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On the Propagation of Singularities of Semi-convex Functions

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0. - Introduction

In a recent paper [1], *upper* bounds on the dimension of singular sets of semi-convex functions were derived by measure-theoretic arguments.

To briefly describe these upper bounds, let $u : \mathbb{R}^n \to \mathbb{R}$ be a semi-convex function (Definition 1.2 below). Define

$$S^k(u) = \{x \in \mathbb{R}^n : \dim(\partial u(x)) = k\},$$

where $k \in [0, n]$ is an integer and $\partial u(x)$ denotes, as usual, the subdifferential of u. Clearly, $\{S^k(u)\}_{k=0}^n$ is a partition of $\mathbb{R}^n$ and $S^0(u)$ is the set of all points of differentiability of u. Since we are interested in first-order singularities, we call a point x singular for u if $x \in S^k(u)$ for some $k \geq 1$.

In [1] it is proved that $S^k(u)$ is countably $\mathcal{H}^{n-k}$-rectifiable. In particular,

$$\mathcal{H} - \dim(S^k(u)) \leq n - k,$$

where $\mathcal{H} - \dim$ is the Hausdorff dimension.

The purpose of the present work is to obtain *lower* bounds on the dimension of $S^k(u)$. More precisely, we will describe the structure of $S^k(u)$ in a neighborhood of x, knowing the geometry of $\partial u(x)$.

A motivating application of these results concerns the analysis of singularities of solutions to the Hamilton-Jacobi-Bellman equation

$$H(x, u, \nabla u) = 0. \tag{1}$$

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In fact, if the data are smooth, viscosity solutions of such PDE's (and, in particular, the solutions that are relevant to optimal control) enjoy well-known semi-concavity properties (see for instance [12], [13], [15] on [16]).

The present work is related to [4] and [5], in which viscosity solutions of (1) are shown not to have any isolated singularity if H is strictly convex with respect to p. In [4] and [5], however, no attention is paid to the dimension of ∂u at such singular points, and no attempt is made to estimate the Hausdorff measure of the singular sets.

Different approaches to the analysis of singularities of Hamilton-Jacobi equations are obtained for the one-dimensional case in [14] and, using characteristics, in [21].

Semi-convexity was the only property used in [1] to prove upper bounds on singular sets. On the contrary, to obtain lower bounds we need additional information. This fact is the essential difference between [1] and the present paper. In order to understand the nature of the additional information, let us consider the set of reachable subgradients

$$\nabla_* u(x) = \Big\{ \lim_{h\to+\infty} \nabla u(x_h) : x_h \in S^0(u)\setminus\{x\},\ \ x_h \to x \Big\}.$$

where ∇u denotes the gradient of u.

The above set is a set of generators of $\partial u(x)$ in the sense of convex analysis. Then, we show that the strict inclusion

$$\nabla_* u(x) \subsetneq \partial u(x) \tag{2}$$

is a sufficient condition for the propagation of any singularity $x \in S^k(u)$, $1 \le k < n$ (see Example 2.1 below). Inclusion (2) is satisfied by any viscosity solution of (1) with a strictly convex Hamiltonian, as $\nabla_* u(x)$ is contained in the zero-level set of $H(x, u(x), \cdot)$.

Moreover, if x is an isolated singularity, by adapting a variational argument of Tonelli (see the proof of the implicit function theorem in [20]), we show that $\nabla_* u(x)$ coincides with $\partial u(x)$, see Theorem 2.1 below.

Furthermore, inserting non-smooth analysis into this procedure, we obtain a more detailed description of the singular sets. In Theorem 2.2 we prove that singularities propagate along directions related to the geometry of $\partial u(x)$. These directions are orthogonal to the exposed faces of $\partial u(x)$. In Theorem 2.3 we give a lower bound on the maximum integer $m \le k$ such that x is a cluster point of

$$\Sigma^m(u) = \bigcup_{i=m}^{n} S^i(u),$$

and in (2.7) we estimate from below the Hausdorff $(n-k)$-dimensional measure of $\Sigma^m(u)$. Roughly speaking, the computation of m takes into account how many vectors in $\nabla_* u(x)$ are necessary to generate $\partial u(x)$.

We conclude with an outline of the paper. The first section contains preliminary material on Hausdorff measures, semi-convex functions, and the estimates of [1]. In Section 2 we develop our main results on propagation of singularities of semi-convex functions. The last section is devoted to applications to Hamilton-Jacobi-Bellman equations and to the discussion of some examples.

1. - Notation and preliminaries

We briefly introduce some notation. We denote by $B_\rho(x)$ the open ball in $\mathbb{R}^n$ centered in x with radius ρ, and we abbreviate $B_\rho = B_\rho(0)$.

For any set $A \subset \mathbb{R}^n$ we denote by co(A) the convex hull of A. Moreover, the following sets of convex combinations of points of A will be often used in the sequel.

$$I_j(A) = \left\{ \sum_{i=1}^{j} \lambda_i p_i : p_i \in A,\ \lambda_i \geq 0,\ \sum_{i=1}^{j} \lambda_i = 1 \right\}$$

for any integer $j \geq 1$. We also define

$$m(A) = \max \{ j \geq 0 : I_j(A) \neq \mathrm{co}(A) \} .$$

Clearly $I_1(A) = A$, hence $m(A) = 0$ if and only if A is a convex set. Moreover, by Carathéodory's Theorem (see for example [18, p. 155]) we know that $I_{k+1}(A) = \mathrm{co}(A)$, where k is the dimension of co(A). Therefore $m(A) \leq \dim[\mathrm{co}(A)]$. However, the integer $m(A)$ does not depend just on the dimension of co(A). For example, if A is a finite set of affinely independent points, then $m(A)$ equals the dimension of co(A). On the other hand, if A is the boundary of a k-dimensional ball, then $m(A) = 1$.

For any set $S \subset \mathbb{R}^n$ we define

$$S^\perp = \{ p \in \mathbb{R}^n : q \mapsto \langle q, p \rangle \text{ is constant on } S \},$$

and

$$T(S, x) = \left\{ r\theta : r \geq 0,\ \theta = \lim_{h \to +\infty} \frac{x_h - x}{|x_h - x|},\ x_h \in S \backslash \{x\},\ x_h \to x \right\}.$$

The set $T(S, x)$ defined above is the so-called *contingent* cone to S at x ([3], [6]). The contingent cone is also known as Whitney's normal cone.

For any real number $r \in]0, n]$ we denote by $\mathcal{H}^r(B)$ the Hausdorff r-dimensional measure of $B \subset \mathbb{R}^n$, defined by

$$\mathcal{H}^r(B) = \frac{\omega_r}{2^r} \sup_{\delta > 0} \inf \left\{ \sum_{i=1}^{\infty} (\mathrm{diam}(B_i))^r : B \subset \bigcup_{i=1}^{\infty} B_i,\ \mathrm{diam}(B_i) < \delta \right\},$$

where ω_r is the Lebesgue measure of the unit ball in $\mathbb{R}^r$ if r is an integer, any positive constant otherwise. We also denote by $\mathcal{H}^0(B)$ the cardinality of B. The

Hausdorff dimension of B is defined by

$$\mathcal{H}-\dim(B)=\inf\{r>0:\mathcal{H}^r(B)=0\}.$$

For an introduction to the properties of Hausdorff measures see for example [10] and [17]. We merely recall that $\mathcal{H}^r$ is a Borel regular measure in $\mathbb{R}^n$, and

$$\mathcal{H}^r(B)<+\infty \Longrightarrow \mathcal{H}^m(B)=0\ \forall m>r. \tag{1.1}$$

We now recall the definition of semi-convexity and the main properties of semi-convex functions.

DEFINITION 1.2. Let $\Omega\subset\mathbb{R}^n$ be an open convex set, and $u:\Omega\to\mathbb{R}$. We say that u is *semi-convex* in Ω if there is a non-decreasing upper semicontinuous function $\omega:[0,+\infty[\to[0,+\infty[$ such that $\omega(0)=0$ and

$$\begin{aligned}&tu(x_1)+(1-t)u(x_2)-u(x_t)\geq -t(1-t)|x_1-x_2|\omega(|x_1-x_2|)\\ &x_t=tx_1+(1-t)x_2,\ \ x_1,x_2\in\Omega,\ \ t\in[0,1].\end{aligned} \tag{1.2}$$

We call *semi-convexity modulus* of u the least function ω satisfying (1.2). If $u:\Omega\to\mathbb{R}$ is semi-convex and $x\in\Omega$, we say that $p\in\mathbb{R}^n$ is a subgradient of u at x if

$$\liminf_{y\to x}\frac{u(y)-u(x)-\langle p,y-x\rangle}{|y-x|}\geq 0.$$

Borrowing the notation of convex analysis, we denote by $\partial u(x)$ the set of subgradients of u at x, call it the subdifferential of u at x. It is easy to see that $\partial u(x)$ is a compact, nonempty, convex set. Moreover,

$$p\in\partial u(x)\Longleftrightarrow u(y)-u(x)-\langle p,y-x\rangle\geq -|y-x|\omega(|y-x|),\ \ \forall y\in\Omega. \tag{1.3}$$

It can also be shown that $\partial u(x)$ is a singleton if an only if u is differentiable at x. Hence, the set of non-differentiability points of u can be classified according to the dimension of the subdifferential at the singular point.

DEFINITION 1.3. Let $x\in\Omega$, and let $k\in\{0,\dots,n\}$ be an integer. We define

$$S^k(u)=\{x\in\Omega:\dim(\partial u(x))=k\},$$

and

$$\Sigma^k(u)=\bigcup_{i=k}^{n}S^i(u)=\{x\in\Omega:\dim(\partial u(x))\geq k\}.$$

In order to find sufficient conditions for the propagation of singularities, it will be useful to consider the set $\nabla_*u(x)$ of reachable subgradients.

DEFINITION 1.4. Let $u:\Omega\to\mathbb{R}$ be a semi-convex function, and let $x\in\Omega$. We define

$$\nabla_*u(x)=\Big\{\lim_{h\to+\infty}\nabla u(x_h):x_h\in S^0(u),\ \ x_h\to x\Big\}.$$

Then, it is known that $\partial u(x)$ is the convex hull of $\nabla_* u(x)$ (see e.g. [4]).

In the following theorem we list some basic properties of semi-convex functions. We recall (see [3]) that a set-valued map $S(x)$ is said to be upper semicontinuous if the following implication holds:

$$p_h \in S(x_h), \quad x_h \to x, \quad p_h \to p \quad \Longrightarrow \quad p \in S(x).$$

THEOREM 1.1. *Let $u : \Omega \to \mathbb{R}$ be a semi-convex function. Then,*

(1) *u is locally Lipschitz continuous in Ω and*

$$\frac{\partial u}{\partial \theta}(x) = \lim_{t \to 0^+} \frac{u(x + t\theta)u(x)}{t} = \max\{\langle p, \theta\rangle : p \in \partial u(x)\}$$

for any $x \in \Omega$ and any $\theta \in \mathbb{R}^n \backslash \{0\}$.

(2) *The set-valued maps $\partial u(x)$ and $\nabla_* u(x)$ are upper semicontinuous at x.*

(3) *If $x \in S^k(u)$, then $I_{k+1}(\nabla_* u(x)) = \partial u(x)$.*

(4) *For any $k \in \{0, \dots, n\}$ and any $\rho > 0$ we have*

$$T(S^k_\rho(u), x) \subset [\partial u(x)]^\perp \qquad \forall x \in S^k_\rho(u),$$

where $S^k_\rho(u)$ denotes the set of all points $x \in S^k(u)$ such that $\partial u(x)$ contains a k-dimensional ball of radius ρ.

(5) *For any integer $k \in \{0, \dots, n\}$ the set $S^k(u)$ is countably $\mathcal{H}^{n-k}$-rectifiable, that is it can be covered, up to a $\mathcal{H}^{n-k}$-negligible set, with a countable sequence of C^1 submanifolds $\Gamma_h \subset \mathbb{R}^n$ of dimension $(n - k)$, i.e.*

$$\mathcal{H}^{n-k}\left(S^k(u) \backslash \bigcup_{h=1}^{\infty} \Gamma_h\right) = 0.$$

Moreover,

$$\int_{S^k(u) \cap \Omega'} \mathcal{H}^k(\partial u(x)) \, d\mathcal{H}^{n-k}(x) < +\infty$$

for any open set $\Omega' \subset\subset \Omega$.

PROOF. (1) See [1] and [4].

(2) The upper semicontinuity of the map $\partial u(x)$ easily follows by (1.3), and the upper semicontinuity of $\nabla_* u(x)$ follows directly from its definition.

(3) Since $\nabla_* u(x)$ is closed and its convex hull equals $\partial u(x)$, the assertion follows by Carathéodory's Theorem.

(4) See [1], Theorem 3.1.

(5) See [1], Theorem 4.1. ∎

REMARK 1.1. Note that (5) provides an upper bound on the Hausdorff dimension of $S^k(u)$, which is not greater than $(n-k)$. It is easy to see that this bound is optimal. Indeed, let

$$u(x_1, \dots, x_n) = |x_1| + \dots + |x_k|.$$

Then, $S^k(u)$ is the $(n-k)$-plane of all $x \in \mathbb{R}^n$ such that $x_i = 0$ for $1 \le i \le k$.

2. - Exposed faces and reachable subgradients

We want to study the structure of the singular set $\Sigma^1(u)$ in the neighborhood of a singular point x.

DEFINITION 2.1. We define the *singularity degree* of $x \in \Sigma^1(u)$ as the unique integer k such that $x \in S^k(u)$. We say that x is an isolated singularity of degree k if $T(\Sigma^k(u), x) = \emptyset$. We say that a singularity propagates if

$$T(\Sigma^1(u), x) \neq \emptyset.$$

Moreover, all vectors $\theta \in T(\Sigma^1(u), x) \cap \partial B_1$ are called directions of propagation of the singularity at x.

Clearly, a convex function may well have an isolated singularity of degree n. Indeed, if $x \in S^n_\rho(u)$ for some $\rho > 0$, then $\partial u(x)$ contains an n-dimensional ball. Hence, by Theorem 1.1, x is not a cluster point of $S^n_\rho(u)$. In other words, $S^n_\rho(u)$ is a discrete set for any $\rho > 0$. Moreover, there are convex functions with isolated singularities of degree $< n$.

EXAMPLE 2.1. Let

$$u(x_1, \dots, x_n) = \sqrt{(x_1^2 + \dots + x_k^2) + (x_{k+1}^4 + \dots + x_n^4)}.$$

Then, u is a convex function in $\mathbb{R}^n$ and $u \in C^1(\mathbb{R}^n \setminus \{0\})$. On the other hand, denoting by B_1^k the k-dimensional unit ball, we have that $\partial u(0) = B_1^k \times \{0\}^{n-k}$. So, 0 is the only point in $S^k(u)$.

Note that, in the above example $\partial u(0) = \nabla_* u(0)$. More generally, we will show that a sufficient condition for the propagation of a singularity of degree $k < n$ at x is the strict inclusion $\nabla_* u(x) \subsetneqq \partial u(x)$. In particular, this condition is satisfied for solutions of some Hamilton-Jacobi equations, see Section 3.

In the remainder of this paper we always assume that $\Omega \subset \mathbb{R}^n$ is a convex open set, $u : \Omega \to \mathbb{R}$ is a semi-convex function and $\omega(t)$ is the semi-convexity modulus of u. Since our statements are local, we assume that u is Lipschitz continuous in Ω and we denote by $[u]_{\mathrm{Lip}}$ its Lipschitz semi-norm.

We will see that the directions of propagation of singularities are related to the geometry of the subdifferential $\partial u(x)$ at the starting point x. To analyze the singular directions we introduce the following sets.

DEFINITION 2.2. Let $x \in \Omega$ and $\theta \in \partial B_1$; we set

$$\partial u(x,\theta) = \left\{ p \in \partial u(x) : \langle p, \theta \rangle = \frac{\partial u}{\partial \theta}(x) = \max_{q \in \partial u(x)} \langle q, \theta \rangle \right\},$$

$$\nabla_* u(x,\theta) = \left\{ \lim_{h \to +\infty} \nabla u(x_h) : x_h \in S^0(u) \backslash \{x\},\ x_h \to x,\ \frac{x_h - x}{|x_h - x|} \to \theta \right\}.$$

The collection $\{\partial u(x,\theta) : \theta \in \partial B_1\}$ consists of all the exposed faces of the convex set $\partial u(x)$. The following theorem is the basis of our singularity propagation argument (see Theorem 2.2 and Theorem 2.3).

THEOREM 2.1. *Let $x \in \Omega$, $p \in \mathbb{R}^n$ and sequences $x_h \to x$, $\partial u(x_h) \ni p_h \to p$ be given. Suppose that*

$$\lim_{h \to +\infty} \frac{x_h - x}{|x_h - x|} = \theta. \tag{2.1}$$

Then, $p \in \partial u(x,\theta)$. In particular,

$$\nabla_* u(x,\theta) \subset \partial u(x,\theta).$$

Conversely, for any $p \in \partial u(x,\theta)$ there are sequences $x_h \to x$ satisfying (2.1), *and $\partial u(x_h) \ni p_h \to p$.*

PROOF. We have to show that $\partial_* u(x,\theta) = \partial u(x,\theta)$, where

$$\partial_* u(x,\theta) = \left\{ \lim_{h \to +\infty} p_h : p_h \in \partial u(x_h),\ x_h \neq x,\ x_h \to x,\ \frac{x_h - x}{|x_h - x|} \to \theta \right\}.$$

Let p_h, x_h be as in the definition of $\partial_* u(x,\theta)$ and set

$$t_h = |x_h - x|, \qquad p = \lim_{h \to +\infty} p_h.$$

We know, by the upper semicontinuity of $\partial u(x)$, that $p \in \partial u(x)$. We will now show that $p \in \partial u(x,\theta)$. Indeed, by the semi-convexity of u we have

$$u(x) - u(x_h) - \langle p_h, x - x_h \rangle \geq -t_h \omega(t_h).$$

Divide both sides by t_h to obtain

$$\left\langle p_h, \frac{x_h - x}{t_h} \right\rangle \geq \frac{u(x + t_h\theta) - u(x)}{t_h} + \frac{u(x_h) - u(x + t_h\theta)}{t_h} - \omega(t_h).$$

Since
$$\frac{|u(x_h) - u(x + t_h\theta)|}{t_h} \leq [u]_{\text{Lip}} \left| \frac{x_h - x}{t_h} - \theta \right| \to 0,$$
by letting $h \to +\infty$ we get
$$\langle p, \theta \rangle \geq \frac{\partial u}{\partial \theta}(x).$$
Thus, $p \in \partial u(x, \theta)$ and $\partial_* u(x, \theta) \subset \partial u(x, \theta)$.

Next, we proceed to show the reverse inclusion. Let us denote by d the dimension of $\partial u(x, \theta)$. Since θ is orthogonal to $\partial u(x, \theta)$, d is strictly less than n. We may assume that $d > 0$, the inclusion being trivial if $\partial u(x, \theta)$ is a singleton.

Since $\partial_* u(x, \theta)$ is compact, it suffices to show that $p \in \partial_* u(x, \theta)$ for any $p \in \text{Int}(\partial u(x, \theta))$, the relative interior of $\partial u(x, \theta)$.

Let θ_i, $1 \leq i \leq (n - d)$ be an orthonormal basis of $[\partial u(x, \theta)]^\perp$, i.e.,
$$\langle \theta_i, \theta_j \rangle = \delta_{ij}, \qquad \langle (p - q), \theta_i \rangle = 0 \quad \forall p, \ q \in \partial u(x, \theta).$$

We can also take θ_1 to be equal to θ. For $r, t > 0$ satisfying the condition $t\sqrt{1 + r^2} < \text{dist}(x, \partial\Omega)$, let $y(r, t)$ be a minimizer of the function
$$u(x + t(\theta_1 + y)) - t\langle p, y \rangle$$
in the compact set K_r defined by
$$K_r = \{y \in \mathbb{R}^n : \langle y, \theta_i \rangle = 0 \ \forall i = 1, \ldots, (n - d), \ |y| \leq r\}.$$

We claim that for any $r > 0$ there is $\tau > 0$ (depending on r) such that for $t < \tau$ any minimizer $y(r, t)$ satisfies the condition $|y(r, t)| < r$. Indeed, if the claim were not true it would be possible to find $r > 0$ and a sequence of minimizers $y_h = y(r, t_h) \in K_r \cap \partial B_r$ corresponding to an infinitesimal sequence t_h. Passing to a subsequence, we may assume that y_h converges to $y \in K_r \cap \partial B_r$. Since y_h is a minimizer, we have
$$u(x + t_h(\theta_1 + y_h)) - t_h\langle p, y_h \rangle \leq u(x + t_h\theta_1).$$

Hence,
$$\frac{u(x + t_h(\theta_1 + y_h)) - u(x)}{t_h} - \frac{u(x + t_h\theta_1) - u(x)}{t_h} \leq \langle p, y_h \rangle.$$

Recalling that
$$\left| \frac{u(x + t_h(\theta_1 + y_h)) - u(x + t_h(\theta_1 + y))}{t_h} \right| \leq [u]_{\text{Lip}} |y_h - y| \to 0,$$

we obtain

(2.2) $$\frac{\partial u}{\partial(\theta_1 + y)}(x) - \frac{\partial u}{\partial \theta_1}(x) \leq \langle p, y \rangle.$$

On the other hand, since the map $\langle \cdot, \theta_1 \rangle$ is constant on $\partial u(x, \theta_1)$, we have that $\partial u/\partial\theta_1(x) = \langle p, \theta_1 \rangle$. Also, since $p \in \mathrm{Int}(\partial u(0))$,

$$\frac{\partial u}{\partial(\theta_1 + y)}(x) \geq \langle p + \epsilon y, \theta_1 + y \rangle = \langle p, \theta_1 \rangle + \langle p, y \rangle + \epsilon r^2$$

for $|\epsilon|$ sufficiently small. We thus obtain a contradiction with (2.2), and the claim is proved.

Now, let $r > 0$ and let $\tau(r) > 0$ be given by the claim. Returning to the definition of $y(r, t)$, by the non-smooth Lagrange multiplier rule (see for instance [6], 6.1.1) we conclude that for any $t \in]0, \tau(r)[$ we can find $\lambda_i(r, t) \in \mathbb{R}$ satisfying

$$0 \in t\{\partial u(x + t(\theta_1 + y(r, t))) - p\} - \sum_{i=1}^{n-d} \lambda_i(r, t)\theta_i,$$

or, equivalently,

$$p + \sum_{i=1}^{n-d} \frac{\lambda_i(r, t)}{t}\theta_i \in \partial u(x + t(\theta_1 + y(r, t))). \tag{2.3}$$

Let $(r_h) \subset]0, +\infty[$ and $t_h \in]0, \tau(r_h)[$ be two sequences converging to 0. By taking scalar products in (2.3) with θ_i it is easy to see that $|\lambda_i(r_h, t_h)|/t_h$ is not greater than $2[u]_{\mathrm{Lip}}$. Hence, by passing to a subsequence if necessary, we may assume that $\lambda_i(r_h, t_h)/t_h$ converges to $\overline{\lambda}_i$ as $h \to +\infty$ for $i = 1, \dots, (n - d)$.

Then, by letting $h \to +\infty$ in (2.3) we get

$$p + \sum_{i=1}^{n-d} \overline{\lambda}_i\theta_i \in \partial_* u(x, \theta_1),$$

as $|y(r_h, t_h)| < r_h$. Moreover,

$$\lim_{h \to +\infty} \frac{\theta_1 + y(r_h, t_h)}{|\theta_1 + y(r_h, t_h)|} = \theta_1.$$

On the other hand, since the vectors θ_i are orthogonal to $\partial u(x, \theta)$, all $\overline{\lambda}_i$'s are equal to 0. Thus, $p \in \partial_* u(x, \theta_1)$ and the proof of the theorem is complete. ∎

THEOREM 2.2. *Let $x \in \Omega$, $\theta \in \partial B_1$ and an integer $m \in [1, n]$ be given. Then,*

$$I_m(\nabla_* u(x, \theta)) \neq \partial u(x, \theta) \quad \Longrightarrow \quad \theta \in \mathrm{Tan}(\Sigma^m(u), x). \tag{2.4}$$

Moreover, $\partial u(x, \theta) = \mathrm{co}(\nabla_* u(x, \theta))$.

REMARK 2.1. In particular, if $\nabla_* u(x, \theta) \neq \partial u(x, \theta)$, then θ is a direction of propagation of the singularity at x. Moreover, (2.4) provides a lower bound on the degree of the singularity near x. Indeed, in view of definition 1.1, (2.4) implies that $\theta \in T(\Sigma^m(u), x)$, where $m = m(\nabla_* u(x, \theta))$. Hence, there are singular points of degree m near x, along the direction θ.

PROOF OF THEOREM 2.2. Let $p \in \partial u(x,\theta) \backslash I_m(\nabla_* u(x,\theta))$. We argue by contradiction. So, suppose that $\theta \notin T(\Sigma^m(u), x)$. By Theorem 2.1, there are a sequence $(x_h) \subset \Omega \backslash \{x\}$, and vectors p_h such that $p_h \in \partial u(x_h)$ and

$$\lim_{h\to+\infty} p_h = p, \quad \lim_{h\to+\infty} x_h = x, \quad \lim_{h\to+\infty} \frac{x_h - x}{|x_h - x|} = \theta.$$

By our assumption, for h large enough, x_h does not belong to $\Sigma^m(u)$. Hence, the dimension of $\partial u(x_h)$ does not exceed $m - 1$. By Theorem 1.1(3), there are vectors $p_{i,h} \in \nabla_* u(x_h)$ and non-negative real numbers $\lambda_{i,h}$ such that

$$p_h = \sum_{i=1}^{m} \lambda_{i,h} p_{i,h}, \qquad \sum_{i=1}^{m} \lambda_{i,h} = 1. \tag{2.5}$$

By passing to a subsequence, we may assume that for any i the m-tuples $\lambda_{i,h}$ converge as $h \to +\infty$ to λ_i and $p_{i,h}$ converge to p_i as $h \to +\infty$. Since $p_{i,h} \in \nabla_* u(x_h)$ a diagonal argument shows that $p_i \in \nabla_* u(x, \theta)$. Now, let $h \to +\infty$ in (2.5) to obtain

$$p = \sum_{i=1}^{m} \lambda_i p_i, \qquad \sum_{i=1}^{m} \lambda_i = 1.$$

Hence, $p \in I_m(\nabla_* u(x,\theta))$ and ths contradiction proves (2.4).

Finally, a similar argument (with $m = n + 1$) shows that each vector $p \in \partial u(x, \theta)$ is the convex combination of at most $(n + 1)$ points of $\nabla_* u(x, \theta)$. ■

Note that (2.4) implies that x is only a cluster point of $\Sigma^m(u)$. However, we will show that, under suitable assumptions, there is a whole continuum of singular points near x, whose size can be estimated from below.

Let S be any plane in $\mathbb{R}^n$ passing through the origin, and let π_S be the orthogonal projection on S. For any $\gamma > 0$ we denote by $C_\gamma(S)$ the cone

$$C_\gamma(S) = \{x \in \mathbb{R}^n : |\pi_S(x)| \le \gamma |\pi_{S^\perp}(x)|\}.$$

We note that $C_\gamma(S) \supset S^\perp$ and $C_\gamma(S)$ approaches $S^\perp$ as $\gamma \to 0^+$.

THEOREM 2.3. *Let $x \in S^k(u)$ with $1 \le k \le n - 1$ be given. Set $m = m(\nabla_* u(x))$. Then,*

$$T(\Sigma^m(u), x) \supset [\partial u(x)]^\perp. \tag{2.6}$$

In addition, we have

$$\liminf_{\rho\to 0^+} \frac{\mathcal{H}^{n-k}(\Sigma^m(u) \cap B_\rho(x) \cap [x + C_\gamma(S)])}{\omega_{n-k}\rho^{n-k}} \ge 1 \tag{2.7}$$

for any $\gamma > 0$, *where* S *is the k-plane parallel to* $\partial u(x)$ *and containing* 0.

PROOF. Observe that $\partial u(x,\theta)$ equals $\partial u(x)$ and $\nabla_* u(x,\theta) \subset \nabla_* u(x)$ for any $\theta \in [\partial u(x)]^\perp$. Hence, (2.6) follows from the previous theorem.

In order to simplify our proof of (2.7), we assume that $x = 0$. Since $\Sigma^m(u) = \Omega$ if $m = 0$, we may also assume that $m > 0$. Let us denote by $\mathbb{S}^\perp$ the unit sphere in $S^\perp$.

Let us pick a vector p in the set $\partial u(0) \setminus I_m(\nabla_* u(0))$, which is not empty. For any $z \in \mathbb{S}^\perp$ and any $r, t > 0$ we denote by $y(r,t,z)$ a minimizer of the function $u(tz + ty) - t\langle p, y\rangle$ in the set

$$K_r = \{y \in S : |y| \leq r\}.$$

We claim that for every $r > 0$ there is $\tau(r) > 0$ such that for any $t \in]0, \tau(r)[$ and any $z \in \mathbb{S}^\perp$ any minimizer $y(r,t,z)$ belongs to the (essential) interior of K_r. This claim can be proved as in Theorem 2.1. Indeed, suppose that the claim is not true. Then, there exist $r > 0$ and a sequence of minimizers $y_h = y(r, t_h, z_h) \in K_r \cap \partial B_r$ corresponding to a sequence $t_h \to 0$. Passing to a subsequence, we may assume that y_h converges to $y \in K_r \cap \partial B_r$ and z_h converges to $z \in \mathbb{S}^\perp$. Since y_h is a minimizer, we infer

$$u(t_h z_h + t_h y_h) - t_h \langle p, y_h\rangle \leq u(t_h z_h).$$

Hence,

$$\frac{u(t_h z_h + t_h y_h) - u(0)}{t_h} - \frac{u(t_h z_h) - u(0)}{t_h} \leq \langle p, y_h\rangle.$$

Recalling that

$$\left|\frac{u(t_h z_h + t_h y_h) - u(t_h z + t_h y)}{t_h}\right| \leq [u]_{\text{Lip}}(|z_h - z| + |y_h - y|) \to 0,$$

and

$$\left|\frac{u(t_h z_h) - u(t_h z)}{t_h}\right| \leq [u]_{\text{Lip}}|z_h - z| \to 0,$$

we obtain

$$\frac{\partial u}{\partial(z+y)}(0) - \frac{\partial u}{\partial z}(0) \leq \langle p, y\rangle. \tag{2.8}$$

On the other hand, since the map $\langle \cdot, z\rangle$ is constant on $\partial u(0)$, we have that

$$\frac{\partial u}{\partial z}(0) = \langle p, z\rangle.$$

Also, since $p \in \text{Int}(\partial u(0))$,

$$\frac{\partial u}{\partial(z+y)}(0) \geq \langle p + \epsilon y, z + y\rangle = \langle p, z\rangle + \langle p, y\rangle + \epsilon r^2$$

for $|\epsilon|$ sufficiently small. We thus obtain a contradiction with (2.8), and the claim is proved.

Next, we claim that there is $\delta > 0$ such that if $r < \delta$ and $t < \inf\{\tau(r), \delta\}$, then for any $z \in \mathbb{S}^{\perp}$, any minimizer $y(r,t,z)$ satisfies the condition

$$tz + ty(r,t,z) \in \Sigma^m(u).$$

Indeed, let us assume that the claim is not true. Then, by the variational argument used in the proof of Theorem 2.1, we construct a sequence of minimizers $y_h = y(r_h, t_h, z_h) \in K_{r_h}$ corresponding to sequences $r_h, t_h \to 0$ and real constants $\lambda_{h,1}, \ldots, \lambda_{h,n-k}$ such that

$$p_h := p + \sum_{i=1}^{n-k} \lambda_{h,i}\theta_i \in \partial u(t_h z_h + t_h y_h), \tag{2.9}$$

$$t_h z_h + t_h y_h \notin \Sigma^m(u), \tag{2.10}$$

and

$$\lim_{h\to+\infty} z_h = z \in \mathbb{S}^{\perp}, \qquad \lim_{h\to+\infty} \lambda_{h,i} = \lambda_i \in \mathbb{R} \quad \forall i = 1, \ldots, (n-k).$$

Passing to the limit as $h \to +\infty$ in (2.9) we get

$$p + \sum_{i=1}^{n-k} \lambda_i \theta_i \in \partial u(0).$$

Hence $\lambda_i = 0$ for any $i = 1, \ldots, (n-k)$ and p_h converges to p as $h \to +\infty$. Moreover, by (2.10) and Theorem 1.1(3) each vector p_h belongs to the convex hull of at most m vectors of $\nabla_* u(x_h)$. Repeating the argument of Theorem 2.2 we obtain a set $A \subset \nabla_* u(0)$ consisting of at most m points, such that $p \in \mathrm{co}(A)$. Hence, $p \in I_m(\nabla_* u(0))$, and this contradiction proves the second claim.

Finally, let $\delta > 0$ be given by the second claim. For any fixed $\gamma > 0$ let $r < \inf\{\gamma, \delta\}$. Then,

$$\Sigma^m(u) \cap C_\gamma(S) \cap B_\rho \supset \left\{ tz + ty(r,t,z) : z \in \mathbb{S}^{\perp},\ 0 \le t < \frac{\rho}{\sqrt{1+r^2}} \right\}$$

provided $\rho < \sqrt{1+r^2}\,\inf\{\tau(r), \delta\}$. Since $\pi_{S^\perp}$ does not increase the Hausdorff measure (see for instance [17], Proposition 3.5), by the inclusion

$$\pi_{S^\perp}(\Sigma^m(u) \cap C_\gamma(S) \cap B_\rho) \supset \left\{ z \in S^{\perp} : |z| < \frac{\rho}{\sqrt{1+r^2}} \right\}$$

we infer

$$\liminf_{\rho\to 0^+} \frac{\mathcal{H}^{n-k}(\Sigma^m(u) \cap B_\rho \cap C_\gamma(S))}{\omega_{n-k}\rho^{n-k}} \ge (1+r^2)^{(k-n)/2}.$$

By letting $r \to 0$, we complete the proof. ■

REMARK 2.2. By (1.1) and Theorem 1.1(5) we infer that $\mathcal{H}^{n-k}(S^i(u)) = 0$ for any $i \geq k+1$. Hence, (2.7) can be written in the equivalent form: for any $x \in S^k(u)$

$$\liminf_{\rho \to 0^+} \sum_{i=m}^{k} \frac{\mathcal{H}^{n-k}(S^i(u) \cap B_\rho(x) \cap [x + C_\gamma(S)])}{\omega_{n-k}\rho^{n-k}} \geq 1,$$

where $m = m(\nabla_* u(x))$. In particular, if $I_k(\nabla_* u(x)) \neq \partial u(x)$ (i.e., $m = k$), we get

$$\liminf_{\rho \to 0^+} \frac{\mathcal{H}^{n-k}(S^k(u) \cap B_\rho(x) \cap [x + C_\gamma(S)])}{\omega_{n-k}\rho^{n-k}} \geq 1,$$

and coupling this estimate with Theorem 1.1(5) we conclude that $\mathcal{H} - \dim(S^k(u)) = (n - k)$.

3. - Hamilton-Jacobi equations

In this section we will apply the general results on the singularities of semi-convex functions to solutions of the Hamilton-Jacobi-Bellman equation

$$F(y, u(y), \nabla u(y)) = 0, \quad y \in \Omega \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open domain. We will assume that

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \text{ is continuous;} \tag{3.2}$$

$$p \mapsto F(y, s, p) \text{ is convex in } \mathbb{R}^N \quad \forall (y, s) \in \Omega \times \mathbb{R}; \tag{3.3}$$

$$u \text{ is semi-concave (i.e. } -u \text{ is semi-convex);} \tag{3.4}$$

$$\text{(3.1) holds at any differentiability point of } u. \tag{3.5}$$

We note that, for a semi-concave function u, the interesting semidifferential is the so-called superdifferential, defined as

$$\partial^+ u(y) = \left\{ p \in \mathbb{R}^N : \limsup_{z \to y} \frac{u(z) - u(y) - \langle p, z - y \rangle}{|z - y|} \leq 0 \right\}.$$

Equivalently, $\partial^+ u(y) = -\partial[-u](y)$. Hence, $\partial^+ u(y) \neq \emptyset$ for any $y \in \Omega$ and the following implication holds

$$\partial u(y) \neq \emptyset \quad \Longrightarrow \quad u \text{ is differentiable at } y. \tag{3.6}$$

Accordingly, the definitions 1.2 and 2.2 will be modified as follows for a semi-concave function u:

$$S^k(u) = \{x \in \Omega : \dim(\partial^+ u(x)) = k\},$$

$$\Sigma^k(u) = \bigcup_{i=k}^{n} S^i(u) = \{x \in \Omega : \dim(\partial^+ u(x)) \geq k\},$$

$$\partial^+ u(x,\theta) = \left\{ p \in \partial^+ u(x) : \langle p, \theta \rangle = \frac{\partial u}{\partial \theta}(x) = \min_{q \in \partial^+ u(x)} \langle q, \theta \rangle \right\}.$$

REMARK 3.1. From (3.2)-(3.5) it follows that u is a viscosity solution in the sense of [8] (see also [7]). Indeed, (3.2) and (3.5) yield

$$F(y, u(y), p) = 0 \quad \forall p \in \nabla_* u(y) \tag{3.7}$$

for any $y \in \Omega$, and so (3.3) implies that

$$F(y, u(y), p) \leq 0 \quad \forall p \in \partial^+ u(y).$$

The converse inequality on the elements of $\partial u(y)$ trivially follows by (3.6).

REMARK 3.2. Semi-concavity is a natural property to expect on viscosity solutions of Hamilton-Jacobi-Bellman equations. Indeed, several existence and uniqueness results were first obtained in classes of semi-concave functions (see [15]). More recently, H-J equations have been studied in the framework of viscosity solutions (see [8] and [7]). Under suitable regularity assumptions on F and on the (Dirichlet) boundary data, viscosity solutions to (3.1) are known to be semi-concave (see [16] and [12]). Similar results are also available for viscosity solution of second order H-J equations, see [13]; hence the result of Section 2 apply to these equations as well. For the sake of simplicity we confine our statements to first-order equations.

For any compact convex set $C \subset \mathbb{R}^N$ we denote by Ext(C) the set of extreme points of C. We say that a set $A \subset \mathbb{R}^N$ is extremal if no $p \in A$ can be written as a convex combination of other points of A, i.e.

$$p \notin \mathrm{co}(A \setminus \{p\}) \quad \forall p \in A.$$

Our terminology is motivated by the following result.

LEMMA 3.1. *Any compact extremal set A coincides with* Ext(co(A)).

PROOF. Let $C = \mathrm{co}(A)$ and let $p \in \mathrm{Ext}(C)$. By Carathéodory's Theorem, we can represent p as a convex combination of $(N+1)$ points $p_i \in A$:

$$p = \sum_{i=1}^{N+1} \lambda_i p_i, \quad \lambda_i > 0.$$

Since p is an extreme point of $C, p = p_i$ for any $i \in \{1, \ldots, N+1\}$, hence $p \in A$.

Conversely, let $p \in A$. By the Krein-Milman theorem (see for instance [18], p. 167) we can represent p as a convex combination of at most $(N+1)$ points $p_i \in \operatorname{Ext}(C)$:

$$p = \sum_{i=1}^{N+1} \lambda_i p_i, \quad \lambda_i > 0, \quad \sum_{i=1}^{N+1} \lambda_i = 1.$$

In turn, each p_i can be represented as a convex combination of at most $(N+1)$ points $p_{ij} \in A$:

$$p_i = \sum_{j=1}^{N+1} \lambda_{ij} p_{ij}, \quad \lambda_{ij} > 0, \quad \sum_{j=1}^{N+1} \lambda_{ij} = 1,$$

so that

$$p = \sum_{i,j=1}^{N+1} \lambda_i \lambda_{ij} p_{ij}.$$

Since A is extremal, $p = p_{ij}$ for any i, j and hence $p = p_i \in \operatorname{Ext}(C)$. ∎

The main result of this section is the following.

THEOREM 3.2. *Assume* (3.2), (3.3), (3.4), (3.5) *and let* $y \in S^k(u)$ *be a singular point. Let us further assume that*

(3.8) $$\{p \in \mathbb{R}^N : F(y, u(y), p) = 0\} \text{ is extremal.}$$

Then

(1) $\nabla_* u(y) = \operatorname{Ext}(\partial^+ u(y))$, *and if* $k < N$ *the singularity propagates. Moreover* $m = m(\nabla_* u(y)) \geq 1$, *and*

(3.9) $$T(\Sigma^m(u), y) \supset [\partial^+ u(y)]^\perp, \qquad \liminf_{\rho \to 0^+} \frac{\mathcal{H}^{N-k}(\Sigma^m(u) \cap B_\rho(y))}{\omega_{N-k} \rho^{n-k}} \geq 1.$$

(2) *Let* $\theta \in \partial B_1$ *and let us assume that* $\partial^+ u(y, \theta)$ *is not a singleton. Then,* $\nabla_* u(y, \theta)$ *coincides with* $\operatorname{Ext}(\partial^+ u(y, \theta))$, $m = m(\nabla_* u(y, \theta)) \geq 1$ *and* $\theta \in T(\Sigma^m(u), y)$.

PROOF. (1) By (3.7) and (3.8), $\nabla_* u(y)$ satisfies the hypotheses of Lemma 3.1, so that $\nabla_* u(y) = \operatorname{Ext}(\partial^+ u(y))$. To show (3.9), we only need to apply Theorem 2.3 to $-u$.

(2) As in (1), Lemma 3.1 yields $\nabla_* u(y, \theta) = \operatorname{Ext}(\partial^+ u(y, \theta))$. The other statements follow from Theorem 2.2 and Remark 2.1. ∎

REMARK 3.3. The extremality condition (3.8) cannot be dropped. In fact, let $N = 2$ and $u(y_1, y_2) = -\sqrt{y_1^2 + y_2^4}$, as in Example 2.1. Then, u is concave in $\mathbb{R}^2$, and has an isolated singularity at $(0,0)$. Moreover, u is a viscosity solution of the equation

$$\sqrt{y_2^2 u_{y_1}^2 + \frac{1}{4} u_{y_2}^2} = |y_2|.$$

REMARK 3.4. Condition (3.8) is trivially satisfied if

$$p \mapsto F(y, s, p) \text{ is strictly convex in } \mathbb{R}^N \quad \forall (y, s) \in \Omega \times \mathbb{R}.$$

Theorem 3.2 also applies to non-stationary H-J equations with strictly convex Hamiltonian. In fact, let $N = n + 1$, $y = (t, x)$ with $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, and $p = (p_t, p_x) \in \mathbb{R} \times \mathbb{R}^n$. Let

$$H((t, x), s, p_x) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

be a continuous function, strictly convex in p_x. Then

$$F(y, s, p) = p_t + H((t, x), s, p_x)$$

satisfies (3.2) and (3.3), and any semi-concave function $u : \Omega \to \mathbb{R}$ satisfying (3.5) is a viscosity solution of the equation

$$u_t + H(t, x, u, \nabla u) = 0. \tag{3.10}$$

Finally, for any $y \in \Omega$ and any $s \in \mathbb{R}$

$$Z(y, s) = \{(p_t, p_x) \in \mathbb{R} \times \mathbb{R}^n : p_t + H(y, s, p_x) = 0\}$$

is extremal, because of the strict convexity of H. Indeed, let

$$Z(y, s) \ni p = \sum_{i=1}^{N+1} \lambda_i p_i$$

with $p_i \in Z(y, s)$, $\lambda_i > 0$ and $\sum_{i=1}^{N+1} \lambda_i = 1$, and let us show that $p_i = p$ for any i. Since

$$p_t + H(y, s, p_x) < \sum_{i=1}^{N+1} (\lambda_i p_{it} + \lambda_i H(y, s, p_{ix})) = 0$$

unless $p_x = p_{ix}$ for any $i \in \{1, \dots, N+1\}$, we have

$$p_{it} = -H(y, s, p_{ix}) = -H(y, s, p_x) = p_t \qquad \forall i \in \{1, \dots, N+1\}$$

and, in particular, $p = p_i$ for any i.

More generally, the same argument of Theorem 3.2 shows that singularities propagate in the direction θ if $\partial^+ u(y, \theta)$ is not a singleton and if the restriction of $F(y, u(y), \cdot)$ to $\partial^+ u(y, \theta)$ is strictly convex, so that $m(\nabla_* u(y, \theta)) \geq 1$.

REMARK 3.5. In Theorem 3.2(1) it is necessary to assume that x is not a singularity of degree N. In fact, $u(y) = -|y|$ is a solution of the eikonal equation $|\nabla u(y)|^2 - 1 = 0$, and the singularity in the origin does not propagate.

However, propagation of singularities of any degree ha been proved for non-stationary H-J equations with strictly convex Hamiltonian (see [4]). Due to the special structure of the equation it has been shown in [5] that for any singularity y there is at least a direction $\theta \in \partial B_1$ such that $\partial u(y, \theta)$ is not a singleton. Note that, once the existence of such a direction has been proved, the propagation of the singularity would follow by Theorem 3.2(2).

In [5] it is also shown that viscosity solutions of (3.10) with strictly convex H are such that any $p \in \nabla_* u(y)$ is exposed, i.e., there exists $\theta \in \partial B_1$ such that $\partial^+ u(y, \theta) = \{p\}$. This condition is stronger than extremality.

REMARK 3.6. We note that the lower bound in Theorem 3.2 on the maximum degree of the singularity near y depends only on the geometry of $\partial^+ u(y)$. To illustrate this phenomenon, we now discuss three examples. In the first example the subdifferential $\partial^+ u(y)$ is a triangle in $\mathbb{R}^3$ and the singularity propagates in singularities of degree two, as implied by Theorem 3.2.

In the second example we show that a singularity y of degree k may well propagate in singularities of degree $m < k$ when $m(\nabla_* u(y)) < k$.

Finally, the third example shows that Theorem 3.2 provides only a sufficient condition for the propagation of singularities of high degree.

EXAMPLE 3.1. Let $\Omega = \mathbb{R}^3$ and let

$$u(t, x, z) = \min\{t, x, z\}.$$

Then, u is a viscosity solution of the equation $-u_t + H(\nabla u) = 0$, where

$$H(p_x, p_z) = (p_x - p_z)^2 + 2(p_x + p_z - 1)^2 - 1$$

is strictly convex. We note that $S^2(u)$ is equal to the line spanned by $(1, 1, 1)$

and

$$\nabla_* u(s,s,s) = \{(1,0,0),(0,1,0),(0,0,1)\} \qquad \forall s \in \mathbb{R}.$$

In this case $m(\nabla_* u(0,0,0)) = 2$. We note that $S^1(u)$ consists of three half-planes intersecting each other in the above singular line, with directions orthogonal to the triangle generated by $\nabla_* u(0,0,0)$. This example describes the typical situation analyzed in Theorem 3.2.

EXAMPLE 3.2. Let $u : \mathbb{R}^3 \to \mathbb{R}$ be the function

$$u(t,x,z) = -\sqrt{x^2 + (|z| + t^2)^2}.$$

The equality

$$\sqrt{\alpha^2 + \beta^2} = \sup\{a\alpha + b\beta : a \geq 0,\ b \geq 0,\ a^2 + b^2 < 1\} \qquad \alpha, \beta \geq 0$$

implies that $\sqrt{\varphi^2 + \psi^2}$ is a convex function whenever φ and ψ are non-negative convex functions. In particular, u is a concave function. The origin belongs to $S^2(u)$ and

$$\partial^+ u(0,0,0) = \{0\} \times \overline{B}_1, \qquad m(\{0\} \times \partial B_1) = 1.$$

The singularity in the origin propagates in singularities of degree 1. In fact, the origin is the only point in $S^2(u)$, $S^1(u) = \{(t,x,0) : t \neq 0\}$ and

$$\partial^+ u(t,x,0) = \left\{ \left(\frac{-2t^3}{u(t,x,0)}, \frac{-x}{u(t,x,0)}, \frac{t^2 \rho}{u(t,x,0)} \right) : |\rho| \leq 1 \right\} \quad \forall (t,x,0) \in S^1(u).$$

Finally, we note that

$$\nabla u(t,x,z) = \left(\frac{-2t(|z| + t^2)}{u(t,x,z)}, \frac{-x}{u(t,x,z)}, \frac{-y(|z| + t^2)}{|z| u(t,x,z)} \right) \qquad \forall (x,z,t) \in S^0(u),$$

so that u is a solution of equation (3.1) with

$$F(t,x,z,p_t,p_x,p_z) = -p_t + |p_x|^2 + |p_z|^2 - 1 + \frac{2t(|z| + t^2)}{\sqrt{x^2 + (|z| + t^2)^2}}.$$

The function F satisfies (3.2), (3.3) and the extremality condition (3.8).

EXAMPLE 3.3. Let $\Omega = \mathbb{R}^3$, $y = (t,x)$ with $t \in \mathbb{R}$ and $x \in \mathbb{R}^2$. The function

$$u(t,x) = \begin{cases} t/2 - |x| - 1 & \text{if } |x| + 2 \geq t \\ \dfrac{|x|^2}{2(2-t)} & \text{if } |x| + 2 < t \end{cases}$$

is a viscosity solution of the equation $-u_t + |\nabla u|^2/2 = 0$. We note that

$(2,0) \in S^3(u)$, and

$$\nabla_* u(2,0) = \left\{ (p_t, p_x) \in \mathbb{R} \times \mathbb{R}^2 : |p_x| \leq 1, \ p_t = \frac{|p_x|^2}{2} \right\}.$$

Moreover, $S^2(u)$ is the half-line $(t,0)$ with $t < 2$. The unit vector $\theta = (-1,0)$ belongs to $T(S^2(u),(2,0))$ even though $m(\nabla_* u((2,0),\theta)) = 1$.

REFERENCES

[1] G. ALBERTI - L. AMBROSIO - P. CANNARSA, *On the singularities of convex functions.* Manuscripta Math. **76** (1992), 421-435.

[2] L. AMBROSIO, *Su alcune proprietà delle funzioni convesse.* Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. (1992) (to appear).

[3] J.P. AUBIN - H. FRANKOWSKA, Set-Valued Analysis. Birkhäuser, Boston, 1990.

[4] P. CANNARSA - H.M. SONER, *On the singularities of the viscosity solutions to Hamilton - Jacobi Bellman equations.* Indiana Univ. Math. J. **36** (1987), 501-524.

[5] P. CANNARSA - H.M. SONER, *Generalized one-sided estimates for solutions of Hamilton-Jacobi equations and applications.* Nonlinear Anal. **13** (1989), 305-323.

[6] F.H. CLARKE, Optimization and Nonsmooth Analysis. Wiley & Sons, New York, 1983.

[7] M.G. CRANDALL - L.C. EVANS - P.L. LIONS, *Some properties of viscosity solutions of Hamilton-Jacobi equations.* Trans. Amer. Mat. Soc. **282** (1984), 487-502.

[8] M.G. CRANDALL - P.L. LIONS, *Viscosity solutions of Hamilton-Jacobi equations.* Trans. Amer. Mat. Soc. **277** (1983), 1-42.

[9] I. EKELAND - R. TEMAM, Convex Analysis and Variational Problems. North-Holland, Amsterdam, 1976.

[10] H. FEDERER, Geometric Measure Theory. Springer Verlag, Berlin, 1969.

[11] W.H. FLEMING, *The Cauchy problem for a nonlinear first order partial differential equation.* J. Differential Equations **5** (1969), 515-530.

[12] H. ISHII, *Uniqueness of unbounded viscosity solutions of Hamilton-Jacobi equations.* Indiana Univ. Math. J. **33** (1984), 721-748.

[13] H. ISHII - P.L. LIONS, *Viscosity solutions of fully nonlinear second-order elliptic partial differential equations.* J. Differential Equations **83** (1990), 26-78.

[14] R. JENSEN - P.E. SOUGANIDIS, *A regularity result for viscosity solutions of Hamilton-Jacobi equations in one space dimension.* Trans. Amer. Math. Soc. **301** (1987), 137-147.

[15] S.N. KRUZKOV, *Generalized solutions of Hamilton-Jacobi equations of eikonal type I.* Math. USSR-Sb. **27** (1975), 406-446.

[16] P.L. LIONS, Generalized solutions of Hamilton-Jacobi equations. Pitman, Boston, 1982.

[17] F. MORGAN, Geometric Measure Theory - A beginner's guide. Academic Press, Boston, 1988.

[18] R.T. ROCKAFELLAR, Convex Analysis. Princeton University Press, Princeton, 1970.

[19] L. SIMON, *Lectures on Geometric Measure Theory. Proceedings of the Centre for Mathematical Analysis*. Australian National University, Camberra, 1983.

[20] L. TONELLI, Lezioni di Analisi Matematica, vol. I. Tacchi Editrice, Pisa, 1940.

[21] M. TSUJI, *Formation of singularities for Hamilton-Jacobi equations II*. J. Math. Kyoto Univ. **26** (1986), 299-308.

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