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Hausdorff combing of groups and π_1^∞ for universal covering spaces of closed 3-manifolds

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Hausdorff Combing of Groups and π_1^∞ for Universal Covering Spaces of Closed 3-manifolds

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1. - Introduction

This paper is part of a general program of connecting the Gromov geometry of the fundamental group of a closed 3-manifold M^3 with the simple connectivity at infinity of its universal covering space. Before we can state our main theorem we have to recall a number facts about combing of groups. We consider a finitely generated group G and a specific finite set of generators $A = A^{-1}$ for G. To this we can attach the Cayley graph $\Gamma = \Gamma(G, A)$. For each $g \in G$ we will denote by $\|g\|$ the minimal length of a word with letters in A expressing g. We also define $d(g, h) = \|g^{-1}h\| = \|h^{-1}g\|$. With respect to this distance, the action of G on Γ is an isometry; path-lengths will be computed with respect to this distance and geodesics will be defined accordingly.

For any positive integer n we can consider the ball of radius n in Γ:

$$B(n) \underset{\text{def}}{=} \{x \in \Gamma : \|x\| \leq n\} \tag{1.1}$$

and the sphere of radius n in Γ:

$$S(n) \underset{\text{def}}{=} \{x \in \Gamma : \|x\| = n\}. \tag{1.2}$$

For any finitely generated group G with a given system of generators $A^{-1} = A$, a *combing* of G is, by definition, a choice for each $g \in G$ of a continuous (not necessarily geodesic) path of $\Gamma(G, A)$ joining 1 to g. It will be convenient to think of this path as a function $\mathbb{Z}_+ \xrightarrow{s_g} G$ such that $s_g(0) = 1$, $d(s_g(t), s_g(t+1)) \leq 1$ and, for all sufficiently large t, $s_g(t) = g$.

Abstracting from the properties of automatic groups defined in [CEHPT], W. Thurston calls a combing *quasi-Lipschitz* if there are constants C_1, C_2 such that, for all $g, h \in G$ and $t \in \mathbb{Z}_+$, we have:

$$d(s_g(t), s_h(t)) \leq C_1 d(g, h) + C_2. \tag{1.3}$$

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The quasi-Lipschitz estimate (1.3) involves the uniform distance between the two paths s_g and s_h, and it is not *a priori* obvious how this kind of estimate behaves under change of system of generators. For this reason we introduce here a more general concept which we call a *Hausdorff combing* of G. Again we consider a combing $g \longmapsto s_g$ as above but now it will be more convenient to think of s_g as being a continuous polygonal path $[0,1] \xrightarrow{s_g} \Gamma(G, A)$ with $s_g(0) = 1 \in G$ and $s_g(1) = g$. We will say that the combing is *Hausdorff* if there exist constants K_1, K_2 such that for any $g, h \in G$, there exists an orientation-preserving homeomorphism $[0,1] \xrightarrow{u=u_{g,h}} [0,1]$ such that, for all $t \in [0,1]$, we have

$$d(s_g(u(t)), s_h(t)) \leq K_1 d(g,h) + K_2. \tag{1.4}$$

One should notice that any quasi-Lipschitz combing is automatically Hausdorff but the converse is not true, which means that the concept of Hausdorff combing is more general that the concept of quasi-Lipschitz combing. In Section 3 of this paper it will be shown that the existence of a Hausdorff combing for a group G is an invariant notion, i.e. it is independent of the specific choice of generators. We can state now our main result.

THEOREM. *Let M^3 be a closed 3-manifold which is such that $\pi_1 M^3$ has the following two properties:*

I) *It admits a Hausdorff combing.*

II) *There exists a system of generators $B = B^{-1}$ and three constants C_3, C_4, and $\varepsilon > 0$, such that the following happens: in the Cayley graph $\Gamma = \Gamma(\pi_1 M^3, B)$ consider $S(n) \subset B(n) \subset \Gamma$; for any $x, y \in S(n)$ with $d(x,y) \leq 3$ consider a path $\gamma = \gamma(x,y) \subset B(n)$ of minimal length joining x to y; then*

$$\text{length}(\gamma) \leq C_3 n^{1-\varepsilon} + C_4. \tag{1.5}$$

Under these two conditions, for any compact subset $K \subset \tilde{M}^3$, we can find a simply-connected compact 3-dimensional submanifold $U^3 \subset \tilde{M}^3$ such that $K \subset U^3$, i.e. $\pi_1^\infty \tilde{M}^3 = 0$.

IMPORTANT REMARKS. A) Condition II is a very mild restriction since on one hand ε is allowed to be arbitrarily close to zero and on the other hand, for an arbitrary group G and an arbitrary $B = B^{-1}$ we *always* have

$$\text{length}(\gamma) \leq 2n, \tag{1.6}$$

this estimate being true for an arbitrary pair $x, y \in S(n)$.

B) We can replace $n^{1-\varepsilon}$ in (1.5) by any function $f(n)$ which is such that for any constant C, we have $\lim\limits_{n=\infty}(n - Cf(n)) = \infty$.

Our Theorem is a partial result concerning the very well-known conjecture in 3-dimensional topology which says that for *any* closed 3-manifold with infinite fundamental groups, one has $\pi_1^\infty \tilde{M}^3 = 0$. In contrast with this conjecture, M. Davis has shown that, in any dimension $n \geq 4$, there are closed manifolds M^n such that $M^n = K(\pi_1 M^n, 1)$ and at the same time $\pi_1^\infty M^n \neq 0$ [Da1]. At this point it should be stressed that our result is purely 3-dimensional; all hyperbolic groups in the sense of Gromov [Gr1], [Gr2], [B], [CDP] as well as the automatic groups of [CEHPT] are Hausdorff combable. But among the groups appearing in the counterexample of M. Davis some are certainly hyperbolic [Da2], [DaS] and this implies both I and II in our theorem. The above Theorem is an extension of Theorem 2 of [Po3]. For the reader's convenience, the next Section will review some background material.

2. - Some preliminaries

If $A \xrightarrow{F} B$ is any map, we will define $M_2(F) \subset A$ by

$$M_2(F) = \{x \in A \text{ such that } \operatorname{card} F^{-1}F(x) > 1\}.$$

One of the ingredients for our Theorem is the following:

DEHN-TYPE LEMMA. *Let X and Y be two simply-connected 3-manifolds. We assume X to be compact, connected, with $\partial X \neq \emptyset$ and Y to be open. We are given a commutative diagram*

$$\begin{tikzcd} K \arrow[rr, "g"] \arrow[dr, "f"'] & & \operatorname{int} X \subset X \arrow[dl, "F"] \\ & Y & \end{tikzcd}$$

where K is a compact connected set, g and f are embeddings and F is a smooth generic immersion. If the following condition is also fulfilled

$$gK \cap M_2(F) = \emptyset$$

then $fK \subset Y$ is contained in a compact simply connected smooth 3-dimensional submanifold $N \subset Y$.

This is proved in [Po2]; the argument mimics the Shapiro-Whitehead [ShWhi] (see also [Do]) approach to the Dehn-Lemma [P].

The next item will be *double point structures*. Let P be a (not necessarily locally-finite) 3-dimensional simplicial complex, M^3 a 3-manifold and $P \xrightarrow{f} M^3$ a non-degenerate simplicial map (i.e. if $\sigma \in P$ is a simplex of P, then dim $f\sigma =$ dim σ). We will denote by $\Phi(f) \subset P \times P$ the subset $\Phi(f) = M^2(f) \cup (\operatorname{diag} P)$;

in other terms $\Phi(f)$ is the equivalence relation on P defined by

$$(x,y) \in \Phi(f) \Longleftrightarrow fx = fy.$$

By definition, $\mathrm{Sing}(f) \subset P$ is the subcomplex whose points $z \in P$ are such that $f \,|\, \mathrm{Star}(z)$ is non immersive; in other words $z \in \mathrm{Sing}(f)$ if and only if there are two *distinct* simplices $\sigma_1, \sigma_2 \in P$ with $z \in \sigma_1 \cap \sigma_2$ and $f(\sigma_1) = f(\sigma_2)$. Clearly the quotient space $P/\Phi(f)$ is isomorphic to the image fP. We are also interested in equivalence relations $R \subset \Phi(f)$ which are such that if $x \in \sigma_1$, $y \in \sigma_2$, where σ_1 and σ_2 are two simplexes of P of the same dimension with $fx = fy$ and $f(\sigma_1) = f(\sigma_2)$, then

$$(x,y) \in R \Longrightarrow \{R \text{ identifies } \sigma_1 \text{ to } \sigma_2\}.$$

Such equivalence relations automatically have the property that P/R is a simplicial complex; the induced map $P/R \to M^3$ is also simplicial. Among these R's, there is a particularly interesting equivalence relation $\Psi(f) \subset \Phi(f)$, the basic features of which are summarized in the following:

LEMMA 2.1. I) *There exists an equivalence relation $\Psi(f) \subset \Phi(f)$ which is completely characterized by the following two properties.*

Ia) *If we consider the natural commutative diagram*

(2.1)
$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "\pi"'] & & M^3 \\ & P/\Psi(f) \arrow[ur, "f_1"'] & \end{tikzcd}$$

then $\mathrm{Sing}(f_1) = \emptyset$ *i.e. f_1 is an immersion.*

Ib) *There is no $R \subset \Phi(f)$, smaller than $\Psi(f)$ having this property. In other words, $\Psi(f)$ is the smallest equivalence relation, compatible with f which kills all the singularities. (We will also say that "$\Psi(f)$ is the equivalence relation which is commanded by the singularities of f".)*

II) *The canonical map $\pi_1(P) \xrightarrow{\pi_*} \pi_1(P/\Psi(f))$ is surjective. In particular, if P is simply connected, then so is $P/\Psi(f)$.*

These facts are proved in [Po1]. We will offer here only some comments about how one effectively constructs $\Psi(f)$. Let $z \in \mathrm{Sing}\, f$; we have two distinct simplices $\sigma_1, \sigma_2 \in P$ with $z \in \sigma_1 \cap \sigma_2$, $\dim \sigma_1 = \dim \sigma_2$ and $f\sigma_1 = f\sigma_2$. Consider the quotient P' of P obtained by identifying σ_1 to σ_2, and the natural diagram

(2.2)
$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "\text{folding map}"'] & & M^3 \\ & P' \arrow[ur, "f'"'] & \end{tikzcd}$$

If $\operatorname{Sing} f' \neq \emptyset$, there exists $z' \in P'$ and two distinct $\sigma'_1, \sigma'_2 \subset P'$ with $z' \in \sigma'_1 \cap \sigma'_2$ and $f'\sigma'_1 = f'\sigma'_2$. We consider the quotient P'' of P' obtained by identifying σ'_1 to σ'_2, and the natural diagram

(2.3)
$$\begin{tikzcd} P' \arrow[rr, "f'"] \arrow[dr, "\text{folding map}"'] & & M^3 \\ & P'' \arrow[ur, "f''"'] & \end{tikzcd}$$

The process continues in a similar way.

If P is a finite simplicial complex, then this process stops when we get to a $P^{(n)} \xrightarrow{f^{(n)}} M^3$ with $\operatorname{Sing}(f^{(n)}) = \emptyset$. The quotient $P^{(n)}$ is in this case $P/\Psi(f)$ and property II) is obvious for $P^{(n)} = P/\Psi(f)$.

If P is not finite, we get from (2.2), (2.3), ... an increasing sequence of equivalence relations

(2.4)
$$\rho_1 \subset \rho_2 \subset \ldots \subset \rho_n \subset \ldots \subset \Phi(f).$$

and we can consider $P^{(\omega)} \underset{\text{def}}{=} P/\bigcup_1^\infty \rho_n \xrightarrow{f^{(\omega)}} M^3$. If $\operatorname{Sing}(f^{(\omega)}) = \emptyset$, then $\Psi(f) = \bigcup_1^\infty \rho_n$ but, in general, we have to go to $P^{(\omega+1)} \xrightarrow{f^{(\omega+1)}} M^3$ and hence to a transfinite sequence continuing our (2.4)

(2.5)
$$\rho_1 \subset \rho_2 \subset \ldots \subset \rho_n \subset \ldots \subset \rho_\omega \subset \rho_{\omega+1} \subset \ldots \Phi(f).$$

The game stops when we get to the first ordinal α such that $P^{(\alpha)} \xrightarrow{f^{(\alpha)}} M^3$ is *non-singular*, and then $\rho_\alpha = \Psi(f)$. In [Po1], it is shown that:

a) this definition is intrinsic (i.e. independent of the various choices made);

b) it verifies condition I of Lemma 2.1;

c) for an appropriate choice of (2.4), we have already $\rho_\omega = \Psi(f)$ (without going to (2.5));

d) using c), one can prove condition II of Lemma 2.1 in full generality.

REMARK. In the passage from P to $P/\Phi(f)$, any kind of topological information gets lost; point II) of our lemma tells us that this is no longer the case when we move from P to $P/\Psi(f)$. We will need to apply the material about double point structures sketched before to a specific situation. This will involve a relatively unusual naive view of universal covering spaces (for closed 3-manifolds, but actually it can be extended to much more general situations).

Let us consider now a 3-manifold M^3. We can always represent M^3 as follows. We start with a polyhedral 3-ball Δ with triangulated $\partial\Delta$, containing an even number of triangles $h_1, h_2, \dots, h_{2p}$; we are given a fixed-point free involution

$$S \underset{\text{def}}{=} \{h_1, h_2, \dots, h_{2p}\} \xrightarrow{j} S,$$

and M^3 is the quotient space Δ/ρ, where the equivalence relation ρ identifies each h_s to jh_s by an appropriate linear isomorphism. We consider the free monoid $\overline{\mathcal{G}}$ generated by S and 1 and the space T obtained from the disjoint union $\sum_{x\in\overline{\mathcal{G}}} x\Delta$ by glueing, for each $x \in \overline{\mathcal{G}}$ and $h_s \in S$, the fundamental domains $x\Delta$ and $(xh_s)\Delta$ along their respective h_s and jh_s faces, in a Cayley graph manner. We do *not* restrict ourselves here to *reduced* words x, which makes T quite complicated already at the local level. There is an obvious tautological map which sends each fundamental domain $x\Delta \subset T$ *identically* onto $\Delta \to M^3$. This map just unrolls indefinitely the fundamental domain $\Delta \to M^3$, along its faces, like the developing map [Th], [SullTh]. In Section 2 of [Po1] it is proved the following:

LEMMA 2.2. *The canonical map* (*see* (2.1))

$$T/\Psi(f) \xrightarrow{f_1} M^3,$$

is the universal covering space of M^3.

Let us look now a little closer at the representation $M^3 = \Delta/\rho$. Let us choose a fundamental domain $\{\overline{g}_1, \dots, \overline{g}_p\} \subset S$ for the action of j on S. This fundamental domain induces a system of generators for $\pi_1 M^3$: we choose as base-point the center $\star \in \Delta$ and we associate to each $\overline{g}_i$ the closed loop of M^3 which, in Δ, joins the center of $(j\overline{g}_i)$ to the center of $\overline{g}_i$. Call $g_i \in \pi_1 M^3 = \pi_1(M^3, \star)$ the corresponding element. This gives a surjective morphism (in the category of semi-groups)

$$\overline{\mathcal{G}} \xrightarrow{\chi} \pi_1 M^3$$

which sends $\overline{g}_i$ to g_i and $j\overline{g}_i$ to g_i^{-1}. We use the notation $\overline{\mathcal{G}} \ni \overline{g} \mapsto g = \chi(\overline{g}) \in \pi_1 M^3$. A complete system of relations for the $\{g_i^{\pm 1}\}$ which generates $\pi_1 M^3$ can be obtained by "going around each edge of Δ".

In [Po3] the following lemma was proved:

LEMMA 2.3. *For any finite system of elements* $\{\gamma_1^{\pm 1}, \dots, \gamma_s^{\pm 1}\} \subset \pi_1 M^3$ *we can choose a representation* $\Delta/\rho = M^3$ *such that for the* $\{g_1^{\pm 1}, \dots, g_p^{\pm 1}\} \subset$

$\pi_1 M^3$ *obtained by the construction above we have* $\{\gamma_1^{\pm 1}, \ldots, \gamma_s^{\pm 1}\} \subset \{g_1^{\pm 1}, \ldots, g_p^{\pm 1}\}$.

3. - Hausdorff combing

We will think from now on of a combing as being a continuous polygonal map $[0,1] \xrightarrow{s_g} \Gamma(G, A)$ defined for each $g \in G$ and such that $s_g(0) = 1$ and $s_g(1) = g$. We remind the reader that the combing will be called *Hausdorff* if, given $g, h \in G$, we can find an orientation preserving homeomorphism $[0,1] \xrightarrow{u=u_{g,h}} [0,1]$ such that, for all $t \in [0,1]$, we have

$$d(s_g(u(t)), s_h(t)) \leq K_1 d(g,h) + K_2. \tag{3.1}$$

It will be useful to give another, equivalent, definition of a Hausdorff combing.

We recall that the term "polygonal" in the definition of a combing means that s_g consists of successive edges in the Cayley graph. We consider two paths s_g, s_h and the successive vertices of s_g and s_h, namely:

$$s_g(t_0 = 0) = 1, s_g(t_1), \ldots, s_g(t_i), \ldots, s_g(t_n = 1) = g$$

and

$$s_h(t'_0 = 0) = 1, s_h(t'_1), \ldots, s_h(t'_j), \ldots, s_h(t'_m = 1) = h.$$

DEFINITION 3.1. *A* comparison *between* s_g *and* s_h *is a set of pairs of parameters* $\{(t_i, t'_j)\}$ *such that:*

1) *each of the above* t_i*'s appears at least once;*
2) *each of the above* t'_j*'s appears at least once;*
3) *there are* no crossings, *which means that if* (t_i, t'_j) *and* (t_h, t'_k) *are part of the comparison and* $t_h > t_i$ *then we cannot have* $t'_j > t'_k$ *(see* Fig. 1).

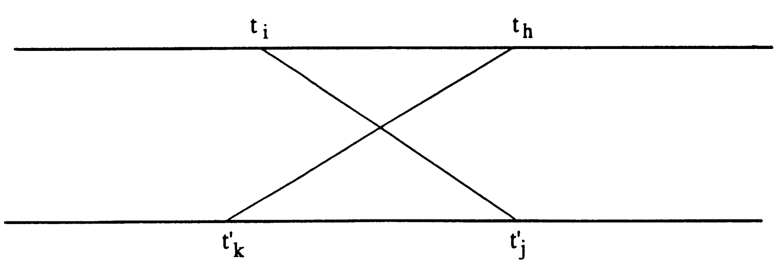

Fig. 1

A comparison will be represented by the abstract (or symbolical) picture described in Fig. 2.

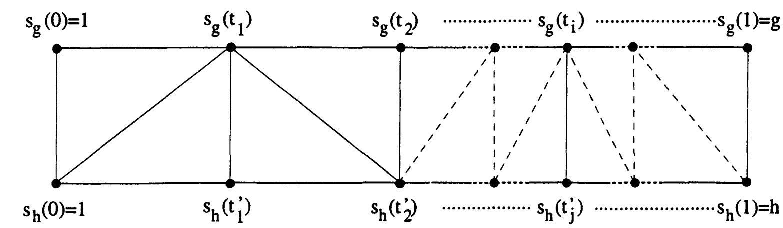

Fig. 2

In Fig. 2 we see an abstract rectangle (abstract meaning not related to the Cayley graph) where:

- all the vertices of s_g appear in order on the upper line,
- all the vertices of s_h appear in order at the lower line,
- each upper (lower) vertex is connect at least once to a lower (upper) vertex,
- there are no crossings.

Hence the whole area of our abstract rectangle is divided into consecutive congigurations which can be either small triangles looking downwards as in Fig. 3-(a), on small triangles looking upwards as in Fig. 3-(b), or small squares as in Fig. 3-(c).

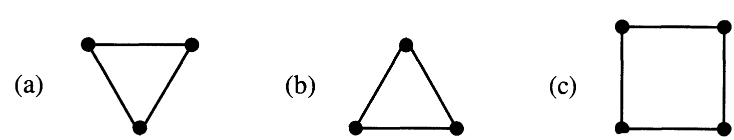

Fig. 3

The word "small" means here that the s_g or s_h vertices appear only as extreme points of our small triangles and/or squares. It should be stressed that this is a completely abstract notion and has no relation (for the time being) to the Cayley graph.

LEMMA 3.1. *We consider a group G and a finite system of generators A such that with respect to A, our group G has a Hausdorff combing $g \longmapsto s_g$.*

Then there exist constants C_1 and C_2 such that for each $g, h \in G$ there exists a comparison $\{(t_i, t'_j)\}$ between s_g and s_h satisfying the estimate

$$d(s_g(t_i), s_h(t'_j)) \leq C_1 d(g, h) + C_2. \tag{3.2}$$

REMARK. The conclusion of this lemma is actually an equivalent definition of Hausdorff combing and this is the definition which will be used from now on.

PROOF OF LEMMA 3.1. Notice that g and h play a symmetric role in (3.1). So we might consider two paths s_g, s_h and an orientation-preserving homeomorphism $[0,1] \xrightarrow{u=u_{h,g}} [0,1]$ such that for all $t \in [0,1]$, we have

$$d(s_g(t), s_h(u(t))) \leq K_1 d(g, h) + K_2$$

(with K_1 and K_2 as in Section 1). We denote by $s_g(0) = 1$, $s_g(t_1), \ldots, s_g(t_i), \ldots$, $s_g(1) = g$ and $s_h(0) = 1$, $s_h(u(t''_1))), \ldots, s_h(u(t''_j))), \ldots, s_h(1) = h$ the successive vertices of the paths s_g and $s_h \circ u$ where $t''_j = u^{-1}(t'_j)$ and hence of course $s_h(u(t''_j)) = s_h(t'_j)$. With this notation we consider the rectangle shown in Fig. 4.

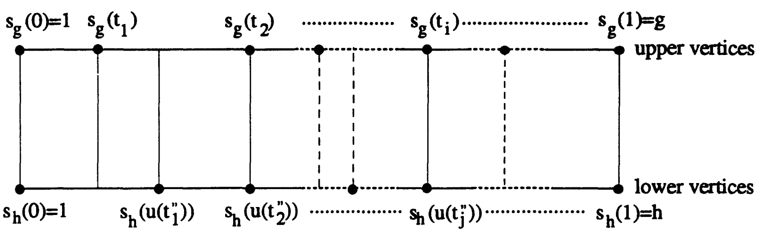

Fig. 4

We consider, in our abstract rectangle, all the vertical lines which pass through an upper or through a lower vertex (or through an upper and a lower vertex). Each vertical line either starts at an upper vertex (which we call situation I) or starts at a point which is not an upper vertex and goes to a point which is a lower vertex (which we call situation II) or starts at an upper vertex and goes to a lower vertex (which we call situation III). In very explicit terms, situation I refers to the case where $u(t_i)$ is of the form $u(t''_j) < u(t_i) < u(t''_{j+1})$, for some j. Situations II and III are defined in a similar way. We leave the vertical lines in situation III as they are, and we transform those in situations I and II as follows. Let us consider a line in situation I going from $s_g(t_i)$ to $s_h(u(t))$ (which is not a vertex) and let us consider the next lower vertex to the left of $s_h(u(t))$ as shown in Fig. 5.

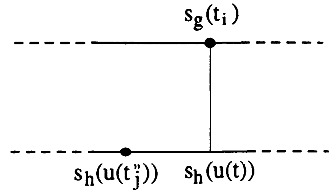

Fig. 5

What we do is to change our vertical line into a line going from $s_g(t_i)$ to $s_h(u(t''_j))$ as shown in Fig. 6.

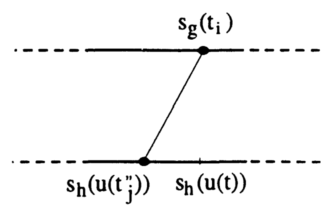

Fig. 6

Lines in situation II are treated similarly and this clearly defines a comparison between s_g and $s_h \circ u$, namely $\{(t_i, t''_j)\}$. Of course this is the same as a comparison between s_g and s_h, namely $\{(t_i, t'_j = u(t''_j)\}$. Notice now that $s_h(u(t))$ is in the Cayley graph a point belonging to the edge $(s_h(u(t''_j)), s_h(u(t''_{j+1})))$ and hence $d(s_h(u(t)), s_h(u(t''_j))) \leq 1$. If follows that the quantities $d(s_g(t_i), s_h(u(t)))$ and $d(s_g(t_i), s_h(u(t''_j)))$ differ by less than 1. This shows that the comparison $\{(t_i, t'_j)\}$ verifies an estimate

$$d(s_g(t_i), s_h(t'_j))) \leq C_1 d(g, h) + C_2$$

with $C_1 = K_1$, $C_2 = K_2 + 1$. Lemma 3.1 is proved. ∎

LEMMA 3.2. *We consider a group G and a system of generators $A = \{g_1^{\pm 1}, \dots, g_n^{\pm 1}\}$ with a Hausdorff combing $g \longmapsto s_g$ for A. We also consider a second system of generators $B = \{h_1^{\pm 1}, \dots, h_m^{\pm 1}\}$ for G. Then there exists a Hausdorff combing $g \longmapsto s'_g$ for B.*

PROOF. We write each generator $g_i^{\pm 1} \in A$ as a word expressed in the letters of $B = \{h_1^{\pm 1}, \ldots, h_m^{\pm 1}\}$, i.e. $g_i^{\pm 1} = g_i^{\pm 1}(h)$. The right-hand side of this formula (namely the word $g_i^{\pm 1}(h)$) can be also viewed as a continuous polygonal path $L_i(\pm) \in \Gamma(G,B)$ joining 1 to $g_i^{\pm 1} \in G \subset \Gamma(G,B)$. At this point we can also consider the left action of G on $\Gamma(G,B)$ and hence for each $x \in G$ the path $xL_i(\pm)$ which goes from x to $xg_i^{\pm}$. We define s'_g by replacing each edge $[x, xg_i^{\pm 1}]$ of s_g (shown in the upper part of Fig. 7) by the polygonal path $xL_i(\pm) \subset \Gamma(G,B)$ (shown in the lower part of Fig. 7). We have to show that $g \longmapsto s'_g$ defines a Hausdorff combing. So we consider the paths s_g and s_h and a comparison $\{(t_i, t'_j)\}$ verifying the estimate

$$d_A(s_g(t_i), s_h(t'_j)) \leq C_1 d_A(g,h) + C_2.$$

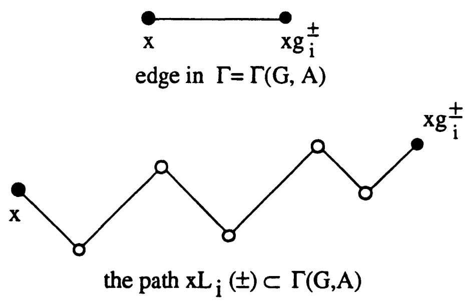

Fig. 7

On s'_g, s'_h we have two kind of vertices: the "old" vertices t_i, t'_j (coming from s_g and s_h), denoted in Fig. 8 by a solid point, and the "new" vertices which are the vertices of the various $xL_i(\pm)$ intermediary between x and $xg_i^{\pm 1}$, denoted in Fig. 8 by a small circle.

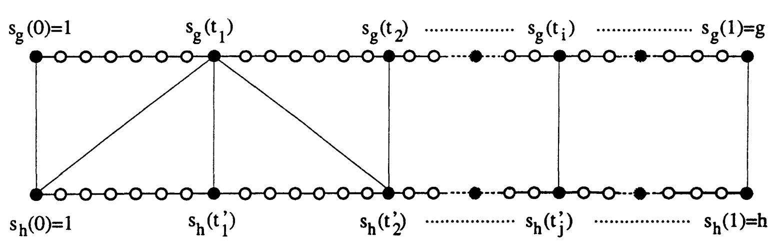

Fig. 8 – *The old comparison* $\{(t_i, t'_j)\}$

In the old comparison, which is represented in Fig. 8, we have three types of "pieces", shown in Fig. 9, 10 and 11.

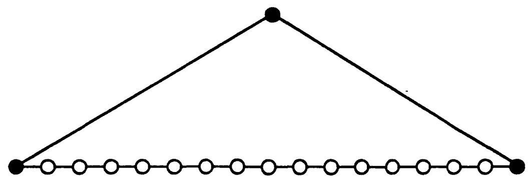

Fig. 9

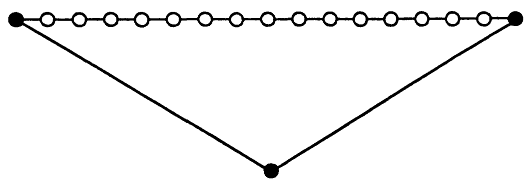

Fig. 10

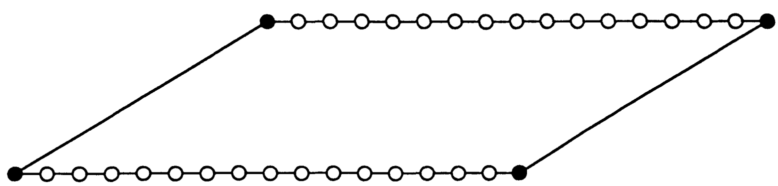

Fig. 11

We can complete Fig. 8 to a new comparison, extending the old one, as described in Fig. 12.

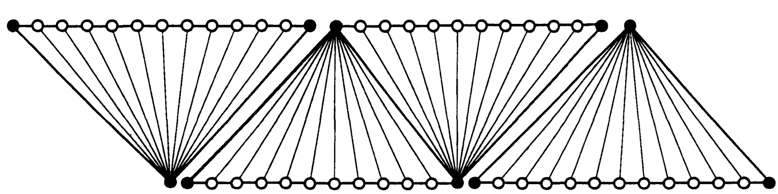

Fig. 12

We claim that one can find new constants for which the new comparison verifies a Hausdorff estimate. This follows easily from the existence of a Hausdorff estimate for the old comparison and from the fact that the length of the paths $xL_i(\pm)$ (which appear as edges in Fig. 12) is bounded. The details are left to the reader. ■

We go back to the notations of the Theorem stated in the introduction: $g \longmapsto s_g$ is a Hausdorff combing of the fundamental group $\pi_1 M^3$ of a 3-manifold M^3, with respect to a set of generators $B = B^{-1}$.

Let $x(t)$ be a finite continuous polygonal path in the Cayley graph $\Gamma(\pi_1 M^3, B)$. We will denote by $\|x\|_B$ the quantity

$$\|x\|_B = \sup_{t_1, t_2} d_B(x(t_1), x(t_2)),$$

where $x(t_1)$, $x(t_2)$ are vertices of $x(t)$.

LEMMA 3.3. *There are constants C_1^*, C_2^* such that for the combing $g \longmapsto s_g$ we have:*

I) *For each g, h there exists a comparison $\{(t_i, t'_j)\}$ with*

$$d_B(s_g(t_i), s_h(t'_j)) \leq C_1^* d_B(g, h) + C_2^*. \tag{3.3}$$

II) *For each combing path s_g we have*

$$\|s_g\|_B \leq C_1^* \|g\|_B + C_2^*. \tag{3.4}$$

PROOF. Lemma 3.1 tells us that for each $g \in \pi_1 M^3$ there is a comparison $\{(t_i, t'_j)\}$ between the combing paths s_g and s_1 satisfying the estimate

$$d(s_g(t_i), s_1(t'_j)) \leq C_1 d(g, 1) + C_2.$$

We have

$$\begin{aligned} d(s_g(0), s_g(t_i)) = d(1, s_g(t_i)) &\leq d(1, s_1(t'_j)) + d(s_1(t'_j), s_g(t_i)) \\ &\leq \|s_1\| + C_1 d(g, 1) + C_2 = \|s_1\| + C_1\|g\| + C_2. \end{aligned}$$

Hence

$$d(s_g(t_1), s_g(t_2)) \leq 2(C_1\|g\| + C_2 + \|s_1\|),$$

from which it is not hard to find constants C_1^*, C_2^* which fulfil conditions I and II. ■

4. - An outline of the proof

We fix a set of generators B as in condition II of our Theorem, and the Cayley graph $\Gamma = \Gamma(\pi_1 M^3, B)$. Whenever the contrary is not explicitly stated, all the norms $\|\,.\,\|$ or distances $d(.\,,\,.)$ will be computed with respect to B (i.e. they will be $\|\,.\,\|_B$ and $d_B(.\,,\,.)$, respectively). As in section 2 we will represent M^3 as the quotient of some fundamental domain Δ. The set of triangles of $\partial\Delta$ is $S = \{h_1, h_2, \dots, h_{2p}\}$ and we choose a fundamental domain $\{\overline{g}_1, \dots, \overline{g}_p\} \subset S$ for the fixed-point free involution $S \xrightarrow{j} S$. As already explained above, we can attach $g_i \underset{\text{def}}{=} \chi(\overline{g}_i) \in \pi_1 M^3$ to g_i^{-1} (and correspondingly g_i^{-1} to $j\overline{g}_i$). Lemma 2.3 tells us that we can assume without loss of generality that $B = \{g_1^{\pm 1}, \dots, g_q^{\pm 1}\}$ for some $q \leq p$. We have a *canonical map*

$$B \longrightarrow \overline{\mathcal{G}} \tag{4.0}$$

which sends, for $i \leq q$, g_i to $\overline{g}_i \in S$ and g_i^{-1} to $j\overline{g}_i \in S$. This is a section of the morphism

$$\overline{\mathcal{G}} \xrightarrow{\chi} \pi_1 M^3.$$

(We emphasize that all the norms $\|g\|$ for $g \in \pi_1 M^3$ in the discussion which follows will be computed with respect to B and *never* with respect to the larger system of generators $\mathcal{B} = \{g_1^{\pm 1}, \dots, g_p^{\pm 1}\}$). We choose once for all a lifting Δ_0 of $\Delta \to M^3$ to $\tilde{M}^3$:

$$\begin{tikzcd} & \tilde{M}^3 \arrow[d, "\pi"] \\ \Delta \arrow[ur, "\Delta_0"] \arrow[r] & M^3 \end{tikzcd} \tag{4.1}$$

The image of Δ_0 will denoted again by Δ, so that Δ is now a fundamental domain for the action of $\pi_1 M^3$ on $\tilde{M}^3$. Once $\Delta \xrightarrow{\Delta_0} \tilde{M}^3$ is fixed by (4.1), we have an obvious commutative diagram, with (T, f) as in section 2:

$$\begin{tikzcd} T \arrow[rr, "F = F(\Delta_0)"] \arrow[dr, "f"'] & & \tilde{M}^3 \arrow[dl, "\pi"] \\ & M^3 & \end{tikzcd} \tag{4.2}$$

where F sends $\overline{g}\Delta$ onto the fundamental domain $g\Delta \to \tilde{M}^3$, with $g = \chi(\overline{g})$. In [Po3] the following lemma is proved.

LEMMA 4.1. *We have* $\Psi(F) = \Phi(F)$.

We will denote by $T_n \subset T = \bigcup_{\bar{x} \in \bar{\mathcal{G}}} \bar{x}\Delta$ the collapsible subspace obtained by restricting ourselves to those $\bar{x}$'s which, expressed as words in $\{h_1, \ldots, h_{2p}\}$, have length $\leq n$. We also consider the equivalence relations $\Psi_n = \Psi(F \mid T_n)$ and $\Phi_n = \Phi(F \mid T_n) = \Phi(F) \mid T_n$. In general, we only have $\Psi_n \subset \Psi(F) \mid T_n$, but we also have the following important consequence of lemma 4.1 (see [Po3] for the proof).

LEMMA 4.2. *For each* $m \in \mathbb{Z}_+$, *there is an* $m_1 = m_1(m) \in Z^+$ *with* $m_1 > m$, *such that* $\Psi_{m_1} \mid T_m = \Phi_m$.

Let us review the properties of the map

$$T \xrightarrow{F} \tilde{M}^3.$$

I) The space T is a tree-like union of fundamental domains Δ, which in particular means that $\pi_1 T = 0$.

II) We have $\Psi(F) = \Phi(F)$ (see lemma 4.1 above).

III) But (unfortunately!) a given compact set $K \subset \tilde{M}^3$ is, in general, touched by the image of T infinitely many times.

While I and II are good properties as far as our Theorem is concerned, III is not, since it does not fit well with our Dehn-type lemma. Once the compact $K \to \tilde{M}^3$ is given, the strategy for proving our Theorem will be to construct a commutative diagram

(4.3)
$$\begin{tikzcd} \overline{T} \arrow[rr, "g"] \arrow[dr, "G"'] & & T \arrow[dl, "F"] \\ & \tilde{M}^3 & \end{tikzcd}$$

with the following list of properties:

I) Like T, the space $\overline{T}$ is a tree-like union of fundamental domains Δ (hence $\pi_1 \overline{T} = 0$) and g is a non-degenerate "simplicial" map, sending fundamental domains of $\overline{T}$ isomorphically onto fundamental domains of T. (For example, $\overline{T}$ could be a sub-tree of fundamental domains of T.)

II) The arrow G is also a non-degenerate surjective simplicial map, just like F, and we have $\Psi(G) = \Phi(G)$.

IV) (Replacing III.) But about $\overline{T} \xrightarrow{G} \tilde{M}^3$ we ask that the given compact set $K \subset \tilde{M}^3$ should be touched only by *finitely* many images of fundamental domains of $\overline{T}$. In other words, $\overline{T}$ touches our given compact subset K of $\tilde{M}^3$ only *finitely* many times.

LEMMA 4.3. *If the given compact K is connected and if we can construct a map $\overline{T} \xrightarrow{G} \tilde{M}^3$ as in diagram* (4.3) *with properties* I, II *and* IV, *then our $K \subset \tilde{M}^3$ can be engulfed inside a bounded simply-connected* 3*-dimensional submanifold $N^3 \subset \tilde{M}^3$.*

PROOF. The proof involves the following steps.

Step I. Consider any exhaustion of $\overline{T}$ by *collapsible finite* unions of fundamental domains

$$Y_1 \subset Y_2 \subset \cdots \subset Y_n \subset \cdots \subset \overline{T}.$$

It follows from property IV that there is a Ψ_n such that

$$G(\overline{T} - Y_n) \cap K = \emptyset.$$

Since $\overline{T}/\Phi(G) = \tilde{M}^3$, this implies, among other things, that K can be lifted to $Y_n/\Phi(G \mid Y_n)$.

Step II. We claim that given our n we can find an $m > n$ such that

$$\Psi(G \mid Y_m) \mid Y_n = \Phi(G \mid Y_n). \tag{4.4}$$

We can prove this fact as follows. Notice first that property II, namely $\Phi(G) = \Psi(G)$, implies (4.4) with $m = \infty$. On the other hand $\Psi(G)$ can be exhausted by a sequence of folding maps modelled on the first transfinite ordinal ω (see point c in Section 2). Since only finitely many of these folding maps involve our given Y_n, we have our result (4.4).

Step III. We consider the inclusion map

$$K \subset Y_n/\Phi(G \mid Y_n) \subset Y_m/\Psi(G \mid Y_m)$$

and the obvious commutative diagram

$$\begin{tikzcd} K \arrow[rr, "i"] \arrow[dr] & & Y_m/\Psi(G|Y_m) \arrow[dl, "g_1 = \text{immersion}"] \\ & \tilde{M}^3 & \end{tikzcd}$$

This diagram has the following properties:

III.1 $Y_m/\Psi(G \mid Y_m)$ is a *simply-connected* finite 3-dimensional polyhedron (see II in lemma 2.1 and point d in Section 2).

III.2 The map g_1 is an immersion.

III.3 If we denote by $M_2(g_1) \subset Y_m/\Psi(G\,|\,Y_m)$ the set of double points of g_1, then

$$K \cap M_2(g_1) = \emptyset.$$

If we replace $Y_m/\Psi(G\,|\,Y_m)$ with a very thin 3-dimensional regular neighbourhood, compatible with g_1, then we are exactly in the conditions of our Dehn-type lemma, which assures us of the existence of an engulfing $K \subset N^3 \subset \tilde{M}^3$ with N^3 compact and simply-connected as desired.

Now since any compact $K_1 \subset \tilde{M}^3$ is contained inside a connected compact $K \subset \tilde{M}^3$, the only thing left in order to prove our Theorem is to exhibit a $\overline{T} \xrightarrow{G} \tilde{M}^3$ with all the desired properties for an arbitrary given compact subset of $\tilde{M}^3$. This will be done in the next Section.

5. - Construction of $\overline{T} \xrightarrow{G} \tilde{M}^3$

We start by noticing that for every $\overline{g} \in \overline{\mathcal{G}}$ there is a map $T \xrightarrow{\overline{g}} T$ with sends each $x\Delta \subset T$ onto $\overline{g}x\Delta \subset T$ (this is clearly compatible with the incidence relations of T). The map $\overline{g}$ is an isomorphism between T and $\overline{g}T \subset T$. We also have an obvious commutative diagram which connects it to the left action $\pi_1 M^3 \times \tilde{M}^3 \to \tilde{M}^3$

$$\begin{tikzcd} T \arrow[r, "\overline{g}"] \arrow[d, "F = F(\Delta_0)"'] & T \arrow[d, "F = F(\Delta_0)"] \\ \tilde{M}^3 \arrow[r, "g = \chi(\overline{g})"] & \tilde{M}^3 \end{tikzcd}$$

The construction of $(\overline{T}, G)$ proceeds now in several stages.

For any $g \in \pi_1 M^3$ we consider the various *geodesic paths* of the Cayley graph $\Gamma = \Gamma(\pi_1 M^3, B)$ joining $1 \in \Gamma$ to $g \in \Gamma$. For a given g there are only *finitely many* such paths. We will denote their number by $\rho(g)$ and write them in the general form

$$\alpha_i(g) = (1, g_{j(1)}, g_{j(1)}g_{j(2)}, \ldots, g_{j(1)}g_{j(2)}\cdots g_{j(n)} \underset{\text{def}}{=} g), \tag{5.1}$$

where $n = \|g\|$, $g_{j(k)} \in B$ and $i = 1, 2, \ldots, \rho(g)$. Using the canonical map $B \to \overline{\mathcal{G}}$ (see (4.0)), we have a canonical lift of $\alpha_i(g)$ to $\overline{\mathcal{G}}$

$$\overline{\alpha}_i(g) = (1, \overline{g}_{j(1)}, \overline{g}_{j(1)}\overline{g}_{j(2)}, \ldots, \overline{g}_{j(1)}\overline{g}_{j(2)}\cdots \overline{g}_{j(n)} \underset{\text{def}}{=} \overline{g}). \tag{5.2}$$

It should be emphasized here that the lift $\overline{g}$ of g, from $\pi_1 M^3$ to $\overline{\mathcal{G}}$, depends on the specific index $i = 1, \ldots, \rho(g)$ in (5.1) and (5.2). To (5.2) we can associate a continuous path of fundamental domains in T

$$(5.3) \qquad \overline{\alpha}_i(g)\Delta \underset{\text{def}}{=} 1 \cdot \Delta \cup \overline{g}_{j(1)}\Delta \cup \overline{g}_{j(1)}\overline{g}_{j(2)}\Delta \cup \ldots \cup \overline{g}\Delta \subset T.$$

We consider the quotient space T_1^∞ of the disjoint union

$$\sum_{g \in \pi_1 M^3,\, i=1,\ldots,\rho(g)} \overline{\alpha}_i(g)\Delta,$$

obtained by identifying all $1 \cdot \Delta \subset \overline{\alpha}_i(g)\Delta$ together; so T_1^∞ is locally finite, except at $1 \cdot \Delta$. The fundamental domains $\overline{g}\Delta \subset \overline{\alpha}_i(g)\Delta \subset T_1^\infty$ which are endpoints of the corresponding paths $\overline{\alpha}_i(g)\Delta$ will be called, by definition, *red fundamental domains*.

Remembering that our construction of $\overline{T}$ depends on an initially given compact subset $K \subset \tilde{M}^3$ we will choose now an $R > 0$ large enough that if for some $g \in \pi_1 M^3$ we have $g\Delta \cap K \neq \emptyset$, then $\|g\| < R$. Let us consider now a positive integer $r \leq \|g\|$; we will denote by $\alpha_i(g) \,|\, r$, $\overline{\alpha}_i(g) \,|\, r$ and $\overline{\alpha}_i(g)\Delta \,|\, r$ the obvious truncations of (5.1), (5.2) and (5.3) respectively. We will also define a quotient-space $T_2^\infty = T_2^\infty(R)$ of T_1^∞ as follows. At any time, we find g_1, $g_2 \in \pi_1 M^3$, $i_1 \leq \rho(g_1)$, $i_2 \leq \rho(g_2)$ and an $r \leq \inf(T, \|g_1\|, \|g_2\|)$ such that $\alpha_{i_1}(g_1) \,|\, r = \alpha_{i_2}(g_2) \,|\, r$, we identify $\overline{\alpha}_{i_1}(g_1)\Delta \,|\, r$ to $\overline{\alpha}_{i_2}(g_2)\Delta \,|\, r$ in the obvious manner. So T_2^∞ is locally finite except along the sphere of radius R. A fundamental domain of T_2^∞ which is the image of a red fundamental domain of T_1^∞ will be, by definition, *red*.

Both T_1^∞ and T_2^∞ are part of an obvious commutative diagram, analogous to (4.3)

$$(5.4) \qquad \begin{tikzcd} T_\varepsilon^\infty \arrow[rr, "g_\varepsilon"] \arrow[dr, "G_\varepsilon"'] & & T \arrow[dl, "F"] \\ & \tilde{M}^3 & \end{tikzcd}$$

with $\varepsilon = 1, 2$. Both these objects T_1^∞ and T_2^∞ verify property I of our strategy and $T_2^\infty \xrightarrow{G_2} \tilde{M}^3$ also verifies IV. But in order to fulfil II (i.e. double points should be commanded by singularities) we will need to enlarge our T_2^∞.

Let $\overline{x}\Delta \subset T_2^\infty$ be a *red* fundamental domain of T_2^∞ corresponding to the element $x \in \pi_1 M^3$; we denote $n = \|x\|_B$. For any $y \in \pi_1 M^3$ which is such that

$$d_B(x, y) \leq C_1^*(C_3 n^{1-\varepsilon} + C_4 + 3) + C_2^* \underset{\text{def}}{=} a n^{1-\varepsilon} + b,$$

we consider the combing arc $\tau(y) \underset{\text{def}}{=} x\sigma_{x^{-1}y}$ which goes from x to y. Via the procedure which by now should be obvious we can change $\tau(y)$ into a continuous chain of fundamental domains going from $\bar{x}\Delta$ to $\bar{y}\Delta$ which we will denote $\bar{\tau}(y)\Delta$. With this we introduce the tree-like object

$$T_3^\infty = T_2^\infty + \sum_{\bar{x}\Delta\,\mathrm{red}} \bar{\tau}(y)\Delta,$$

where $\bar{x}\Delta$ runs through all red fundamental domains of T_2^∞, y runs through all elements with $d(x,y) \leq an^{1-\varepsilon} + b$ and one identifies the red $\bar{x}\Delta \subset T_2^\infty$ to the initial $\bar{x}\Delta \subset \bar{\tau}(y)\Delta$. So we go from T_2^∞ to T_3^∞ by adding for each red $\bar{x}\Delta$ a whole collection of *long tails* starting at $\bar{x}\Delta$, whose length goes to infinity as $\|\bar{x}\| \to \infty$ but is controlled by $an^{1-\varepsilon} + b$.

At this point we consider a parameter M, which for the time being will be left free, and depending on this parameter we construct an extension of T_3^∞ denoted by $\overline{T}(M)$. Eventually our desired $\overline{T}$ will be $\overline{T}(M)$ with M sufficiently large; how large, it will be explained later. Anyway for a given M, lemma 4.2 fixes an $M_1 = M_1(M)$ such that $\Psi_{M_1} \mid T_M = \Phi_M$. With this we define

$$\overline{T}(M) = T_3^\infty + \sum_{\bar{g}\Delta \subset \bar{\tau}(y)\Delta} \bar{g}T_{M_1},$$

where $T_{M_1} \subset T$ is defined as in Section 4, $\bar{\tau}(y)\Delta$ is an arbitrary long tail, $\bar{g}\Delta$ is an arbitrary fundamental domain of $\bar{\tau}(y)\Delta$ and $\bar{g}\Delta \subset \bar{\tau}(y)\Delta \subset T_3^\infty$ is identified to the initial $\bar{g}\Delta \subset \bar{g}T_{M_1}$. So we attach now to each $\bar{g}\Delta$ belonging to the *long tails* (in particular to each *red fundamental* domain) an arborescent *short tail* $\bar{g}T_{M_1}$ of bounded length. The object $\overline{T}(M)$ comes equipped with a natural non-degenerate simplicial map $\overline{T}(M) \xrightarrow{g} T$ and with a tautological map $\overline{T}(M) \xrightarrow{G} \tilde{M}^3$, which enter into the obvious commutative diagram

$$\begin{tikzcd} \overline{T}(M) \arrow[rr, "g"] \arrow[dr, "G"'] & & T \arrow[dl, "F"] \\ & \tilde{M}^3 & \end{tikzcd}$$

Clearly $(\overline{T}(M), G)$ verifies property I of our strategy. We also have the following lemma, which holds for any value of M.

LEMMA 5.1. *The map $\overline{T} \xrightarrow{\overline{G}} \tilde{M}^3$ touches only finitely many times our given compact $K \subset \tilde{M}^3$. In other words property* IV *of our strategy is also fulfilled.*

PROOF. We know that this is already true for the part $T_2^\infty \subset \overline{T}(M)$. We go from T_2^∞ to $\overline{T}(M)$ by adding for each *red* $\bar{x}\Delta \subset T_2^\infty$ the finite arborescent contribution, based at $\overline{x}\Delta$,

$$\sum \overline{\tau}(y)\Delta + \sum_{\overline{g}\Delta \subset \overline{\tau}(y)\Delta} \overline{g}T_{M_1}, \tag{5.5}$$

where the first sum is over all the *long tails* attached to the red $\overline{x}\Delta$, and the second sum is over all the *short tails* pertaining to the red $\overline{x}\Delta$. We will denote (5.5) by $t(\overline{x}\Delta) \subset \overline{T}(M)$. If $||x|| = n$ then for an arbitrary $\overline{z}\Delta \subset t(\overline{x}\Delta)$ we have the estimate

$$d(x, z) \leq C_1^*(C_3 n^{1-\varepsilon} + C_4 + 3) + C_2^* + M_1 \underset{\text{def}}{=} a n^{1-\varepsilon} + b_1.$$

But for a given n there are only *finitely* many *red* fundamental domains $\overline{x}\Delta \subset T_2^\infty$ with $||x|| \leq n$, and since $\lim_{n\to\infty} (n - (an^{1-\varepsilon} + b_1)) = \infty$, only *finitely* many $\overline{z}\Delta$'s (appearing in *all* the possible $t(\overline{x}\Delta)$'s) have images which can touch K. ■

6. - The property $\Phi(G) = \Psi(G)$

It remains to be shown that we can choose the parameter M so as to fulfil requirement II from our strategy.

CONDITIONS ON M.

Here are two lower bounds which we will impose on M.

C.1) Let us consider the constants C_1^* and C_2^* from Lemma 3.3; with this our M will be such that $M \geq 2(C_1^* + C_2^* + 1)$.

C.2) A second lower bound we will impose to M is the following. Remember that we had $B = \{g_1^{\pm 1}, \ldots, g_q^{\pm 1}\} \subset \{g_1^{\pm 1}, \ldots, g_p^{\pm 1}\} = \mathcal{B}$, where $\mathcal{B}$ corresponds to *all* the faces of the fundamental domain Δ. We will require that:

$$M \geq \sup_{g_i \in \mathcal{B}} ||g_i|| \underset{\text{def}}{=} \gamma. \tag{6.0}$$

(Remember that all the norms $||\,.\,||$ are computed with respect to $B \subset \pi_1 M^3$.)

We assume from now on that M satisfies these two conditions, and the corresponding $\overline{T}(M)$ will be simply denoted by $\overline{T}$.

LEMMA 6.1. (Main technical lemma. Closing property for the *very short tails*).

1) *As an immediate consequence of Lemma* 4.2, *for any fundamental domain* $\overline{y}\Delta \subset$ {*the union of all the* long tails *corresponding to all the red fundamental domains* $\overline{x}\Delta$}, *we have* $\Psi(G) \mid \overline{y}T_M = \Phi_M$ *and the map*

$$\overline{y}T_M/\Psi(G) = \overline{y}(T_M/\Phi_M) \xrightarrow{G} y(FT_M) \subset \tilde{M}^3$$

is an isomorphism (here $\overline{y}T_M \subset \overline{y}T_{M_1} \subset \overline{T}$).

2) *Let* $\overline{x}_1\Delta$ *and* $\overline{x}_2\Delta$ *be two red fundamental domains of* $\overline{T}$ *such that in* $\pi_1 M^3$ *we have* $x_1 = x_2 = x$. *Then the equivalence relation* $\Psi(G)$ *identifies* $\overline{x}_1\Delta$ *to* $\overline{x}_2\Delta$.

3) *Let* $\overline{x}_1\Delta$ *and* $\overline{x}_2\Delta$ *be two red fundamental domains of* $\overline{T}$ *such that in* $\pi_1 M^3$ *we have* $x_2 = x_1 g_i^{\pm 1}$ *with* $i \leq q$ *(i.e.* $g_i^{\pm 1} \in B$). *Then the equivalence relation* $\Psi(G)$ *identifies the* $\overline{g}_i$*-face of* $\overline{x}_1\Delta$ *to the* $j\overline{g}_i$*-face of* $\overline{x}_2\Delta$.

3′) *(Generalization of* 3 *from* B *to* $\mathcal{B}$.) *Let* $\overline{x}_1\Delta$ *and* $\overline{x}_2\Delta$ *be two red fundamental domains of* $\overline{T}$ *such that in* $\pi_1 M^3$ *we have* $x_2 = x_1 g_k^{\pm 1}$ *with* $k \leq p$. *Then the equivalence relation* $\Psi(G)$ *identifies the* $\overline{g}_k$*-face of* $\overline{x}_1\Delta$ *to the* $j\overline{g}_k$*-face of* $\overline{x}_2\Delta$.

4) *As an immediate consequence of* 2 *and* 3′, *the subset* $\mathcal{R} \subset \overline{T}/\Psi(G)$ *defined by*

$$\mathcal{R} \underset{\text{def}}{=} \{\textit{the union of the red fundamental domains of } \overline{T}\}/\Psi(G) \subset \overline{T}/\Psi(G)$$

is isomorphic to $\tilde{M}^3$, *via the map* G.

Before proving this main technical fact, we will show how we can deduct from it that $\Psi(G) = \Phi(G)$, i.e. requirement IV of our strategy. In the context of formula (4.3) we consider the obvious commutative diagram

$$\begin{tikzcd} \overline{T} \arrow[r] \arrow[dr] & \overline{T}/\Psi(G) \arrow[d, "G_1 = \text{immersion}"] \\ & \tilde{M}^3 \end{tikzcd}$$

LEMMA 6.2. *For* $\overline{T}$ *we have* $\Psi(G) = \Phi(G)$, *and hence* G_1 *is an isomorphism between* $\overline{T}/\Psi(G)$ *and* $\tilde{M}^3$.

PROOF. We consider the commutative diagram

$$\begin{tikzcd} \mathcal{R} \arrow[rr, "i"] \arrow[dr, "\approx"'] & & \overline{T}/\Psi(G) \arrow[dl, "G_1 = \text{immersion}"] \\ & \tilde{M}^3 & \end{tikzcd}$$

where i is the obvious inclusion map. It suffices to show that i is surjective. If not, we could find a fundamental domain $\Delta \subset \overline{T}/\Psi(G)$ such that $(\operatorname{int}\Delta) \cap \operatorname{Im} i = \emptyset$. But $\overline{T}$ is connected and hence so is $\overline{T}/\Psi(G)$. This means that we could also find a Δ with $(\operatorname{int} D) \cap \operatorname{Im} i = \emptyset \neq \partial\Delta \cap \operatorname{Im} i$. but since $\mathcal{R} \xrightarrow{\approx} \tilde{M}^3$ is a homeomorphism, any $x \in \partial\Delta \cap \operatorname{Im} i$ would be a singularity for G_1, which is absurd. ∎

PROOF OF LEMMA 6.1. We consider the "Statement 2(n)" obtained by restriction of conclusion 2 in 6.1 to x_1, x_2 such that $\|x_1\|, \|x_2\| \leq n$, and similarly the "Statement 3(n)" for conclusion 3. To begin with, we prove that:

$$\text{Statements } 2(n-1) \text{ and } 3(n-1) \Longrightarrow \text{Statement } 2(n).$$

Let $x \in \pi_1 M^3$ be such that $\|x\| = n$ and, as in (5.1), consider $\alpha_1(x)$, $\alpha_2(x)$ such that our $\overline{x}_1\Delta$, $\overline{x}_2\Delta$ are the endpoints of the corresponding $\overline{\alpha}_1(x)\Delta$, $\overline{\alpha}_2(x)\Delta$. So, in $\overline{T}$ we have a continuous path of fundamental domains $\overline{\alpha}_1(x)\Delta \bigcup\limits_{1\cdot\Delta} \overline{\alpha}_2(x)\Delta$ with endpoints $\overline{x}_1\Delta$, $\overline{x}_2\Delta$, and what we want to show is that $\Psi(G)$ forces this path to close. We will consider the elements $x(\varepsilon) \in \pi_1 M^3$ (with $\varepsilon = 1, 2$) which are the last ones in $\alpha_\varepsilon(x)$ before x; so $\|x(\varepsilon)\| = n - 1$. For $x(\varepsilon)$ there is in $\Gamma = \Gamma(\pi_1 M^3, B)$ a geodesic path $a(x(\varepsilon))$ isomorphic to $a_\varepsilon(x) \mid (n-1)$. Clearly, $\Psi(G)$ forces the identification of $\overline{\alpha}(x(\varepsilon))\Delta$ to $(\overline{\alpha}_\varepsilon(x)\Delta) \mid (n-1)$. Now $d_B(x(1), x(2)) \leq 2$ so point 2 of our Theorem tells us that inside the ball of radius $(n-1)$, i.e. in $B(n-1) \subset \Gamma(\pi_1 M^3, B)$, we can join $x(1)$ to $x(2)$ by a path L of length $\leq C_3(n-1)^{1-\varepsilon} + C_4$. In Fig. 13 the path L appears as the fat polygonal line

$$L = (y_0 = x(1), y_1, y_2, \ldots, y_{\overline{m}-1}, y_{\overline{m}} = x(2))$$

with $\overline{m} \leq C_3(n-1)^{1-\varepsilon} + C_4$. Each of the straight lines we see in Fig. 13 joining 1 respectively to $x(1), y_1, y_2, \ldots, y_{\overline{m}-1}, y_{\overline{m}} = x(2)$ is a geodesic of Γ. In Fig. 13, we also see a *closed* path $\lambda \subset \Gamma$ of length $\leq C_3(n-1)^{1-\varepsilon} + C_4 + 2$, namely $(x, x(1), y_1, y_2, \ldots, y_{\overline{m}-1}, y_{\overline{m}} = x(2), x)$. Since every directed edge of λ corresponds canonically to a generator $g_i^{\pm 1} \in B$, by (3.0) λ defines a continuous path of fundamental domains in T, starting at $1 \cdot \Delta$, which we will denote by $\overline{\lambda}\Delta \subset T$.

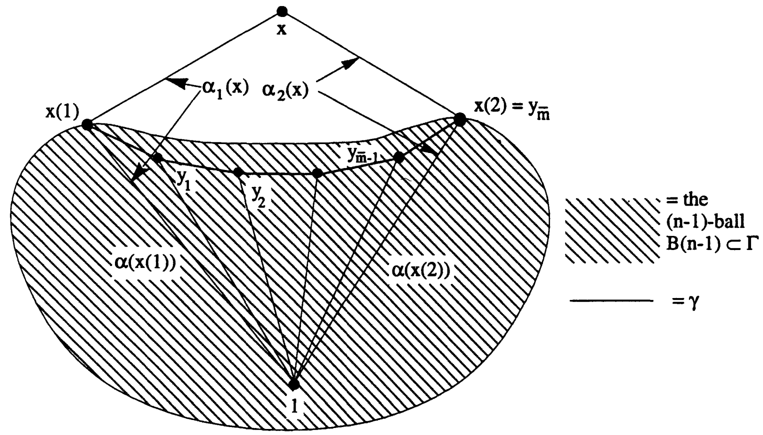

Fig. 13 – *This figure represents, symbolically, a piece of the Cayley graph* Γ. *The fat points represent elements of* $\pi_1 M^3$. *The number* $\overline{m}$ *is* $\leq C_3(n-1)^{1-\varepsilon} + C_4$.

To each vertex $v \in \pi_1 M^3$ of λ (with the two endpoints counting as distinct vertices) we attach a red fundamental domain in $\overline{T}/\Psi(G)$, which we will denote by $\overline{v}\Delta$, in the following way:

α) For the endpoints, we take simply the images of $\overline{x}_1\Delta$, $\overline{x}_2\Delta \subset \overline{T}$ in $\overline{T}/\Psi(G)$.

β) As far as the other points $y_i \in B(n-1)$ are concerned, we use the fact that the inductive hypothesis $2(n)$ implies that for any $w \in B(n-1) \subset \pi_1 M^3$ there exists a *unique red* representative in $\overline{T}/\Psi(G)$, which we will denote by $\overline{w}\Delta \subset \overline{T}/\Psi(G)$.

(*Caution.* Here $\overline{w}$ is not unambiguously defined as an element of the monoid $\overline{\mathcal{G}}$; it is only the red fundamental domain $\overline{w}\Delta$ which is well-defined in $\overline{T}/\Psi(G)$).

As we have already remarked, if v_{s-1} and v_s are two consecutive vertices of λ then $[v_{s-1}, v_s]$ corresponds to a well-defined element in $S \subset \overline{\mathcal{G}}$ which we will denote by $h[v_{s-1}, v_s]$. With this, we claim that in $\overline{T}/\Psi(G)$, the $h[v_{s-1}, v_s]$ - face of $\overline{v}_{s-1}\Delta$ is *identified* to the $j(h[v_{s-1}, v_s])$ - face of $\overline{v}_s\Delta$, the glueing pattern being exactly the same as for the path $\overline{\lambda}\Delta \subset T$. For the extremal edges $[x, x(1)]$ and $[x(2), x]$ our claim follows directly from the way in which $\alpha_\varepsilon(x)$, $\alpha(x(\varepsilon))$ were constructed. For all the other edges, the triangle $(v_{s-1}, v_s, 1)$ (see Fig. 13) is completely contained in the ball $B(n-1)$, and our *claim* follows hence from the inductive hypothesis $3(n-1)$. So, we get a continuous path Λ of fundamental domains in $\overline{T}/\Psi(G)$, isomorphic to $\overline{\lambda}\Delta$, joining $\overline{x}_1\Delta$ to $\overline{x}_2\Delta$.

We have length$(\Lambda) \leq C_3(n-1)^{1-\varepsilon} + C_4 + 2$, and we claim that the equivalence relation $\Psi(G) \subset \overline{T} \times \overline{T}$ *forces* Λ *to close.*

For $x \in \pi_1 M^3$ and y_s (see Fig. 13), we have the combing path $\tau(y_s) = x\sigma_{x^{-1}y_s}$. In $\overline{T}$ to the red $\overline{x}_1\Delta$ there is attached, for every y_s, a long tail $\overline{\tau}(y_s)\Delta$.

Notice that the distance between x and y_s in the Cayley graph is dominated by $C_3 n^{1-\varepsilon} + C_4 + 2$ and hence the length of the long tail $\overline{\tau}(y_s)\Delta$ is dominated by $an^{1-\varepsilon} + b$ (with a and b universal constants, independent of n). Fig. 14 lives in the Cayley graph and in this figure we see the closed path Λ and the various combing paths $\tau(y_s)$ and $\tau(y_{s+1})$. Our hypothesis tells us that given $\tau(y_s)$ and $\tau(y_{s+1})$ there is a comparison (between the two combing arcs) $\{(t_i, t'_j), (t_\alpha, t'_\beta)\ldots\}$ such that

$$d(\tau(y_s)(t_i), \tau(y_{s+1})(t'_j)) \leq C_1^* + C_2^*.$$

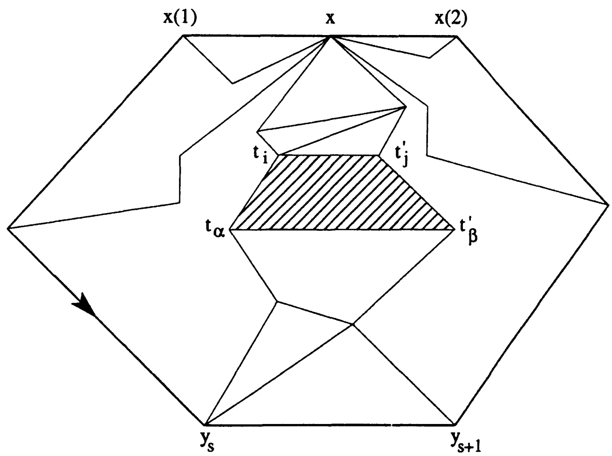

Fig. 14

Here we have denoted by $\tau(y_s)(t_i)$, $(\tau(y_{s+1})(t'_j))$ the vertex of $\tau(y_s)$ (respectively, $\tau(y_{s+1})$) corresponding to the parameter t_i (respectively, t'_j). In the Cayley graph we consider a geodesic path $[\tau(y_s)(t_i), \tau(y_{s+1})(t'_j)]$ joining $\tau(y_s)(t_i)$ to $\tau(y_{s+1})(t'_j)$. So, the intermediary region between the combing paths $\tau(y_s)$ and $\tau(y_{s+1})$, which we see in Fig. 14, is a geometric realization of the corresponding abstract rectangle of Section 3. In particular the triangles and the rectangles connecting $\tau(y_s)$ to $\tau(y_{s+1})$ in Fig. 14 are the geometric realizations of the abstract small triangles and small rectangles of Section 3.

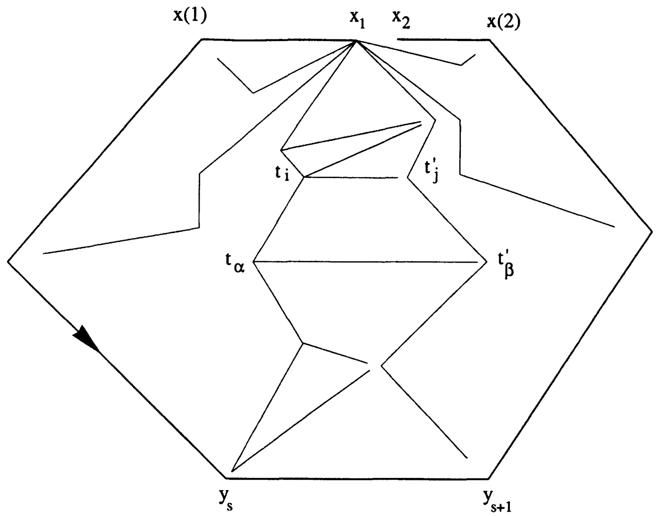

Fig. 15

Of course our geodesics might meet in a more complicated way than the figure suggests, but this is immaterial. The important point is that each of the geometric small rectangles and small triangles of Fig. 14 (as for example the perimeter of the hatched region $[t_i, t_\alpha, t'_\beta, t'_j]$ has length $\leq 2(C_1^* + C_2^* + 1)$. All this was happening in the Cayley graph. At the level of $\overline{T}$, Fig. 14 becomes Fig. 15 and all our closed paths are now open. In particular Λ becomes a polygonal path going from $\overline{x}_1\Delta$ to $\overline{x}_2\Delta$, each $\overline{\tau}(y_s)\Delta$ starts at $\overline{x}_1\Delta$ without quite making it to $\overline{y}_s\Delta$, while each $[\tau(y_s)(t_i), \tau(y_{s+1})(t'_j)]$ becomes a continuous chain of fundamental domains starting at $\overline{\tau}(y_s)(t_i)\Delta$ and not quite making it to $\overline{\tau}(y_{s+1})(t'_j)\Delta$. Assuming inductively that the equivalence relation $\Psi(G)$ forces the endpoint of $\overline{\tau}(y_s)\Delta$ to be identified to $\overline{y}_s\Delta$, we will show that it also forces the endpoint of $\overline{\tau}(y_{s+1})\Delta$ to be identified to $\overline{y}_{s+1}\Delta$. This is proved by a second induction, running along the small triangles and small rectangles contained between $\tau(y_s)$ and $\tau(y_{s+1})$, going from $\overline{x}_1\Delta$ all the way down to the arc $[y_s, y_{s+1}]$. Assuming that the induction has reached the hatched region of Fig. 14, we can assume already that to this hatched region corresponds in $\overline{T}/\Psi(G)$ a continuous path of fundamental domains looking (at least) like in Fig. 16. This symbolical figure lives in $\overline{T}/\Psi(G)$ and represents a continuous path of fundamental domains which we will denote by P. We want to show that $\Psi(G)$ actually forces this path to close. The fundamental domain $\overline{\tau}(y_s)(t_i)\Delta$ belongs to one of the long tails starting at the red $\overline{x}_1\Delta$ and hence it is the origin of a short tail $\overline{\tau}(y_s)(t_i)T_{M_1}$. If we consider the obvious commutative diagram

$$
(6.2) \qquad
\begin{tikzcd}
\overline{\tau}(y_s)(t_i)T_{M_1} \bigcup\limits_{\overline{\tau}(y_s)(t_i)\Delta} P \arrow[rr, "\varsigma"] \arrow[dr] & & \overline{T}/\Psi(G) \arrow[dl, "G_1"] \\
& \tilde{M}^3 &
\end{tikzcd}
$$

we can make the following remarks. Our P which is of length $\leqq M$ corresponds canonically to a piece of $\overline{\tau}(y_s)(t_i)T_M$ and at the level of $\overline{T}/\Psi(G)$ the two corresponding ς-images *have to be identified* to each other. Point 1 of Lemma 6.1 tells us, on the other hand, that $G_1 \mid \varsigma(\overline{\tau}(y_s)(t_i)T_M)$ is injective. This means that in $\overline{T}/\Psi(G)$ the image $\varsigma(P)$ is a *closed path.*

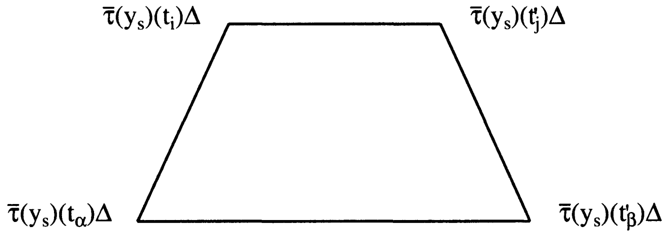

Fig. 16

This proves the implication:

Statements $2(n-1)$ and $3(n-1) \Longrightarrow$ Statement 2

and the same line of argument can be used to prove that:

Statements $2(n-1)$ and $3(n-1) \Longrightarrow$ Statement 3.

We leave it to the reader to complete the proof of 3 following this line of argument. We will show now how 2 and 3 together imply 3′. So we consider the red fundamental domains $\overline{x}_1\Delta$, $\overline{x}_2\Delta \subset \overline{T}$ with $x_2 = x_1 g_k (q < k \leq p)$ in $\pi_1 M^3$. Notice that at the level of $\tilde{M}^3$ the fundamental domains $G_1(\overline{x}_1\Delta)$ and $G_1(\overline{x}_2\Delta)$ touch along their respective (g_k and g_k^{-1})-faces. We want to show that $\Psi(G)$ forces them actually to be *glued* together at that site, at the level of $\overline{T}/\Psi(G)$, source of the map G_1. Now, in $\pi_1 M^3$ we can write $g_k = h_1, h_2, \dots, h_c$ with $h_j \in B$ and $c \leq \gamma$ (see (6.0)). The $h_1, h_2, \dots, h_c$ lift via (4.0) to $\overline{h}_1, \dots, \overline{h}_c \in S \subset \overline{\mathcal{G}}$.

We consider in $\pi_1 M^3$ the sequence of elements

$$y_0 \underset{\text{def}}{=} x_1, \quad y_1 = x_1 h_1, \quad y_2 = x_1 h_1 h_2, \quad \ldots \quad y_c = x_1 h_1 h_2 \ldots h_c = x_2.$$

It follows from 2 that for each $j = 0, \ldots, c$ there exists a *unique* red fundamental domain $\overline{y}_j\Delta$ in $\overline{T}/\Psi(G)$. For $j = 0$ and $j = c$ the corresponding $\overline{y}_j\Delta$ are exactly our original $\overline{x}_1\Delta$ and $\overline{x}_2\Delta$ (actually their images in $\overline{T}/\Psi(G)$). It follows from 3 that we can define in $\overline{T}/\Psi(G)$ a continuous path of fundamental domains

$$\Lambda' \underset{\text{def}}{=} \overline{x}_1\Delta \cup \overline{y}_1\Delta \cup \ldots \cup \overline{y}_{c-1}\Delta \cup \overline{x}_2\Delta,$$

with $\overline{y}_{s-1}\Delta$ and $\overline{y}_s\Delta$ glued together along their respective $(\overline{h}_s, j\overline{h}_s)$-faces; this path is suggested in Fig. 17.

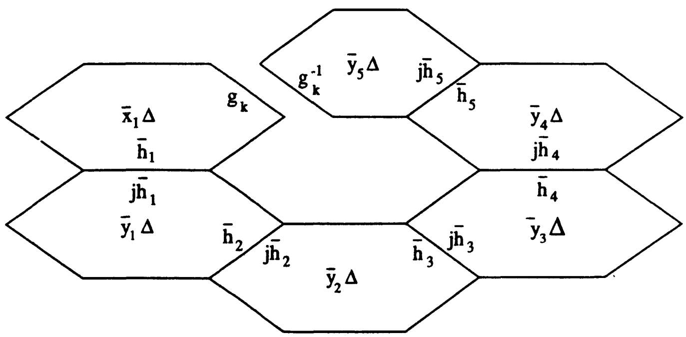

Fig. 17 – *A continuous path of fundamental domains which projects onto the path Λ' of $\overline{T}/\Psi(G)$. Here $x_2 = x_1 g_k$ and $g_k = y_1 y_2 y_3 y_4 y_5$.*

Now, by (6.0), we have length $(\Lambda') \leq \gamma \leq M$ and we can consider the obvious commutative diagram, analogous to (6.1) above

$$\begin{tikzcd}
\overline{x}_1 T_{M_1} \bigcup\limits_{\overline{x}_i\Delta} \Lambda' \arrow[rr, "\xi'"] \arrow[dr] & & \overline{T}/\Psi(G) \arrow[dl, "G_1"] \\
& \tilde{M}^3 &
\end{tikzcd}$$

An analysis which is completely similar to our treatment of (6.1) shows that $\xi'(\Lambda') \subset \overline{T}/\Psi(G)$ is a closed path and this finishes the proof of our theorem. ∎

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