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0. - Introduction

Let $f \mapsto \tilde{f}$ denote the conjugate function operator on the torus $\mathbf{T} \simeq [-1/2, 1/2)$, and consider the weighted L^2-inequality

$$\int_{\mathbf{T}} |\tilde{f}(x)|^2 u(x) dx \leq C \int_{\mathbf{T}} |f(x)|^2 v(x) dx. \tag{0.1}$$

The question, raised for the first time by Muckenhoupt, is the following:

Find all $v(x)$ (resp. all $u(x)$) such that (0.1) holds for some $u(x)$ (resp. all $v(x)$).

Using complex-variable methods the following simple answer was given by Koosis: (0.1) holds for some non-trivial $u(x)$ if and only if $v^{-1} \in L^1(\mathbf{T})$, and it holds for some non-trivial $v(x)$ if and only if $u \in L^1(\mathbf{T})$; see [K].

A more systematic study of this kind of problem was made later on by different authors. The general setting is the following.

Let $\left(\sum_1, \mathcal{A}_1, m_1\right)$, $\left(\sum_2, \mathcal{A}_2, m_2\right)$ be two measure spaces and T be a linear operator. Find conditions on $v(x)$ (resp. $u(x)$) such that

$$\left(\int_{\Sigma_2} |Tf(x)|^q u(x) dm_2(x) \right)^{1/q} \leq C \left(\int_{\Sigma_1} |f(x)|^p v(x) dm_1(x) \right)^{1/p} \tag{0.2}$$

is satisfied for some $u(x)$ (resp. $v(x)$ where u and v are positive measurable functions).

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Essentially two methods are used to deal with this problem. The first one is a constructive method, which means the following: given a weight v a new weight u is constructed such that (0.2) is satisfied. The second one uses non-constructive techniques of factorizarion of operators and then the existence of a certain u satisfying (0.2) is proved. The first method together with the A_p-weights theory was used in [C-J] in order to prove the following results.

0.3 THEOREM. *Let M be the Hardy-Littlewood maximal operator on $\mathbb{R}^n$ and assume $1 < p < \infty$. The following conditions are equivalent:*

(i) *There exists $u \not\equiv 0$ such that*

$$\int_{\mathbb{R}^n} |Mf(x)|^p u(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x)dx.$$

(ii) *v belongs to the class D_p^*, i.e.*

$$\sup_{r\geq 1} \frac{1}{r^{np'}} \int_{|x|\leq r} v^{-p'/p}(x)dx < +\infty.$$

0.4 THEOREM. *Let $1 < p < \infty$; on $\mathbb{R}^n$ the following conditions are equivalent:*

(i) *There exists $u \not\equiv 0$ such that*

$$\int_{\mathbb{R}^n} |Sf(x)|^p u(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x)dx$$

for all singular integral operators S.

(ii) *v belongs to the class D_p, i.e.*

$$\int_{\mathbb{R}^n} \frac{v^{-p'/p}(x)dx}{(1+|x|)^{np'}} \leq C < +\infty.$$

The following theorems were also proved using the constructive method.

0.5 THEOREM. ([G-G]). *Let $1 < p < \infty$; on $\mathbb{R}^n$ the following conditions are equivalent:*

(i) *There exists $v \not\equiv 0$ such that*

$$\int_{\mathbb{R}^n} |Mf(x)|^p u(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x)dx.$$

(ii) *u belongs to the class Z_p, i.e.*

$$\int_{\mathbb{R}^n} \frac{u(x)}{(1+|x|)^{np}}\, dx < +\infty.$$

0.6 THEOREM. ([H-M-S]). *Let I_γ denote the fractional integral operator*

$$I_\gamma f(x) = \int_{\mathbb{R}^n} f(y)|x-y|^{\gamma-n}dy, \qquad 0<\gamma<n;$$

and assume $1 < p,q < \infty$, $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n}$. *Then the following conditions are equivalent:*

(i) *There exists* $u \not\equiv 0$ *such that*

$$\left(\int_{\mathbb{R}^n} |I_\gamma f(x)|^q u(x)dx\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p v(x)dx\right)^{1/p}.$$

(ii) v *belongs to the class* D_p^γ, *i.e.*

$$\int_{\mathbb{R}^n} \frac{v(x)^{-p'/p}}{(1+|x|)^{(n-\gamma)p'}}\,dx \leq C < +\infty.$$

0.7 THEOREM. *Let M_γ denote the fractional maximal operator*

$$M_\gamma f(x) = \sup_{r>0} \frac{1}{r^{n-\gamma}} \int_{|x-y|<r} |f(y)|dy, \qquad 0<\gamma<n$$

and assume $1 < p,q < \infty$, $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n}$. *Then the following conditions are equivalent:*

(i) *There exists* $u \not\equiv 0$ *such that*

$$\left(\int_{\mathbb{R}^n} |M_\gamma f(x)|^q u(x)dx\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p v(x)dx\right)^{1/p}.$$

(ii) v *belongs to the class* $D_p^{*\gamma}$, *i.e.*

$$\sup_{r\geq 1} \frac{1}{r^{(n-\gamma)p'}} \int_{|x|\leq r} v(x)^{-p'/p}dx \leq C < +\infty.$$

The non-constructive method was developed by Rubio de Francia and with this method he proved Theorems 0.3, 0.4, 0.5, 0.6 and 0.7 for $p=q$ (see [GC-R de F], Chapt. VI] for a complete description of the method). The method of Rubio cannot be applied in the cases $p \neq q$ but, being non-constructive, it can be applied to a huge class of operators and measure spaces; in particular the A_p-theory is not needed.

This paper grew out of an effort to understand better the two methods mentioned above, and, as a consequence, to solve the general problem (0.2) for operators acting on the upper half-plane (see Section 1), in which case the A_p-theory is not available. We also look at both methods from the point of vew of vector-valued function theory: then maximal operators can be handled as ℓ^∞-valued linear operators.

The organization of the paper is as follows. In Section 1 we introduce the operators in the upper half-plane for which we want to solve problem (0.2); we also prove some estimates (see Proposition 1.7) that have interest in themselves and that we shall use later. Section 2 contains the basic lemmas that are needed for the methods mentioned above; both methods have a common part of strategy: given a function f, decompose it as $f_1 + f_2$, where $f_1 = f\chi_B$ and B is a certain ball, while Tf_2 is estimated in both methods with a local L^∞-bound. In this estimate, when working in Banach lattices X, it is natural to consider a new class $D^\gamma_{p,X}$ of weights (see (2.15)); the class $D^\gamma_{p,X}$ solves problem (0.2) for fractional maximal operators acting on lattice-valued functions; the classes D^*_p, D^γ_p, $0 < \gamma < n$, solve the problem for the Poisson integral and the fractional integral operator in the upper half-plane (even with Banach-valued functions), whereas the class D_p solves the problem for the generalization of the Riesz transform with U.M.D.-valued functions. All these results are proved in Section 3 (Theorems 3.5, 3.9, 3.1 and 3.7). In Section 4 we see that the class $D^\gamma_{p,X}$ is an intermediate class between D^γ_p and $D^{*,\gamma}_p$.

1. - Notation and background

We shall consider the following operators:

$$\mathcal{M}_\gamma f(x,t) = \sup_{r\geq t} \frac{1}{r^{n-\gamma}} \int_{B(x,r)} |f(y)|dy, \qquad (x,t) \in \mathbb{R}^{n+1}_+, \quad 0 \leq \gamma < n,$$

$$T_\gamma f(x,t) = \int_{\mathbb{R}^n} \frac{f(y)}{(|x-y|+t)^{n-\gamma}}\,dy, \qquad (x,t) \in \mathbb{R}^{n+1}_+, \quad 0 < \gamma < n,$$

$$Q_i f(x,t) = c_n \int_{\mathbb{R}^n} \frac{(x_i - y_i)}{(|x-y|^2+t^2)^{\frac{n+1}{2}}}\, f(y)dy, \quad (x,t) \in \mathbb{R}^{n+1}_+, \quad i = 1,2,\ldots,n,$$

$$Pf(x,t) = c_n \int_{\mathbb{R}^n} \frac{t}{(|x-y|^2+t^2)^{\frac{n+1}{2}}}\, f(y)dy, \quad (x,t) \in \mathbb{R}^{n+1}_+$$

where $\mathbb{R}^{n+1}_+$ denotes the upper half-plane $\mathbb{R}^n \times [0,\infty)$, c_n is a constant depending on the dimension and $B(x,r)$ is the ball $\{y \in \mathbb{R}^n : |x-y| < r\}$.

The operator P is the Poisson-Integral, $\mathcal{M}_0$ was introduced in [F-S] and $\mathcal{M}_\gamma$ and T_γ were studied in [R-T].

Given a measure $d\mu$ on $\mathbb{R}^{n+1}_+$ we shall say, as usual, that $d\mu$ is a Carleson measure if there exists a constant C such that for any ball $B = B(x,r)$ in $\mathbb{R}^n$

we have $\mu(\hat{B}) \leq C|B|$ where $\hat{B} = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| + t < r\}$ and $|B|$ stands for the Lebesgue measure of B.

1.1 REMARK. It is known (see [R-T]) that if μ is a Carleson measure then $\mathcal{M}_\gamma$ and T_γ map $L^1(\mathbb{R}^n, dx)$ into weak-$L^{\frac{n}{n-\gamma}}(\mathbb{R}^{n+1}_+, d\mu)$ and P and Q_i map $L^1(\mathbb{R}^n, dx)$ into weak-$L^1(\mathbb{R}^{n+1}_+, d\mu)$.

Moreover if a weight v belongs to D_p then there exists a weight u such that $\mathcal{M}_0$ maps $L^p(v)$ into $L^p(u, d\mu)$ (see [F-T]).

On the other hand it is known that if for an open set $\theta \subset \mathbb{R}^n$ one defines

$$T(\theta) = \{(x,t) \in \mathbb{R}^{n+1}_+ : B(x,t) \subset \theta\}$$

then $d\mu$ is a Carleson measure if and only if $\mu(T(\theta)) \leq C|\theta|$ (see [A-B] and [J]).

1.2 REMARK. The operators defined above generalize several known operators. In particular:

- the fractional maximal operator of order γ

$$M_\gamma f(x) = \sup_{r>0} \frac{1}{r^{n-\gamma}} \int_{B(x,r)} |f(y)|dy = \mathcal{M}_\gamma f(x,0),$$

(observe that $M_0 = M$ is the Hardy-Littlewood maximal operator);

- the fractional integral

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}\, dy = T_\gamma f(x,0);$$

- the Riesz transforms

$$R_i f(x) = p.v. \int_{\mathbb{R}^n} \frac{x_i - y_i}{|x-y|^{n+1}} f(y)dy = Q_i f(x,0).$$

Now we shall consider vector-valued extensions of the above operators.

We observe that T_γ and P are positive ($f \geq 0 \Rightarrow T_\gamma f \geq 0$ and $Pf \geq 0$) and linear operators. Therefore given any Banach space E we can consider the vector-valued extensions $\tilde{T}_\gamma$ and $\tilde{P}$ such that

$$\begin{aligned} \langle \tilde{T}_\gamma f(x,t), e^* \rangle &= T_\gamma(\langle f, e^* \rangle)(x,t) \\ \langle \tilde{P} f(x,t), e^* \rangle &= P(\langle f, e^* \rangle)(x,t) \end{aligned} \tag{1.3}$$

where f is an E-valued strongly measurable function and $e^* \in E^*$. It is well known (see [L-T]) that the extensions $\tilde{T}_\gamma$ and $\tilde{P}$ have the same boundedness properties as T_γ and P. In particular, if $d\mu$ is a Carleson measure, $\tilde{T}_\gamma$ maps $L^1_E(\mathbb{R}^n, dx)$ into weak-$L_E^{\frac{n}{n-\gamma}}(\mathbb{R}^{n+1}_+, d\mu)$ and $\tilde{P}$ maps $L^1_E(\mathbb{R}^n, dx)$ into weak-$L^1_E(\mathbb{R}^{n+1}_*, d\mu)$ for any Banach space E.

We recall that the class of the Banach spaces E such that the Riesz transforms $\tilde{R}_i$ are bounded from $L^2_E(\mathbb{R}^n, dx)$ into $L^2_E(\mathbb{R}^n, dx)$ has been characterized by Burkholder and Bourgain (see [Bk] and [B]) ant it is denoted by U.M.D. Therefore choosing the Carleson measure $d\mu(x,t) = dx \otimes \delta_0(t)$, where δ_0 is Dirac's delta, it is clear that if the $\tilde{Q}_i$'s are bounded from $L^1_E(\mathbb{R}^n, dx)$ into weak-$L^1_E(\mathbb{R}^{n+1}_+, d\mu)$ then E must be in the U.M.D. class. On the other hand

$$\{(x,t) : \|\tilde{Q}_i f(x,t)\|_E > \lambda\} \subset T(\{x : \|\tilde{R}_i f(x)\|_E > 2^{\frac{n+1}{2}}\lambda\})$$

and therefore if E is U.M.D., using Remark 1.1, for any Carleson measure $d\mu$ we have

$$\mu(\{(x,t) : \|\tilde{Q}_i f(x,t)\|_E > \lambda\}) \le \mu(T(\{x : \|\tilde{R}_i f(x)\|_E > 2^{\frac{n+1}{2}}\lambda\}))$$
$$\le C|\{x : \|\tilde{R}_i f(x)\|_E > 2^{\frac{n+1}{2}}\lambda\}| \le \frac{C}{\lambda}\|f\|_{L^1_E}.$$

1.4 Definition. *Let $(\Omega, \mathcal{A}, \nu)$ be a complete σ-finite measure space. A Banach space X consisting of equivalence classes, modulo equality almost everywhere, of locally-integrable real-valued functions on Ω is called a* Köthe function space *if the following two conditions hold:*

(1) *If $|f(\omega)| \le |g(\omega)|$ a.e. on Ω with f measurable and $g \in X$, then $f \in X$ and $\|f\| \le \|g\|$.*

(2) *For every $E \in \mathcal{A}$ with $\nu(E) < +\infty$ the characteristic function χ_E of E belongs to X.*

Every Köthe function space is a Banach lattice with the obvious order ($f \ge 0$ if $f(\omega) \ge 0$ a.e.).

Given a measurable function g on Ω such that $gf \in L^1(\nu)$ for every $f \in X$, one defines an element x^*_g in X^* by

$$x^*_g(f) = \int_\Omega f(\omega)g(\omega)d\nu(\omega).$$

The linear space of these x^*_g is denoted by X'. It is known (see [L-T]) that X' is a norming subspace of X^* if and only if whenever $\{f_n\}_{n=1}^\infty$ and f are non-negative elements of X such that $f_n(\omega) \uparrow f(\omega)$ a.e. we have $\|f_n\| \to \|f\|$.

Let X be a Banach lattice and let J be a finite subset of the set $\mathbb{Q}_+$ of positive rational numbers. Given a locally-integrable function $f : \mathbb{R}^n \to X$ (this means, of course, that f is strongly measurable and that the $\mathbb{R}$-valued function $y \mapsto \|f(y)\|_X$ is locally-integrable) we define

$$\tilde{\mathcal{M}}_{\gamma,J} f(x,t) = \sup_{\substack{r\in J\\ r\ge t}} \frac{1}{r^{n-\gamma}} \int_{B(x,r)} |f(y)|dy, \tag{1.5}$$

where the sup is taken in the lattice X.

When X is a Köthe function space, as it will be the case in the sequel, it is clear that $\tilde{\mathcal{M}}_{\gamma,J}f(x,t)$ is a function of ω given by

$$\tilde{\mathcal{M}}_{\gamma,J}f(x,t)(\omega) = \sup_{\substack{r\in J \\ r\geq J}} \frac{1}{r^{n-\gamma}} \int_{B(x,r)} |f(y,\omega)|dy,$$

where the sup is now taken with respect to the order of $\mathbb{R}$.

In this situation we can consider f and $\tilde{\mathcal{M}}_{\gamma,J}f$ as functions on $\mathbb{R}^n \times \Omega$ and $\mathbb{R}^{n+1}_+ \times \Omega$ respectively. If $\gamma > 0$ we have

$$\mathcal{M}_\gamma f(x,t) \leq T_\gamma f(x,t), \qquad (x,t) \in \mathbb{R}^{n+1}_+$$

and therefore for any Banach lattice X and any finite subset J of $\mathbb{Q}_+$, we have

$$\tilde{\mathcal{M}}_{\gamma,J}f(x,t) \leq \tilde{T}_\gamma f(x,t), \qquad (x,t) \in \mathbb{R}^{n+1}_+$$

where now the inequality holds with respect to the order of the lattice X. By the above results about $\tilde{T}_\gamma$ we have that given a Banach lattice X, a Carleson measure $d\mu$ and $\gamma > 0$, the inequality

$$\mu(\{(x,t) : \|\tilde{\mathcal{M}}_{\gamma,J}f(x,t)\|_X > \lambda\}) \leq C\left(\frac{1}{\lambda}\int \|f(x)\|_X dx\right)^{\frac{n}{n-\gamma}}$$

holds with the constant C depending on $d\mu$ and γ but not on J.

1.6 DEFINITION. *We shall say that a Banach lattice X satisfies the* Hardy-Littlewood (H.L.) property *if the inequality*

$$|\{x \in \mathbb{R}^n : \|\tilde{M}_J f(x)\|_X > \lambda\}| \leq \frac{C}{\lambda}\int_{\mathbb{R}^n} \|f(x)\|_X dx$$

holds, with C independent of J (see [GC-M-T]*); $\tilde{M}_J$ is defined by*

$$\tilde{M}_J f(x) = \sup_{r\in J} \frac{1}{r^n}\int_{B(x,r)} |f(y)|dy,$$

the sup being taken with respect to the lattice order.

It is easy to check that for every for any finite subset J of $\mathbb{Q}_+$ we have

$$\{(x,t) : \|\tilde{\mathcal{M}}_{0,J}f(x,t)\|_X > \lambda\} \subset T(\{x : \|\tilde{M}_J f(x)\|_X > \lambda\}),$$

therefore if X has the H.L. property, using Remark 1.1, we have for any Carleson measure $d\mu$

$$\begin{aligned}&\mu(\{(x,t) : \|\tilde{\mathcal{M}}_{0,J}f(x,t)\|_X > \lambda\})\\ &\le \mu(T(\{x : \|\tilde{M}_J f(x)\|_X > \lambda\}) \le C|\{x : \|\tilde{M}_J f(x)\|_X > \lambda\}|\\ &\le \frac{C}{\lambda}\|f\|_{L^1_X}.\end{aligned}$$

On the other hand, choosing $d\mu(x,t) = dx \otimes \delta_0(t)$, it is clear that if for a Banach lattice X we have the inequality

$$\mu\{(x,t) : \|\tilde{\mathcal{M}}_{0,J}f(x,t)\|_X > \lambda\} \le \frac{C}{\lambda}\int_{\mathbb{R}^n} \|f(x)\|_X dx$$

with C independent of J, then, Definition 1.6, X must have the H.L. property. Now we collect the above results for further reference.

1.7 PROPOSITION.

(1.8) *Let E be an arbitrary Banach space and $\tilde{T}_\gamma$, $\tilde{P}$ be the E-valued extensions defined in* (1.3)*; then for any Carleson measure $d\mu$ in $\mathbb{R}^{n+1}_+$, we have*

$$\mu(\{(x,t) : \|\tilde{T}_\gamma f(x,t)\|_E > \lambda\}) \le C\left(\frac{1}{\lambda}\int_{\mathbb{R}^n} \|f(x)\|_E dx\right)^{\frac{n}{n-\gamma}}$$

and

$$\mu(\{(x,t) : \|\tilde{P}f(x,t)\|_E > \lambda\}) \le \frac{C}{\lambda}\int_{\mathbb{R}^n} \|f(x)\|_E dx.$$

(1.9) *Let X be an arbitrary Banach lattice and $\tilde{\mathcal{M}}_{\gamma,J}$ be the operator defined in* (1.5)*; then for any Carleson measure $d\mu$ in $\mathbb{R}^{n+1}_+$ and any $\gamma > 0$ we have*

$$\mu(\{(x,t) : \|\tilde{\mathcal{M}}_{\gamma,J}f(x,t)\|_X > \lambda\}) \le C\left(\frac{1}{\lambda}\int_{\mathbb{R}^n} \|f(x)\|_X dx\right)^{\frac{n}{n-\gamma}}$$

with C independent of J.

(1.10) *Let E be a Banach space and $\tilde{Q}_i$ be the vector-valued extension of Q_i. For any Carleson measure $d\mu$ in $\mathbb{R}^{n+1}_+$ we have*

$$\mu(\{(x,t) : \|\tilde{Q}_i f(x,t)\|_E > \lambda\}) \le \frac{C}{\lambda}\int_{\mathbb{R}^n} \|f(x)\|_E dx \qquad i = 1,2,\dots,n$$

if and only if E is U.M.D.

(1.11) *Let X be a Banach lattice and $\tilde{\mathcal{M}}_{0,J}$ be the operator defined in* (1.5). *For any Carleson measure $d\mu$ in $\mathbb{R}^{n+1}_+$ we have*

$$\mu(\{(x,t) : \|\tilde{\mathcal{M}}_{0,J} f(x,t)\|_X > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} \|f(x)\|_X dx$$

with C independent of J, if and only if X has the H.L. *property.*

2. - Technical lemmas

2.1 LEMMA. *Assume that $d\mu$ is a Carleson measure in $\mathbb{R}^{n+1}_+$. Let $1 < p < \infty$, $B_1 = \{x : |x| \leq 1\}$ and let v be a non-negative measurable function in $\mathbb{R}^n$ such that $\mathcal{M}_0(v^{-p'/p}\chi_{B_1})(x,t) < +\infty$ a.e. in (x,t) and $\int_{B_1} v^{-p'/p} dx > 0$. Under these conditions the weight $w(x,t) = [\mathcal{M}_0(v^{-p'/p}\chi_{B_1})(x,t)]^{-\beta}$ with $\beta > p-1$ satisfies*

(2.2) *For any $\gamma \geq 0$ and $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n} \geq 0$*

$$\left(\int_{\hat{B}_1} \mathcal{M}_\gamma(f\chi_{B_1})(x,t)^q w(x,t)^{q/p} d\mu(x,t) \right)^{1/q} \leq C_{\gamma,p,q} \left(\int_{B_1} |f(x)|^p v(x) dx \right)^{1/p}.$$

(2.3) *For any $\gamma > 0$ and $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n} \geq 0$*

$$\left(\int_{\hat{B}_1} T_\gamma(f\chi_{B_1}(x,t)^q w(x,t)^{q/p} d\mu(x,t) \right)^{1/q} \leq C_{\gamma,p,q} \left(\int_{B_1} |f(x)|^p v(x) dx \right)^{1/p}.$$

PROOF. If $\frac{1}{r} > \frac{1}{p} - \frac{\gamma}{n} = \frac{1}{q}$, by Hölder's inequality, we have

$$\left(\int_{\hat{B}_1} \mathcal{M}_\gamma f(x,t)^r w^{r/p}(x,t) d\mu(x,t) \right)^{1/r}$$

$$\leq \mu(\hat{B}_1)^{1/(q/r)'} \left(\int_{\hat{B}_1} \mathcal{M}_\gamma f(x,t)^q w^{q/p}(x,t) d\mu(x,t) \right)^{1/q}$$

Then we need to consider only the case $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$. Let

$$E_k = \{(x,t) \in \mathbb{R}^{n+1}_+ : \mathcal{M}_0(v^{-p'/p}\chi_{B_1})(x,t) \leq 2^k\},$$

$k = 0, 1, 2, \dots$. We define

$$T_k f(x,t) = \mathcal{M}_\gamma(f v^{-p'/p}\chi_{B_1})(x,t)\chi_{E_k}(x,t);$$

as the operator $\mathcal{M}_\gamma$ maps $L^1(\mathbb{R}^n, dx)$ into weak-$L^{\frac{n}{n-\gamma}}(\mathbb{R}^{n+1}_+, d\mu)$, see Remark 1.1, we have

$$\mu(\{(x,t) : T_k f(x,t) > \lambda\}) \leq C \left(\frac{1}{\lambda} \int_B f(x) v^{-p'/p}(x) dx \right)^{\frac{n}{n-\gamma}}. \tag{2.4}$$

On the other hand, for any $r > 0$, Hölder's inequality gives

$$\frac{1}{r^{n-\gamma}} \int_{B(x,r)} |f(y)| \chi_{B_1}(y) v^{-p'/p}(y) dy$$

$$\leq \left(\frac{1}{r^{n-\gamma}} \int_{B(x,r)} \chi_{B_1} v^{-p'/p} dx \right)^{1-\gamma/n} \left(\int_{B_1} |f|^{n/\gamma} v^{-p'/p} dx \right)^{\gamma/n};$$

therefore, by the definition of E_k, we obtain

$$\|T_k f\|_\infty \leq C 2^{k(1-\gamma/n)} \left(\int_{B_1} |f|^{n/\gamma} v^{-p'/p} dx \right)^{\gamma/n}. \tag{2.5}$$

Applying Marcinkiewicz's interpolation theorem to (2.4) and (2.5) we obtain

$$\left(\int_{\mathbb{R}^{n+1}_+} |T_k f|^q d\mu \right)^{1/q} \leq C_{\alpha,p} 2^{k/p'} \left(\int_{\mathbb{R}^n} |f|^p v^{-p'/p} dx \right)^{1/p},$$

for $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$.

Replacing, in the last inequality, $f v^{-p'/p}$ by f it follows that

$$\left(\int_{E_k} \mathcal{M}_\gamma(f\chi_{B_1})^q d\mu \right)^{1/q} \leq C 2^{k/p'} \left(\int_{B_1} |f|^p v dx \right)^{1/p} \tag{2.6}$$

The assumption that $\mathcal{M}_0(v^{-p'/p}\chi_{B_1})(x,t) < +\infty$ a.e. in (x,t) implies that $\mu(\mathbb{R}^{n+1}_+ \setminus \cup E_k) = 0$; therefore:

$$\int_{\hat{B}_1} \mathcal{M}_\gamma(f\chi_{B_1})^q w^{q/p} d\mu = \left(\int_{\hat{B}_1 \cap E_0} + \sum_{k=1}^{\infty} \int_{\hat{B}_1 \cap (E_k \setminus E_{k-1})} \right) \mathcal{M}_\gamma(f\chi_{B_1})^q w^{q/p} d\mu. \tag{2.7}$$

On the other hand if $(x,t) \in \hat{B}_1$ then

$$\int_{B_1} v^{-p'/p}(y)dy \leq C \sup_{r \geq t} \frac{1}{r^n} \int_{B(x,r)} v^{-p'/p}(y)\chi_{B_1}(y)dy;$$

therefore, as $\int_{B_1} v^{-p'/p} > 0$, we have that $\omega(x,t)$ is bounded for $(x,t) \in \hat{B}_1 \cap E_0$. Then, using (2.6) and (2.7), we get

$$\int_{\hat{B}_1} \mathcal{M}_\gamma(f\chi_{B_1})^q w^{q/p} d\mu \leq \sum_{k=0}^{\infty} 2^{-k\beta q/p} \int_{E_k} \mathcal{M}_\gamma(f\chi_{B_1})^q d\mu$$

$$\leq C_{\gamma,p} \left(\int_{B_1} |f|^p v dx \right)^{q/p} \sum_{k=0}^{\infty} 2^{-k\beta q/p/} 2^{kq/p'};$$

since $\beta > p-1$, the above geometric series is convergent and this completes the proof of (2.2).

In order to estimate $T_\gamma(f\chi_{B_1})$ we shall use the following inequality, see [R-T], valid for $\varepsilon > 0$ small enough

$$|T_\gamma f(x,t)| \leq C[\mathcal{M}_{\gamma-\varepsilon} f(x,t) \mathcal{M}_{\gamma+\varepsilon} f(x,t)]^{1/2} \tag{2.8}$$

where C depends on ε and γ.

Let $\varepsilon > 0$, $\gamma_1 = \gamma - \varepsilon$ and $\gamma_2 = \gamma + \varepsilon$; if ε is small enough we get $0 < \gamma_i < \frac{n}{p}$. Let q_1 and q_2 be such that $\frac{1}{q_1} = \frac{1}{q} + \frac{\varepsilon}{n}$ and $\frac{1}{q^2} = \frac{1}{q}\frac{\varepsilon}{n}$; in particular $\frac{1}{q_1} = \frac{1}{p} - \frac{\gamma_1}{n}$ and $\frac{1}{q_2} = \frac{1}{p} - \frac{\gamma_2}{n}$.

By (2.2) we have

$$\left(\int_{\hat{B}_1} \mathcal{M}_{\gamma_i}(f\chi_{B_1})^{q_i} w^{q_i/p} d\mu \right)^{1/q_i} \leq C \left(\int_{B_1} |f|^p v dx \right)^{1/p}, \qquad i = 1,2.$$

Then applying (2.8) and Hölder's inequality with exponents $\frac{2q_i}{p}$ we obtain

$$\left(\int_{\hat{B}_1} T_\gamma(f\chi_{B_1})^q w^{q/p} d\mu\right)^{1/q}$$

$$\leq C\left(\int_{\hat{B}_1} \mathcal{M}_{\gamma_1}(f\chi_{B_1})^{q_1} w^{q_1/p} d\mu\right)^{1/2q_1}\left(\int_{\hat{B}_1} \mathcal{M}_{\gamma_2}(f\chi_{B_1})^{q_2} w^{q_2/p} d\mu\right)^{1/2q_2}$$

$$\leq C\left(\int_{B_1} |f|^p v dx\right)^{1/p}. \qquad \square$$

Now we state the vector-valued version of Lemma 2.1.

2.9 LEMMA. *Assume that $d\mu$ is a Carleson measure in $\mathbb{R}^{n+1}_+$. Let $1 < p < \infty$ and let v be a non-negative measurable function in $\mathbb{R}^n$ such that $\mathcal{M}_0(v^{-p'/p}\chi_{B_1})(x,t) < \infty$ a.e. in (x,t) and $\int_{B_1} v^{-p'/p}dx > 0$. Under these conditions the weight w defined in Lemma 2.1 satisfies the following properties:*

(2.10) *For any Köthe function space X, any finite subset J of $\mathbb{Q}_+$, any $\gamma > 0$ and $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n}$*

$$\left(\int_{\hat{B}_1} \|\tilde{\mathcal{M}}_{\gamma,J}f(x,t)\|_X^q w(x,t)^{q/p} d\mu(x,t)\right)^{1/q} \leq C_{\gamma,p,q}\left(\int_{B_1} \|f(x)\|_X^p v(x)dx\right)^{1/p}$$

with $C_{\gamma,p,q}$ independent of J and X;

(2.11) *For any Banach space E, any $\gamma > 0$ and $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n}$*

$$\left(\int_{\hat{B}_1} \|\tilde{T}_\gamma f(x,t)\|_E^q w(x,t)^{q/p} d\mu(x,t)\right)^{1/q} \leq C_{\gamma,p,q}\left(\int_{B_1} \|f(x)\|_E^p v(x)dx\right)^{1/p}$$

with $C_{\gamma,p,q}$ independent of E.

PROOF. Since $\|\tilde{T}_\gamma f(x,t)\|_E \leq T_\gamma(\|f\|_E)(x,t)$, (2.11) is a direct consequence of (2.3). On the other hand if X is a Banach space of classes of equivalence of measurable functions on $(\Omega, \mathcal{A}, \nu)$ we have for any $\omega \in \Omega$ and $\gamma > 0$

$$\tilde{\mathcal{M}}_\gamma f(x,t)(\omega) = \sup_{r\geq t} \frac{1}{r^{n-\gamma}} \int_{B(x,r)} |f(y,\omega)|dy \leq C_\gamma T_\gamma(|f(\cdot,\omega)|)(x,t)$$

and therefore, by definition (1.4), we have

$$\|\tilde{M}_\gamma f(x,t)\|_X \le \|\tilde{T}_\gamma(|f|)(x,t)\|_X \le T_\gamma(\||f|\|_X)(x,t) = T_\gamma(\|f\|_X)(x,t);$$

now (2.10) follows from (2.11). □

The following Lemma can be found in [F-T], Theorem 1. We state if here for further reference.

2.12 LEMMA. *Let $(Y, d\nu)$ be a measure space, F and G be Banach spaces and $\{A_k\}_{k=0}^\infty$ be a sequence of disjoint sets in Y such that $\bigcup_{k=0}^{\infty} A_k = Y$. Assume that $0 < s < p < \infty$ and that $(T_j)_{j\in I}$ is a family of sublinear operators which satisfies*

$$\left\| \left(\sum_j \|T_j f_j\|_F^p \right)^{1/p} \right\|_{L^s(A_k, d\nu)} \le C_k \left(\sum_j \|f_j\|_G^p \right)^{1/p}, \qquad k \in \mathbb{N}$$

where, for each $k \in \mathbb{N}$, C_k is a constant depending on G, F, p and s. Then there exists a positive function $u(x)$ on Y such that

$$\left(\int_Y \|T_j f(x)\|_F^p u(x) d\nu(x) \right)^{1/p} \le C\|f\|_G \qquad (j \in I)$$

where C is a constant depending on G, F, p and s.

2.13 DEFINITION. *Given $0 \le \gamma < n$ and a finite subset L of $\mathbb{Q}_+ \cap [1, \infty)$ we define the function $\varphi_{\gamma,L}$ on $\mathbb{R}^n \times \Omega$ as*

$$\varphi_{\gamma,L}(x,\omega) = \frac{\chi_{B_\omega}(x)}{|B_\omega|^{1-\frac{\gamma}{n}}}$$

where B_ω is a ball centered at the origin with radius $r_\omega \in L$.

2.14 DEFINITION. *Let v be a weight in $\mathbb{R}^n$ (i.e. a real-valued, locally-integrable function with $v(x) \ge 0$ a.e. in $\mathbb{R}^n$). Let X be a Köthe function space with X' norming and $1 < p < \infty$. We shall say that v belongs to the class $D_{p,X}^\gamma$ if the inequality*

$$\int_{\mathbb{R}^n} \|a\varphi_{\gamma,L}(x)\|_{X^*}^{p'} v(x)^{-p'/p} dx \le C$$

holds for any $a \in X'$ with $\|a\|_{X^*} \le 1$ and with the constant C independent of $\varphi_{\gamma,L}$.

2.15 LEMMA. *Assume $0 \le \gamma < n$. Let X be a Köthe function space with X' norming and let $1 < p < \infty$. For a weight v in $\mathbb{R}^n$ the following properties are equivalent:*

(i) $v \in D^{\gamma}_{p,X}$.

(ii) *Given a ball* $B = B(0,R) = \{x : |x| \le R\}$, $R \ge 1$, *and a* X*-valued locally-integrable function* f *with support in the complement of* $2B = B(0,2R)$, *inequality*

$$\sup_{(x,t)\in\hat{B}} \|\tilde{\mathcal{M}}_{\gamma,J} f(x,t)\|_X \le C\|f\|_{L^p_X(v)}$$

holds with the constant C *independent of* J.

PROOF. (i) $\Rightarrow$ (ii) Let f be a locally-integrable function with support in $\mathbb{R}^n \backslash B(0,2R)$. Given $(x,t) \in \hat{B}$ and a rational number r, the integral $\int_{B(x,r)} |f(y)|dy$ is equal to zero unless $B(x,r) \cap \mathbb{R}^n \backslash B(0,2R)) \neq \emptyset$; in this case $r > R$ and then $B(x,r) \subset B(0,2r)$. Hence if $J \subset \mathbb{Q}_+$ we have

$$\begin{aligned}
\mathcal{M}_{\gamma,J} f(x,t)(\omega) &= \sup_{\substack{r\in J\\ r\ge t}} \frac{1}{r^{n-\gamma}} \int_{B(x,r)} |f(y,\omega)|dy \\
&= C \sup_{\substack{r\in J\\ r\ge \max(t,R)}} \frac{1}{r^{n-\gamma}} \int_{B(x,r)} |f(y,\omega)|dy \\
&\le C \sup_{\substack{r\in J\\ r\ge \max(t,R)}} \frac{1}{r^{n-\gamma}} \int_{B(0,2r)} |f(y,\omega)|dy \\
&\le C \sup_{\substack{r\in J\\ r\ge \max(1,t)}} \frac{1}{r^{n-\gamma}} \int_{B(0,2r)} |f(y,\omega)|dy \\
&= C \int_{\mathbb{R}^n} |f(y,\omega)| \varphi_{\gamma,L_t}(y,\omega)dy,
\end{aligned}$$

where

$$L_t = \{2r : r \in J, r \ge \max(1,t)\}, \qquad \varphi_{\gamma,L_t}(y,\omega) = \frac{\chi_{B_\omega}(y)}{r_\omega^{n-\gamma}},$$

and for each ω, $r_\omega = s \in L_t$ if

$$\sup_{\substack{r\in J\\ r\ge \max(1,t)}} \frac{1}{(2r)^{n-\gamma}} \int_{B(0,2R)} |f(y,\omega)|dy = \frac{1}{s^{n-\gamma}} \int_{B(0,s)} |f(y,\omega)|dy.$$

In the last equality we have used the hypothesis that J is finite and the consequence that for each (ω,t) the sup is in fact a maximum.

As X' is norming, we have

$$\|\mathcal{M}_{\gamma,J}f(x,t)\|_X \le C \sup_{\substack{a\in X'\\ \|a\|\le 1}} \int_\Omega \int_{\mathbb{R}^n} |f(y,\omega)|\varphi_{\gamma,L_t}(y,\omega)a(\omega)dyd\omega.$$

Now we observe that for each y, the function $\omega \mapsto \varphi_{\gamma,L_t}(y,\omega)$ is a step function and therefore the function $\omega \mapsto a(\omega)\varphi_{\gamma,L_t}(y,\omega)$ is an element of X'; therefore by duality and Hölder's inequality, we have

$$\|\mathcal{M}_{\gamma,J}f(x,t)\|_X \le C \sup_{\substack{a\in X'\\ \|a\|\le 1}} \left(\int_{\mathbb{R}^n} \|f(y)\|_X^p v(y)dy \right)^{1/p}$$

$$\times \left(\int_{\mathbb{R}^n} \|a\varphi_{\gamma,L_t}(y)\|_{X^*}^{p'} v(y)^{-p'/p}dy \right)^{1/p'} \le C\|f\|_{L^p_X(v)}.$$

(ii) $\Rightarrow$ (i) Let $B_1 = \{x : |x| < 1\}$ be the unit ball in $\mathbb{R}^n$. By duality we have

$$\int_{\mathbb{R}^n} \|a\varphi_{\gamma,L}(y)\|_{X^*}^{p'} v(y)^{-p'/p}dy$$

$$= \sup_{\substack{g\in L^p_X\\ \|g\|\le 1}} \left(\int_{\mathbb{R}^n} \int_\Omega a(\omega)\varphi_{\gamma,L}(y,\omega)g(y,\omega)v(y)^{-1/p}d\omega dy \right)^{p'}$$

$$\le \sup_{\substack{g\in L^p_X\\ \|g\|\le 1}} \left(\int_{\mathbb{R}^n} \int_\Omega a(\omega)\varphi_{\gamma,L}(y,\omega)g(y,\omega)\chi_{\mathbb{R}^n\setminus 2B_1}(y)v(y)^{-1/p}d\omega dy \right.$$

$$\left. + \int_{\mathbb{R}^n} \int_\Omega a(\omega)\varphi_{\gamma,L}(y,\omega)g(y,\omega)\chi_{2B_1}(y)v(y)^{-1/p}d\omega dy \right)^{p'}$$

$$\le \sup_{\substack{g\in L^p_X\\ \|g\|\le 1}} \left(\int_\Omega a(\omega)\mathcal{M}_{\gamma,L}(g\chi_{\mathbb{R}^n\setminus 2B_1}v^{1/p})(0,t,\omega)d\omega \right.$$

$$\left. + \int_{2B_1} \int_\Omega a(\omega)\varphi_{\gamma,L}(y,\omega)g(y,\omega)v(y)^{-1/p}d\omega dy \right)^{p'}$$

where $0 \le t < 1$.

To estimate the first summand we observe that $(0,t) \in \hat{B}_1$ and therefore we have

$$\|\mathcal{M}_{\gamma,L}(g\chi_{\mathbb{R}^n\setminus 2B_1}v^{-1/p})(0,t)\|_X \leq C\|g\chi_{\mathbb{R}^n\setminus 2B_1}v^{-1/p}\|_{L^p_X(v)} \leq C$$

On the other hand since $\varphi_{\gamma,L}(y,\omega) \leq 1$ we have

$$\left|\int_{2B_1}\int_{\Omega} a(\omega)\varphi_{\gamma,L}\varphi_{\gamma,L}(y,\omega)g(y,\omega)v(y)^{-1/p}d\omega dy\right|$$
$$\leq \int_{2B_1}\int_{\Omega} |a(\omega)|\,|g(y,\omega)|v(y)^{-1/p}d\omega dy$$
$$\leq \|a\|_{X^*}\|g\|_{L^p_X}\left(\int_{2B_1} v(y)^{-p'/p}dy\right)^{1/p'} \leq C.$$

This completes the proof. □

2.16 LEMMA. *Let $0 \leq \gamma < n$, E and F be Banach spaces and v be a weight in $\mathbb{R}^n$ such that $v \in D^\gamma_p$ (see 0.6). Assume that τ_γ is an operator from $L^p_E(\mathbb{R}^n, dx)$ into $L^q_F(\mathbb{R}^{n+1}_+, d\mu)$ such that*

$$\|\tau_\gamma f(x,t)\|_F \leq C\int \frac{\|f(y)\|_E}{(|x-y|+t)^{n-\gamma}}\,dy = CT_\gamma(\|f\|_E)(x,t).$$

Then given a ball $B = B(0,R)$, $R \geq 1$, and a E-valued locally-integrable function f with support in the complement of $2B$, inequality

$$\sup_{(x,t)\in\hat{B}} \|\tau_\gamma f(x,t)\|_F \leq C_\gamma\|f\|_{L^p_E(v)}$$

holds with C independent of τ_γ.

PROOF. It is sufficient to prove that

$$\sup_{(x,t)\in\hat{B}} T_\gamma(\|f\|_E)(x,t) \leq C_\gamma\|f\|_{L^p_E(v)}.$$

If $(x,t) \in \hat{B}$, we have

$$T_\gamma(\|f\|_E)(x,t) = \int\limits_{|x|+t<R<2|y|} \frac{\|f(y)\|_E}{(|x-y|+t)^{n-\gamma}}\,dy$$

$$\le \int\limits_{|x|<R<2|y|} \frac{\|f(y)\|_E}{|x-y|^{n-\gamma}}\,dy \le \int\limits_{\mathbb{R}^n} \frac{\|f(y)\|_E}{(|y|+1)^{n-\gamma}}\,dy$$

$$\le \left(\int\limits_{\mathbb{R}^n} \|f(y)\|_E^p v(y)dy\right)^{1/p} \left(\int\limits_{\mathbb{R}^n} \frac{v(y)^{-p'/p}}{(1+|y|)^{n-\gamma}}\,dy\right)^{1/p'}$$

$$\le C\|f\|_{L^p_E(v)}.$$ □

3. - Main results

In this section we shall state and prove the main results of the paper. We recall that in the Introduction (0.3 and 0.6) we have defined the classes D_p^* and D_p^γ.

3.1 THEOREM. *Let E be a Banach space, $d\mu$ be a Carleson measure in $\mathbb{R}^{n+1}_+$ and v be a weight in $\mathbb{R}^n$.*

(3.2) *If $v \in D_p^*$, $1 < p < \infty$, then there exists a weight u in $\mathbb{R}^{n+1}_+$ such that*

$$\int\limits_{\mathbb{R}^{n+1}_+} \|\tilde{P}f(x,t)\|_E^p u(x,t)d\mu(x,t) \le C \int\limits_{\mathbb{R}^n} \|f(x)\|_E^p v(x)dx.$$

(3.3) *v belongs to D_p^γ with $1 < p < \infty$ and $0 < \gamma < n$, if and only if there exists a weight u in $\mathbb{R}^{n+1}_+$ such that*

$$\left(\int\limits_{\mathbb{R}^{n+1}_+} \|\tilde{T}_\gamma f(x,t)\|_E^q u(x,t)d\mu(x,t)\right)^{1/q} \le C_\gamma \left(\int\limits_{\mathbb{R}^n} \|f(x)\|_E^p v(x)dx\right)^{1/p}$$

with $\frac{1}{q} \ge \frac{1}{p} - \frac{\gamma}{n}$ and $1 < q < \infty$.

3.4 COROLLARY. *Let $0 < \gamma < n$, E and F be Banach spaces, $d\mu$ be a Carleson measure in $\mathbb{R}^{n+1}_+$ and v be a weight in $\mathbb{R}^n$ such that $v \in D_p^\gamma$. Assume that τ_γ is a bounded sublinear operator from $L^p_E(\mathbb{R}^n, dx)$ into $L^q_F(\mathbb{R}^{n+1}_+, d\mu)$ such that*

$$\|\tau_\gamma f(x,t)\|_F \le C T_\gamma(\|f\|_E)(x,t).$$

Then there exists a weight u in $\mathbb{R}^{n+1}_+$ such that

$$\left(\int\limits_{\mathbb{R}^{n+1}} \|\tau_\gamma f(x,t)\|_F^q u(x,t)d\mu(x,t)\right)^{1/q} \le C_\gamma \left(\int\limits_{\mathbb{R}^n} \|f(x)\|_E^p v(x)dx\right)^{1/p}$$

with $\frac{1}{q} \ge \frac{1}{p} - \frac{\gamma}{n}$ *and* $1 < q < \infty$.

3.5 THEOREM. *Let X be a Banach lattice, $d\mu$ be a Carleson measure in $\mathbb{R}^{n+1}_+$ and v be a weight in $\mathbb{R}^n$; then v belongs to $D^\gamma_{p,X}$, (see 2.14) $0 < \gamma < n$ and $1 < p < \infty$, if and only if there exists a weight u in $\mathbb{R}^{n+1}_+$ such that*

$$\left(\int\limits_{\mathbb{R}^{n+1}_+} \|\tilde{\mathcal{M}}_{\gamma,J} f(x,t)\|_X^q u(x,t)d\mu(x,t)\right)^{1/q} \le C_\gamma \left(\int\limits_{\mathbb{R}^n} \|f(x)\|_X^p v(x)dx\right)^{1/p} \tag{3.6}$$

where the constant C_γ is independent of the finite set J, $\frac{1}{q} \ge \frac{1}{p} - \frac{\gamma}{n}$ *and* $1 < q < \infty$.

3.7 THEOREM. *Let E be a Banach space with the U.M.D. property, $d\mu$ be a Carleson measure in $\mathbb{R}^{n+1}_+$ and v be a weight in $\mathbb{R}^n$. Then v belongs to D_p with $1 < p < \infty$, if and only if there exists u in $\mathbb{R}^{n+1}_+$ such that*

$$\int\limits_{\mathbb{R}^{n+1}_+} \|\tilde{Q}_i f(x,t)\|_E^p u(x,t) \le C \int\limits_{\mathbb{R}^n} \|f(x)\|_E^p v(x)dx, \qquad i = 1, \dots, n. \tag{3.8}$$

3.9 THEOREM. *Let X be a Banach lattice with the* H.L. *property, $d\mu$ be a Carleson measure in $\mathbb{R}^{n+1}_+$ and v be a weight in $\mathbb{R}^n$. Then v belongs to $D_{p,X}$ (see 2.14) with $1 < p < \infty$, if and only if there exists u in $\mathbb{R}^{n+1}_+$ such that*

$$\int\limits_{\mathbb{R}^{n+1}_+} \|\tilde{\mathcal{M}}_J f(x,t)\|_X^p u(x,t)d\mu(x,t) \le C \int\limits_{\mathbb{R}^n} \|f(x)\|_X^p v(x)dx \tag{3.10}$$

where the constant C is independent of the set J.

PROOF OF 3.1. We first observe that for every $(x,t) \in \mathbb{R}^{n+1}_+$ we have

$$\|\tilde{P} f(x,t)\|_E \le P(\|f\|_E)(x,t)$$

and

$$\|\tilde{T}_\gamma f(x,t)\|_E \le T_\gamma(\|f\|)_E(x,t).$$

Therefore it is enough to prove the result for the special case $E = \mathbb{R}$ and $f \ge 0$.

In order to see that condition D_p^* is sufficient we observe that, by Remark 1.1 there exists a weight u satisfying

$$\int_{\mathbb{R}^{n+1}} \mathcal{M}_0 f(x,t)^p u(x,t) d\mu(x,t) \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x) dx$$

and then (3.2) follows since

$$Pf(x,t) \leq \mathcal{M}_0 f(x,t), \qquad (x,t) \in \mathbb{R}^{n+1}_+.$$

Now we shall show that condition D_p^γ is sufficient in (3.3). We assume first that $\frac{1}{p} - \frac{\gamma}{n} > 0$. Without loss of generality we may assume that $v^{-p'/p}$ is not equal to zero a.e. on $B(0,1)$. Given a function f we write $f = f_1 + f_2$, where $f_1 = f\chi_{B(0,2)}$. To deal with f_1, remark that

$$\int_{B(0,1)} v(y)^{-p'/p} dy \leq C \int_{\mathbb{R}^n} \frac{v(y)^{-p'/p}}{(1+|y|)^{(n-\gamma)p'}} \leq C.$$

Therefore as $\mathcal{M}_0$ maps $L^1(\mathbb{R}^n, dx)$ into weak-$L^1(\mathbb{R}^{n+1}_+, d\mu)$, see [F-S], we have that

$$\mathcal{M}_0(v^{-p'/p}\chi_{B(0,1)})(x,t) < +\infty \text{ a.e. in } (x,t)$$

and then we are in the hypothesis of Lemma 2.1. Therefore, if u is defined as

$$u^{p/q} = \mathcal{M}_0(v^{-p'/p}\chi_{B_1})^{-\beta'}\chi_{\hat{B}_1}$$

we get by (2.3)

$$\left(\int_{\hat{B}_1} |T_\gamma f_1(x,t)|^q u(x,t) d\mu(x,t)\right)^{1/q} \leq C_{\gamma,p,q} \left(\int_{B_1} |f_1(x)|^p v(x) dx\right)^{1/p}$$
$$\leq C_{\gamma,p,q} \left(\int_{\mathbb{R}^n} |f(x)|^p v(x) dx\right)^{1/p}.$$

In order to consider f_2 we apply Lemma 2.16 and get

$$\left(\int_{\hat{B}_1} |T_\gamma f_2(x,t)|^q u(x,t) d\mu(x,t) \right)^{1/q}$$

$$\leq C_{\gamma,p,q} \left(\int_{\mathbb{R}^n} |f_2(x)|^p v(x) dx \right)^{1/p} \left(\int_{\hat{B}_1} u(x,t) d\mu(x,t) \right)^{1/q}$$

$$\leq C_{\gamma,p,q} \left(\int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{1/p}$$

where for the last inequality we have used the hypotheses that u is a bounded function and that μ is a Carleson measure.

Suppose now that $\frac{1}{p} - \frac{\gamma}{n} \leq 0$. As before we write $f = f_1 + f_2$ where $f_1 = f\chi_{B(0,2)}$.

To deal with f_1 we first remark that there always exists δ with $0 < \delta < n$, such that $\frac{1}{q} \geq \frac{1}{p} - \frac{\delta}{n} > 0$. It turns out that $\delta < \gamma$; then $D_p^\gamma \subset D_p^\delta$ and hence we can apply the previous case obtaining a weight u supported in $\hat{B}_1 = \hat{B}(0,1)$ and satisfying

$$\left(\int_{\hat{B}_1} |T_\delta f_1|^q(x,t) u(x,t) d\mu(x,t) \right)^{1/q} \leq C \left(\int |f_1(x)|^p v(x) dx \right)^{1/p}.$$

Then the desired estimate for $T_\gamma f_1$ follows from inequality

$$\chi_{\hat{B}_1}(x,t) T_\gamma f_1(x,t) \leq C T_\delta f_1(x,t).$$

The estimate for f_2 is proved using the same method as in the previous case.

We finally show that condition D_p^γ is necessary in (3.3).

Given $(z,t) \in \hat{B}(0,1)$ we have

$$T_\gamma f(z,t) = \int_{\mathbb{R}^n} \frac{f(y)}{(|z-y|+t)^{n-\gamma}} dy \geq \int_{\mathbb{R}^n} \frac{f(y)}{(|y|+1)^{n-\gamma}} dy.$$

Therefore, using the hypothesis, we have

$$\int_{\mathbb{R}^n} \frac{f(y)}{(|y|+1)^{n-\gamma}} dy \left(\int_{\hat{B}(0,1)} u(z,t) d\mu(z,t) \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(y)|^p v(y) dy \right)^{1/p}.$$

Now, if we take $f(y) = g(y)v^{-1/p}(y)$, we can conclude that

$$\int_{\mathbb{R}^n} g(y)\frac{v^{-1/p}(y)}{(|y|+1)^{n-\gamma}}\,dy \leq C\left(\int_{\mathbb{R}^n} g^p(y)dy\right)^{1/p}$$

and this implies that

$$\int_{\mathbb{R}^n} \frac{v^{-p'/p}(y)}{(|y|+1)^{(n-\gamma)p'}}\,dy \leq C.$$

This concludes the proof of Theorem 3.1. □

PROOF OF 3.5. Assume that $v \in D^{\gamma}_{p,X}$ and suppose that $\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n} > 0$. Given a function f, write $f = f_1 + f_2$, where $f_1 = f\chi_{B(0,2)}$, and remark that

$$\int_{B(0,1)} v^{-p'/p}(x)dx = \int_{\mathbb{R}^n} \|a\varphi_{\gamma,L}(x)\|^{p'}_{X^*}v(x)^{-p'/p}dx \leq C$$

where $\varphi_{\gamma,L}(x,\omega) = \dfrac{\chi_{B(0,1)}(x)}{|B(0,1)|^{1-\gamma/n}}$ and $\|a\|_{X^*} = 1$. Therefore, as in the proof of (3.3), since $\mathcal{M}_0$ maps $L^1(\mathbb{R}^n, dx)$ into weak-$L^1(\mathbb{R}^{n+1}_+, d\mu)$, we have that $\mathcal{M}_0(v^{-p'/p}\chi_{B(0,1)})(x,t) < +\infty$ a.e. in (x,t) and then we are in the hypothesis of Lemma 2.9. Therefore, if u is defined as

$$u^{p/q} = \mathcal{M}_0(v^{-p'/p}\chi_{B_1})^{-\beta'}\chi_{\hat{B}_1},$$

we get

$$\left(\int_{\hat{B}_1} \|\tilde{\mathcal{M}}_{\gamma,J}f_1(x,t)\|^q_X u(x,t)d\mu(x,t)\right)^{1/q} \leq C_{\gamma,p,q}\left(\int_{B_1} \|f_1(x)\|^p_X v(x)dx\right)^{1/p}$$

$$\leq C_{\gamma,p,q}\left(\int \|f(x)\|^p_X v(x)dx\right)^{1/p}.$$

In order to consider f_2 we apply Lemma 2.15 and get

$$\left(\int_{\hat{B}_1} \|\tilde{\mathcal{M}}_{\gamma,J}f_2(x,t)(x,t)\|^q_X u(x,t)d\mu(x,t)\right)^{1/q}$$

$$\leq C_{\gamma,p,q}\|f\|_{L^p_X(v)}\cdot\left(\int_{\hat{B}_1} u(x,t)d\mu(x,t)\right)^{1/q}$$

$$\leq C_{\gamma,p,q}\|f\|_{L^p_X(v)};$$

for the last inequality we have used the hypotheses that u is a bounded function and that μ is a Carleson measure.

If $\frac{1}{p} - \frac{\gamma}{n} \leq 0$ we continue as in the corresponding proof of (3.3).

In order to see that condition $D^{\gamma}_{p,X}$ is necessary we shall prove that (3.6) implies (ii) of 2.15. We first observe that if f is a X-valued locally-integrable function with support in the complement of a ball $2B = B(0, 2R)$, then for any (x, t), $(z, t) \in \hat{B} = \hat{B}(0, R)$, we have

$$\tilde{\mathcal{M}}_{\gamma,J} f(x, t) \leq C \tilde{\mathcal{M}}_{\gamma,L} f(0, R) \leq C' \mathcal{M}_{\gamma,K} f(z, t)$$

where the sets L and K are defined by

$$L = \{2r : r \in J, r \geq R\}, \;\; K = \{3r : r \in J, r \geq R\}.$$

Then if u is the weight whose existence is given by the hypothesis we have

$$\sup_{(x,t)\in\hat{B}} \|\tilde{\mathcal{M}}_{\gamma,J} f(x,t)\|_X \leq C \|\tilde{\mathcal{M}}_{\gamma,L} f(0,R)\|_X$$

$$= C \left(\int_{\hat{B}} u(x,t) d\mu(x,t) \right)^{-1} \left(\int_{\hat{B}} \|\tilde{\mathcal{M}}_{\gamma,L} f(0,R)\|_X^q u(x,t) d\mu(x,t) \right)^{1/q}$$

$$\leq C \left(\int_{\hat{B}(0,1)} u(x,t) d\mu(x,t) \right)^{-1} \left(\int_{\mathbb{R}^n} \|\tilde{\mathcal{M}}_{\gamma,K} f(x,t)\|_X^q u(x,t) d\mu(x,t) \right)^{1/q}$$

$$\leq C \|f\|_{L^q_X(v)}.$$

□

PROOF OF 3.9. The necessity of condition $D_{p,X}$ can be proved as in Theorem 3.5. In order to see that condition $D_{p,X}$ is sufficient, we shall apply Lemma 2.12. We consider the following sequence of subsets of $\mathbb{R}^{n+1}_+$:

$$S_0 = \{(x,t) \in \mathbb{R}^{n+1}_+ : |x| + t \leq 1\},$$
$$S_k = \{(x,t) \in \mathbb{R}^{n+1}_+ : 2^{k-1} \leq |x| + t < 2^k\} \qquad k = 1, 2, \ldots.$$

Given $k \geq 0$ we decompose each function f as $f' + f''$, where $f' = f\chi_{B_{k+1}}$ and $B_k = \{x : |x| < 2^k\}$.

Using Lemma 2.15 we have

$$\sup_{(x,t)\in S_k} \|\tilde{\mathcal{M}}_J f''(x,t)\| \leq C \|f''\|_{L^p_X(v)}$$

with the constant C independent of J. Thus

$$\sup_{(x,t)\in S_k}\left(\sum_j \|\tilde{\mathcal{M}}_J f_j''(x,t)\|_X^p\right)^{1/p} \le C\left(\sum_j \|f''\|_{L_X^p(v)}^p\right)^{1/p}.$$

Therefore, the vector-valued inequality of the hypothesis of Lemma 2.12 for the functions (f_j'') follows immediately with $G = L_X^p(v)$, $F = X$ and $A_k = S_k$.

On the other hand we use Cotlar's inequality (see V.2.8 in [GC-R de F]), and (1.11) to obtain for $0 < s < 1$

$$\left\|\left(\sum_j \|\tilde{\mathcal{M}}_J f_j'\|_X^p\right)^{1/p}\right\|_{L^s(S_k,d\mu)}$$

$$\le C_s\mu(S_k)^{1/s-1}\sup_{\lambda>0}\lambda\mu\left(\left\{(x,t): \sum_j \|\tilde{\mathcal{M}}_J f_k'(x,t)\|_X^p > \lambda^p\right\}\right)$$

$$\le C_s|S_k|^{1/s-1}\int_{|x|<2^{k+1}}\left(\sum_j \|f_j(x)\|_X^p\right)^{1/p}dx$$

$$\le C_s|S_k|^{1/s}\left(\int_{\mathbb{R}^n}\sum_j \|f_j(x)\|_X^p v(x)dx\right)^{1/p}\left(\frac{1}{|B_k|^{p'}}\int_{|x|<2^{k+1}} v(x)^{-p'/p}dx\right)^{1/p'}$$

$$\le C_s|S_k|^{1/s}\left(\sum_j \|f_j\|_G^p\right)^{1/p};$$

for the last inequality we have used the fact that

$$\frac{1}{|B_k|^{p'}}\int_{B_k} v^{-p'/p} = \int_{\mathbb{R}^n}\|a\varphi_L(x)\|_{X^*}^{p'} v(x)^{-p'/p}dx \le C$$

with $\varphi_L(x,\omega) = \dfrac{\chi_{B_k}(x)}{|B_k|}$ and $\|a\|_{X^*} = 1$.

Now a direct application of Lemma 2.12 shows the existence of a function u satisfying (3.10). □

PROOF OF 3.7. Sufficiency of condition D_p can be established as in Theorem 3.7.

For necessity, we remark that if $d\mu(x,t) = dx \otimes \delta_0(t)$ then inequality (3.8) becomes

$$\int_{\mathbb{R}^n} \|R_i f(x)\|_E^p u(x)dx \leq C \int_{\mathbb{R}^n} \|f(x)\|_E^p v(x)dx$$

and in this case it is known (see [GC-R de F], p. 561 for a proof) that $v \in D_p$. For a general Carleson measure μ the mentioned proof can be easily adapted. □

3.11 REMARK. Given a Carleson measure $d\mu$ in $\mathbb{R}^{n+1}_+$ we define for $1 < p < \infty$ and $0 \leq \gamma < n$ the following classes of weights in $\mathbb{R}^{n+1}_+$:

$$D_p^{*\gamma}(d\mu) = \left\{ v : \sup_{R \geq 1} R^{(\gamma-n)p'} \int_{|x|+t \leq R} v(x,t)^{-p'/p} d\mu(x,t) < +\infty \right\},$$

$$D_p^{\gamma}(d\mu) = \left\{ v : \int_{\mathbb{R}^{n+1}_+} v(x,t)^{-p'/p} (|x| + t + 1)^{(\gamma-n)p'} d\mu(x,t) < +\infty \right\}.$$

Then we have the following result.

3.12 THEOREM. *Let $d\mu$ be a Carleson measure in $\mathbb{R}^{n+1}_+$ and v be a weight in $\mathbb{R}^{n+1}_+$.*

(3.13) *v belongs to $D_p^*(d\mu)$ with $1 < p < \infty$ if and only if there exists a weight u in $\mathbb{R}^n$ such that for any Banach space E*

$$\int_{\mathbb{R}^{n+1}_+} \|\tilde{P}^* g(x)\|_E^p u(x)dx \leq C \int_{\mathbb{R}^{n+1}_+} \|g(x,t)\|_E^p v(x,t) d\mu(x,t)$$

where $\tilde{P}^*$ is the vector-valued extension of the positive operator

$$P^* g(x) = \int_{\mathbb{R}^{n+1}_+} \frac{g(y,t) d\mu(y,t)}{(|x-y|^2 + t^2)^{\frac{n+1}{2}}}.$$

(3.14) *v belongs to $D_p^{\gamma}(d\mu)$ with $1 < p < \infty$ and $0 < \gamma < n$ if and only if there exists a weight u in $\mathbb{R}^n$ such that for any Banach E*

$$\left(\int_{\mathbb{R}^n} \|\tilde{T}_\gamma^* g(x)\|_E^q u(x)dx \right)^{1/q} \leq C_\gamma \left(\int_{\mathbb{R}^{n+1}_+} \|g(x,t)\|_E^p v(x,t) d\mu(x,t) \right)^{1/p},$$

$$\frac{1}{q} \geq \frac{1}{p} - \frac{\gamma}{n}, \qquad 1 < q < \infty.$$

(3.15) *Let $0 \leq \gamma < n$, E and F be Banach spaces and suppose that $v \in D_p^{\gamma}(d\mu)$. Assume that τ_γ^* is a bounded sublinear operator from $L^p_E(\mathbb{R}^{n+1}_+, d\mu)$ into*

$L^q_F(\mathbb{R}^n, dx)$ such that

$$\|\tau^*_\gamma g(x)\|_F \le C \int\limits_{\mathbb{R}^{n+1}_+} \frac{\|g(y,t)\|_E d\mu(y,t)}{(|x-y|+t)^{n-\gamma}}.$$

Then there exists a weight u in $\mathbb{R}^n$ such that

$$\left(\int\limits_{\mathbb{R}^n} \|\tau^*_\gamma g(x)\|^q_F u(x)dx\right)^{1/q} \le C_\gamma \left(\int\limits_{\mathbb{R}^{n+1}_+} \|g(x,t)\|^p_E v(x,t)d\mu(x,t)\right)^{1/p}.$$

(3.16) *Let E be a Banach space with the U.M.D. property. The weight v belongs to $D_p(d\mu)$ with $1 < p < \infty$ if and only if there exists u in $\mathbb{R}^n$ such that*

$$\int\limits_{\mathbb{R}^n} \|\tilde{Q}^*_i g(x)\|^p_E u(x)dx \le C \int\limits_{\mathbb{R}^{n+1}_+} \|g(x,t)\|^p_E v(x,t)d\mu(x,t)$$

where $\tilde{Q}^*_i$ is the vector-valued extension of the operator

$$\tilde{Q}^*_i g(x) = \int\limits_{\mathbb{R}^{n+1}_+} \frac{y_i - x_i}{(|y-x|^2+t^2)^{\frac{n+1}{2}}}\, g(y,t)d\mu(y,t).$$

PROOF. Technical results analogous to Proposition 1.7 and Lemmas 2.1, 2.9 and 2.16 hold also for the operators $\tilde{P}^*$, $\tilde{Q}^*_i$, $\tilde{T}^*_\gamma$ and τ^*_γ, and this theorem is therefore established following the lines of the proofs of Theorems 3.1, 3.4 and 3.7. □

4. - Duality of the two-weight problem

Let $(Y, d\mu)$ and $(Z, d\nu)$ be measure spaces, E and F be Banach spaces and assume that T is a linear operator such that for some pair of exponents (p,q), $1 < p\ q < \infty$, T is bounded from $L^p_E(d\nu)$ into $L^q_F(d\mu)$ and there exists a pair of weights (u,v) such that

$$\left(\int\limits_Y \|Tf(y)\|^q_F u(y)d\mu(y)\right)^{1/q} \le C \left(\int\limits_Z \|f(z)\|^p_E v(z)d\nu(z)\right)^{1/p}. \tag{4.1}$$

Then can consider the traspose operator T^* from $L^{q'}_{F^*}(d\mu)$ into $(L^p_E(d\nu))^*$ defined by

$$\langle T^*g, f\rangle = \int\limits_Y \langle Tf(y), g(y)\rangle\, d\mu(y).$$

If we know that T^*g is a E^*-valued function then we actually have that

T^* maps $L^{q'}_{F^*}(d\mu)$ into $L^{p'}_{E^*}(d\nu)$ and moreover the following inequality holds:

$$(4.2)\qquad \left(\int\limits_Z \|T^*g(z)\|^{p'}_{E^*} v^{1-p'}(z)d\nu(z)\right)^{1/p'} \le C\left(\int\limits_Y \|g(y)\|^{q'}_{F^*} u(y)^{1-q'} d\mu(y)\right)^{1/q'}.$$

In this case we also have that the pair (u, v) satisfies (4.1) if and only if the pair $(v^{1-p'}, u^{1-q'})$ satisfies (4.2).

In other words finding necessary and sufficient conditions for a weigth u to satisfy inequality (4.1) is equivalent to finding necessary and sufficient conditions for a weight $u^{1-q'}$ to satisfy (4.2).

4.3 DEFINITION. *Let $d\mu$ be a Carleson measure on $\mathbb{R}^{n+1}_+$. We shall say that a weight u in $\mathbb{R}^{n+1}_+$ satisfies condition $Z^\gamma_p(d\mu)$ with $1 < p < \infty$ and $0 \le \gamma < n$, if and only if*

$$\int\limits_{\mathbb{R}^{n+1}_+} \frac{u(x,t)}{(|x|+t+1)^{(n-\gamma)p}}\, d\mu(x,t) \le C < +\infty.$$

4.4 THEOREM. *Let E be a Banach space, $d\mu$ be a Carleson measure in $\mathbb{R}^{n+1}_+$ and u be a weight in $\mathbb{R}^{n+1}_+$. The following conditions are equivalent for $1 < p < \infty$:*

(i) $u^{1-p'} \in D^*_{p'}(d\mu)$;

(ii) *There exists a weight v in $\mathbb{R}^n$ such that*

$$\int\limits_{\mathbb{R}^{n+1}_+} \|\tilde{P}f(x,t)\|^p_E u(x,t) \le C \int\limits_{\mathbb{R}^n} \|f(x)\|^p_E v(x)dx.$$

4.5 THEOREM. *Let $d\mu$ a be Carleson measure in $\mathbb{R}^{n+1}_+$ and u be a weight in $\mathbb{R}^{n+1}_+$. Assume $1 < p < \infty$. The following conditions are equivalent:*

(i) $u \in Z_p(d\mu)$;

(ii) *For any U.M.D. Banach space E there exists a weight v such that*

$$\int\limits_{\mathbb{R}^{n+1}_+} \|\tilde{Q}_i f(x,t)\|^p_E u(x,t)d\mu(x,t) \le C_p \int\limits_{\mathbb{R}^n} \|f(x)\|^p_E v(x)dx, \qquad i = 1, 2, \ldots, n;$$

(iii) *For any Banach lattice X with the* H.L. *property there exists a weight v such that*

$$\int\limits_{\mathbb{R}^{n+1}_+} \|\tilde{\mathcal{M}}_{0,L} f(x,t)\|^p u(x,t)d\mu(x,t) \le C \int\limits_{\mathbb{R}^n} \|f(x)\|^p v(x)dx$$

with the constant C independent of the finite set L.

4.6 THEOREM. *Let $d\mu$ be a Carleson measure in $\mathbb{R}^{n+1}_+$ and u be a weight in $\mathbb{R}^{n+1}_+$. Assume $0 < \gamma < n$, $\dfrac{1}{q} \ge \dfrac{1}{p} - \dfrac{\gamma}{n}$ and $1 < q < \infty$. The following conditions* *are equivalent:*

(i) $u \in Z_p^\gamma(d\mu)$;

(ii) *For any Banach space E there exists a weight v such that*

$$\left(\int_{\mathbb{R}^{n+1}_+} \|\tilde{T}_\gamma f(x,t)\|_E^q u(x,t)d\mu(x,t)\right)^{1/q} \leq C_\gamma \left(\int_{\mathbb{R}^n} \|f(x)_E^p v(x)dx\right)^{1/p};$$

(iii) *For any Banach lattice X there exist a weight v such that*

$$\left(\int_{\mathbb{R}^{n+1}_+} \|\tilde{\mathcal{M}}_{\gamma,L} f(x,t)\|_X^q u(x,t)d\mu(x,t)\right)^{1/q} \leq C_\gamma \left(\int_{\mathbb{R}^n} \|f(x)_E^p\|v(x)dx\right)^{1/p}$$

with the constant C independent of the finite set L.

PROOF OF 4.4, 4.5 AND 4.6. The results for the operators $\tilde{P}$, $\tilde{T}_\gamma$ and $\tilde{Q}_i$ follow directly using duality and (3.13), (3.14) and (3.16).

For the operator $\tilde{\mathcal{M}}_{\gamma,L}$ we consider the $\ell^\infty(X)$-valued linear operator

$$\tilde{T}_{\gamma,L} f(x,t) = \left\{\frac{1}{r^{n-\gamma}} \int_{B(x,r)} f(y)dy\right\}_{r\in L}.$$

We have

$$\tilde{\mathcal{M}}_{\gamma,L} f(x,t) = |\tilde{T}_{\gamma,L} f(x,t)|,$$

where $|\cdot|$ denotes the absolute value in the lattice

$$\ell^\infty(X) = \{(x_n) \in X : \|(x_n)\| = \|\sup|x_n|\,\|_X\},$$

and

$$\|\tilde{\mathcal{M}}_{\gamma,L} f(x,t)\|_X = \|\tilde{T}_{\gamma,L} f(x,t)\|_{\ell^\infty(X)}.$$

Therefore statement (iii) is equivalent to the following:

There exists a weight v such that

$$\left(\int_{\mathbb{R}^{n+1}_+} \|\tilde{T}_{\gamma,L} f(x,t)\|_{\ell^\infty(X)}^q u(x,t)d\mu(x,t)\right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} \|f(x)\|_X^p v(x)dx\right)^{1/p}.$$

But this fact can be obtained in a standard way using duality and Theorem 3.5. This completes the proof of Theorems 4.4, 4.5 and 4.6. □

5. - Properties of the classes $D^{\gamma}_{p,X}$

5.1 PROPOSITION. *Let X be a Köthe function space with X' norming and $1 < p < \infty$. The following relations hold:*

(i) $D^{\gamma}_p \subset D^{\delta}_p \subset D^0_p$ *for* $0 < \delta < \gamma$.

(ii) $D^{\gamma}_p \subset D^{\gamma}_{p,X} \subset D^{*,\gamma}_p$ *for* $0 < \gamma < n$.

PROOF. (i) is obvious.

In order to show (ii) we remark that for every ball $B = B(0,r)$ (with $r \geq 1$)

$$\frac{\chi_B(x)}{|B|^{1-\gamma/n}} \leq \frac{C}{(1+|x|)^{n-\gamma}};$$

then if $a \in X'$ and $\varphi_{\gamma,L}$ is a function of the type defined in 2.13 we have

$$a(\omega)\varphi_{\gamma,L}(x,\omega) \leq \frac{Ca(\omega)}{(1+|x|)^{n-\gamma}}.$$

Therefore $\|a\varphi_{\gamma,L}(x)\|_{X^*} \leq C\|a\|_{X^*} \dfrac{1}{(1+|x|)^{n-\gamma}}$ which implies that $D^{\gamma}_p \subset D^{\gamma}_{p,X}$.

Now we shall see that $D^{\gamma}_{p,X} \subset D^{*,\gamma}_p$. Given a ball $B(0,R)$ with $R \geq 1$ we consider

$$\varphi_{\gamma,L}(x,\omega) = \frac{\chi_B(x)}{|B|^{1-\gamma/n}}$$

where $L = \{R\}$; then for $a \in X'$ with $\|a\|_{X^*} = 1$, we have

$$\|a\varphi(x)\|_{X^*} = \frac{\chi_B(x)}{|B|^{1-\gamma/n}}.$$

Therefore

$$\frac{1}{R^{(n-\gamma)p'}} \int_{|x|\leq R} v(x)^{-p'/p}dx = C\frac{1}{|B|^{(1-\gamma/n)p'}} \int_B v(x)^{-p'/p}dx$$

$$= C\int_{\mathbb{R}^n} \|a\varphi(x)\|^{p'}_{X^*} v(x)^{-p'/p}dx \leq C'. \qquad \square$$

In case $X = \ell^r$ with $1 \leq r \leq \infty$ we denote D^{γ}_{p,ℓ^r} by $D^{\gamma}_{p,r}$. For this case we have the following result.

5.2 COROLLARY. *Assume $1 < p < \infty$ and $0 \leq \gamma < n$. Then $D^{\gamma}_{p,1} = D^{\gamma}_p$ and $D^{\gamma}_{p,\infty} = D^{*\gamma}_p$.*

PROOF. It is clear that $v \in D^{\gamma}_{p,r}$ with $1 < r \leq \infty$ if and only if

$$\int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} \left| a_j \frac{\chi_{B_{r_j}}(x)}{r_j^{n-\gamma}} \right|^{r'}\right)^{p'/r'} v(y)^{-p'/p}dy \leq C$$

for any sequence $(a_j) \in \ell^{r'}$ with $\|(a_j)\|_{\ell^{r'}} \leq 1$ and any sequence of balls (B_{r_j}) centered at the origin with rational radii r_j bigger than one.

If $v \in D_p^*$ using Minkowski's inequality, we have

$$\int_{\mathbb{R}^n} \left(\sum_j \left| a_j \frac{\chi_{B_{r_j}}(x)}{r_j^{n-\gamma}} \right| \right)^{p'} v(x)^{-p'/p} dx$$

$$\leq \left(\sum_j \left(\int_{\mathbb{R}^n} \frac{\chi_{B_{r_j}}(x)}{|r_j|^{(n-\gamma)p'}} v(x)^{-p'/p} dx \right)^{1/p'} a_j \right)^{p'}$$

$$\leq C \left(\sum_j a_j \right)^{p'} \leq C'.$$

Therefore $D_p^{*,\gamma} \subset D_{p,\infty}^{\gamma}$.

On the other hand $v \in D_{p,1}^{\gamma}$ if and only if

$$\int_{\mathbb{R}^n} \sup_j \left| a_j \frac{\chi_{B_{r_j}}(y)}{r_j^{n-\gamma}} \right|^{p'} v(y)^{-p'/p} dy \leq C$$

for any $(a_j) \in \ell^{\infty}$ with $\|(a_j)\|_{\ell^{\infty}} \leq 1$ and any sequence of balls (B_{r_j}) centered at the origin with rational radii (r_j) bigger than one.

Now if $v \in D_{p,1}^{\gamma}$ and if we denote by B_k the ball centered at the origin with radius 2^k for $k = 0, 1, 2, \ldots$ then

$$\int_{\mathbb{R}^n} \frac{v(x)^{-p'/p}}{(1+|x|)^{(n-\gamma)p'}} dx = \int_{B_0} \frac{v(x)^{-p'/p} dx}{(1+|x|)^{(n-\gamma)p'}} + \sum_{k=1}^{\infty} \int_{B_k \setminus B_{k-1}} \frac{v(x)^{-p'/p}}{(1+|x|)^{(n-\gamma)p'}} dx$$

$$\leq C \left(\int_{B_0} v(x)^{-p'/p} dx + \sum_{k=1}^{\infty} \int_{B_k} \frac{v(x)^{-p'/p}}{2^{k(n-\gamma)p'}} dx \right)$$

$$\leq C \int_{\mathbb{R}^n} \sup_k \frac{\chi_{B_k}(x)}{2^{k(n-\gamma)p'}} v(x)^{-p'/p} dx \leq C'.$$

Therefore $D_{p,1}^{\gamma} \subset D_p^{\gamma}$. □

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