}} \operatorname{grad} f_{k(\varepsilon)}^{n(k(\varepsilon))})$$

weakly converges in $L^2(D)^{n+1}$ to (μ,λ), and the domain $\Omega_{k(\varepsilon)}^{n(k(\varepsilon))}$ converges, as ε goes to zero, to Ω in BPS(D). Actually, (μ,λ) belongs to $L^2(\Omega)^{n+1}$; this is easily checked by passing to the limit in the identities $0 = (1-\chi_{\Omega_k})\chi_{\Omega_k} f_k$ and $0 = (1-\chi_{\Omega_k})\chi_{\Omega_k} h_k$, which gives $0 = (1-\chi_\Omega)\mu$ and $0 = (1-\chi_\Omega)\lambda$. We conclude that (μ,λ) belongs to $H^1(\Omega)$.

In the limit we also have

$$\begin{aligned}&\int_D (\lambda^2 + \mu^2)\, dx - \int_D F\lambda\, dx + \sigma P_D(\Omega)\\ &\leq \liminf \int_{\Omega_k} 1/2(|h^k|^2 + (f^k)^2)\, dx - \int_{\Omega_k} F f^k\, dx + \sigma P_D(\Omega_k);\end{aligned}$$

as (λ,μ) belongs to $H^1(\Omega)$ we have

$$\int_D (\lambda^2 + \mu^2)\, dx - \int_D F\lambda\, dx = \int_\Omega (\lambda^2 + \mu^2)\, dx - \int_\Omega F\lambda\, dx,$$

so that Ω is a solution of $(\mathcal{P}N_\sigma^\alpha)$ and (λ,μ) is the associated solution of problem (5.4). □

REMARK 5.4. In the definition of $H^1(\Omega)$ the sequence Ω_n of polyhedral open sets in D can be replaced by a sequence of C^∞ smooth open domains in D. We know from [2] that any set in BPS(D) can be approached in $L^2(D)$ (in the sense of $L^2(D)$ convergence of characteristic functions and weak convergence of the gradients as bounded measures on D) by a sequence of polyhedral open sets in D.

5.2. - *Max Inf formulation*

We turn now to the easiest situation:

$$\max_{\Omega} \inf_{\phi} \{L(\Omega, \phi) | \Omega \subset D, \ \text{meas}(\Omega) = \alpha, \ \phi \in H_0^1(D)\} \tag{5.9}$$

where

$$L(\Omega, \phi) = \int_{\Omega} (1/2\, |\text{grad}\, \phi|^2 - f\phi)\, dx + \int_{\Omega} G\, dx - \sigma P_D(\Omega). \tag{5.10}$$

The mapping $\Omega \to \inf\{L(\Omega, \phi) | \phi \in H_0^1(D)\}$ is upper semi-continuous on BPS(D), so that the maximum is attained in (5.9).

6. - Free interface with transmission conditions

The optimization problem associated to the trasmission condition, which we shall define in a similar way as above, is the easiest possible. It could be compared to the problem $(\mathcal{P})$ we studied at the beginning for $\partial\Omega$, with an interface on which now we have first order differential conditions satisfied by the solution y. Let a and b, $a < b$, be two given positive real numbers and let F be in $H^1(D)$. We consider the following problem:

$$(\mathcal{T}_\sigma^{\alpha}) \qquad \min\{E_\sigma(\Omega) | \Omega \subset D \text{ measurable}, \ \text{meas}(\Omega) = \alpha\}$$

where

$$E_\sigma(\Omega) = \min\left\{ \int_D (1/2\kappa(a, b, \Omega) |\text{grad}\, \phi|^2 - F\phi)\, dx | \phi \in H_0^1(D) \right\} + \sigma P_D(\Omega) \tag{6.1}$$

with

$$\kappa(a, b, \Omega) = b\chi_{\Omega} + a\chi_{D\setminus\Omega} = a + (b - a)\chi_{\Omega}; \tag{6.2}$$

for brevity we shall write $\kappa(\Omega)$ or simply κ.

This problem can be re-written as

$$\min\left\{ \int_D (1/2\kappa(\Omega)\, |\text{grad}\, \phi|^2 - F\phi)\, dx | \phi \in H_0^1(D), \quad \Omega \text{ measurable in } D \right\} + \sigma P_D(\Omega). \tag{6.3}$$

Let (ϕ_n, Ω_n) be a minimizing sequence: as we assume $\sigma > 0$, it is immediate to verify that it remains bounded in $H_0^1(D) \times \text{BPS}(D)$. So, after extracting a subsequence, we can assume that this sequence weakly converges in $H_0^1(D) \times \mathrm{BPS}(D)$ to (y, Ω). The element ϕ_n can be chosen as a solution of $E_\sigma(\Omega_n)$, so we have:

$$\int_D (\kappa(\Omega_n)\,\mathrm{grad}\,\phi_n \cdot \mathrm{grad}\,\psi - F\psi)\,dx = 0 \quad \forall y \in H_0^1(D) \tag{6.4}$$

As $\kappa(\Omega_n)$ strongly converges in $L^2(D)$ to $\kappa(\Omega)$, for $\psi \in C^1(\mathrm{cl}(D))$ we have that $\kappa(\Omega_n)\psi$ strongly converges in $L^2(D)$ to $\kappa(\Omega)\psi$. On the other hand $\mathrm{grad}\,\phi_n$ weakly converges in $L^2(D)$ to $\mathrm{grad}\,y$. Then in the limit we get that (Ω, y) verifies (6.4), so that y is the solution of (6.1).

Actually, we can show that ϕ_n strongly converges to ϕ in $H_0^1(D)$. Taking $\psi = \phi_n$ in (6.4) we get $E_\sigma(\Omega_n) = -1/2 \int_D F\phi_n\,dx$, which converges to $-1/2 \int_D Fy\,dx = E_\sigma(\Omega)$, so that $E_\sigma(\Omega)$ attains its minimum in $(\mathcal{T}_\sigma{}^\alpha)$. The strong $H_0^1(D)$ convergence of ϕ_n is checked as follows: of course $\mathrm{grad}\,\phi_n = (\kappa(\Omega_n))^{-1}(\kappa(\Omega_n)\,\mathrm{grad}\,\phi_n)$; the sequence $(\kappa(\Omega_n))^{-1}$ converges strongly in $L^2(D)$ to $(\kappa(\Omega))^{-1}$, while $\kappa(\Omega_n)\,\mathrm{grad}\,\phi_n$ converges strongly in $L^2(D)$ to $\kappa(\Omega)\,\mathrm{grad}\,y$, as it weakly converges and its norm, whose square is $-\int_D F\phi_n\,dx$, converges. We can state this result as follows.

PROPOSITION 6.1. *For any $s > 0$ and $0 < a < \mathrm{meas}(D)$ there exists (at least) a solution Ω of problem $(\mathcal{T}_\sigma{}^\alpha)$. If Ω_n converges to Ω in the sense that χ_{Ω_n} converges in the $L^2(D)$ norm, and if $y(\Omega_n)$ and $y(\Omega)$ are the solutions of $E_\sigma(\Omega_n)$ and $E_\sigma(\Omega)$ respectively, then $y(\Omega_n)$ strongly converges to $y(\Omega)$ in $H_0^1(D)$.*

7. - First order necessary conditions

7.1. - *The flow mapping T_t of a speed vector field $\mathcal{V}$*

We shall show that the solutions of problems $(\mathcal{D}_0)$, $(\mathcal{D}_0^\alpha)$, $(\mathcal{D}_\sigma^\alpha)$, $(\mathcal{D}\Sigma\mathcal{N}_\sigma^\alpha)$, $(\mathcal{N}_\sigma^\alpha)$ and $(\mathcal{T}_\sigma{}^\alpha)$ provide weak solutions of the Free Boundary Problems under consideration.

We need to introduce the perturbations Ω_t, for small t, of an optimal solution of one of the previous problems. The simplest idea is to use perturbations of the identity mapping as follows: let V belong to $C^\infty(\mathrm{cl}(D), \mathbb{R}^n)$ with $V(x) = 0$ on ∂D; for t small enough, $0 \le t \le \tau(V)$, the mapping $T_t = \mathrm{Id} + tV$ is one-to-one from D onto D and from ∂D onto ∂D. We define the perturbed domain as $\Omega_t = T_t(\Omega)$ with $\Omega_0 = \Omega$. Since T_t and T_t^{-1} belong to $C^\infty(\mathrm{cl}(D), \mathbb{R}^n)$, the boundary $\partial\Omega_t$ is given by $\partial\Omega_t = T_t(\partial\Omega)$.

The Jacobian matrix will be denoted by $DT_t = \mathrm{Id} + tDV$; its determinant $j(t) = \det(DT_t) = \det(\mathrm{Id}+tDV)$ satisfies $j'(t) = j(t)\,\mathrm{div}(\mathcal{V}(t))$, where $\mathcal{V}(t) = V \circ T_t^{-1}$ is the velocity field associated to the transformation T_t. The vector field V is autonomous, while $\mathcal{V}(t)$ is not. Then we get $j(t) = \exp\left(\int_0^t \mathrm{div}(\mathcal{V}(s))\,ds\right)$. To handle the constraint on the measure in the problems indexed by α, we need transformations which preserve the measure of Ω. This condition is achieved when $j(t) = 1$ and to obtain it we shall consider the transformation T_t as being built by a divergence-free vector field $\mathcal{V}$: for any t, $\mathrm{div}(\mathcal{V}(t)) = 0$ implies $j(t) = 1$. Let $\mathcal{V}$ be given in $C^\infty([0,t[, C^\infty(\mathrm{cl}(D), \mathbb{R}^n))$ with $\mathcal{V}(t,x)\cdot n(x) = 0$ on ∂D at each point x of ∂D where the normal field n exists, and $\mathcal{V}(t,x) = 0$ at the other points of ∂D. Then the transformation T_t is the flow of the field $\mathcal{V}$, given by

$$T_t(x) = x + \int_0^t \mathcal{V}(s, T_s(x))\,ds, \quad \text{or}$$
$$\partial_t(T_t(x)) = \mathcal{V}(t, T_t(x)), \quad (T_t(x))_{t=0} = x \tag{7.1}$$

while the field V is given by $V = \mathcal{V}(0)$. In this definition the field $\mathcal{V}$ can be chosen to be autonomous, i.e. independent of time t, which we shall always do in the sequel; then the transformation T_t cannot be written in the form $\mathrm{Id} + tV$, but $T_t = \mathrm{Id} + tV(t,\cdot)$. That is to say, when $\mathcal{V}$ is autonomous V is not (and conversely, as we have seen). When there is no constraint on the measure of the optimal domain Ω the transformation $T_t = \mathrm{Id} + tV$, with V autonomous, is suitable to obtain the first-order necessary condition, but it will be definitely not suitable for the first-order necessary conditions of the problems indexed by α. On the other hand the transformations

$$T_t(x) = x + \int_0^t \mathcal{V}(s, T_s(x))\,ds$$

are suitable in all situations, including second-order necessary conditions. For this purpose it will be sufficient to consider the transformations associated to autonomous vector fields V. Of course when both $T_t(x) = x + \int_0^t \mathcal{V}(s, T_s(x))\,ds$ and $T_t = I_d + tV$ are suitable to obtain the first-order necessary conditions (which occurs only for the problem $(\mathcal{D}_0)$), these two families of transformations lead to the same first-order conditions, i.e. to the same free-boundary problem solved by the optimal domain Ω. This is false for second-order conditions.

To impose the constraint that Ω should contain a given measurable subset $\mathfrak{C}$ of D, it is sufficient to take $\mathcal{V}(t,\cdot) = 0$ on $\mathfrak{C}$.

REMARK. It is important to note that the choice of the transformations T_t (obtained as the flow of a field $\mathcal{V}$) also allows to move the smooth parts of the boundary of D (as it is enough to impose the normal component of the field $\mathcal{V}$ to be zero on the smooth parts of the boundary in order to define a mapping from D onto itself). With the transformations $\mathrm{Id} + tV$ we were obliged to impose the condition $V = 0$ on the whole boundary of D.

7.2. *Free boundary conditions in weak form*

In these problems optimality of the measurable set Ω can be written as follows. There exist a measurable set Ω in D and $y = y(\Omega)$ in $H(\Omega)$, y being uniquely associated to Ω, such that for any admissible field $\mathcal{V}$, for any $0 \leq t \leq \tau(\mathcal{V})$ and for any ϕ_t in $H(\Omega_t)$ with $\Omega_t = T_t(\Omega)$ and T_t the flow mapping of $\mathcal{V}$:

$$\begin{aligned} &\int_{\Omega} \{1/2\,|\mathrm{grad}\, y|^2 - fy + G\}\, dx + \delta \int_{\Sigma_0 \cap \mathrm{cl}(\Omega)} gy\, d\mu + \sigma P_D(\Omega) \\ &\leq \int_{\Omega_t} \{1/2\,|\mathrm{grad}\, \phi_t|^2 - f\phi_t + G\}\, dx + \delta \int_{\Sigma_0 \cap \mathrm{cl}(\Omega_t)} g\phi_t\, d\mu + \sigma P_D(\Omega_t) \end{aligned} \tag{7.2}$$

where $\delta \geq 0$, f and G could be zero and $H(\Omega)$ is the Hilbert space under consideration in each problem: respectively $H_0^1(\Omega)$, $\tilde{H}_0^1(\Omega)$. In these two problems the "natural" boundary condition attached to $H(\Omega)$ is that $y = 0$ on $\partial\Omega$. The additive optimality condition (7.2) implies an extra boundary condition solved by $y(\Omega)$ on $\partial\Omega$. This extra condition is given by (7.6) of Proposition 7.5 below. In the next section we shall explicit the boundary expression for (7.6) when we assume Ω and y to be smooth enough.

PROPOSITION 7.1. *Let Ω be a measurable subset of D. When $H(\Omega)$ is one of the Hilbert spaces $H_0^1(\Omega)$ or $\tilde{H}_0^1(\Omega)$, we have that ϕ belongs to $H(\Omega)$ if and only if $\phi_t = \phi \circ T_t^{-1}$ belongs to $H(\Omega_t)$.*

The proof follows from the next result, which is immediate.

LEMMA 7.2. *Let f_n be a sequence in $L^2(D)$ which converges weakly to f in $L^2(D)$. Then the sequence $f_n \circ T_t^{-1}$ converges weakly to $f \circ T_t^{-1}$ in $L^2(D)$.*

In view of Proposition 7.1 we can take $\phi_t = y \circ T_t^{-1}$ in (7.2). We get that for any admissible field $\mathcal{V}$ and for any t, $0 \leq t \leq \tau(\mathcal{V})$, if T_t is the flow mapping of $\mathcal{V}$, the following holds:

$$\begin{aligned} &\int_{\Omega} \{1/2\,|\mathrm{grad}\, y|^2 - fy + G\}\, dx \\ &\leq \int_{\Omega_t} \{1/2\,|\mathrm{grad}\, y \circ T_t^{-1}|^2 - fy \circ T_t^{-1} + G\}\, dx. \end{aligned} \tag{7.3}$$

Using the change of variable $\chi = T_t(x)$ in the right-hand side of (7.3) we get:

$$\begin{aligned}&\int_\Omega \{1/2\,|\operatorname{grad} y|^2 - fy + G\}\,dx \\ &\le \int_\Omega \{1/2\,\langle A(t)\cdot \operatorname{grad} y,\ \operatorname{grad} y\rangle - j(t)\,f\circ T_t y + j(t) G\circ T_t\}\,dx\end{aligned} \tag{7.4}$$

where n is the unitary normal field on ∂D (which is assumed to exist on Σ_0) and $A(t)$ is the symmetric matrix $j(t)DT_t^{-1,*}(DT_t)^{-1}$.

LEMMA 7.4 ([11], [12], [13], [14]). *Let $2\varepsilon(\mathcal{V}) = D\mathcal{V} + {}^*D\mathcal{V}$; then for all integers k we have*

$$\|t^{-1}(A(t) - I_d) - (\operatorname{div}(\mathcal{V})I_d - 2\varepsilon(\mathcal{V})\|_{W^{k,\infty}(\mathrm{cl}(D),R^n)} \to 0,\ \ as\ t \to 0.$$

Let us define the 'Eulerian semi-derivative' of the perimeter $P_D(\Omega)$ at Ω in the direction V by:

$$d_-P_D(\Omega; \mathcal{V}) = \liminf_{t>0,\, t\to 0} t^{-1}(P_D(\Omega_t) - P_D(\Omega)) \tag{7.5}$$

where $\Omega_t = T_t(\Omega)$, T_t being the flow mapping of the field $\mathcal{V}$. This lim inf can be expressed as follows:

$$d_-P_D(\Omega; \mathcal{V}) = \inf\left\{\liminf_{n\to\infty} t_n^{-1}(P_D(\Omega_{t_n}) - P_D(\Omega)) | \{t_n\} \in \mathbb{R}^{\mathbb{N}}, t_n > 0, t_n \to 0\right\}.$$

From Lemma 2.1 and (7.4) we get the following result.

PROPOSITION 7.5. *Let f and G be elements of $H^1(D)$ and let Ω be an optimal solution of one of the problems $(\mathcal{D}_0)$, $(\mathcal{D}_0^\alpha)$, $(\mathcal{DC}_0)$, $(\mathcal{DC}_0^\alpha)$ or $(\mathcal{D}_\sigma^\alpha)$. Then for any admissible field $\mathcal{V}$ we have*

$$\begin{aligned}&\int_\Omega \{1/2\,\langle[\operatorname{div}(\mathcal{V})I_d - 2\varepsilon(\mathcal{V})]\cdot \operatorname{grad} y,\ \operatorname{grad} y\rangle - \operatorname{div}(f\mathcal{V})y + \operatorname{div}(G\mathcal{V})\}\,dx \\ &\qquad + \sigma\, d_-P_D(\Omega; \mathcal{V}) \ge 0\end{aligned} \tag{7.6}$$

and then $d_-P_D(\Omega; \mathcal{V}) > -\infty$.

7.3 *Weak condition at Ω having curvature $\mathfrak{H}$*

The two variational inequalities (7.2) and (7.6) become equalities as soon and $\sigma = 0$ or $\sigma > 0$ but Ω has curvature $\mathfrak{H}$.

DEFINITION 7.8. We say that an element $\mathfrak{H}$ of $\mathcal{E}'(\mathbb{R}^n, \mathbb{R}^n)$, i.e. a vectorial distribution over $\mathbb{R}^n$, having support contained in $\partial\Omega$, is the curvature of a measurable set Ω if Ω belongs to BPS(D), the mapping $\mathcal{V} \mapsto d_{-}P_D(\Omega; \mathcal{V})$ is linear and continuous on $C^\infty(\mathrm{cl}(D), \mathbb{R}^n)$ and $\mathfrak{H}$ is such that

$$d_{-}P_D(\Omega; \mathcal{V}) = \langle \mathfrak{H}, \mathcal{V} \rangle_{\mathcal{D}'(\mathbb{R}^n, \mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n)}. \tag{7.10}$$

PROPOSITION 7.9. *Assume that Ω is an optimal solution of either $(\mathcal{D}_0)$, or $(\mathcal{D}_0^\alpha)$ or $(\mathcal{D}_\sigma^\alpha)$. Then for any admissible field $\mathcal{V}$ we have:*

$$\langle -1/2\, \mathrm{grad}(\chi_\Omega |\mathrm{grad}\, y|^2) + \mathrm{div}(\chi_\Omega(\mathrm{grad}\, y^* \mathrm{grad}\, y)) + (fy - G)\, \mathrm{grad}\, \chi_\Omega, \mathcal{V} \rangle = 0$$

where the brackets $\langle\, , \rangle$ stand for the duality pairing between $\mathcal{D}'(\mathbb{R}^n, \mathbb{R}^n)$ and $\mathcal{D}(\mathbb{R}^n, \mathbb{R}^n)$, ${}^*\gamma_{\partial D}$ is the transposed of the trace operator $\gamma_{\partial D}$ on ∂D and $\mathrm{grad}\, y^* \mathrm{grad}\, y$ *is the matrix $\partial_i y \partial_j y$.*

7.4. - *Optimality condition when Ω is a smooth open set*

We assume now that $\partial\Omega$ is a sufficiently smooth manifold and that Ω is placed on one side of $\partial\Omega$. Hence, when $s > 0$, the mean curvature H exists a.e. on $\partial\Omega$ and $\mathfrak{H} = {}^*\gamma_{\partial\Omega}(Hn)$, n being the normal field on $\partial\Omega$. From the smoothness of Ω we obtain that $H^1(\Omega)$, $H_0^1(\Omega)$ etc. are defined in the classical way; by the classical regularity results for the solutions of elliptic boundary value problems we obtain that for problems $(\mathcal{D}_0)$, $(\mathcal{D}_0^\alpha)$ and $(\mathcal{D}_\sigma^\alpha)$ the solution $y = y(\Omega)$ belongs to $H^2(\Omega)$. Using Stoke's formula we easily obtain from (7.11) the free boundary condition verified by y in each problem: for any admissible field $\mathcal{V}$, y verifies

$$\int_\Omega (\Delta y + F) \langle \mathrm{grad}\, y, \mathcal{V} \rangle \, dx + \int_{\partial\Omega} \{[1/2\, |\mathrm{grad}\, y|^2 - fy + G]\, \mathcal{V} \cdot n - \partial_n y \, \langle \mathrm{grad}\, y, \mathcal{V} \rangle\}\, d\mu + \sigma \int_{\partial\Omega} H \mathcal{V} \cdot n \, d\mu = 0. \tag{7.12}$$

In the Dirichlet situations y is zero on $\partial\Omega$ or constant on each connected component of $\partial\Omega$, so that in both cases we get $\mathrm{grad}\, y = \partial_n y n$ and (7.12) reduces to

$$\int_{\partial\Omega} [-1/2\, |\mathrm{grad}\, y|^2 - fy + G]\, \mathcal{V} \cdot n \, d\mu + \sigma \int_{\partial\Omega} H \mathcal{V} \cdot n \, d\mu = 0. \tag{7.13}$$

In the first problem there is no constraint on the measure of the domain so there is no constraint on the divergence of the admissible field $\mathcal{V}$: we only have

$\mathcal{V} \cdot n = 0$ a.e. on ∂D ($\mathcal{V} = 0$ at exceptional points where the normal n does not exist on ∂D). So from (7.13) we get:

$$-1/2|\partial_n y|^2 + G + \sigma H = 0 \quad \text{on } \partial\Omega \cap D \tag{7.14}$$

and, when $\sigma = 0$, i.e. for $(\mathcal{D}_0)$:

$$\partial_n y = 0 \quad \text{on } \partial\Omega \cap D. \tag{7.15}$$

In problems $(\mathcal{D}_0^\alpha)$ and $(\mathcal{D}_\sigma^\alpha)$ the admissible field is divergence-free, which implies that the normal component $\mathcal{V} \cdot n$ on $\partial\Omega$ verifies the following constraint:

$$\int_{\partial\Omega} \mathcal{V} \cdot n \, d\mu = 0 \tag{7.16}$$

so that from (7.13) we get that there exists a constant c such that

$$-1/2\,|\partial_n y|^2 - fy + G + \sigma H = c \quad \text{on } \partial\Omega \cap D. \tag{7.17}$$

8. - Second order necessary conditions

8.1. - *Second order derivative of* $E(\Omega)$

We shall assume now that Ω is an optimal solution of one of the previous problems and that Ω is a smooth (enough) domain contained in D. We shall define the second order derivative of the cost function $E(\Omega)$ in a general way, and then use it to obtain the second order necessary conditions.

Under smoothness assumptions we have seen in the previous section that $E(\cdot)$ possesses a gradient at Ω: there exists $g = g(\Gamma)$ in $L^1(\Gamma)$, $\Gamma = \partial\Omega$, (g is the density gradient in the terminology of [10] and [13]) such that

$$dE(\Omega; \mathcal{V}) := \lim_{t>0,\, t\to 0} (E(\Omega_t(\mathcal{V})) - E(\Omega))/t = \int_\Gamma g(\Gamma)\, \mathcal{V} \cdot n \, d\mu. \tag{8.1}$$

We define the boundary shape derivative of g (as in [16]), if it exists, as the following element of $\mathcal{D}'(\Gamma)$:

$$g'(\Gamma; \mathcal{V}) = [\partial/\partial t\{g(T_t(\Gamma)) \circ T_t\}]_{t=0}[\text{grad}_\Gamma g(\Gamma)] \cdot \mathcal{V} \tag{8.2}$$

where T_t is the flow mapping associated to the autonomous vector field $\mathcal{V}$.

Following [3], let us define the second order derivative. Given two autonomous vector fields $\mathcal{V}$ and $\mathcal{W}$:

$$d^2E(\Omega; \mathcal{V}, \mathcal{W}) = \lim_{t>0,\, t\to 0} (dE(\Omega_t(\mathcal{W}); \mathcal{V}) - dE(\Omega; \mathcal{V}))/t. \tag{8.3}$$

LEMMA 8.1. *This second derivative has the following expression:*

$$d^2E(\Omega; \mathcal{V}, \mathcal{W}) = \int_\Gamma g'(\Gamma; \mathcal{W})\, \mathcal{V} \cdot n \, d\mu + \int_\Gamma (\mathrm{grad}_\Gamma g(\Gamma) \cdot \mathcal{W} + g(\Gamma) \,\mathrm{div}\, \mathcal{W})\, \mathcal{V} \cdot n \, d\mu + \int_\Gamma g \,\langle D\mathcal{V} \cdot \mathcal{W} - D\mathcal{W} \cdot \mathcal{V}, n\rangle \, d\mu. \tag{8.4}$$

We are going to use (8.4) for specific fields; namely, let A and B be two real functions defined on D such that A and B are both constant on each connected components of $\partial\Omega$ and

$$\partial^2/\partial n^2 A = \partial^2/\partial n^2 B = 0 \text{ on } \partial\Omega, \tag{8.5}$$

where $\partial^2/\partial n^2 A = \langle D^2 A \cdot n, n\rangle$ and $\partial^2/\partial n^2 B = \langle D^2 B \cdot n, n\rangle$.

Let us define on Γ the two real functions v and w by $v = \partial/\partial_n A$ and $w = \partial/\partial_n B$.

COROLLARY 8.2. *If H is the mean curvature on $\partial\Omega$ and $H = \mathrm{div}_\Gamma n$ we have:*

$$d^2E(\Omega; \mathrm{grad}\, A, \mathrm{grad}\, B) = \int_\Gamma g'(\Gamma; \mathrm{grad}\, B)\, v \, d\mu + \int_\Gamma H g(\Gamma)\, vw \, d\mu. \tag{8.6}$$

PROOF. The conclusion follows from the fact that

$$(\Delta A)\big|_\Gamma = \Delta_\Gamma(A\big|_\Gamma) + H\partial/\partial_n A + \partial^2/\partial n^2 A = Hv \tag{8.7}$$

(where Δ_Γ is the Laplace-Beltrami operator on $\partial\Omega$, see [11], [12] and [13]) and from integration by parts on $\partial\Omega$ in (8.4). □

8.2. - *Second order necessary condition for Dirichlet problems*

We know, see [14], that the derivative of E at any smooth domain Ω is given by:

- when f is $L^2(D)$,

$$dE(\Omega; V) = \int_{\partial\Omega} (1/2\, |\mathrm{grad}\, y|^2 - fy)\, \mathcal{V} \cdot n \, d\mu - \int_{\partial\Omega} \partial_n y \,\mathrm{grad}\, y \cdot \mathcal{V} \, d\mu;$$

- when f is in $H^{-1}(D)$ with its support in $\mathcal{C}$, and $\mathcal{C} \cap \partial\Omega$ is empty,

$$dE(\Omega; V) = \int_{\partial\Omega} 1/2\, |\mathrm{grad}\, y|^2 \, d\mu - \int_{\partial\Omega} \partial_n y \,\mathrm{grad}\, y \cdot \mathcal{V} \, d\mu.$$

But y is constant on $\partial\Omega$, so $\operatorname{grad} y \cdot \mathcal{V} = \partial_n y \mathcal{V} \cdot n$ and we get, with the convention that f is zero on $\partial\Omega$ in the second situation:

$$dE(\Omega; V) = \int_{\partial\Omega} (-1/2\,|\operatorname{grad} y|^2 - fy)\,\mathcal{V} \cdot n\, d\mu. \tag{8.8}$$

Then the density gradient of E is $g(\Gamma) = -1/2\,|\operatorname{grad} y|^2 - fy$, where y is the solution of the Dirichlet problem $-\Delta y = f$ in Ω, $y = 0$ on $\partial\Omega$. Recalling (8.6), in order to get $d^2E(\Omega; \cdot, \cdot)$, we only need to compute the boundary shape derivative $g'(\Gamma; \operatorname{grad} B)$. The density gradient g is in fact the restriction to $\partial\Omega$ of the function $\tilde{g} = -1/2|\operatorname{grad} y|^2 - fy$ defined on Ω and for which we can compute the shape derivative $\tilde{g}'(\Omega; \mathcal{W}) = -\operatorname{grad} y \cdot \operatorname{grad} y'(\Omega; \mathcal{W}) - fy'(\Omega; \mathcal{W})$. But the trace on $\partial\Omega$ of the shape derivative of $\tilde{g}$ and the boundary shape derivative $g'(\Gamma; \mathcal{W})$ (g being the restriction to $\partial\Omega$ of $\tilde{g}$) are related by:

$$g'(\Gamma; \mathcal{W}) = \tilde{g}'(\Omega; \mathcal{W})\big|_{\partial\Omega} + \partial_n \tilde{g}\, W \cdot n. \tag{8.9}$$

The shape derivative y' is a solution of the following Dirichlet BVP, see [11], [12] and [13]:

$$-\Delta y' = 0 \text{ in } \Omega, \quad y' = -\partial_n y \quad \text{on } \partial\Omega. \tag{8.10}$$

LEMMA 8.3.

$$g'(\Gamma; \mathcal{W}) = -\partial_n y(\partial_n y'(\Omega; \mathcal{W})\big|_{\partial\Omega} - \langle \partial^2/\partial n^2\, y \cdot n, n\rangle\, \mathcal{W} \cdot n). \tag{8.11}$$

PROOF. It follows from (8.9), (8.10) and classical arguments. □

PROPOSITION 8.4. *Let Ω be an optimal solution of one of the problems $(\mathcal{D}_0)$ or $(\mathcal{DC}_0)$. Assume that Ω is a smooth open domain in D and that y is smooth enough. Then for all v and w smoothly defined on $\partial\Omega$, we have: $d^2E(\Omega; \operatorname{grad} A, \operatorname{grad} B) = 0$.*

PROOF. As Ω is optimal, from (7.14) we get $\partial_n y = 0$ on $\partial\Omega$, so that $y' = 0$. Then $g'(\Gamma; \mathcal{W}) = 0$ and $g(\Gamma) = 0$, and the result follows from (8.6). □

When the measure of Ω is imposed to be equal to α, i.e. when we consider the problems $(\mathcal{D}_0^\alpha)$ and $(\mathcal{DC}_0^\alpha)$, the second order derivative is not zero because the first order necessary condition (7.17), with $\sigma = 0$, $G = 0$ and $y = 0$ on $\partial\Omega$, leads to $(\partial_n y)^2 = c^2$ on $\partial\Omega$. Then y' is not zero and, from (8.7), we get

$$\langle \partial^2/\partial n^2\, y \cdot n, n\rangle = -H\partial_n y \tag{8.11}$$

when f is equal to zero in a neighborhood of $\partial\Omega$.

Hence we can write

$$g'(\Gamma; \mathcal{W}) = -\partial_n y\, \partial_n y'(\Omega; \mathcal{W})\big|_{\partial\Omega} - Hc^2 \mathcal{W} \cdot n \tag{8.12}$$

and we get the following second order necessary condition.

PROPOSITION 8.5. *Let* Ω *be an optimal solution of problem* $(\mathcal{D}\mathcal{C}_0^\alpha)$ *with* $G = 0$. *Assume that* Ω *is smooth. Then we have:* $(\partial_n y)^2 = c^2$ *(constant) on* $\partial\Omega$ *and for all smooth real functions* v *defined on* $\partial\Omega$ *with* $\int_\Gamma v\,d\mu = 0$ *we have:*

$$\int_\Gamma \partial_n y\,\partial_n y'(\Omega;\operatorname{grad} A)\big|_{\partial\Omega} v\,d\mu + \int_\Gamma 1/2\,Hc^2v^2\,d\mu \geq 0. \tag{8.13}$$

9. - Regularity result

Assume that Ω is an optimal solution of problem $(\mathcal{D}_0^\alpha)$. Then from (7.11), as $\delta = 0$ (i.e. we have no boundary term in this problem), the necessary first order optimality condition can be written as follows: for any admissible field $\mathcal{V}$ we have

$$\begin{aligned}\langle -1/2\operatorname{grad}(\chi_\Omega|\operatorname{grad} y|^2) + \operatorname{div}(\chi_\Omega(\operatorname{grad} y\cdot {}^*\operatorname{grad} y))\\ + (fy - G)\operatorname{grad}\chi_\Omega, \mathcal{V}\rangle = 0.\end{aligned} \tag{9.1}$$

As $\mathcal{V}$ can be chosen divergence-free over D, we obtain in a classical way from (9.1) that the gradient $\mathcal{G}(\Omega)$ is equal to $\operatorname{grad} p$. In fact we can here give more precise informations on this "pressure term p", which appears to be *proportional to the characteristic function* χ_Ω *of the optimal domain.*

PROPOSITION 9.1. *Let* Ω *be an optimal solution of problem* $(\mathcal{D}_0^\alpha)$. *Then there exist a real number* β *such that*

$$(\Delta y + f)\operatorname{grad} y = (\beta + G - yf)\operatorname{grad}\chi_\Omega. \tag{9.2}$$

PROOF. Since $\operatorname{grad} y = 0$ a.e. in $D\backslash\Omega$ (so that $\chi_\Omega \operatorname{grad} y = \operatorname{grad} y$ a.e. in D), the gradient

$$\mathcal{G}(\Omega) = -1/2\operatorname{grad}(\chi_\Omega|\operatorname{grad} y|^2) + \operatorname{div}(\chi_\Omega(\operatorname{grad} y\cdot {}^*\operatorname{grad} y)) + (fy - G)\operatorname{grad}\chi_\Omega$$

can be re-written as

$$\mathcal{G}(\Omega) = -1/2\operatorname{grad}(|\operatorname{grad} y|^2) + \operatorname{div}((\operatorname{grad} y\cdot {}^*\operatorname{grad} y)) + (fy - G)\operatorname{grad}\chi_\Omega. \tag{9.3}$$

But $-1/2\operatorname{grad}(|\operatorname{grad} y|^2) + \operatorname{div}((\operatorname{grad} y\cdot {}^*\operatorname{grad} y)) = \Delta y\operatorname{grad} y$.

According to Lemma 9.2, which we state below, to check that the field $\mathcal{V}$ is an admissible field it is sufficient to verify the following condition:

$$\int_{\partial\Omega} \mathcal{V}\cdot n\,d\mu = \int_\Omega \operatorname{div}\mathcal{V}\,dx = 0. \tag{9.4}$$

The orthogonality condition (9.1) for fields verifying condition (9.4) leads to (9.3), where b is obtained as follows: let V_0 be a vector field over D such that $\int_\Omega \operatorname{div}(V_0)\,dx \neq 0$, then for any field V, the field $V - \gamma V_0$ verifies condition (9.4) when

$$\gamma = \left(\int_\Omega \operatorname{div}(V_0)\,dx\right)^{-1} \left(\int_\Omega \operatorname{div}(V)\,dx\right).$$

The optimality condition $\langle \mathcal{G}(\Omega), V - \gamma V_0\rangle = 0$ is equivalent to

$$\langle \mathcal{G}(\Omega), V\rangle = \gamma\,\langle \mathcal{G}(\Omega), V_0\rangle,$$

but $\gamma = \left(\int_\Omega \operatorname{div}(V_0) dx\right)^{-1} \langle -\operatorname{grad}\chi_\Omega, V\rangle$, so that we get:

$$\langle \mathcal{G}(\Omega), V\rangle = \langle \mathcal{G}(\Omega), V_0\rangle \left(\int_\Omega \operatorname{div}(V_0)\,dx\right)^{-1} \langle -\operatorname{grad}\chi_\Omega, V\rangle$$

for any field V. Then relation (9.2) holds with

$$\beta = -\langle \mathcal{G}(\Omega), V_0\rangle \left(\int_\Omega \operatorname{div}(V_0)\,dx\right)^{-1}. \qquad \square$$

LEMMA 9.2. *Let Ω be a measurable subset of D and V be given in $W^{1,\infty}(D)$ such that $\int_\Omega \operatorname{div} V\,dx = 0$. Then there exists $\tau > 0$ and $\mathcal{V}$ in $L^\infty(0,\tau,W^{1,\infty}(D))$ such that $\int_{\Omega_t(V)} \operatorname{div}\mathcal{V}(t)\,dx = 0$ for all t with $0 \le t \le \tau$.*

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