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Semicontinuity and relaxation properties of a curvature depending functional in 2D

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 20, nº 2 (1993), p. 247-297
<http://www.numdam.org/item?id=ASNSP_1993_4_20_2_247_0>

Semicontinuity and Relaxation Properties of a Curvature Depending Functional in 2D

G. BELLETTINI - G. DAL MASO - M. PAOLINI

1. - Introduction

The aim of this paper is to study the functional

$$\mathcal{F}(E) = \int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z), \tag{1.1}$$

where $E \subseteq \mathbb{R}^2$ is a bounded open set of class C^2, $p > 1$ is a real number, $\kappa(z) = \kappa_{\partial E}(z)$ is the curvature of ∂E at the point z, and $\mathcal{H}^1$ denotes the one dimensional Hausdorff measure in $\mathbb{R}^2$.

We are interested in the study of the minimum problem

$$\inf_E \left\{ \int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z) + \int_E g(z) \, dz \right\}, \tag{1.2}$$

where $g \in L^\infty(\mathbb{R}^2)$ is a given function. This can be considered as a simplified version of a variational problem proposed in [15] as a criterion for segmentation of images in computer vision. Moreover, it was recently conjectured by E. De Giorgi [4] that problem (1.2) is connected with the asymptotic behaviour of some singular perturbations of minimum problems arising in the theory of phase transitions.

Assume, for simplicity, that g is non-negative for $|z|$ large enough. If we apply the direct method of the calculus of vatiations to problem (1.2), we are led to consider a minimizing sequence $\{E_h\}_h$, whose elements, according to our hypothesis on g, are contained in a suitable ball independent of h and satisfy

$$\sup_h \mathcal{H}^1(\partial E_h) < +\infty.$$

Pervenuto alla Redazione il 10 Marzo 1992.

By a well known compactness theorem there exists a subsequence $\{E_{h_k}\}_k$ which converges in $L^1(\mathbb{R}^2)$ to some bounded set E of finite perimeter. We shall prove that the functional $\mathcal{F}$ is lower semicontinuous with respect to the convergence in $L^1(\mathbb{R}^2)$. This allows us to show that, if the limit set E is of class C^2, then E is a minimizer of (1.2). Since, in general, it is hard to prove directly that the limit of a minimizing sequence is of class C^2, we want to extend the functional $\mathcal{F}$ to the set $\mathcal{M}$ of all Lebesgue measurable subsets of $\mathbb{R}^2$, in such a way that the extended functional $\overline{\mathcal{F}}$ is still lower semicontinuous. As usual in the relaxation method (see [2]), we define $\overline{\mathcal{F}}\colon \mathcal{M} \to [0,+\infty]$ as the lower semicontinuous envelope of $\mathcal{F}$ with respect to the $L^1(\mathbb{R}^2)$-topology, i.e.,

$$\overline{\mathcal{F}}(E) = \inf\left\{\liminf_{h\to+\infty} \mathcal{F}(E_h)\colon E_h \to E \text{ in } L^1(\mathbb{R}^2)\right\}.$$

The main purpose of this paper is to study the functional $\overline{\mathcal{F}}$ and to determine the family of sets E for which $\overline{\mathcal{F}}(E) < +\infty$. The study of the minimum problem (1.2) led us to consider the functional

$$\int_{\partial E} [1 + |\kappa(z)|^p]\, d\mathcal{H}^1(z) \tag{1.3}$$

as a function of E rather than of ∂E. Indeed, the compactness properties of a minimizing sequence of (1.2) ensure a good convergence for the sets E_h, but the corresponding weak convergence of the boundaries ∂E_h does not seem to be appropriate for variational purposes. The main difficulties that we encountered were due essentially to the lack of good contintuity properties of the map $E \mapsto \partial E$.

Lower semicontinuity results and existence of minimizers under suitable boundary conditions are much easier to reach for the functional

$$\mathcal{F}(\gamma) = \int_0^{l(\gamma)} [1 + |\ddot{\gamma}(s)|^p]\, ds, \tag{1.4}$$

related to (1.3), where γ varies now over all curves of class C^2 satisfying prescribed boundary conditions and s is the corresponding arc lenght parameter. In the case $p = 2$, this problem is classical and the minimizers, discovered by Euler [8] in 1744, are called *elastica* because of their application to the theory of flexible inextensible rods. For a complete treatment of the elastica we refer to [14, 10]. Unfortunately, these results cannot be applied directly to the study of (1.2).

In this paper we have considered the problem in the plane. The extension of our results to the n-dimensional case is a difficult open problem and seems to require the methods of geometric measure theory [9, 18]. We want to stress that all our proofs are obtained by using only elementary tools. We cannot exclude that some of these results could also be obtained in a more direct (but less elementary) way by using varifolds theory [13, 6, 7].

We describe now in detail the content of the paper.

In Section 2 we fix the notation and introduce the problem.

Section 3 is devoted to the study of the lower semicontinuity of $\mathcal{F}$. Precisely, we prove (Theorem 3.2) that *given a sequence* $\{E_h\}_h$ *of bounded open sets of class* C^2 *converging in* $L^1(\mathbb{R}^2)$ *to a bounded open set* E *of class* C^2, *then*

$$\int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z) \leq \liminf_{h \to +\infty} \int_{\partial E_h} [1 + |\kappa_h(z)|^p] \, d\mathcal{H}^1(z),$$

where κ *and* κ_h *are the curvatures of* ∂E *and of* ∂E_h, *respectively.*

The definition (1.1) of $\mathcal{F}$ makes sense if E is a set whose boundary can be parametrized, locally, by arcs of regular curves of class $H^{2,p}$. The previous semicontinuity theorem holds also in this case (Corollary 3.2).

We emphasize that the semicontinuity of $\mathcal{F}$ and the definition of $\overline{\mathcal{F}}$ are considered with respect to the $L^1(\mathbb{R}^2)$-topology. This means that the sequence $\{E_h\}_h$ approximates E if and only if $|E_h \Delta E| \to 0$ as $h \to +\infty$, where $|\cdot|$ denotes the Lebesgue measure and Δ is the symmetric difference of sets. No further conditions are required on ∂E_h and ∂E.

Simple examples show that there exist sets $E \in \mathcal{M}$ with $\overline{\mathcal{F}}(E) < +\infty$, whose boundary is not smooth. In particular, let us consider the set E of Figure 1.1 and suppose that ∂E is parametrized by arcs of curves of class C^∞, except for the cusp points.

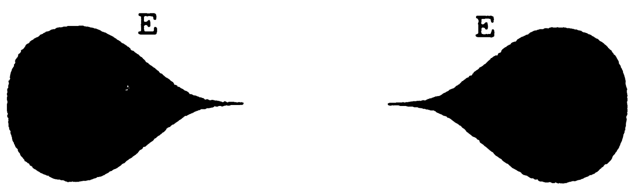


Fig. 1.1: *A set* E *with* $\overline{\mathcal{F}}(E) < +\infty$, *and whose boundary is not smooth.*

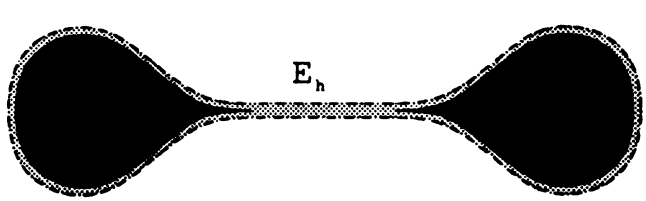


Fig. 1.2: *This approximating sequence* $\{E_h\}_h$ *shows that* $\overline{\mathcal{F}}(E) < +\infty$.

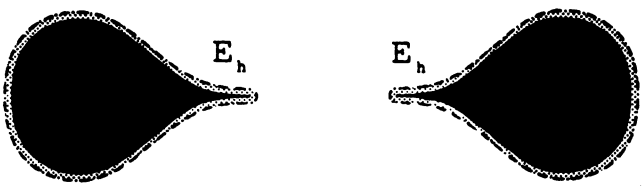


Fig. 1.3: *This approximating sequence* $\{E_h\}_h$ *is such that* $\sup_h \int_{\partial E_h} |\kappa_h(z)|^p \, d\mathcal{H}^1(z) = +\infty$.

Figure 1.2 shows that $\overline{\mathcal{F}}(E) < +\infty$. Indeed, the sequence $\{E_h\}_h$ described in Figure 1.2 converges to the set E in the $L^1(\mathbb{R}^2)$ norm and $\sup_h \mathcal{F}(E_h) < +\infty$. On the other hand, the approximating sequence $\{E_h\}_h$ of Figure 1.3 gives rise to an infinite energy. Here the cusp points are smoothed by circular arcs, and it is easy to prove that, if $p > 1$, then $\int_{\partial E_h} |\kappa_h(z)|^p \, d\mathcal{H}^1(z) \to +\infty$ as $h \to +\infty$. Note that Figure 1.2 shows that, in the limit, the sequence $\{\partial E_h\}_h$ creates a hidden arc (with multiplicity two), given by the segment joining the two cusp points.

The main issue is obviously to characterize those subsets E of $\mathbb{R}^2$ such that $\overline{\mathcal{F}}(E) < +\infty$. We find some necessary conditions and a number of different sufficient conditions, but the complete characterization of this class of sets still remains an open problem.

In Section 4 we present some regularity properties of the sets E such that $\overline{\mathcal{F}}(E) < +\infty$. To be precise, we prove the following result (Theorem 4.1, Proposition 4.3): *Let* $E \subseteq \mathbb{R}^2$ *be a measurable set such that* $\overline{\mathcal{F}}(E) < +\infty$. *Then, up to a modification of* E *on a set of measure zero, we have that* E *is bounded, open, and* $\mathcal{H}^1(\partial E) < +\infty$. *Moreover, there exists a system of curves* $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ *of class* $H^{2,p}$ *such that* ∂E *is contained in the union* (Γ) *of the traces of the curves* γ^i *and* $E = \mathrm{int}(A_\Gamma \cup (\Gamma))$, *where* A_Γ *is the set of all points of* $\mathbb{R}^2 \setminus (\Gamma)$ *of index* 1 *with respect to* Γ. *Finally,* ∂E *has a continuous unoriented tangent and can have at most a countable set of cusp points.*

At the end of Section 4 we classify the points of ∂E according to the local behaviour of the normal line with respect to ∂E.

In Section 5 we show examples of sets E such that $\overline{\mathcal{F}}(E) < +\infty$, despite of their boundaries being very irregular. Precisely, in Example 1 a set E is described whose irregular boundary points have positive one dimensional Hausdorff measure. In Example 2 a set having an infinite number of cusp points is shown.

In Section 6 we deal with the following problem: which conditions must be satisfied by the boundary of a set E in order to have $\overline{\mathcal{F}}(E) < +\infty$? To answer this question, we first study those systems of curves that can be obtained as limits, in the $H^{2,p}$ norm, of a sequence of boundaries of smooth sets. To this aim we introduce the definition of system of curves satisfying the finiteness property (Definition 6.1), and the compatibility condition (Definition 6.6). Then the following result holds (Theorem 6.2): *Let* Γ *be a system of curves of class* $H^{2,p}$ *without crossings and satisfying the finiteness property, and define* E *as the set of all points of* $\mathbb{R}^2\backslash(\Gamma)$ *of odd index with respect to* Γ. *Then* $\partial E \subseteq (\Gamma)$. *Moreover* $\overline{\mathcal{F}}(E) < +\infty$, *i.e., there exists a sequence* $\{E_h\}_h$ *of bounded open sets of class* C^∞ *such that* $E_h \to E$ *in* $L^1(\mathbb{R}^2)$ *as* $h \to +\infty$ *and* $\sup_h \mathcal{F}(E_h) < +\infty$. *In addition, up to a suitable surgery operation on the parameter space of* Γ, *we have that* $\partial E_h \to \Gamma$ *in* $H^{2,p}$ *as* $h \to +\infty$.

Then, quite surprisingly, using only elementary properties of graphs, we prove one of the main results of the paper (Theorem 6.5): *Suppose that* ∂E *is smooth except for a finite number* n *of cusp points. Then*

$$\overline{\mathcal{F}}(E) < +\infty \iff n \text{ is even.}$$

In Section 7 we localize the definition of $\mathcal{F}$ to all open subsets of $\mathbb{R}^2$, i.e., we consider the functional

$$\mathcal{F}(E,\Omega) = \int_{\Omega\cap\partial E} [1 + |\kappa(z)|^p]\, d\mathcal{H}^1(z),$$

where $\Omega \subseteq \mathbb{R}^2$ is an open set and E is a bounded open subset of $\mathbb{R}^2$ such that $\Omega \cap \partial E$ is of class C^2. We prove (Theorem 7.1) that $\mathcal{F}(\cdot,\Omega)$ is $L^1(\Omega)$-lower semicontinuous.

Let $\overline{\mathcal{F}}(\cdot,\Omega)\colon \mathcal{M} \to [0,+\infty]$ denote the lower semicontinuous envelope of $\mathcal{F}(\cdot,\Omega)$ with respect to the $L^1(\Omega)$-topology. One of the main results of this section is that, as conjectured by E. De Giorgi in a slightly different context [4, Conjecture 5], there are sets E such that $\overline{\mathcal{F}}(E,\cdot)$, if considered as a set function, is not a measure. Precisely, we construct an example of a set E whose boundary is smooth except for two cusp points and such that

$$\overline{\mathcal{F}}(E,\Omega_1) + \overline{\mathcal{F}}(E,\Omega_2) < \overline{\mathcal{F}}(E,\mathbb{R}^2) < +\infty,$$

where Ω_1, $\Omega_2 \subseteq \mathbb{R}^2$ are two open sets, with $\Omega_1 \cup \Omega_2 = \mathbb{R}^2$.

This shows that $\overline{\mathcal{F}}(E,\Omega)$ cannot be represented by an integral of the form (1.3). Finally, the value of $\overline{\mathcal{F}}(E,\mathbb{R}^2)$ is computed explicitly in this example (Theorem 7.2).

Acknowledgements

We are indebted to Prof. E. De Giorgi for having suggested us the study of this problem.

2. - Notations and preliminaries

A plane curve $\gamma: [0,1] \to \mathbb{R}^2$ of class C^1 is said to be regular if $\frac{d\gamma(t)}{dt} \neq 0$ for every $t \in [0,1]$. Each closed regular curve $\gamma: [0,1] \to \mathbb{R}^2$ will be identified, in the usual way, with a map $\gamma: \mathbf{S}^1 \to \mathbb{R}^2$, where $\mathbf{S}^1$ denotes the oriented unit circle. By $(\gamma) = \gamma([0,1]) = \{\gamma(t): t \in [0,1]\}$ we denote the trace of γ and by $l(\gamma)$ its length; s denotes the arc length parameter, and $\dot{\gamma}$, $\ddot{\gamma}$ the first and the second derivative of γ with respect to s. Let us fix a real number $p > 1$ and let $p' > 1$ be its conjugate exponent, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. If the second derivative $\ddot{\gamma}$ in the sense of distributions belongs to L^p, then the curvature $\kappa(\gamma)$ of γ is given by $|\ddot{\gamma}|$, and

$$\|\kappa(\gamma)\|_{L^p}^p = \int_0^{l(\gamma)} |\kappa(\gamma)|^p \, ds < +\infty;$$

in this case we say that γ is a curve of class $H^{2,p}$, and we write $\gamma \in H^{2,p}$. Moreover, we put

$$\mathcal{F}(\gamma) = l(\gamma) + \|\kappa(\gamma)\|_{L^p}^p.$$

If $z \in \mathbb{R}^2 \setminus (\gamma)$, $I(\gamma, z)$ is the index of z with respect to γ [3].

$\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure in $\mathbb{R}^2$ [9]; for any $z_0 \in \mathbb{R}^2$, $\varrho > 0$, $B_\varrho(z_0) = \{z \in \mathbb{R}^2: |z - z_0| < \varrho\}$ is the ball centered at z_0 with radius ϱ. We remark that if f is a positive Borel function defined on (γ), then (see for instance [9, Theorem 3.2.6])

$$\int_{\gamma(B)} f(z) \, d\mathcal{H}^1(z) \leq \int_B f(\gamma(s)) \, ds \tag{2.1}$$

for any Borel set $B \subseteq [0, l(\gamma)]$.

Given a measurable set $E \subseteq \mathbb{R}^2$, χ_E denotes its characteristic function, that is $\chi_E(z) = 1$ if $z \in E$, $\chi_E(z) = 0$ if $z \notin E$; $|E|$ is the Lebesgue measure of E. For any subset C of $\mathbb{R}^2$, we denote by $\text{int}(C)$ the interior of C, by $\overline{C}$ the closure of C, and by ∂C the boundary of C. Let $E \subseteq \mathbb{R}^2$; we say that E is of class $H^{2,p}$ (respectively C^2) if E is open, and, near each point $z \in \partial E$, the set E is the subgraph of a function of class $H^{2,p}$ (respectively C^2) with respect to a suitable orthogonal coordinate system. Note that, if ∂E can be parametrized, locally, by arcs of regular curves of class $H^{2,p}$ (respectively C^2), and E lies locally on one side of its boundary, then E is of class $H^{2,p}$ (respectively C^2).

If E is a bounded subset of $\mathbb{R}^2$ of class $H^{2,p}$, then ∂E can be viewed, locally, as the graph of a function of class $H^{2,p}$. This allows us to define, locally, the notion of curvature of ∂E at $\mathcal{H}^1$-almost every point of ∂E, by using the classical formulas involving second derivatives. It is easy to see that the function $\kappa(z)$ does not depend on the choice of the coordinate system used to describe ∂E as a graph, and belongs to $L^p(\partial E, \mathcal{H}^1)$.

If $\mathcal{M}$ denotes the class of all Lebesgue measurable subsets of $\mathbb{R}^2$, we define the map $\mathcal{F} : \mathcal{M} \to [0, +\infty]$ by

$$\mathcal{F}(E) = \begin{cases} \displaystyle\int_{\partial E} [1 + |\kappa(z)|^p] \, \mathrm{d}\mathcal{H}^1(z) & \text{if } E \text{ is a bounded open set of class } C^2, \\ +\infty & \text{elsewhere on } \mathcal{M}, \end{cases} \tag{2.2}$$

where $\kappa(z)$ is the curvature of ∂E at the point z.

Let $g \in L^\infty(\mathbb{R}^2)$ be a function such that $\{z \in \mathbb{R}^2 : g(z) < 0\} \subseteq B_R(0)$, for a suitable $R > 0$. Let us consider the minimum problem

$$\inf_E \left\{ \int_{\partial E} [1 + |\kappa(z)|^p] \, \mathrm{d}\mathcal{H}^1(z) + \int_E g(z) \, \mathrm{d}z \right\}, \tag{2.3}$$

where E varies over all bounded open subsets of $\mathbb{R}^2$ with boundary of class C^2. It is clear that (2.3) is equivalent to the problem

$$\inf_{E \in \mathcal{M}} \mathcal{G}(E), \quad \text{where } \mathcal{G}(E) = \mathcal{F}(E) + \int_E g(z) \, \mathrm{d}z, \tag{2.4}$$

in the sense that (2.3) and (2.4) have the same minimum values and the same minimizers. If the region where g is less than a prescribed negative number is large enough, then it is immediate to verify that the minimum value of problem (2.3) is negative, hence every solution of (2.3) must be non-empty.

We shall identify $\mathcal{M}$ with a closed subset of $L^1(\mathbb{R}^2)$ by means of the map $E \mapsto \chi_E$. The $L^1(\mathbb{R}^2)$-topology on $\mathcal{M}$ is, therefore, the topology on $\mathcal{M}$ induced by the distance $\mathrm{d}(E_1, E_2) = |E_1 \Delta E_2|$, where E_1, $E_2 \in \mathcal{M}$ and Δ is the symmetric difference of sets.

Let us prove that any minimizing sequence $\{E_h\}_h$ of $\mathcal{G}$ is relatively compact in $L^1(\mathbb{R}^2)$. Let $\{E_h\}_h$ be such a minimizing sequence; clearly, we can assume that $\sup_h \mathcal{G}(E_h) < +\infty$, hence E_h is a bounded open set of class C^2 for any h. Then, since

$$\int_{\partial E_h} [1 + |\kappa_h(z)|^p] \, \mathrm{d}\mathcal{H}^1(z) \leq \sup_h \mathcal{G}(E_h) + \|g\|_\infty |B_R(0)|,$$

it follows that

$$H = \sup_h \mathcal{H}^1(\partial E_h) < +\infty.$$

Now we will show that there exists $h_0 \in \mathbf{N}$ such that $E_h \subseteq B_{R+H}(0)$ for any $h \geq h_0$. If this condition is not satisfied, then there exists a subsequence of $\{E_h\}_h$, still denoted by $\{E_h\}_h$, such that $E_h \backslash B_{R+H}(0) \neq \emptyset$ for every $h \in \mathbf{N}$. As the total length of ∂E_h is bounded by H, each set E has a connected component C_h which does not meet $B_R(0)$. Denoting by E'_h the set $E'_h = E_h \backslash C_h$, we get

$\mathcal{G}(E'_h) < \mathcal{G}(E_h)$ (recall that $g \geq 0$ on $\mathbb{R}^2 \setminus B_R(0)$). Since also $\{E'_h\}_h$ is a minimizing sequence, it follows that necessarily $\mathcal{G}(C_h) = \mathcal{G}(E_h) - \mathcal{G}(E'_h) \to 0$ as $h \to +\infty$. On the other hand, one can show (see (3.5)) that C_h gives a positive contribution to the energy $\mathcal{F}$ independent of h, that is $\mathcal{F}(C_h) \not\to 0$, which gives a contradiction.

Hence there exists $h_0 \in \mathbf{N}$ such that

$$E_h \subseteq B_{R+H}(0) \qquad \text{for any } h \geq h_0.$$

We deduce that

$$\sup_h [\mathcal{H}^1(\partial E_h) + |E_h|] < +\infty, \qquad E_h \subseteq B_{R+H}(0), \quad \text{for any } h.$$

Using the Rellich Compactness Theorem in BV (see [12, Theorem 1.19]), it follows that there exist a bounded set E of finite perimeter and a subsequence $\{E_{h_k}\}_k$ such that $E_{h_k} \to E$ in $L^1(\mathbb{R}^2)$ as $k \to +\infty$, and this shows that any minimizing sequence of $\mathcal{G}$ is relatively compact in $L^1(\mathbb{R}^2)$.

We denote by $\overline{\mathcal{F}}$ the lower semicontinuous envelope of $\mathcal{F}$ with respect to the topology of $L^1(\mathbb{R}^2)$. It is known that, for every $E \in \mathcal{M}$, we have

$$\overline{\mathcal{F}}(E) = \inf \left\{ \liminf_{h \to +\infty} \mathcal{F}(E_h) : E_h \to E \text{ in } L^1(\mathbb{R}^2) \text{ as } h \to +\infty \right\}.$$

For the main properties of the relaxed functional we refer to [2]. In particular, one can prove that

$$\inf_{E \in \mathcal{M}} \left\{ \mathcal{F}(E) + \int_E g(z)\,\mathrm{d}z \right\} = \min_{E \in \mathcal{M}} \left\{ \overline{\mathcal{F}}(E) + \int_E g(z)\,\mathrm{d}z \right\}.$$

In addition, every minimizing sequence of $\mathcal{F}(E) + \int_E g(z)\,\mathrm{d}z$ has a subsequence converging to a minimum point of $\overline{\mathcal{F}}(E) + \int_E g(z)\,\mathrm{d}z$ and every minimum point of $\overline{\mathcal{F}}(E) + \int_E g(z)\,\mathrm{d}z$ is the limit of a minimizing sequence of $\mathcal{F}(E) + \int_E g(z)\,\mathrm{d}z$.

The main purpose of this paper is to study the properties of the functional $\overline{\mathcal{F}}$. From the definition, it follows immediately that $\overline{\mathcal{F}}(E) < +\infty$ if and only if there exists a sequence $\{E_h\}_h$ of bounded open sets of class C^2 such that $E_h \to E$ in $L^1(\mathbb{R}^2)$ as $h \to +\infty$ and

$$\sup_h \mathcal{H}^1(\partial E_h) < +\infty, \qquad \sup_h \int_{\partial E_h} |\kappa_h(z)|^p \,\mathrm{d}\mathcal{H}^1(z) < +\infty, \tag{2.5}$$

where κ_h denotes the curvature of ∂E_h.

DEFINITION 2.1. By a system of curves we mean a finite family $\Gamma =$

$\{\gamma^1, \ldots, \gamma^m\}$ of closed regular curves of class C^1 such that $\left|\frac{d\gamma^i}{dt}\right|$ is constant on $[0,1]$ for any $i = 1, \ldots, m$. Denoting by S the disjoint union of m unit circles $\mathbf{S}^1_1, \ldots, \mathbf{S}^1_m$, we shall identify Γ with the map $\Gamma: S \to \mathbb{R}^2$ defined by $\Gamma_{|\mathbf{S}^1_i} = \gamma^i$, for $i = 1, \ldots, m$. The trace (Γ) of Γ is the union of the traces of the curves γ^i, i.e., $(\Gamma) = \bigcup_{i=1}^m (\gamma^i)$.

DEFINITION 2.2. If $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ is a system of curves of class $H^{2,p}$, we define

$$l(\Gamma) = \sum_{i=1}^m l(\gamma^i), \quad \|\kappa(\Gamma)\|^p_{L^p} = \sum_{i=1}^m \|\kappa(\gamma^i)\|^p_{L^p} = \sum_{i=1}^m \int_0^{l(\gamma^i)} |\ddot{\gamma}^i(s)|^p \, ds,$$

and

$$\mathcal{F}(\Gamma) = \sum_{i=1}^m \mathcal{F}(\gamma^i) = \sum_{i=1}^m l(\gamma^i) + \|\kappa(\gamma^i)\|^p_{L^p}.$$

If $z \in \mathbb{R}^2 \setminus (\Gamma)$ we define the index of z with respect to Γ as $I(\Gamma, z) = \sum_{i=1}^m I(\gamma^i, z)$.

As $\left|\frac{d\gamma^i}{dt}\right|$ is constant on $[0,1]$, we have $s(t) = tl(\gamma^i)$, hence

$$\int_0^{l(\gamma^i)} |\ddot{\gamma}^i(s)|^p \, ds = l(\gamma^i)^{1-2p} \int_0^1 \left|\frac{d^2\gamma^i}{dt^2}\right|^p dt. \tag{2.6}$$

3. - Semicontinuity of $\mathcal{F}$

DEFINITION 3.1. By a disjoint system of curves we mean a system of curves $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ such that $(\gamma^i) \cap (\gamma^j) = \emptyset$ for any $i, j = 1, \ldots, m$, $i \neq j$.

Let $\{E_h\}_h$ be a sequence of sets satisfying (2.5). Note that, for any h, there exists a suitable parametrization Γ_h of ∂E_h such that the sequence $\{\Gamma_h\}_h$ satisfies

$$\sup_h l(\Gamma_h) < +\infty, \qquad \sup_h \|\kappa(\Gamma_h)\|^p_{L^p} < +\infty. \tag{3.1}$$

LEMMA 3.1. *Let $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ be a system of curves of class $H^{2,p}$. Then*

$$m \leq l(\Gamma) \|\kappa(\Gamma)\|^{p'}_{L^p} (2\pi)^{-p'}. \tag{3.2}$$

PROOF. Let γ be a simple closed regular curve of class $H^{2,p}$; let us prove that

$$l(\gamma) \geq (2\pi)^{p'} \|\kappa(\gamma)\|_{L^p}^{-p'}. \tag{3.3}$$

If $\gamma \in \mathcal{C}^2$, then [5, Theorem 5.7.3]

$$\int_0^{l(\gamma)} |\ddot{\gamma}(s)|\, \mathrm{d}s \geq 2\pi. \tag{3.4}$$

By a standard approximation argument, inequality (3.4) holds for any curve γ of class $H^{2,p}$. Then (3.3) follows from the Hölder inequality, since, if $\gamma \in H^{2,p}$,

$$2\pi \leq \int_0^{l(\gamma)} |\ddot{\gamma}(s)|\, \mathrm{d}s \leq l(\gamma)^{\frac{1}{p'}} \left(\int_0^{l(\gamma)} |\ddot{\gamma}(s)|^p\, \mathrm{d}s \right)^{\frac{1}{p}} = l(\gamma)^{\frac{1}{p'}} \|\kappa(\gamma)\|_{L^p}.$$

Hence $l(\gamma^i) \geq (2\pi)^{p'} \|\kappa(\gamma^i)\|_{L^p}^{-p'}$ for any $i = 1, \dots, m$. Then, recalling Definition 2.2, it follows that

$$l(\Gamma) = \sum_{i=1}^m l(\gamma^i) \geq \sum_{i=1}^m (2\pi)^{p'} \|\kappa(\gamma^i)\|_{L^p}^{-p'} \geq (2\pi)^{p'} \|\kappa(\Gamma)\|_{L^p}^{-p'} m. \qquad \square$$

The following generalization of inequality (3.3), as well as Corollary 3.1, will be useful in Section 7.

LEMMA 3.2. *Let $\gamma: [0,1] \to \mathbb{R}^2$ be a regular curve of class $H^{2,p}$ and let $\theta: [0,1] \to \mathbb{R}$ be a continuous function such that $\theta(t)$ is the oriented angle between the x-axis and the tangent vector of γ at t. Then*

$$l(\gamma) \geq |\theta_1 - \theta_0|^{p'} \|\kappa(\gamma)\|_{L^p}^{-p'}, \qquad \textit{where } \theta_0 = \theta(0) \textit{ and } \theta_1 = \theta(1).$$

PROOF. Let us write with obvious notation $\dot{\gamma}(s) = (\cos\theta(s), \sin\theta(s))$. Then, using the Hölder inequality, it follows that

$$\begin{aligned} \|\kappa(\gamma)\|_{L^p}^p &= \int_0^{l(\gamma)} |\dot{\theta}(s)|^p\, \mathrm{d}s \geq l(\gamma)^{-\frac{p}{p'}} \left(\int_0^{l(\gamma)} |\dot{\theta}(s)|\, \mathrm{d}s \right)^p \\ &\geq l(\gamma)^{-\frac{p}{p'}} \left(\left| \int_0^{l(\gamma)} \dot{\theta}(s)\, \mathrm{d}s \right| \right)^p = l(\gamma)^{-\frac{p}{p'}} \left(\int_0^1 \frac{\mathrm{d}\theta}{\mathrm{d}t}(t)\, \mathrm{d}t \right)^p \\ &= |\theta_1 - \theta_0|^p l(\gamma)^{-\frac{p}{p'}}. \end{aligned} \qquad \square$$

COROLLARY 3.1. *Let γ, θ_0, θ_1 be as in Lemma* 3.2. *Then*

$$l(\gamma) + \|\kappa(\gamma)\|_{L^p}^p \geq |\theta_1 - \theta_0| \left[\left(\frac{p}{p'}\right)^{\frac{1}{p}} + \left(\frac{p'}{p}\right)^{\frac{1}{p'}} \right]. \tag{3.5}$$

PROOF. Lemma 3.2 implies that

$$l(\gamma) + \|\kappa(\gamma)\|_{L^p}^p \geq l(\gamma) + l(\gamma)^{-\frac{p}{p'}} |\theta_1 - \theta_0|^p.$$

Hence, since the minimum point of the function $\lambda + \lambda^{-\frac{p}{p'}} |\theta_1 - \theta_0|^p$ is reached at $\lambda = |\theta_1 - \theta_0| \left(\frac{p}{p'}\right)^{\frac{1}{p}}$, we obtain

$$\begin{aligned} l(\gamma) + \|\kappa(\gamma)\|_{L^p}^p &\geq |\theta_1 - \theta_0| \left(\frac{p}{p'}\right)^{\frac{1}{p}} + |\theta_1 - \theta_0|^p |\theta_1 - \theta_0|^{-\frac{p}{p'}} \left(\frac{p}{p'}\right)^{-\frac{1}{p'}} \\ &= |\theta_1 - \theta_0| \left[\left(\frac{p}{p'}\right)^{\frac{1}{p}} + \left(\frac{p'}{p}\right)^{\frac{1}{p'}} \right]. \end{aligned}$$

□

DEFINITION 3.2. Let $E \subseteq \mathbb{R}^2$ be a bounded open set of class C^1. We say that a disjoint system of curves Γ is an oriented parametrization of ∂E if each curve of the system is simple, $(\Gamma) = \partial E$, and, in addition,

$$E = \{z \in \mathbb{R}^2 : I(\Gamma, z) = 1\}, \qquad \mathbb{R}^2 \setminus \overline{E} = \{z \in \mathbb{R}^2 : I(\Gamma, z) = 0\}.$$

PROPOSITION 3.1. *Each bounded subset E of $\mathbb{R}^2$ of class $H^{2,p}$ (respectively C^2) admits an oriented parametrization of class $H^{2,p}$ (respectively C^2).*

PROOF. Since E is of class $H^{2,p}$ each connected component of ∂E can be parametrized by a regular closed curve of class $H^{2,p}$. As ∂E is compact and locally connected, it has a finite number of connected components. The statement about the index follows from Jordan's Theorem. □

DEFINITION 3.3. We say that a sequence $\{\Gamma_h\}_h$ of systems of curves of class $H^{2,p}$ converges weakly in $H^{2,p}$ to a system of curves $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ of class $H^{2,p}$ if the number of curves of each system Γ_h equals the number of curves of Γ for h large enough, i.e., $\Gamma_h = \{\gamma_h^1, \ldots, \gamma_h^m\}$, and, in addition, $\gamma_h^i \rightharpoonup \gamma^i$ weakly in $H^{2,p}$ as $h \to +\infty$ for any $i = 1, \ldots, m$.

Note that, if $\{\Gamma_h\}_h$ converges to $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ weakly in $H^{2,p}$, then

$$\gamma_h^i \to \gamma^i \qquad \text{in } C^1 \text{ as } h \to +\infty, \tag{3.6}$$

for any $i = 1, \ldots, m$. In particular, $l(\gamma_h^i) \to l(\gamma^i)$ as $h \to +\infty$.

THEOREM 3.1. *Let* $\{\Gamma_h\}_h$ *be a sequence of systems of curves of class* $H^{2,p}$ *satisfying* (3.1) *such that the traces* (Γ_h) *are contained in a bounded subset of* $\mathbb{R}^2$ *independent of* h. *Then* $\{\Gamma_h\}_h$ *has a subsequence which converges weakly in* $H^{2,p}$ *to a system of curves* Γ.

PROOF. By (3.2), using (3.1) it follows that the number m_h of curves of the system Γ_h is bounded uniformly with respect to h. Hence, for a subsequence (still denoted by $\{\Gamma_h\}_h$), there exists an integer m such that $\Gamma_h = \{\gamma_h^1, \ldots, \gamma_h^m\}$ for any h. Fix $i \in \{1, \ldots, m\}$; using (3.3) and (3.1) we get that there exist two positive constants c_1, c_2 such that

$$c_1 \le l(\gamma_h^i) \le c_2 \qquad \text{for any } h.$$

Then, using (2.6) we obtain that there exists a positive constant c_3 such that $\int_0^1 \left| \frac{d^2\gamma_h^i}{dt^2} \right|^p dt \le l(\gamma_h^i)^{2p-1} \|\kappa(\gamma_h^i)\|_{L^p}^p \le c_3\ c_2^{2p-1}$, whence, as (Γ_h) are bounded uniformly with respect to h by the hypothesis, the family $\{\gamma_h^i\}_h$ is equibounded in $H^{2,p}$. Then, for a subsequence, there exist m curves $\gamma^1, \ldots, \gamma^m$ of class $H^{2,p}$ such that $\gamma_h^i \rightharpoonup \gamma^i$ weakly in $H^{2,p}$ as $h \to +\infty$, for any $i = 1, \ldots, m$. This shows that $\{\Gamma_h\}_h$ converges to the system of curves $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ weakly in $H^{2,p}$.
□

DEFINITION 3.4. We say that Γ is a limit system of curves of class $H^{2,p}$ if Γ is the weak $H^{2,p}$ limit of a sequence $\{\Gamma_h\}_h$ of oriented parametrizations of bounded open sets of class $H^{2,p}$.

The following remark is an easy consequence of (3.6).

REMARK 3.1. If Γ is a limit system of curves of class $H^{2,p}$, then $I(\Gamma, z) \in \{0, 1\}$ for any $z \in \mathbb{R}^2 \setminus (\Gamma)$.

In order to prove the semicontinuity Theorem 3.2, we need the following Lemma.

LEMMA 3.3. *Let* $E \subseteq \mathbb{R}^2$ *be a measurable set, let* $\{E_h\}_h$ *be a sequence of bounded open sets of class* $H^{2,p}$ *satisfying* (2.5), *and suppose that* $E_h \to E$ *in* $L^1(\mathbb{R}^2)$ *as* $h \to +\infty$. *Let us define*

$$E^\star = \{z \in \mathbb{R}^2 : \exists r > 0\ |B_r(z) \setminus E| = 0\},$$

$$F^\star = \{z \in \mathbb{R}^2 : \exists r > 0\ |B_r(z) \cap E| = 0\}.$$

Then $E^\star$ *and* $F^\star$ *are open,* $E^\star$ *is bounded,* $|E \Delta E^\star| = 0$, $E^\star = \mathrm{int}(\mathbb{R}^2 \setminus F^\star)$, $F^\star = \mathrm{int}(\mathbb{R}^2 \setminus E^\star)$, *and*

$$\partial E^\star = \partial F^\star = \{z \in \mathbb{R}^2 : 0 < |B_r(z) \cap E| < |B_r(z)|\ \forall r > 0\}. \tag{3.7}$$

Moreover there exists a limit system Γ *of curves of class* $H^{2,p}$ *with the following properties:*

(i) $E^\star = \text{int}(A_\Gamma \cup (\Gamma)) \supseteq A_\Gamma$, *where* $A_\Gamma = \{z \in \mathbb{R}^2 \backslash (\Gamma) : I(\Gamma, z) = 1\}$;

(ii) $F^\star = \text{int}(B_\Gamma \cup (\Gamma)) \supseteq B_\Gamma$, *where* $B_\Gamma = \{z \in \mathbb{R}^2 \backslash (\Gamma) : I(\Gamma, z) = 0\}$;

(iii) $\partial E^\star = \partial F^\star = \partial A_\Gamma \cap \partial B_\Gamma \subseteq (\Gamma)$.

PROOF. It follows immediately from the definition that $E^\star$ and $F^\star$ are open. For any h, by (3.1) and by Lemma 3.1, ∂E_h admits an oriented parametrization Δ_h of class $H^{2,p}$ and the number m_h of curves of the system Δ_h is bounded uniformly with respect to h. Hence, for a subsequence (still denoted by $\{\Delta_h\}_h$), there exists an integer m such that $\Delta_h = \{\gamma_h^1, \ldots, \gamma_h^m\}$ for any h. In order to apply compactness arguments, we shall transform the sequence $\{\Delta_h\}_h$, which is not necessarily bounded, (see Figure 3.1), into a sequence $\{\Gamma_h\}_h$ of oriented parametrizations whose traces are contained in a bounded set independent of h.

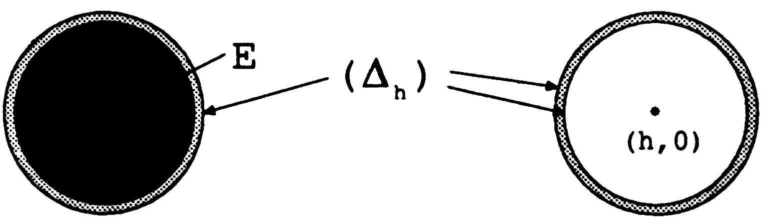


Fig. 3.1: *The sequence* $\{\Delta_h\}_h$ *is not bounded: two connected components of* (Δ_h) *approaches infinity as* $h \to +\infty$ *with length and curvature uniformly bounded with respect to h.*

Let us consider first the sequence of curves $\{\gamma_h^1\}_h$. If the sequence $a_h^1 = \sup_{t\in[0,1]} |\gamma_h^1(t)|$ converges to $+\infty$ as $h \to +\infty$, then we eliminate γ_h^1, i.e., we replace Δ_h by the system $\Delta_h^1 = \{\gamma_h^2, \ldots, \gamma_h^m\}$. Note that, as $l(\gamma_h^1)$ is uniformly bounded with respect to h, the behaviour of $\{a_h^1\}_h$ gives that, for any $r > 0$, there exists $h_r \in \mathbf{N}$ such that $I(\gamma_h^1, z) = 0$ (hence $I(\Delta_h^1, z) = I(\Delta_h, z)$) for any $h \geq h_r$ and for any $z \in B_r(0) \backslash (\Delta_h)$. If $\{a_h^1\}_h$ does not tend to $+\infty$, there exists a subsequence, still denoted by $\{\gamma_h^1\}_h$, such that the traces (γ_h^1) are bounded uniformly with respect to h. In this case we define $\Delta_h^1 = \Delta_h$. Starting from $\{\Delta_h^1\}_h$, we repeat the same procedure for $\{\gamma_h^2\}_h$, obtaining a new sequence of systems of curves $\{\Delta_h^2\}_h$. After m steps, we end up with a sequence of systems $\{\Delta_h^m\}_h$, which we shall denote by $\{\Gamma_h\}_h$. By construction, for every h, Γ_h is a disjoint system of curves of class $H^{2,p}$, and for every $r > 0$ there exists $h_r \in \mathbf{N}$ such that

$$I(\Gamma_h, z) = I(\Delta_h, z) \tag{3.8}$$

for any $h \geq h_r$ and for any $z \in B_r(0) \backslash (\Delta_h)$. It is clear also by construction that the traces (Γ_h) are bounded uniformly with respect to h, i.e., there exists $R > 0$

such that

$$(\Gamma_h) \subseteq B_R(0) \qquad \text{for any } h. \tag{3.9}$$

From (3.8) and (3.9) it follows easily that, for h large enough, for any $z \in \mathbb{R}^2 \setminus (\Gamma_h)$ we have $I(\Gamma_h, z) \in \{0,1\}$. Let us define $A_h = \{z \in \mathbb{R}^2 \setminus (\Gamma_h): I(\Gamma_h, z) = 1\}$. Since Γ_h is a disjoint system of class $H^{2,p}$, A_h is a bounded open set and $\partial A_h = (\Gamma_h)$. Using Theorem 3.1, there exists a subsequence $\{\Gamma_h\}_h$ which converges weakly in $H^{2,p}$ to a limit system Γ of curves of class $H^{2,p}$. By (3.9) we have $(\Gamma) \subseteq \overline{B_R(0)}$. By Remark 3.1, $I(\Gamma, z) \in \{0,1\}$ for any $z \in \mathbb{R}^2 \setminus (\Gamma)$. Let A_Γ be the open set defined in (i). It is clear that $A_\Gamma \subseteq B_R(0)$. Since $\chi_{A_h}(z) = I(\Gamma_h, z)$ for any $z \in \mathbb{R}^2 \setminus (\Gamma_h)$, and $\chi_{A_\Gamma}(z) = I(\Gamma, z)$ for any $z \in \mathbb{R}^2 \setminus (\Gamma)$, by the continuity property of the index and by the Dominated Convergence Theorem we have that $A_h \to A_\Gamma$ in $L^1(\mathbb{R}^2)$ as $h \to +\infty$. Let us prove that

$$|E \Delta A_\Gamma| = 0. \tag{3.10}$$

By (3.8) and (3.9), for any $r \geq R$, we have that $A_h = E_h \cap B_r(0)$ for h large enough; passing to the limit as $h \to +\infty$, we obtain that $|A_\Gamma \Delta (E \cap B_r(0))| = 0$. Since r is arbitrary, we get (3.10).

Let us prove that $|E \Delta E^\star| = 0$. By (3.10), it is enough to prove that

$$|A_\Gamma \Delta E^\star| = 0. \tag{3.11}$$

Since A_Γ is open and $|E \Delta A_\Gamma| = 0$, for any $z \in A_\Gamma$ there exists $r > 0$ such that $B_r(z) \subseteq A_\Gamma$, so that $|B_r(z) \setminus E| = |B_r(z) \setminus A_\Gamma| = 0$, hence $z \in E^\star$. Therefore

$$A_\Gamma \subseteq E^\star. \tag{3.12}$$

To prove (3.11), being $|(\Gamma)| = 0$, it is sufficient to show that

$$E^\star \subseteq A_\Gamma \cup (\Gamma). \tag{3.13}$$

If $z \notin A_\Gamma \cup (\Gamma)$, then $I(\Gamma, z) = 0$. Since the index is locally constant, there exists $r > 0$ such that $I(\Gamma, w) = 0$ for any $w \in B_r(z)$. As $|B_r(z) \cap E| = |B_r(z) \cap A_\Gamma| = 0$, we have $z \notin E^\star$. This proves (3.13), so we can conclude that $|E \Delta E^\star| = 0$. Moreover (3.13) shows that $E^\star$ is bounded.

Let us prove (i). The inclusion $\mathrm{int}(A_\Gamma \cup (\Gamma)) \supseteq A_\Gamma$ follows from the fact that A_Γ is open. The inclusion $E^\star \subseteq \mathrm{int}(A_\Gamma \cup (\Gamma))$ follows from (3.13) and from the fact that $E^\star$ is open. To prove the opposite inclusion, let $z \in \mathrm{int}(A_\Gamma \cup (\Gamma))$. Then there exists $r > 0$ such that $B_r(z) \subseteq A_\Gamma \cup (\Gamma)$. Hence $|B_r(z) \setminus E| = |B_r(z) \setminus A_\Gamma| = 0$ by (3.10), and, therefore, $z \in E^\star$. This shows that $E^\star = \mathrm{int}(A_\Gamma \cup (\Gamma))$.

Let us prove (ii). The inclusion $\mathrm{int}(B_\Gamma \cup (\Gamma)) \supseteq B_\Gamma$ follows from the fact that B_Γ is open. Since $E^\star$ and $F^\star$ are disjoint open sets, we have $F^\star \subseteq \mathrm{int}(\mathbb{R}^2 \setminus E^\star)$.

As $B_\Gamma \cup (\Gamma) = \mathbb{R}^2 \backslash A_\Gamma$, by (3.12) we have $\mathbb{R}^2 \backslash E^\star \subseteq B_\Gamma \cup (\Gamma)$, hence

$$F^\star \subseteq \mathrm{int}(\mathbb{R}^2 \backslash E^\star) \subseteq \mathrm{int}(B_\Gamma \cup (\Gamma)). \tag{3.14}$$

To prove the opposite inclusion, let $z \in \mathrm{int}(B_\Gamma \cup (\Gamma))$. Then there exists $r > 0$ such that $B_r(z) \subseteq B_\Gamma \cup (\Gamma)$, hence $|B_r(z) \cap E| = |B_r(z) \cap A_\Gamma| = 0$ (see (3.10)), and, therefore, $z \in F^\star$. This shows that $\mathrm{int}(B_\Gamma \cup (\Gamma)) \subseteq F^\star$, which, together with (3.14), gives

$$F^\star = \mathrm{int}(\mathbb{R}^2 \backslash E^\star) = \mathrm{int}(B_\Gamma \cup (\Gamma)), \tag{3.15}$$

and concludes the proof of (ii).

Since $E^\star$ and $F^\star$ are disjoint open sets, we have $E^\star \subseteq \mathrm{int}(\mathbb{R}^2 \backslash F^\star)$. As $A_\Gamma \cup (\Gamma) = \mathbb{R}^2 \backslash B_\Gamma$, by (ii) we have $\mathbb{R}^2 \backslash F^\star \subseteq \mathbb{R}^2 \backslash B_\Gamma = A_\Gamma \cup (\Gamma)$, hence $E^\star \subseteq \mathrm{int}(\mathbb{R}^2 \backslash F^\star) \subseteq \mathrm{int}(A_\Gamma \cup (\Gamma))$. By (i) we conclude that

$$E^\star = \mathrm{int}(\mathbb{R}^2 \backslash F^\star), \tag{3.16}$$

which, together with (3.15), gives

$$\begin{aligned} \partial E^\star &= \mathbb{R}^2 \backslash (E^\star \cup \mathrm{int}(\mathbb{R}^2 \backslash E^\star)) = \mathbb{R}^2 \backslash (E^\star \cup F^\star) \\ &= \mathbb{R}^2 \backslash (\mathrm{int}(\mathbb{R}^2 \backslash F^\star) \cup F^\star) = \partial F^\star \end{aligned}$$

and proves (3.7).

We will now prove (iii). Since A_Γ and B_Γ are disjoint open sets, we have $\partial A_\Gamma \cup \partial B_\Gamma \subseteq \mathbb{R}^2 \backslash (A_\Gamma \cup B_\Gamma) = (\Gamma)$. Let us prove that $\partial E^\star \subseteq \overline{A}_\Gamma$. For every $z \in \partial E^\star$ and for every neighbourhood U of z we select $w \in U \cap E^\star = U \cap \mathrm{int}(A_\Gamma \cup (\Gamma))$. Then there exists $r > 0$ such that $B_r(w) \subseteq U$ and $B_r(w) \subseteq A_\Gamma \cup (\Gamma)$. As (Γ) has Lebesgue measure zero, we have $B_r(w) \cap A_\Gamma \neq \emptyset$, hence $U \cap A_\Gamma \neq \emptyset$. This implies that $z \in \overline{A}_\Gamma$.

Let us prove that $\partial E^\star \subseteq \overline{B}_\Gamma$. For every $z \in \partial E^\star$ and for every neighbourhood U of z we select $w \in U \backslash E^\star = U \backslash \mathrm{int}(A_\Gamma \cup (\Gamma))$. Then there exists $r > 0$ such that $B_r(w) \subseteq U$ and $B_r(w) \cap B_\Gamma = B_r(w) \backslash (A_\Gamma \cup (\Gamma)) \neq \emptyset$. This shows that $U \cap B_\Gamma \neq \emptyset$, hence $z \in \overline{B}_\Gamma$.

Therefore $\partial E^\star \subseteq \overline{A}_\Gamma \cap \overline{B}_\Gamma$. Since A_Γ and B_Γ are disjoint open sets, we have $\overline{A}_\Gamma \cap \overline{B}_\Gamma = \partial A_\Gamma \cap \partial B_\Gamma$, hence $\partial E^\star \subseteq \partial A_\Gamma \cap \partial B_\Gamma$.

Let us prove that $\partial A_\Gamma \cap \partial B_\Gamma \subseteq \partial E^\star$. By (i) we have $\partial A_\Gamma \subseteq \overline{A}_\Gamma \subseteq \overline{E^\star}$, by (ii) we have $\partial B_\Gamma \subseteq \overline{B}_\Gamma \subseteq \overline{F^\star}$, and by (3.16) we have $E^\star = \mathbb{R}^2 \backslash \overline{F^\star}$, hence $\overline{F^\star} = \mathbb{R}^2 \backslash E^\star$. It follows that

$$\partial A_\Gamma \cap \partial B_\Gamma \subseteq \overline{F^\star} \cap \overline{E^\star} = \overline{E^\star} \backslash E^\star = \partial E^\star$$

hence $\partial E^\star = \partial A_\Gamma \cap \partial B_\Gamma$. From (3.7) we obtain $\partial E^\star = \partial F^\star$, which concludes the proof of (iii). □

LEMMA. 3.4. *Let $E \subseteq \mathbb{R}^2$ be a bounded open set of class $H^{2,p}$, and let Γ be a system of curves of class $H^{2,p}$ such that $\partial E \subseteq (\Gamma)$. Then*

$$\int_{\partial E} |\kappa(z)|^p \, d\mathcal{H}^1(z) \leq \|\kappa(\Gamma)\|_{L^p}^p. \tag{3.17}$$

Hence

$$\int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z) \leq \mathcal{F}(\Gamma)$$

(see Definition 2.2).

PROOF. Let $\Gamma = \{\gamma^1, \dots, \gamma^m\}$. For any measurable set $T \subseteq \partial E$ and any $i = 1, \dots, m$ let

$$V_i = \{s \in [0, l(\gamma^i)] : \gamma^i(s) \in T\},$$

$$T_i = \{z \in \mathbb{R}^2 : z = \gamma^i(s), \ s \in V_i\} = (\gamma^i) \cap T.$$

Since $\partial E \subseteq (\Gamma)$, we have $T = \bigcup_{i=1}^m T_i$. To prove the thesis it will be enough to show that

$$\int_{T_i} |\kappa(z)|^p \, d\mathcal{H}^1(z) \leq \int_{V_i} |\ddot{\gamma}^i(s)|^p \, ds \qquad \text{for any } i = 1, \dots, m, \tag{3.18}$$

where T varies over a suitable finite measurable partition $\mathcal{P}$ of ∂E. In fact, from (3.18) it follows that

$$\int_T |\kappa(z)|^p \, d\mathcal{H}^1(z) \leq \sum_{i=1}^m \int_{T_i} |\kappa(z)|^p \, d\mathcal{H}^1(z) \leq \sum_{i=1}^m \int_{V_i} |\ddot{\gamma}^i(s)|^p \, ds,$$

hence

$$\int_{\partial E} |\kappa(z)|^p \, d\mathcal{H}^1(z) = \sum_{T \in \mathcal{P}} \int_{T \cap \partial E} |\kappa(z)|^p \, d\mathcal{H}^1(z) \leq \sum_{i=1}^m \int_0^{l(\gamma^i)} |\ddot{\gamma}^i(s)|^p \, ds = \|\kappa(\Gamma)\|_{L^p}^p,$$

that proves the assertion. We choose the finite partition $\mathcal{P}$ of ∂E as follows. Any element of $\mathcal{P}$ must be contained in a rectangle in which ∂E is a cartesian graph. This means that for any $T \in \mathcal{P}$ we can choose a suitable orthogonal coordinate system, two bounded open intervals I, J, and a function $f: I \to J$ of class $H^{2,p}$ such that

$$(I \times J) \cap E = \{(x, y) \in I \times J : y < f(x)\},$$

$$(I \times J) \cap \partial E = \{(x, f(x)) : x \in I\},$$

$$T \subseteq (I \times J) \cap \partial E.$$

Let us fix $i \in \{1, \dots, m\}$ and let $\gamma^i(s) = (\gamma_1^i(s), \gamma_2^i(s))$. For any $s \in V_i$ we have $\gamma_2^i(s) = f(\gamma_1^i(s))$. Since the functions γ_2^i and $f(\gamma_1^i)$ are of class $H^{2,p}$ on an open set containing V_i and coincide on V_i, we have [11]

$$\dot\gamma_2^i(s) = \frac{\mathrm{d}}{\mathrm{d}s} f(\gamma_1^i(s)) \text{ on } V_i, \qquad \ddot\gamma_2^i(s) = \frac{\mathrm{d}^2}{\mathrm{d}s^2} f(\gamma_1^i(s)) \text{ for a.e. } s \in V_i.$$

Then, since $|\dot\gamma^i(s)| = 1$, we obtain

$$\dot\gamma_1^i(s) = \frac{\pm 1}{\sqrt{1 + (f'(\gamma_1^i(s)))^2}}, \qquad \dot\gamma_2^i(s) = \frac{\pm f'(\gamma_1^i(s))}{\sqrt{1 + (f'(\gamma_1^i(s)))^2}} \text{ for any } s \in V_i,$$

and

$$\ddot\gamma_1^i(s) = \frac{-f'(\gamma_1^i(s)) f''(\gamma_1^i(s))}{(1 + (f'(\gamma_1^i(s)))^2)^2}, \qquad \ddot\gamma_2^i(s) = \frac{f''(\gamma_1^i(s))}{(1 + (f'(\gamma_1^i(s)))^2)^2} \text{ for a.e. } s \in V_i.$$

It follows that

$$|\ddot\gamma^i(s)| = \frac{|f''(\gamma_1^i(s))|}{(1 + (f'(\gamma_1^i(s)))^2)^{\frac{3}{2}}} = |\kappa(\gamma^i(s))| \qquad \text{for a.e. } s \in V_i.$$

Hence, using (2.1),

$$\int_{V_i} |\ddot\gamma^i(s)|^p \,\mathrm{d}s = \int_{V_i} |\kappa(\gamma^i(s))|^p \,\mathrm{d}s \geq \int_{T_i} |\kappa(z)|^p \,\mathrm{d}\mathcal{H}^1(z),$$

that is (3.17). This concludes the proof. □

THEOREM 3.2. *The functional $\mathcal{F}(\cdot)$ is $L^1(\mathbb{R}^2)$-lower semicontinuous on the class of all bounded open subsets of $\mathbb{R}^2$ of class C^2, i.e., given a sequence $\{E_h\}_h$ of bounded open sets of class C^2 converging in $L^1(\mathbb{R}^2)$ to a bounded open set E of class C^2, then*

$$\int_{\partial E} (1 + |\kappa(z)|^p) \,\mathrm{d}\mathcal{H}^1(z) \leq \liminf_{h \to +\infty} \int_{\partial E_h} (1 + |\kappa_h(z)|^p) \,\mathrm{d}\mathcal{H}^1(z), \tag{3.19}$$

where κ and κ_h are the curvatures of ∂E and of ∂E_h, respectively.

PROOF. Let $\{E_h\}_h$ be a sequence of bounded open sets of class C^2 such that $E_h \to E$ in $L^1(\mathbb{R}^2)$ as $h \to +\infty$. We can suppose that the right hand side in (3.19) is finite, otherwise the result is trivial. Let $\{E_{h_k}\}_k$ be a subsequence of $\{E_h\}_h$ with the property that

$$\lim_{k \to +\infty} \mathcal{F}(E_{h_k}) = \liminf_{h \to +\infty} \mathcal{F}(E_h) < +\infty.$$

For simplicity, this subsequence (and any further subsequence) will be denoted by $\{E_k\}_k$. For any k, ∂E_k has a finite number of connected components. Let Δ_k be an oriented parametrization of ∂E_k. Since $\{E_k\}_k$ satisfies (2.5), the sequence $\{\Delta_k\}_k$ satisfies (3.1). As in the proof on Theorem 3.1, we can suppose that $\Delta_k = \{\gamma_k^1, \ldots, \gamma_k^m\}$ with m independent of k. Let $\{\Gamma_k\}_k = \{\gamma_k^{i_1}, \ldots, \gamma_k^{i_n}\}$ be the equibounded sequence and let Γ be the limit system of curves of class $H^{2,p}$ constructed in the proof of Lemma 3.3. Then, by construction, $n \le m$, $\Gamma = \{\gamma^{i_1}, \ldots, \gamma^{i_n}\}$, and for any $j = 1, \ldots, n$ we have that $\gamma_k^{i_j} \rightharpoonup \gamma^{i_j}$ weakly in $H^{2,p}$ as $k \to +\infty$. Using (iii) of Lemma 3.3 and the weak lower semicontinuity of the L^p norm, we have

$$
\begin{aligned}
\liminf_{h\to+\infty} \mathcal{F}(E_h) &= \lim_{k\to+\infty} \mathcal{F}(E_k) \ge \lim_{k\to+\infty} \sum_{j=1}^{n} l(\gamma_k^{i_j}) \\
&\quad + \liminf_{k\to+\infty} \sum_{j=1}^{n} l(\gamma_k^{i_j})^{1-2p} \int_0^1 \left| \frac{\mathrm{d}^2\gamma_k^{i_j}}{\mathrm{d}t^2} \right|^p \mathrm{d}t \\
&\ge \sum_{j=1}^{n} l(\gamma^{i_j}) + \sum_{j=1}^{n} l(\gamma^{i_j})^{1-2p} \liminf_{k\to+\infty} \int_0^1 \left| \frac{\mathrm{d}^2\gamma_k^{i_j}}{\mathrm{d}t^2} \right|^p \mathrm{d}t \\
&\ge l(\Gamma) + \sum_{j=1}^{n} l(\gamma^{i_j})^{1-2p} \int_0^1 \left| \frac{\mathrm{d}^2\gamma^{i_j}}{\mathrm{d}t^2} \right|^p \mathrm{d}t \\
&= l(\Gamma) + \sum_{j=1}^{n} \int_0^{l(\gamma^{i_j})} |\ddot{\gamma}^{i_j}(s)|^p \,\mathrm{d}s = l(\Gamma) + \|\kappa(\Gamma)\|_{L^p}^p.
\end{aligned}
$$

Since $\|\kappa(\Gamma)\|_{L^p}^p \ge \int_{\partial E} |\kappa(z)|^p \,\mathrm{d}\mathcal{H}^1(z)$ by Lemma 3.4 and $l(\Gamma) \ge \mathcal{H}^1(\partial E)$ by (2.1), we conclude that

$$
\liminf_{h\to+\infty} \mathcal{F}(E_h) \ge \int_{\partial E} [1 + |\kappa(z)|^p] \,\mathrm{d}\mathcal{H}^1(z) = \mathcal{F}(E),
$$

which is the assertion of the theorem. □

The following result generalizes Theorem 3.2.

COROLLARY 3.2. *Let $E \subseteq \mathbb{R}^2$ be a bounded open set of class $H^{2,p}$. Then*

$$
\overline{\mathcal{F}}(E) = \int_{\partial E} [1 + |\kappa(z)|^p] \,\mathrm{d}\mathcal{H}^1(z). \tag{3.20}
$$

In particular, $\overline{\mathcal{F}}(E) < +\infty$.

PROOF. Theorem 3.2 holds with the same proof if E is of class $H^{2,p}$, hence, passing to the infimum with respect to the approximating sequence $\{E_h\}_h$ in (3.19), we infer that

$$\int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z) \leq \overline{\mathcal{F}}(E).$$

Let us prove the opposite inequality. Proposition 3.1 implies that there exists an oriented parametrization $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ of ∂E of class $H^{2,p}$. Hence in particular $\partial E = \bigcup_{i=1}^m (\gamma^i)$, where $\gamma^i : [0,1] \to \mathbb{R}^2$ are closed simple regular disjoint curves of class $H^{2,p}$. For any $i = 1, \ldots, m$, let us consider a sequence $\{\gamma_h^i\}_h$ of curves of class $C^\infty([0,1])$ such that $\gamma_h^i \to \gamma^i$ strongly in $H^{2,p}$ as $h \to +\infty$. It follows that, for h large enough, the approximating system $\Gamma_h = \{\gamma_h^1, \ldots, \gamma_h^m\}$ is a disjoint system of curves. Moreover, the curves γ_h^i are simple for h large enough. If not, there exist a subsequence, still denoted by $\{\gamma_h^i\}_h$, and two sequences $\{s_h\}_h$, $\{t_h\}_h$ of points, with $0 \leq s_h < t_h < 1$, such that $\gamma_h^i(s_h) = \gamma_h^i(t_h)$. By compactness, we may also assume that $s_h \to s$ and $t_h \to t$. Then $\gamma^i(s) = \gamma^i(t)$. If $t < 1$, this implies $s = t$. By the Mean Value Theorem, there exist points ξ_h and η_h between s_h and t_h such that the first component of the derivative of γ_h^i vanishes on ξ_h and the second component vanishes on η_h. As $\xi_h \to s = t$ and $\eta_h \to s = t$, we conclude that the derivative of γ vanishes at s, which contradicts our assumption on γ (see Definition 2.1). If $t = 1$, then either $s = 1$ or $s = 0$. The former case leads to the same contradiction obtained before. The latter case can be treated with obvious modifications. Hence the curves γ_h^i are simple for h large enough.

By construction, the traces (Γ_h) are equibounded uniformly with respect to h. For any h, let us define $E_h = \{z \in \mathbb{R}^2 \setminus (\Gamma_h) : I(\Gamma_h, z) = 1\}$. Then $\partial E_h = \bigcup_{i=1}^m (\gamma_h^i)$, $\{E_h\}_h$ converges to $\{z \in \mathbb{R}^2 \setminus (\Gamma) : I(\Gamma, z) = 1\} = E$ in $L^1(\mathbb{R}^2)$ as $h \to +\infty$, and $\int_{\partial E_h} [1 + |\kappa_h(z)|^p] \, d\mathcal{H}^1(z) = \sum_{i=1}^m \int_0^{l(\gamma_h^i)} [1 + |\ddot{\gamma}_h^i(s)|^p] \, ds$. It follows that

$$\overline{\mathcal{F}}(E) \leq \lim_{h \to +\infty} \sum_{i=1}^m \int_0^{l(\gamma_h^i)} [1 + |\ddot{\gamma}_h^i(s)|^p] \, ds = \int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z),$$

by construction. This concludes the proof of (3.20). □

4. - Some properties of the sets E with $\overline{\mathcal{F}}(E) < +\infty$

We recall the definition of tangent cone for an arbitrary subset of $\mathbb{R}^2$.

DEFINITION 4.1. Whenever $C \subseteq \mathbb{R}^2$, $z \in \overline{C}$, we define the (unoriented) tangent cone of C at z, denoted by $T_C(z)$, as the set of all $v \in \mathbb{R}^2$ such that for every $\epsilon > 0$ there exist $w \in C$ and $r \in \mathbb{R}$ with $|w - z| < \epsilon$ and $|r(w - z) - v| < \epsilon$. Such vectors v will be called tangent vectors of C at z.

By the definition, it follows that $T_C(z)$ is a closed subset of $\mathbb{R}^2$ and that, if $v \neq 0$ is an element of $T_C(z)$, then the straight line through the origin containing v is contained in $T_C(z)$.

If $T_C(z)$ is reduced to a straight line, we can write $T_C(z) = \{c\tau(z): c \in \mathbb{R}\}$, where $\tau(z)$ is a tangent unit vector of C at z, determined up to the sign. In this case we define the normal line $N_C(z)$ of C at z as the straight line through the origin orthogonal to $T_C(z)$. We will make the following convention: once $\tau(z)$ has been chosen, $\nu(z)$ is taken in such a way that $\{\tau(z), \nu(z)\}$ is oriented as the canonical basis of $\mathbb{R}^2$. Of course, we have $N_C(z) = \{c\nu(z): c \in \mathbb{R}\}$, where $\nu(z)$ is a unit vector orthogonal to $\tau(z)$, determined up to the sign.

REMARK 4.1. Let C be a subset of $\mathbb{R}^2$, and let $z \in C$ be a point such that the tangent cone $T_C(z)$ is a straight line. It follows immediately from the definition and the uniqueness of the tangent line that there exists $r > 0$ such that $B_r(z) \cap N_C(z) \cap C = \{z\}$.

Note that if Γ is a system of curves, $z \in (\Gamma)$ and $\Gamma^{-1}(z) = \{t_1, \dots, t_k\}$, then

$$T_{(\Gamma)}(z) = \bigcup_{i=1}^{k} \left\{ c\,\frac{\mathrm{d}\Gamma}{\mathrm{d}t}(t_i): c \in \mathbb{R} \right\}. \tag{4.1}$$

DEFINITION 4.2. We say that a system of curves Γ is without crossings if $\dfrac{\mathrm{d}\gamma^i(t_1)}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}\gamma^j(t_2)}{\mathrm{d}t}$ are parallel, whenever $\gamma^i(t_1) = \gamma^j(t_2)$ for some $i \neq j$ and $t_1,\ t_2 \in [0,1]$.

Let Γ be a system of curves without crossings, and let $z \in (\Gamma)$; note that by (4.1) the tangent cone of (Γ) at z is a straight line. Observe also that if Γ is a limit system of curves of class $H^{2,p}$ then Γ is without crossings. This follows easily from the fact that Γ is the limit in $\mathcal{C}^1$ of a sequence of disjoint systems of simple curves (see (3.6)).

We want now to list some regularity properties of those sets E such that $\overline{\mathcal{F}}(E) < +\infty$. We shall identify the real projective space $\mathbb{P}^1$ with the set of all one dimensional linear subspaces of $\mathbb{R}^2$.

DEFINITION 4.3. Let C be a subset of $\mathbb{R}^2$. We say that C has a continuous unoriented tangent if at each point $z \in C$ the tangent cone $T_C(z)$ of C at z is a straight line and the map $T_C: z \to T_C(z)$ from C into $\mathbb{P}^1$ is continuous.

PROPOSITION 4.1. *Let Γ be a system of curves of class $\mathcal{C}^1$ without crossings and let C be a closed subset of (Γ) with no isolated points. Then C has a continuous unoriented tangent.*

PROOF. It is clear that the tangent cone $T_C(z)$ of C is a straight line at each point $z \in C$. We have to prove that the map T_C is continuous. Let $z \in C$, $\Gamma^{-1}(z) = \{t_1, \dots, t_k\}$. As $\frac{\mathrm{d}\Gamma}{\mathrm{d}t}(t_1), \dots, \frac{\mathrm{d}\Gamma}{\mathrm{d}t}(t_k)$ are proportional, we have $T_C(z) = \pi\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}t}(t_1)\right) = \cdots = \pi\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}t}(t_k)\right)$, where π denotes the canonical projection of $\mathbb{R}^2 \backslash \{0\}$ into $\mathbb{P}^1$, i.e., π associates with each vector $v \in \mathbb{R}^2 \backslash \{0\}$ the straight line through the origin containing v. Let $U \subseteq \mathbb{P}^1$ be an open neighbourhood of $T_C(z)$. The map $t \to T_C(\Gamma(t))$ is continuous because $T_C(\Gamma(t)) = \pi\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}t}(t)\right)$. Hence for any $\varepsilon > 0$, there exist positive numbers $\delta_1, \dots, \delta_k$ such that $T_C(\Gamma(t)) \in U$ if $|t - t_i| < \delta_i$, for some $i = 1, \dots, k$, so that, if $V = \bigcup_{i=1}^k \Gamma(t_i - \delta_i, t_i + \delta_i)$, then $T_C(V \cap C) \subseteq U$. Let us take δ_i so small that the intervals $]t_i - \delta_i, t_i + \delta_i[$ are pairwise disjoint; since Γ is a system of curves parametrized with constant velocity, the Implicit Function Theorem implies that V is a neighbourhood of z in (Γ) for the induced topology from $\mathbb{R}^2$. This concludes the proof. □

THEOREM 4.1. *Let $E \subseteq \mathbb{R}^2$ be a measurable set such that $\overline{\mathcal{F}}(E) < +\infty$, and let $E^\star = \{z \in \mathbb{R}^2 : \exists r > 0\ |B_r(z) \backslash E| = 0\}$. Then $E^\star$ satisfies the following properties:*

(i) *$E^\star$ is bounded, open, and $|E \Delta E^\star| = 0$;*

(ii) *$\mathcal{H}^1(\partial E^\star) < +\infty$;*

(iii) *there exists a limit system of curves Γ of class $H^{2,p}$ such that $(\Gamma) \supseteq \partial E^\star$ and $E^\star = \operatorname{int}(A_\Gamma \cup (\Gamma))$, where $A_\Gamma = \{z \in \mathbb{R}^2 \backslash (\Gamma) : I(\Gamma, z) = 1\}$;*

(iv) *$\partial E^\star$ has a continuous unoriented tangent;*

(v) *$\overline{\mathcal{F}}(E) \geq \inf\{\mathcal{F}(\Gamma) : \Gamma \in \mathcal{A}(E)\}$, where $\mathcal{A}(E)$ is the collection of all limit systems of curves Γ of class $H^{2,p}$ satisfying* (iii).

PROOF. Assertions (i), (ii), and (iii) follow from (2.5) and Lemma 3.3. Assertion (iv) follows from Proposition 4.1, and from Lemma 3.3 (iii) (note that (3.7) guarantees that $\partial E^\star$ has no isolated points). Let us prove (v). Since $\overline{\mathcal{F}}(E) < +\infty$, there exists a sequence $\{E_h\}_h$ of bounded open sets of class C^2 satisfying (2.5) and such that $E_h \to E$ in $L^1(\mathbb{R}^2)$ as $h \to +\infty$. Using the same notation as in the proof of Theorem 3.2, for a subsequence $\{E_k\}_k$ we have, from the weak lower semicontinuity of the L^p norm,

$$\liminf_{h \to +\infty} \mathcal{F}(E_h) = \lim_{k \to +\infty} \mathcal{F}(E_k) = l(\Gamma) + \liminf_{k \to +\infty} \int_{\partial E_k} |\kappa_k(z)|^p \,\mathrm{d}\mathcal{H}^1(z)$$

$$\geq l(\Gamma) + \|\kappa(\Gamma)\|_{L^p}^p \geq \inf\{\mathcal{F}(\Gamma) : \Gamma \in \mathcal{A}(E)\}.$$

Taking the infimum with respect to the approximating sequence $\{E_h\}_h$, we get (v). □

DEFINITION 4.4. Let C be a subset of $\mathbb{R}^2$, let $z \in \partial C$, and suppose that the tangent cone of ∂C at z is a straight line. Let $\nu(z)$ be a normal unit vector of ∂C at z, and, for any $\varrho > 0$, let

$$N_\varrho^+(z) = \{z + s\nu(z): 0 < s < \varrho\}, \qquad N_\varrho^-(z) = \{z - s\nu(z): 0 < s < \varrho\}.$$

Then we define

$$\partial_{00}C = \{z \in \partial C: \exists r > 0 \ N_r^-(z) \cup N_r^+(z) \subseteq \mathbb{R}^2 \setminus \overline{C}\},$$

$$\partial_{11}C = \{z \in \partial C: \exists r > 0 \ N_r^-(z) \cup N_r^+(z) \subseteq \mathrm{int}(C)\},$$

$$\partial_{01}C = \{z \in \partial C: \exists r > 0 \ N_r^-(z) \subseteq \mathbb{R}^2 \setminus \overline{C}, \ N_r^+(z) \subseteq \mathrm{int}(C) \text{ or conversely}\}.$$

REMARK 4.2. Let C be a subset of $\mathbb{R}^2$ such that for any $z \in \partial C$ the tangent cone of ∂C at z is a straight line. Then $\partial C = \partial_{00}C \cup \partial_{11}C \cup \partial_{01}C$. The inclusion $\partial_{00}C \cup \partial_{11}C \cup \partial_{01}C \subseteq \partial C$ is obvious. The opposite inclusion follows from Remark 4.1. In particular, if $E \subseteq \mathbb{R}^2$ is measurable and $\overline{\mathcal{F}}(E) < +\infty$, then $\partial E^\star = \partial_{00}E^\star \cup \partial_{11}E^\star \cup \partial_{01}E^\star$.

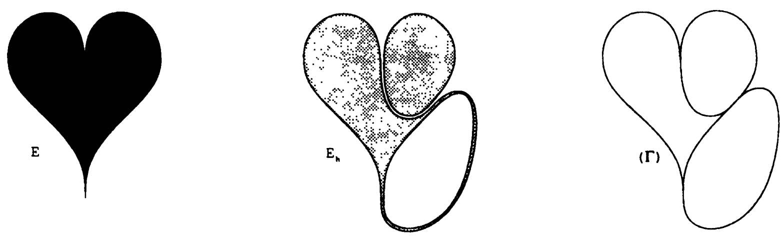
PROPOSITION 4.2. *Let $E \subseteq \mathbb{R}^2$ be a measurable set such that $\overline{\mathcal{F}}(E) < +\infty$, and let Γ and A_Γ be as Theorem* 4.1 (iii). *Then the sets $\partial_{00}E^\star$, $\partial_{11}E^\star$, and $\partial_{01}E^\star$ remain unchanged if we replace in their definition* $\mathrm{int}(E^\star)$ *by A_Γ and $\mathbb{R}^2 \setminus E^\star$ by $B_\Gamma = \{z \in \mathbb{R}^2 \setminus (\Gamma): I(\Gamma, z) = 0\}$.*

PROOF. Let $z \in \partial E$, let $z \in \partial_{11}E$, and let $r > 0$ be such that $N_r^-(z) \cup N_r^+(z) \subseteq \mathrm{int}(E^\star) = E^\star$. We will show that $N_r^-(z) \cup N_r^+(z) \subseteq A_\Gamma$. As the tangent cone of (Γ) is a straight line, by Remark 4.1 there exists $r > 0$ such that $B_r(z) \cap N_{(\Gamma)}(z) \cap (\Gamma) = \{z\}$. Since $E^\star = \mathrm{int}(A_\Gamma \cup (\Gamma))$, the assertion follows. The other cases are analogous. □

DEFINITION 4.5. Let Γ be a system of curves without crossings, let $z \in (\Gamma)$, let $\tau(z)$ be a tangent unit vector of (Γ) at z and let $\nu(z)$ be the corresponding oriented normal unit vector. If $t \in S = \mathbf{S}_1^1 \cup \cdots \cup \mathbf{S}_m^1$ satisfies $\Gamma(t) = z$ and $\frac{\mathrm{d}\Gamma}{\mathrm{d}t}(t) \cdot \tau(z) > 0$ (respectively $\frac{\mathrm{d}\Gamma}{\mathrm{d}t}(t) \cdot \tau(z) < 0$), then we say that Γ points to the right (respectively to the left) with respect to $\nu(z)$ at t. If there are k points $t_1, \ldots, t_k \in S$ such that $\Gamma(t_i) = z$ and Γ points to the right (respectively to the left) with respect to $\nu(z)$ at each point t_i, then we say that Γ points k times to the right (respectively to the left) with respect to $\nu(z)$ at z.

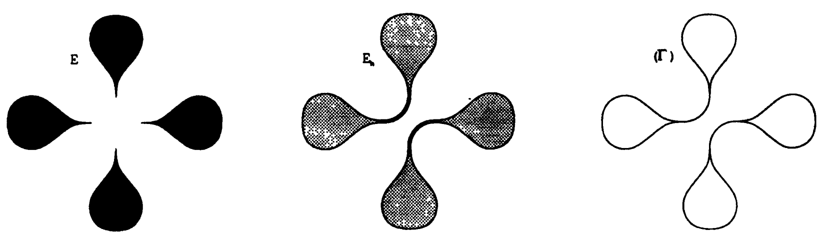
A particular case of the next Lemma (the case in which $k = 0$, $\mathrm{d} = 1$) can be found in [1, Lemma 9.2.5]. The general case can be proved using the same methods.

LEMMA 4.1. *Let Γ be a system of curves without crossings, let $z \in (\Gamma)$, and let z_1, $z_2 \neq z$ be points on $N_{(\Gamma)}(z)$, one on each side of z, and close enough to z. Suppose that Γ points k-times to the right, and* d*-times to the left with respect to $\nu(z)$ at z. Then $|I(\Gamma, z_1) - I(\Gamma, z_2)| = |k - \mathrm{d}|$.*



a. *b.* *c.*

Fig. 4.1: *a. A set E with* $\overline{\mathcal{F}}(E) < +\infty$.
b. An example of approximating sequence $\{E_h\}_h$ *satisfying* (2.5).
c. The trace of the limit system Γ *of curves.*



a. *b.* *c.*

Fig. 4.2: *a. A set E with* $\overline{\mathcal{F}}(E) < +\infty$.
b. An example of approximating sequence $\{E_h\}_h$ *satisfying* (2.5).
c. The trace of the limit system Γ *of curves.*

Let $E \subseteq \mathbb{R}^2$ be a measurable set such that $\overline{\mathcal{F}}(E) < +\infty$, let $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ be one of the limit systems of curves satisfying condition (iii) of Theorem 4.1, and let $z \in \partial_{01}E^\star$. Using Proposition 4.1, we deduce that, if $\Gamma^{-1}(z)$ has k elements, then k is odd. More precisely, we get that $k = 2n + 1$, and Γ points n-times to the right with respect to $\nu(z)$ at z, and $(n+1)$-times to the left (or viceversa). Conversely, if $z \in \partial_{00}E^\star$ or $z \in \partial_{11}E^\star$, then $k = 2n$, $n \geq 1$, and Γ points n-times to the right with respect to $\nu(z)$ at z, and n-times to the left.

Let $z \in (\Gamma)\backslash\partial E^\star$; then from Theorem 4.1 (iii) there exists $r > 0$ such that either $I(\Gamma, w) = 0$ for any $w \in N_{(\Gamma)}(z) \cap (B_r(z)\backslash(\Gamma))$, or $I(\Gamma, w) = 1$ for any $w \in N_{(\Gamma)}(z) \cap (B_r(z)\backslash(\Gamma))$. This shows that (Γ) has even multiplicity at any point $z \in (\Gamma)\backslash\partial E$.

In Figures 4.1 and 4.2 we show some examples of sets E with $\overline{\mathcal{F}}(E) < +\infty$, their approximation in the $L^1(\mathbb{R}^2)$ norm by a sequence $\{E_h\}_h$ satisfying (2.5), and the limit systems of curves Γ of Theorem 4.1 (iii).

DEFINITION 4.6. Let $E \subseteq \mathbb{R}^2$ be a measurable set such that $\overline{\mathcal{F}}(E) < +\infty$, let $q \in \partial E^\star$, and let $\tau(q)$ be a tangent unit vector of $\partial E^\star$ at q. We say that q is a cusp point of $\partial E^\star$ if there exists $r > 0$ such that

$$\text{either } B_r^+(q) \cap \partial E^\star = \emptyset, \text{ or } B_r^-(q) \cap \partial E^\star = \emptyset,$$

where

$$B_r^+(q) = \{z \in B_r(q) : (z - q) \cdot \tau(q) > 0\}, \quad B_r^-(q) = \{z \in B_r(q) : (z - q) \cdot \tau(q) < 0\}.$$

PROPOSITION 4.3. *Let $E \subseteq \mathbb{R}^2$ be a measurable set such that $\overline{\mathcal{F}}(E) < +\infty$. Then the set of the cusp points of $\partial E^\star$ is at most countable.*

PROOF. Let $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ be one of the limit systems of curves satisfying condition (iii) of Theorem 4.1, and let C be the set of the cusp points of $\partial E^\star$ belonging to (γ^1). For any $q \in C$ let $t_q \in [0, 1]$ be such that $\gamma^1(t_q) = q$. Assume for simplicity that $t_q \in]0, 1[$. Since $\dfrac{d\gamma^i(t_q)}{dt} \neq 0$, by the definition of cusp point there exists $\varepsilon_q > 0$ such that either $\gamma^1(I_q^-) \cap \partial E^\star = \emptyset$, or $\gamma^1(I_q^+) \cap \partial E^\star = \emptyset$, where $I_q^- =]t_q - \epsilon_q, t_q[$, $I_q^+ =]t_q, t_q + \epsilon_q[$. Let $C^- = \{q \in C : \gamma^1(I_q^-) \cap \partial E^\star = \emptyset\}$, $C^+ = \{q \in C : \gamma^1(I_q^+) \cap \partial E^\star = \emptyset\}$. Note that in particular $\gamma^1(I_q^-) \cap C^- = \emptyset$ for any $q \in C^-$, and $\gamma^1(I_q^+) \cap C^+ = \emptyset$ for any $q \in C^+$. Hence, for any $q_1, q_2 \in C^+$, $q_1 \neq q_2$, $I_{q_2}^+ \cap I_{q_1}^+ = \emptyset$, and a similar property holds for C^-. This implies that C^+ and C^- are at most countable. □

We will see at the end of the next section that there exist measurable sets $E \subseteq \mathbb{R}^2$ such that $\overline{\mathcal{F}}(E) < +\infty$ whose boundary has an infinite number of cusp points.

DEFINITION 4.7. Let $E \subseteq \mathbb{R}^2$ be a measurable set such that $\overline{\mathcal{F}}(E) < +\infty$, let $q \in \partial E$, and let $T(q)$ be the tangent line of ∂E at q. We say that q is a branch point of ∂E if there exist $r > 0$ and a tangent unit vector $\tau(q)$ of ∂E at q such that $B_r^+(q) \cap \partial E$ is a cartesian graph with respect to $T(q)$ and $B_r^-(q) \cap \partial E$ is not a cartesian graph with respect to $T(q)$.

An example of branch point is shown in Figure 6.2.

The proof of the following proposition is similar to the proof of Proposition 4.3.

PROPOSITION 4.4. *Let $E \subseteq \mathbb{R}^2$ be a measurable set such that $\overline{\mathcal{F}}(E) < +\infty$. Then $\partial E^\star$ can have at most a countable number of branch points.*

5. - Some critical examples

In this section we show two pathological examples of measurable sets E such that the boundary of the set $E^\star$ defined in Theorem 4.1 is very irregular and, despite of this fact, $\overline{\mathcal{F}}(E) < +\infty$.

EXAMPLE 1. *There exists a measurable set* $E \subseteq \mathbb{R}^2$ *such that* $\overline{\mathcal{F}}(E) < +\infty$ *and* $\mathcal{H}^1(\partial_{00}E^\star \cup \partial_{11}E^\star) > 0$.

Let A be a dense, open subset of $[0,1]$ such that $A = \bigcup_{k=1}^{\infty} I_k$, where $I_k =]a_k, b_k[$ are pairwise disjoint, and $\mathcal{H}^1([0,1]\setminus A) > 0$.

For any k, let $x_k = \dfrac{a_k + b_k}{2}$, and let $\phi: \mathbb{R} \to [0,1]$ be a function of class C^∞, such that $\phi(0) = 1$, $\phi(x) > 0$ if $|x| < 1$, $\phi(x) = 0$ if $|x| \geq 1$, $\phi(x) = \phi(-x)$ for any $x \in [0, +\infty]$.

For any $x \in \mathbb{R}$, let $\phi_k(x) = \phi\left(\dfrac{2}{b_k - a_k}(x - x_k)\right)$. It is possible to find a sequence of positive real numbers $\{c_k\}_k$ such that the function

$$\Phi(x) = \sum_{k=1}^{\infty} c_k \phi_k(x)$$

is of class $C^\infty(\mathbb{R})$. Finally, let us define (see Figure 5.1)

$$E = \{(x, y): x \in A, \ -\Phi(x) < y < \Phi(x)\}.$$

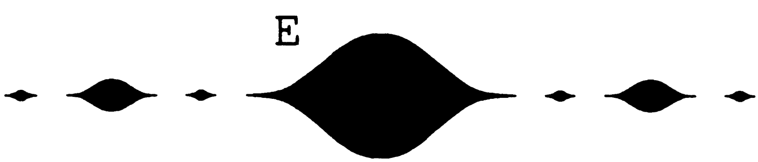


Fig. 5.1: *A pathological set* E *with* $\overline{\mathcal{F}}(E) < +\infty$.

By construction, $\partial_{00}E^\star = ([0,1]\setminus A) \times \{0\}$, $\partial_{11}E^\star = \emptyset$; hence $\mathcal{H}^1(\partial_{00}E^\star \cup \partial_{11}E^\star) = \mathcal{H}^1([0,1]\setminus A) > 0$. Note that, as $\partial_{01}E^\star = (A \times \mathbb{R}) \cap \partial E^\star$ and A is dense in $[0,1]$, we have that $\overline{\partial_{01}E^\star} = \partial E^\star$.

Let us prove that $\overline{\mathcal{F}}(E) < +\infty$. We will show that there exists a sequence $\{E_h\}_h$ of bounded open sets of class C^∞ such that $E_h \to E$ in $L^1(\mathbb{R}^2)$ as $h \to +\infty$, and $\sup_h \mathcal{F}(E_h) < +\infty$. Let γ be any simple closed curve of class

C^∞ whose trace (γ) contains the segment $[0,1]\times\{0\}$, and, for any $h\in\mathbf{N}$, let

$$E_h = \left\{(x,y): x\in[0,1],\ \ -\Phi(x)-\frac{1}{h}<y<\Phi(x)+\frac{1}{h}\right\}$$
$$\bigcup\left\{(x,y): \mathrm{dist}((x,y),\ (\gamma))<\frac{1}{h}\right\}$$

(see Figure 5.2).

Then E_h has a boundary of class C^∞ for h large enough, and the sequence $\{E_h\}_h$ satisfies the required properties. □

EXAMPLE 2. *There exists a measurable set E such that $\overline{\mathcal{F}}(E)<+\infty$ and $\partial E^\star$ has an infinite number of cusp points.*

Consider the family of intervals $I_k=\left]\frac{1}{2k-1},\frac{1}{2k}\right[=\,]a_k,b_k[$, $k\in\mathbf{N}$, $k\geq 1$. Using the same notation of Example 1, we can construct the functions ϕ_k and Φ, and we can define E as follows:

$$E=\{(x,y): x\in A,\ \ -\Phi(x)<y<\Phi(x)\}.$$

Then E verifies the required properties. □

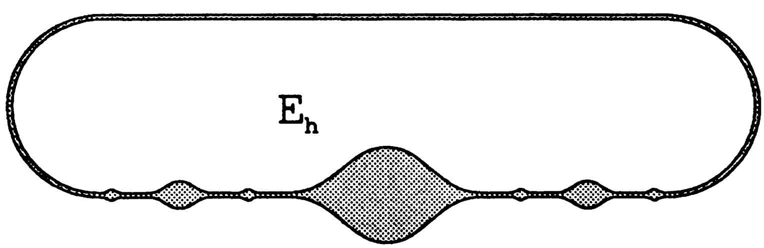


Fig. 5.2: *The approximation of E in the $L^1(\mathbb{R}^2)$ norm by a sequence $\{E_h\}_h$ satisfying* (2.5).

6. - Systems of curves that can be approximated by boundaries of sets

Let $E\subseteq\mathbb{R}^2$ be a bounded measurable set. The aim of this section is to find which conditions must be satisfied by ∂E in order to have that $\overline{\mathcal{F}}(E)<+\infty$. To do that, we need the following definition.

DEFINITION 6.1. We say that a system of curves Γ satisfies the finiteness property if there exists a finite set F such that $(\Gamma)\backslash F$ is a one dimensional submanifold of $\mathbb{R}^2$ of class $\mathcal{C}^1$.

As the curves γ^i of Γ have constant velocity, it is easy to see that $(\Gamma)\backslash F$ has a finite number of connected components. By adding, if necessary, a finite number of points to F, we may assume that each connected component of $(\Gamma)\backslash F$ is diffeomorphic to an open interval.

REMARK 6.1. Let S be the disjoint union of m unit circles $\mathbf{S}^1_1, \ldots, \mathbf{S}^1_m$, and let $\Gamma: S \to \mathbb{R}^2$ be a system of curves. By the elementary properties of one dimensional submanifolds and of their parametrizations with constant velocity, it follows that Γ satisfies the finiteness property if and only if there exists a finite number of points $t_1, \ldots, t_n$ in S such that each circle $\mathbf{S}^1_i$ contains at least two of these points, and the unique finite partition $\mathcal{P}$ of $S\backslash\{t_1, \ldots, t_n\}$ composed of open arcs of circles having endpoints in $\{t_1, \ldots, t_n\}$ satisfies the following properties:

(i) for any I, $H \in \mathcal{P}$ either $\Gamma(I) \cap \Gamma(H) = \emptyset$ or $\Gamma(I) = \Gamma(H)$;

(ii) Γ is injective on the closure $\overline{I}$ of any element I of $\mathcal{P}$;

(iii) $\Gamma(t_i) \notin \Gamma(H)$ for every $H \in \mathcal{P}$ and for every $i = 1, \ldots, n$.

Note that if $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ is a system of curves satisfying the finiteness property, then no curve γ^i is closed.

DEFINITION 6.2. Let Γ be a system of curves satisfying the finiteness property, and let $t_1, \ldots, t_n$ and $\mathcal{P}$ be as in Remark 6.1. For every $I \in \mathcal{P}$ the set $\Gamma(I)$ will be called a branch of Γ, and for every $i = 1, \ldots, n$ the point $\Gamma(t_i)$ will be called a node of Γ. The set of all branches will be denoted by $\mathcal{B}(\Gamma)$ and the set of all nodes will be denoted by $\mathcal{N}(\Gamma)$.

DEFINITION 6.3. Let Γ be a system of curves satisfying the finiteness property, and let J be an open and connected subset of $S\backslash\{t_1, \ldots, t_n\}$. The set $\Gamma(J)$ will be called an arc of Γ.

Let A be an arc of Γ, and let $z \in A$. Then the number of elements of the $\Gamma^{-1}(z)$ depends only on A and does not depend on z. This number will be called the multiplicity of the arc A. In particular, since A is contained in a branch B, the multiplicity of A is the multiplicity of B.

We will consider now the problem of the approximation, in the $H^{2,p}$ norm, of a system of curves Γ by a sequence $\{\Gamma_h\}_h$ whose elements are boundaries of smooth bounded open sets (see also [19, Chapter 8.9.4] and [17] for a similar approach, in a very different context). We begin with the following Lemma.

LEMMA 6.1. *Let $\Gamma = \{\gamma^1, \ldots, \gamma^m\}: S \to \mathbb{R}^2$ be a system of curves of class $H^{2,p}$ without crossings and satisfying the finiteness property, and let $\{t_1, \ldots, t_n\} = \Gamma^{-1}(\mathcal{N}(\Gamma))$. Then there exist $\varepsilon_0 > 0$ and a sequence $\Gamma_\varepsilon: S \to \mathbb{R}^2$, $0 < \varepsilon < \varepsilon_0$, of systems of curves of class $H^{2,p}$ without crossings, satisfying the finiteness property, such that for every $0 < \varepsilon < \varepsilon_0$*

(i) $\Gamma_\varepsilon^{-1}(\mathcal{N}(\Gamma_\varepsilon)) = \{t_1^\varepsilon, \dots, t_n^\varepsilon\}$, *and* $t_i^\varepsilon \to t_i$ *as* $\varepsilon \to 0$, *for any* $i = 1, \dots, n$;

(ii) $\Gamma_\varepsilon \to \Gamma$ *strongly in* $H^{2,p}(S, \mathbb{R}^2)$ *as* $\varepsilon \to 0$;

(iii) $\Gamma_\varepsilon(t_i^\varepsilon) = \Gamma(t_i)$ *for any* $i = 1, \dots, n$;

(iv) $\Gamma_{\varepsilon|S\setminus\{t_1^\varepsilon,\dots,t_n^\varepsilon\}}: S\setminus\{t_1^\varepsilon, \dots, t_n^\varepsilon\} \to \Gamma_\varepsilon(S\setminus\{t_1^\varepsilon, \dots, t_n^\varepsilon\})$ *is a homeomorphism.*

Lemma 6.1 shows that the system Γ can be approximated in the $H^{2,p}$ norm by a sequence of systems of curves $\{\Gamma_\varepsilon\}_\varepsilon$ of class $H^{2,p}$ satisfying the following properties: the systems Γ_ε are defined on the same parameter space S, the nodes of Γ_ε coincide with the nodes of Γ (condition (iii)), all branches of Γ_ε have multiplicity one (condition (iv)), and $\Gamma_\varepsilon^{-1}(\mathcal{N}(\Gamma_\varepsilon))$ is a small perturbation of $\Gamma^{-1}(\mathcal{N}(\Gamma))$ (condition (i)). In particular, any branch of Γ with multiplicity k is approximated by k different branches of Γ_ε with multiplicity one.

PROOF. We will construct the approximation for a pair (q_1, q_2) of consecutive nodes of Γ, and for one of the branches joining q_1 and q_2. Then it will be sufficient to repeat this procedure for all pairs of consecutive nodes and for all the branches.

Let q_1, $q_2 \in \mathcal{N}(\Gamma)$ be two consecutive nodes, let B be a branch joining q_1 and q_2, and let $\nu: B \to \mathbb{R}^2$ be a continuous unit normal vector field on B. As $\overline{B}$ is homeomorphic to a segment, we can extend ν as a uniformly continuous vector field defined on $\mathbb{R}^2$ such that $|\nu(z)| = 1$ for any $z \in \mathbb{R}^2$. To overcome regularity problems for the approximating curves, we regularize ν as follows. We take a function $\tilde{\nu} \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ such that $|\tilde{\nu}(z)| \le 1$ and $\tilde{\nu}(z) \cdot \nu(z) \ge \frac{1}{2}$ for any $z \in \mathbb{R}^2$. Such a function exists: in fact, defining $\nu_\eta = \nu \star \varrho_\eta$, where $\{\varrho_\eta\}_\eta$ is a sequence of mollifiers, we get $\nu_\eta \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ and $\nu_\eta \cdot \nu \to 1$ uniformly on $\mathbb{R}^2$ as $\eta \to 0$. Then it is enough to define $\tilde{\nu} = \nu_\eta$ for η small enough.

Let $\mathrm{d}: \mathbb{R}^2 \to [0, +\infty[$ be the function defined by

$$\mathrm{d}(z) = \min\{\mathrm{dist}(z, B'): B' \in \mathcal{B}(\Gamma),\ B' \ne B\}.$$

From the definition it follows that d is continuous, $\mathrm{d}(z) > 0$ for any $z \in B$, and $\mathrm{d}(z) = 0$ if $z = q_1$ or $z = q_2$. In order to guarantee that the approximating sequence has the same regularity as the original system of curves Γ, and that approximations of different branches do not overlap, we introduce a function $h: \mathbb{R}^2 \to [0, +\infty[$ of class C^∞ having the following properties: $h(q_1) = h(q_2) = 0$, $0 < h(z) < \mathrm{d}(z)$ for any $z \in B$, and all the derivatives of h at the points q_1, q_2 vanish. Let k be the multiplicity of B. Then $\Gamma^{-1}(B) = \bigcup_{j=1}^k I_j$, where $I_j \in \mathcal{P}$ for any $j = 1, \dots, k$. We define the k approximating branches $B_\varepsilon^1, \dots, B_\varepsilon^k$ of B as follows: for any $j = 1, \dots, k$ and $\varepsilon > 0$ let

$$B_\varepsilon^j = \Gamma_\varepsilon(I_j), \qquad \text{where } \Gamma_\varepsilon(t) = \Gamma(t) + \varepsilon \frac{j}{k} h(\Gamma(t))\tilde{\nu}(\Gamma(t)), \qquad t \in \overline{I_j}.$$

Let us consider the function $g : \overline{I_j} \times \mathbb{R} \to \mathbb{R}^2$ defined by

$$g(t, \varrho) = \Gamma(t) + \varrho\tilde{\nu}(\Gamma(t)).$$

Since the jacobian of g is strictly positive at the points of the form $(t,0)$, by the Implicit Function Theorem it follows that g is locally invertible. As $\overline{I_j}$ is compact, there exists $\varepsilon_0 > 0$ such that the function g is injective on $\overline{I_j}\times[-\varepsilon_0,\varepsilon_0]$. Since $\Gamma_\varepsilon(t) = g(t, \varepsilon \frac{j}{k} h(\Gamma(t)))$ for any $t \in I_j$, it follows that, if $\varepsilon < \varepsilon_0$, then Γ_ε is injective on $\overline{I_j}$. This implies that $\Gamma_{\varepsilon|I_j}: I_j \to \Gamma_\varepsilon(I_j)$ is a homeomorphism. Moreover $\Gamma_\varepsilon(I_j) \cap \Gamma_{\varepsilon'}(I_j) = \emptyset$ for any $0 < \varepsilon$, $\varepsilon' < \varepsilon_0$, $\varepsilon \neq \varepsilon'$. Note that, if $\varepsilon < \frac{1}{2}$ and $B' \in \mathcal{B}(\Gamma)$, with $B' \neq B$, then

$$\text{dist}(z, B) < \text{dist}(z, B') \qquad \forall z \in B^j_\varepsilon. \tag{6.1}$$

In fact, if $z = \Gamma_\varepsilon(t) \in B^j_\varepsilon$ and $B' \neq B$, we have $\mathrm{d}(\Gamma(t)) \leq \text{dist}(\Gamma(t), B') \leq \text{dist}(\Gamma(t), z) + \text{dist}(z, B') \leq \varepsilon h(\Gamma(t)) + \text{dist}(z, B') < \varepsilon\, \mathrm{d}(\Gamma(t)) + \text{dist}(z, B')$. Hence

$$\text{dist}(z, B') > \mathrm{d}(\Gamma(t)) - \varepsilon\, \mathrm{d}(\Gamma(t)). \tag{6.2}$$

Moreover, if $\varepsilon < \frac{1}{2}$, then $\mathrm{d}(\Gamma(t)) - \varepsilon\, \mathrm{d}(\Gamma(t)) > \varepsilon\, \mathrm{d}(\Gamma(t))$. Since $\varepsilon \mathrm{d}(\Gamma(t)) \geq \text{dist}(z, B)$, using (6.2) we get (6.1).

Let us repeat this procedure for all pairs of consecutive nodes of Γ and for all branches. From inequality (6.1) it follows that the approximating branches corresponding to different branches of Γ are disjoint. At the end, we obtain a family $\Gamma_\varepsilon = \{\gamma^1_\varepsilon, \dots, \gamma^m_\varepsilon\}$ of regular closed curves, which satisfy all conditions of the lemma except the condition that $\left|\frac{\mathrm{d}\gamma^i_\varepsilon}{\mathrm{d}t}\right|$ is constant. This last requirement can be fulfilled by taking a suitable reparametrization of Γ_ε. □

Let $q \in \mathcal{N}(\Gamma)$; we denote by $\mathcal{B}(q)$ the set of all branches of Γ having endpoint q.

DEFINITION 6.4. Let $q \in \mathcal{N}(\Gamma)$ and B_1, $B_2 \in \mathcal{B}(q)$. We say that B_1 and B_2 lie on the same side with respect to q if there exist a tangent unit vector $\tau(q)$ of (Γ) at q and a neighbourhood U of q such that $(z - q) \cdot \tau(q) > 0$ for every $z \in (B_1 \cup B_2) \cap U$. If this does not happen, we say that B_1 and B_2 lie on opposite sides with respect to q.

DEFINITION 6.5. Let B, B' be two branches of Γ having multiplicity one. We say that B and B' are consecutive if there exist two consecutive arcs of circles I, $H \in \mathcal{P}$ in S such that $B = \Gamma(I)$ and $B' = \Gamma(H)$.

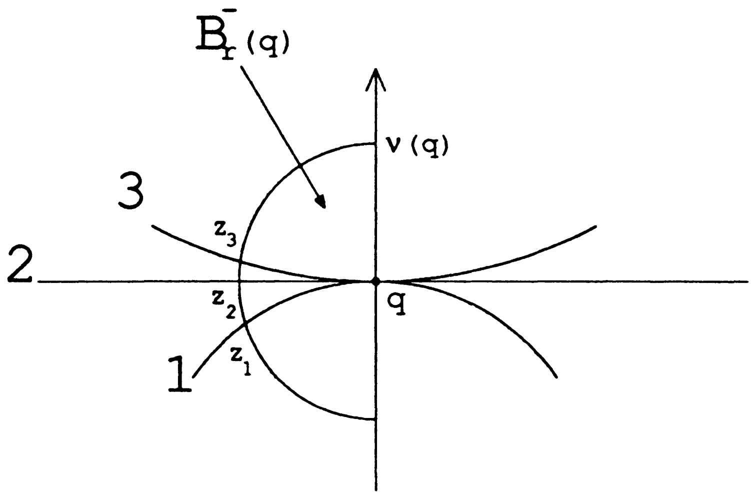


Fig. 6.1: *The ordering number of the elements of* $\mathcal{B}^-(q)$.

Note that two consecutive branches have a common endpoint and lie on opposite sides with respect to it. Clearly the converse is not true in general.

Suppose that any branch of Γ has multiplicity one; it is easy to prove that, given $B \in \mathcal{B}(q)$, the number of the branches of $\mathcal{B}(q)$ lying on the same side of B with respect to q is the same of the number of the branches of $\mathcal{B}(q)$ lying on the opposite side.

We want now to order the branches of $\mathcal{B}(q)$ which lie on the same side of B with respect to q. Let us fix a tangent unit vector $\tau(q)$ of (Γ) at q such that $z \cdot \tau(q) > 0$ for every $z \in B$ near q, and let $\nu(q)$ be a normal unit vector of (Γ) at q (this time we do not require that $\{\tau(q),\ \nu(q)\}$ has a particular orientation). Let $B_r^+(q) = \{z \in B_r(q): (z-q)\cdot\tau(q) > 0\}$. Using the Implicit Function Theorem, one can prove that, when r is sufficiently small, the set $B_r^+(q) \cap ((\Gamma)\setminus\{q\})$ is composed of k different arcs $A_1, \dots, A_k$ having q as an endpoint. For any $i = 1, \dots, k$ let $z_i = (\partial B_r^+(q) \cap \partial B_r(q)) \cap \overline{A_i}$ (see Figure 6.1). We say that $z_i \prec z_j$ if and only if $z_i \cdot \nu(q) < z_j \cdot \nu(q)$, and that $A_i \prec A_j$ if and only if $z_i \prec z_j$. Since each branch B' of $\mathcal{B}(q)$ lying on the same side of B with respect to q contains one and only one arc A_i, we can associate with B' the ordering number corresponding to A_i. This number will be called the ordering number of B' with respect to q. Of course, the ordering number depends on the orientation of the normal vector $\nu(q)$.

DEFINITION 6.6. Suppose that any branch of Γ has multiplicity one. We say that Γ satisfies the compatibility condition if consecutive arcs have the same ordering number.

Note that, while the ordering number of a branch depends on the choice of $\nu(q)$, the compatibility condition is independent of the choice of the unit normal vectors of (Γ) at its nodes.

DEFINITION 6.7. Let Γ be a system of curves without crossings and satisfying the finiteness property. The finite undirected graph $\mathcal{G}_\Gamma$ whose vertices are the nodes of Γ, and whose edges are the branches of Γ, counted with the corresponding multiplicity, will be called the graph associated to Γ.

DEFINITION 6.8. Let $\Gamma_1: S_1 \to \mathbb{R}^2$ and $\Gamma_2: S_2 \to \mathbb{R}^2$ be two systems of curves without crossings, satisfying the finiteness property and having all branches with multiplicity one. We say that Γ_1 is equivalent to Γ_2 if $\mathcal{G}_{\Gamma_1} = \mathcal{G}_{\Gamma_2}$.

We now prove the following crucial result.

THEOREM 6.1. *Let $\Gamma: S \to \mathbb{R}^2$ be a system of curves without crossings, satisfying the finiteness property and having all branches with multiplicity one. Then there exists a system of curves equivalent to Γ and satisfying the compatibility condition.*

PROOF. For any $q \in \mathcal{N}(\Gamma)$ let us fix a unit normal vector $\nu(q)$ of (Γ) at q. To prove the theorem, one has to show that there exists a finite number of cyclic paths of edges of the graph $\mathcal{G}_\Gamma$ with the following properties:

(i) each edge of $\mathcal{G}_\Gamma$ appears once and only once in the family of paths;

(ii) if A and B are two edges of the same path with a common vertex q, then A and B lie on opposite sides with respect to q and have the same ordering number.

We shall construct the cyclic paths using the following algorithm. Let us choose $B_1 \in \mathcal{B}(\Gamma)$ in an arbitrary way, and let q_0 and q_1 be the vertices of B_1. Suppose that $B_1, \dots, B_i$ and $q_0, \dots, q_i$ have been defined. If $q_i = q_0$ and B_i lies on the opposite side of B_1 with respect to q_0 and the ordering number of B_i with respect to q_i coincides with the ordering number of B_1 with respect to q_0, then $B_1, \dots, B_i$ is a cyclic path, and the algorithm stops. Otherwise, we define B_{i+1} as the unique branch of $\mathcal{B}(q_i)$ lying on the opposite side of B_i with respect to q_i and having the same ordering number. Next, we define q_{i+1} as the vertex of B_{i+1} different from q_i. Let us show that $B_i \neq B_j$ for any $j = 1, \dots, i-1$. We argue by contradiction. Let $i \geq 1$ be the smallest integer such that there exists $j \in \mathbf{N}$, $1 \leq j < i$ such that $B_i = B_j$. Note that $i \geq 2$. There are two possibilities: either $q_{i-1} = q_j$ or $q_{i-1} = q_{j-1}$.

Suppose that $q_{i-1} = q_j$. Let $q = q_{i-1} = q_j$ and $B = B_i = B_j$. By the definition of the algorithm, we have that B_{i-1} and B_{j+1} lie on the opposite side of B with respect to q and have the same ordering number of B, hence they coincide. By the minimality of i, this implies that $i - 1 = j + 1$. The endpoints of B_{j+1}

are $q_j = q_{i-1}$ and $q_{j+1} = q_{i-1}$, so that B_{j+1} is a loop, in a contradiction with the hypothesis.

Suppose that $q_{i-1} = q_{j-1}$, and suppose that $j > 1$. Let $q = q_{i-1} = q_{j-1}$ and $B = B_i = B_j$. Then B_{i-1} and B_{j-1} lie on the opposite side of B with respect to q and have the same ordering number of B, hence they coincide, and this contradicts the minimality of i. If $j = 1$ we have $q_{i-1} = q_0$. Let $q = q_{i-1} = q_0$ and $B = B_i = B_1$. Then B_{i-1} and B_1 lie on the opposite side of B_1 with respect to q and have the same ordering number, so that the algorithm would have stopped at the step $i-1$, a contradiction.

In this way we get a cyclic path satisfying property (ii), and such that no edge of $\mathcal{G}_\Gamma$ appears twice in the path. Then we repeat the algorithm, starting from an edge not contained in the first path. It is clear that after a finite number of implementations of the algorithm all edges are reached, since one can prove, arguing as above, that all the cyclic paths obtained are pairwise disjoint.

For any $i = 1, \ldots, m$ let $r_i = \dfrac{l(\gamma^i)}{2\pi}$, let S_i be an oriented circle of radius r_i and let $S(r_1, \ldots, r_m)$ be the disjoint union of the circles S_i's. Let us reparametrize Γ on $S(r_1, \ldots, r_m)$ in such a way that $\left|\dfrac{d\Gamma}{ds}(s)\right| = 1$ for any $s \in S(r_1, \ldots, r_m)$. Let us still denote by Γ this parametrization, by $\mathcal{P}$ the partition of $S(r_1, \ldots, r_m)$ into arcs of circles, and by $I_1, \ldots, I_k$ the arcs of circles of $S(r_1, \ldots, r_m)$ corresponding to $B_1, \ldots, B_k$.

The family of cyclic paths obtained above corresponds to the following surgery operations on the parameter space S of Γ. Let $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ and $S = \mathbf{S}_1^1 \cup \cdots \cup \mathbf{S}_m^1$. Let $B_1, \ldots, B_k$ represent a cyclic path of edges of $\mathcal{G}_\Gamma$, and let $I_1, \ldots, I_k$ be the corresponding arcs of circles in S (recall that there exists a bijection between $\mathcal{B}(\Gamma)$ and the partition $\mathcal{P}$ of S). For any $j = 1, \ldots, k$, let q_{j-1}, q_j be the vertices of B_j, and let $I_j = (a_{j-1}, b_{j-1})$. Obviously, q_0, $q_1, \ldots, q_k = q_0$ are ordered in the cyclic path. Let us possibly change the orientation of some I_j (and let us still denote it by $I_j = (a_{j-1}, b_{j-1})$) in such a way that a_{j-1} corresponds to q_{j-1} and b_{j-1} to q_j. Next, for any $j = 1, \ldots, k$, let us glue together a_j with b_{j-1}. Note that, as $\left|\dfrac{d\gamma^i}{ds}\right| = 1$, $\dfrac{d\Gamma}{ds}(a_j) = \dfrac{d\Gamma}{ds}(b_{j-1})$, for any j. What we get is an oriented closed circle. Repeating this procedure for all the cyclic paths, we obtain a new parameter space formed by a disjoint union of circles having different radii. With a reparametrization, we get the system of curves of the statement. □

We stress that the system of curves of Theorem 6.1 has the same vertices, branches, and multiplicities (in this case all equal to one) of Γ although, in general, it is defined on a different parameter space. In particular $l(\Gamma)$ and $\|\kappa(\Gamma)\|_{L^p}$ remain unchanged when we pass to the new equivalent system of curves. Note also that the orientation of some branches of Γ can be reversed. Let $z \notin (\Gamma)$ and let α be a continuous curve connecting z with ∞ such that all the intersections between (Γ) and (α) are transversal. Since $I(\Gamma, z) \pmod 2$ can be computed using the parity of the number of the intersections of (Γ) with (α), we also deduce that the set of points of odd index with respect to the new system of curves coincides with the set $\{z \in \mathbb{R}^2 \backslash (\Gamma): I(\Gamma, z) \equiv 1 \pmod 2\}$.

LEMMA 6.2. *Let $\Gamma: S \to \mathbb{R}^2$ be a system of curves of class $H^{2,p}$ without crossings, satisfying the finiteness property, having all branches with multiplicity one and satisfying the compatibility condition. Then there exist $\eta_0 > 0$ and a sequence $\Gamma^\eta: S \to \mathbb{R}^2$, $0 < \eta < \eta_0$, of disjoint systems of simple curves of class C^∞ such that $\Gamma^\eta \to \Gamma$ strongly in $H^{2,p}$ as $\eta \to 0$.*

PROOF. We will construct the approximation near a fixed node q, and this procedure must be repeated for any $q \in \mathcal{N}(\Gamma)$.

Let $q \in \mathcal{N}(\Gamma)$, $\Gamma^{-1}(q) = \{t_1, \ldots, t_k\}$. Without loss of generality, we shall suppose that $q = 0$ and $\nu(q) = \mathbf{e}_2$. Let $B_1^-, \ldots, B_k^-$ (respectively $B_1^+, \ldots, B_k^+$) be the branches of Γ lying on the left (respectively on the right) of q, labelled by their ordering number with respect to 0. Let $R(q) = R = \{(x, y) \in \mathbb{R}^2: |x| < c_1,\ |y| < c_2\}$ be a rectangle centered at 0, and let $R^- = \{(x, y) \in R: x < 0\}$, $R^+ = \{(x, y) \in R: x > 0\}$. If R is sufficiently small, using the Implicit Function Theorem one can show that, for any $i = 1, \ldots, k$, $R^- \cap B_i^- = A_i^-$, $R^+ \cap B_i^+ = A_i^+$, where $A_1^-, \ldots, A_k^-$ and $A_1^+, \ldots, A_k^+$ are cartesian graph of a function of class $H^{2,p}$ with respect to the x-axis. Hence, if $i < j$, then A_j lies above A_i. For any $j = 1, \ldots, k$, let $J_i^- = \Gamma^{-1}(A_i^-)$, $J_i^+ = \Gamma^{-1}(A_i^+)$, $J_i = J_i^- \cup J_i^+ \cup \{t_i\}$. Note that, since Γ satisfies the compatibility condition, J_i is an open interval. Let $\pi: R \to]-c_1, c_1[$ be the projection of R onto $]-c_1, c_1[$. Let $\phi:]-c_1, c_1[\to [0, 1]$ be a C^∞ function with the following properties: $\phi(0) = 1$, $\phi(x) = 0$ for any $\frac{c_1}{2} \le x < c_1$, $\phi'(x) < 0$ for any $0 < x < \frac{c_1}{2}$, and $\phi(-x) = \phi(x)$ for any $x \in [0, c_1[$. For any $\eta > 0$ let us define

$$\Gamma^\eta(t) = \begin{cases} \Gamma(t) + \eta \mathbf{e}_2 \dfrac{i}{l} \phi(\pi(\Gamma(t))) & \text{if } t \in J_i \text{ for some } i = 1, \ldots, k, \\ \Gamma(t) & \text{elsewhere on } S. \end{cases}$$

Note that, if η is sufficiently small, the functions $\Gamma_{|J_i}$ are injective on J_i and $\Gamma^\eta(J_i) \cap \Gamma^\eta(J_j) = \emptyset$ for any $i \neq j$. Repeating this procedure for any node, and taking the rectangles $R(q)$ pairwise disjoint, after a reparametrization, we obtain that $\Gamma^\eta: S \to \mathbb{R}^2$ is a disjoint system of simple curves of class $H^{2,p}$ such that $\Gamma^\eta \to \Gamma$ strongly in $H^{2,p}$ as $\eta \to 0$. Furthermore, any system Γ^η can be approximated in the $H^{2,p}$ norm by a disjoint system of simple curves of class C^∞ (see the proof of Corollary 3.2). By a diagonal argument, we get the assertion. □

Let $\Gamma: S \to \mathbb{R}^2$ be a system of curves of class $H^{2,p}$ without crossings and satisfying the finiteness property. Let

$$\begin{aligned} E &= \{z \in \mathbb{R}^2 \backslash (\Gamma):\ I(\Gamma, z) \equiv 1 \pmod 2\}, \\ F &= \{z \in \mathbb{R}^2 \backslash (\Gamma):\ I(\Gamma, z) \equiv 0 \pmod 2\}, \end{aligned} \tag{6.3}$$

$$E^\star = \{z \in \mathbb{R}^2\colon\ \exists r > 0\ |B_r(z)\backslash E| = 0\},$$
(6.4)
$$F^\star = \{z \in \mathbb{R}^2\colon\ \exists r > 0\ |B_r(z)\backslash F| = 0\}.$$

Then E, $E^\star$, F, $F^\star$ are open, $E^\star$ is bounded,

(6.5) $$|E\Delta E^\star| = 0, \quad E^\star = \mathrm{int}(\mathbb{R}^2\backslash F^\star), \quad F^\star = \mathrm{int}(\mathbb{R}^2\backslash E^\star),$$
$$\partial E^\star = \partial F^\star = \{z \in \mathbb{R}^2\colon 0 < |B_r(z) \cap E| < |B_r(z)|\ \forall r > 0\} = \partial E \cap \partial F \subseteq (\Gamma).$$

The fact that E, $E^\star$, F, and $F^\star$ are open is trivial. To prove that $|E\Delta E^\star| = 0$ it is enough to observe that $E \subseteq E^\star \subseteq E \cup (\Gamma)$ (see (3.12), (3.13)). In particular, $E^\star$ is bounded. To prove that $E^\star = \mathrm{int}(\mathbb{R}^2\backslash F^\star)$, $F^\star = \mathrm{int}(\mathbb{R}^2\backslash E^\star)$ and $\partial E^\star = \partial F^\star = \{z \in \mathbb{R}^2\colon 0 < |B_r(z) \cap E| < |B_r(z)|\ \forall r > 0\}$, it is sufficient to repeat the proof of (3.15), (3.16) and (3.7), respectively. Moreover, the relation $\partial E^\star = \partial F^\star = \partial E \cap \partial F \subseteq (\Gamma)$ can be proved as in Lemma 3.3 (iii).

THEOREM 6.2. *Let $\Gamma\colon S \to \mathbb{R}^2$ be a system of curves of class $H^{2,p}$ without crossings and satisfying the finiteness property. Let E, $E^\star$, F, and $F^\star$ be defined as in* (6.3), (6.4). *Then*

(6.6) $$\overline{\mathcal{F}}(E) = \overline{\mathcal{F}}(E^\star) < +\infty,$$

hence there exists a sequence $\{E_h\}_h$ of bounded open sets of class C^∞ such that $E_h \to E$ in $L^1(\mathbb{R}^2)$ as $h \to +\infty$ and $\sup_h \mathcal{F}(E_h) < +\infty$. In addition, there exist oriented parametrizations Γ_h of ∂E_h defined on the same parameter space $\tilde{S}$, such that $\{\Gamma_h\}_h$ converges strongly in $H^{2,p}$ to a system of curves equivalent to Γ, defined on $\tilde{S}$, and whose trace contains $\partial E^\star$. Finally,

(6.7) $$\overline{\mathcal{F}}(E) \leq \inf\{\mathcal{F}(\Delta)\colon \Delta \in \mathcal{Q}(E)\},$$

where $\mathcal{Q}(E)$ is the collection of those systems of curves Δ of class $H^{2,p}$ without crossings, satisfying the finiteness property and such that $E^\star = \mathrm{int}(A_\Delta \cup (\Delta))$, where $A_\Delta = \{z \in \mathbb{R}^2\backslash(\Delta)\colon I(\Delta, z) \equiv 1 \pmod 2\}$.

PROOF. Let $\{t_1, \dots, t_n\} = \Gamma^{-1}(\mathcal{N}(\Gamma))$, let $\Gamma_\varepsilon\colon S \to \mathbb{R}^2$, $0 < \varepsilon < \varepsilon_0$, be the approximating sequence of systems of curves of Lemma 6.1, and let $\{t_1^\varepsilon, \dots, t_n^\varepsilon\} = \Gamma_\varepsilon^{-1}(\mathcal{N}(\Gamma_\varepsilon))$. For any $0 < \varepsilon < \varepsilon_0$, there exists, by Theorem 6.1, a system of curves equivalent to Γ_ε and satisfying the compatibility condition. Let us prove that these systems of curves are defined on a same parameter space. It is easy to see that the graphs associated to these systems of curves are isomorphic. If we identify isomorphic graphs, then the cyclic paths of edges constructed in the proof of Theorem 6.1 do not depend on ε, as well as the ordering numbers of any node $q \in \mathcal{N}(\Gamma_\varepsilon) = \mathcal{N}(\Gamma)$. It follows that the parameter space obtained at the end of the proof of Theorem 6.1, on which the new systems of curves are defined, does not depend on ε, and we shall denote it by $\tilde{S}$. Repeating on $S\backslash\{t_1,\dots,t_n\}$ the same surgery operations made on $S\backslash\{t_1^\varepsilon,\dots,t_n^\varepsilon\}$ (recall that $t_i^\varepsilon \to t_i$ for any $i = 1,\dots,n$ as $\varepsilon \to 0$), we get a system of curves defined on $\tilde{S}$ and equivalent to Γ. Let us still denote this system by the symbol Γ. Then $\Gamma_\varepsilon \to \Gamma$ strongly in $H^{2,p}(\tilde{S},\mathbb{R}^2)$ as $\varepsilon \to 0$.

For any $\varepsilon > 0$, let $\{\Gamma_\varepsilon^\eta\}_\eta$ be the sequence of disjoint systems of simple curves of Lemma 6.2 converging to Γ_ε strongly in $H^{2,p}(\tilde{S},\mathbb{R}^2)$ as $\eta \to 0$. By a diagonal argument, there exists a sequence $\Gamma_h: \tilde{S} \to \mathbb{R}^2$ of disjoint systems of simple curves of class $\mathcal{C}^\infty$ such that $\Gamma_h \to \Gamma$ strongly in $H^{2,p}(\tilde{S},\mathbb{R}^2)$ as $h \to +\infty$. For any h, let us define

$$E_h = \{z \in \mathbb{R}^2\backslash(\Gamma_h): I(\Gamma_h, z) \equiv 1 \pmod 2\}.$$

Note that this definition does not depend on the orientation of each circle composing the parameter space $\tilde{S}$. By the continuity property of the index and by the Dominated Convergence Theorem, it follows that $\{E_h\}_h$ converges to the set $\{z \in \mathbb{R}^2\backslash(\Gamma): I(\Gamma, z) \equiv 1 \pmod 2\}$, as $h \to +\infty$. Since this set coincides with the set of all points of $\mathbb{R}^2\backslash(\Gamma)$ of odd index with respect to the original system of curves, by the definition of E we get that $E_h \to E$ in $L^1(\mathbb{R}^2)$ as $h \to +\infty$. Moreover, by construction, $\sup_h \mathcal{F}(E_h) < +\infty$; it follows that $\overline{\mathcal{F}}(E) < +\infty$. Since $|E\Delta E^\star| = 0$ (see (6.5)), we have that $\overline{\mathcal{F}}(E) = \overline{\mathcal{F}}(E^\star)$, and (6.6) is proven.

In addition, since $\Gamma_h \to \Gamma$ strongly in $H^{2,p}(\tilde{S},\mathbb{R}^2)$ as $h \to +\infty$, it follows that

$$l(\Gamma) + \|\kappa(\Gamma)\|_{L^p}^p = \lim_{h\to+\infty} \int_{\partial E_h} [1 + |\kappa_h(z)|^p]\, d\mathcal{H}^1(z) \geq \overline{\mathcal{F}}(E).$$

Using the same arguments, we deduce that, if $\Delta \in \mathcal{Q}(E)$, then $\overline{\mathcal{F}}(E) = \overline{\mathcal{F}}(A_\Delta) \leq l(\Delta) + \|\kappa(\Delta)\|_{L^p}^p$ (recall that $|A_\Delta \Delta E| = 0$). Whence, passing to the infimum with respect to $\Delta \in \mathcal{Q}(E)$, we get (6.7), and this concludes the proof. □

Let $\mathcal{G}$ be a finite graph. We denote by $\mathcal{B}(\mathcal{G})$ the edges of $\mathcal{G}$ and by $\mathcal{N}(\mathcal{G})$ the vertices of $\mathcal{G}$. For any $q \in \mathcal{N}(\mathcal{G})$, we denote by $\mathcal{B}(q)$ the edges of $\mathcal{G}$ having endpoint q. We shall designate by $\varrho_\mathcal{G}(q)$ the number of elements of $\mathcal{B}(q)$, counted with their multiplicity. In this definition, any loop at q will be counted as a double edge.

Suppose that $\mathcal{G}$ (considered as a subset of $\mathbb{R}^2$) has a continuous unoriented tangent, and let $\nu(q)$ be a unit normal vector of $\mathcal{G}$ at q. We denote by $\mathcal{B}^-(q)$ (respectively $\mathcal{B}^+(q)$) the edges of $\mathcal{B}(q)$ lying on the left (respectively on the right) of q with respect to $\nu(q)$. We shall designate by $\varrho_\mathcal{G}^-(q)$ (respectively by $\varrho_\mathcal{G}^+(q)$) the number of elements of $\mathcal{B}^-(q)$ (respectively of $\mathcal{B}^+(q)$) counted with their multiplicity.

DEFINITION 6.9. Let $\mathcal{G}$ be a finite graph. We say that $\mathcal{G}$ is regular if $\varrho_\mathcal{G}^-(q) = \varrho_\mathcal{G}^+(q)$ for any $q \in \mathcal{N}(\mathcal{G})$.

Let us observe that if Γ is a system of curves without crossings and satisfying the finiteness property, and $\mathcal{G}_\Gamma$ is the graph associated to Γ, then $\mathcal{G}_\Gamma$ is regular.

DEFINITION 6.10. We say that a bounded open set $E \subseteq \mathbb{R}^2$ has a piecewise $H^{2,p}$ boundary if $\partial E = \bigcup_{i=1}^k (\alpha^i)$, where $\alpha^1, \dots, \alpha^k : [0,1] \to \mathbb{R}^2$ are regular simple curves of class $H^{2,p}$ such that, for any $i,\ j \in \{1, \dots, k\}$, $i \neq j$, $\alpha^i(]0,1[) \cap \alpha^j(]0,1[) = \emptyset$, $\alpha^i(0) \neq \alpha^i(1)$, and for any $t \in]0,1[$ there exists $r > 0$ such that $B_r(\alpha^i(t)) \setminus \partial E$ has exactly two connected components, $B_r(\alpha^i(t)) \cap E$ and $B_r(\alpha^i(t)) \setminus \overline{E}$.

Note that, if $E \subseteq \mathbb{R}^2$ is a bounded open set with a piecewise $H^{2,p}$ boundary, then $\alpha^i(0)$, $\alpha^i(1) \notin \alpha^j(]0,1[)$, for any $i \neq j$. If, in addition, ∂E has a continuous unoriented tangent, then, for any $t_1,\ t_2 \in \{0,1\}$ such that $\alpha^i(t_1) = \alpha^j(t_2)$, the vectors $\dfrac{d\alpha^i}{dt}(t_1)$ and $\dfrac{d\alpha^j}{dt}(t_2)$ are parallel.

Note also that the curves $\alpha^1, \dots, \alpha^k$ can be reordered and suitably parametrized and glued together to form a family of curves $\{\gamma^1, \dots, \gamma^m\}$, with $1 \leq m < k$, satisfying the following properties: for any $i = 1, \dots, m$, γ^i is continuous, closed, and of class $H^{2,p}$ up to a finite number of points, and

$$E = \left\{ z \in \mathbb{R}^2 \setminus \bigcup_{i=1}^m (\gamma^i) : \sum_{i=1}^m I(\gamma^i, z) \equiv 1 \pmod 2 \right\}. \tag{6.8}$$

As a notation, for any $i = 1, \dots, k$, the sets $\alpha^i(]0,1[)$ will be called the branches of ∂E and will be denoted by $\mathcal{B}(\partial E)$, and the points $\alpha^i(0)$, $\alpha^i(1)$ will be called the vertices of ∂E and will be denoted by $\mathcal{N}(\partial E)$. For any $q \in \mathcal{N}(\partial E)$, we denote by $\mathcal{B}(q)$ the branches of ∂E having endpoint q. Assume now that ∂E has a continuos unoriented tangent, and let $\nu(q)$ be a unit normal vector of ∂E at q. We denote by $\mathcal{B}^-(q)$ (respectively $\mathcal{B}^+(q)$) the branches of $\mathcal{B}(q)$ lying on the left (respectively on the right) of q with respect to $\nu(q)$, and by $\varrho^-_{\partial E}(q)$ (respectively $\varrho^+_{\partial E}(q)$) the number of elements of $\mathcal{B}^-(q)$ (respectively of $\mathcal{B}^+(q)$). Note that, if $\overline{\mathcal{F}}(E) < +\infty$, then the cusp and the branch points of ∂E (see Definitions 4.6 and 4.7) are vertices. It is clear that there can be vertices in a neighbourhood of which ∂E is regular (some of them must be artificially introduced because of the condition $\alpha^i(0) \neq \alpha^i(1)$). These can be regarded as "artificial" vertices. The simplest "genuine" vertex is the *simple cusp point*. This is defined as a vertex q such that $\varrho^+_{\partial E}(q) = 2$ and $\varrho^-_{\partial E}(q) = 0$ for a suitable choice of the normal vector $\nu(q)$. Note that the simple cusp points are cusp points according to Definition 4.6.

Observe that $|\varrho^+_{\partial E}(q) - \varrho^-_{\partial E}(q)|$ is even, for any $q \in \mathcal{N}(\partial E)$. In fact, given $q \in \mathcal{N}(\partial E)$, there exists $r > 0$ such that $B_r(q)$ contains both points of E and points of $\mathbb{R}^2 \setminus E$, $\partial B_r(q)$ meets transversally all branches of ∂E, hence the number of points of the set $\partial E \cap \partial B_r(q)$ is even. If r is sufficiently small, this number coincides with $\varrho^+_{\partial E}(q) + \varrho^-_{\partial E}(q)$, which has the same parity of $|\varrho^+_{\partial E}(q) - \varrho^-_{\partial E}(q)|$.

Intuitively, more complex vertices can be seen as the result of the collapse of a finite number of simple cusp points, with the possible addition of some artificial vertices. We now introduce the balanced multiplicity $\omega_{\partial E}(q)$ of a vertex

q, whose intuitive meaning is the difference between the number of simple cusps having vertex at q and lying on opposite sides of q with respect to $\nu(q)$. The formal definition is

$$\omega_{\partial E}(q) = \frac{|\varrho^+_{\partial E}(q) - \varrho^-_{\partial E}(q)|}{2}.$$

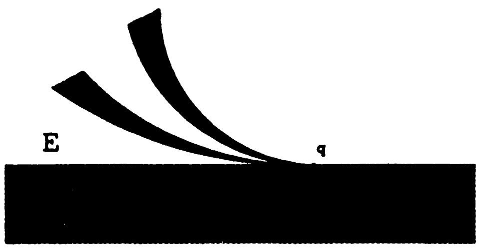


Fig. 6.2: $\omega_{\partial E}(q) = 2$.

Note that if q is a simple cusp point of ∂E, then $\omega_{\partial E}(q) = 1$, and if q is a vertex of ∂E in a neighbourhood of which ∂E is regular, then $\omega_{\partial E}(q) = 0$.

DEFINITION 6.11. Let $E \subseteq \mathbb{R}^2$ be a bounded open set with a piecewise $H^{2,p}$ boundary ∂E having a continuous unoriented tangent. The undirected graph $\mathcal{G}_{\partial E}$ whose vertices are the vertices of ∂E, and whose edges are the branches of ∂E with multiplicity one, will be called the graph associated to ∂E.

Note that, in general, $\mathcal{G}_{\partial E}$ is not a regular graph.

THEOREM 6.3. *Let $E \subseteq \mathbb{R}^2$ be a bounded open set with a piecewise $H^{2,p}$ boundary ∂E having a continuous unoriented tangent. Then*

$$\sum_{q \in \mathcal{N}(\partial E)} \omega_{\partial E}(q) \textit{ is even } \Longrightarrow \overline{\mathcal{F}}(E) < +\infty.$$

PROOF. We shall define a regular graph $\mathcal{G}$ whose set of edges contains $\mathcal{B}(\partial E)$ and such that each edge of $\mathcal{B}(\mathcal{G}) \setminus \mathcal{B}(\partial E)$ has even multiplicity.

For any vertex q of ∂E, let us fix a normal unit vector $\nu(q)$ of ∂E at q in such a way that $\varrho^+_{\partial E}(q) \geq \varrho^-_{\partial E}(q)$. Denote by $\{q_1, \ldots, q_m\}$ the set of all vertices of ∂E with $\omega_{\partial E}(q) \geq 2$. We construct m graphs $\mathcal{G}'_1, \ldots, \mathcal{G}'_m$ inductively. Let $\mathcal{G}'_1 = \mathcal{G}_{\partial E}$, and suppose that the graph $\mathcal{G}'_{i-1}$ has been defined. If $i \leq m$, we define the graph $\mathcal{G}'_i$ as follows. Let $\gamma^i: [0, 1] \to \mathbb{R}^2$ be a smooth curve lying on the left of q with respect to $\nu(q)$, such that $\gamma^i(0) = \gamma^i(1) = q_i$, $\dfrac{d\gamma^i}{dt}(0) = -\dfrac{d\gamma^i}{dt}(1)$, $(\gamma^i) \cap \partial E = \{q_i\}$, and such that the tangent lines of ∂E and (γ^i) at q_i coincide.

Let us associate with (γ^i) a multiplicity defined by $\omega_{\partial E}(q_i)$, if $\omega_{\partial E}(q_i)$ is even, and by $\omega_{\partial E}(q_i) - 1$, if $\omega_{\partial E}(q_i)$ is odd. To eliminate loops, add a new vertex $z_i \neq q_i$ on (γ^i). Let $\mathcal{G}'_i$ be the graph whose vertices are the vertices of $\mathcal{G}'_{i-1}$ together with z_i, and whose edges are the edges of $\mathcal{G}'_{i-1}$ together with the two arcs on the curve (γ^i) determined by the point z_i, with the even multiplicity previously defined. Clearly, $\varrho^+_{\mathcal{G}'_i}(z_i) = \varrho^-_{\mathcal{G}'_i}(z_i)$.

At the end of this algorithm, we get a graph $\mathcal{G}' = \mathcal{G}'_m$ whose set of edges contains $\mathcal{B}(\partial E)$, such that each edge of $\mathcal{B}(\mathcal{G}')\setminus\mathcal{B}(\partial E)$ has even multiplicity, and such that

$$\varrho^+_{\mathcal{G}'}(q) - \varrho^-_{\mathcal{G}'}(q) = \begin{cases} 0 & \text{if } \omega_{\partial E}(q) \text{ is even,} \\ 2 & \text{if } \omega_{\partial E}(q) \text{ is odd,} \end{cases}$$

for any $q \in \mathcal{N}(\partial E)$.

By the hypothesis, the number of the set of vertices q of ∂E with $\omega_{\partial E}(q)$ odd is even. Let us denote by $\{p_1, \dots, p_{2k}\}$ these vertices. We shall construct $\mathcal{G}$ inductively using the following algorithm.

Define $\mathcal{G}_1 = \mathcal{G}'$, and suppose that the graph $\mathcal{G}_{i-1}$ has been defined. If $i \leq k$, we define the graph $\mathcal{G}_i$ as follows. Let $\gamma^i : [0, 1] \to \mathbb{R}^2$ be a smooth curve such that $\gamma^i(0) = p_{2i-1}$, $\gamma^i(1) = p_{2i}$, the direction of $\frac{d\gamma^i}{dt}(0)$ (respectively of $\frac{d\gamma^i}{dt}(1)$) coincides with the direction of $T_{\partial E}(p_{2i-1})$ (respectively $T_{\partial E}(p_{2i})$), $(\gamma^i) \cap \mathcal{N}(\partial E) = \{p_{2i-1},\ p_{2i}\}$ and such that (γ^i) meets the edges of $\mathcal{G}'$ in a finite number of points, say $\{z_i, \dots, z_{h(i)}\}$, in such a way that the tangent lines of (γ^i) and of the branches of $\mathcal{G}'$ coincide at the intersection points.

Let us associate with (γ^i) the multiplicity two. Let $\mathcal{G}_i$ be the graph whose vertices are the vertices of $\mathcal{G}_{i-1}$ together with $\{z_1, \dots, z_{h(i)}\}$, and whose edges are the edges of $\mathcal{G}_{i-1}$ together with the arcs on the curve (γ^i) determined by the points $z_1, \dots, z_{h(i)}$ with the multiplicity two.

At the end of this algorithm, we get a regular graph $\mathcal{G}$ whose set of edges contains $\mathcal{B}(\partial E)$ and such that each edge of $\mathcal{B}(\mathcal{G})\setminus\mathcal{B}(\partial E)$ has even multiplicity. As $\mathcal{G}$ is regular, there exists a system of curves Γ of class $H^{2,p}$ associated to $\mathcal{G}$ (see Definition 6.7). Since, by construction, all branches of Γ having even multiplicity do not meet the boundary of E, using (6.8) we deduce that

$$|E \Delta \{z \in \mathbb{R}^2 \setminus (\Gamma) : I(\Gamma, z) \equiv 1 \pmod 2\}| = 0.$$

The thesis now follows from Theorem 6.2. □

Let $E \subseteq \mathbb{R}^2$ be a bounded open set with a piecewise $H^{2,p}$ boundary ∂E having a continuous unoriented tangent. Assume that $\overline{\mathcal{F}}(E) < +\infty$. Let Γ be one of the limit systems of curves of Theorem 4.1 (iii), and suppose that Γ satisfies the finiteness property. Observe that, for any $B \in \mathcal{B}(\Gamma)$, we can assume that either $B \subseteq \partial E$ or $B \cap \partial E = \emptyset$. Moreover, by Theorem 4.1 (iii) and Lemma 4.1, it follows that, if $B \subseteq \partial E$ then B has odd multiplicity, and if $B \cap \partial E = \emptyset$, then B has even multiplity. Now we can prove the following result.

THEOREM 6.4. *Let $E \subseteq \mathbb{R}^2$ be a bounded open set with a piecewise $H^{2,p}$ boundary ∂E having a continuous unoriented tangent. Let $\mathcal{N}(\partial E) = \{q, \dots, q_n\}$,* *and suppose that each* q_i *is a simple cusp point of* ∂E. *Then*

$$\overline{\mathcal{F}}(E) < +\infty \Rightarrow n \text{ is even.}$$

PROOF. Let Γ be one of the limit systems of curves satisfying (iii) of Theorem 4.1, and let us suppose first that Γ satisfies the finiteness property. We shall define an auxiliary graph $\mathcal{G}$ such that $\mathcal{N}(\partial E)$ coincides with the set of those vertices q of $\mathcal{G}$ such that $\varrho_{\mathcal{G}}(q)$ is odd. We construct $\mathcal{G}$ in such a way that the vertices of $\mathcal{G}$ are the vertices of $\mathcal{G}_\Gamma$, and whose edges are those edges B of $\mathcal{G}_\Gamma$ having multiplicity $m_{\mathcal{G}_\Gamma}(B) > 1$. Precisely, let $B \in \mathcal{B}(\mathcal{G}_\Gamma)$; the multiplicity $m_{\mathcal{G}}(B)$ of any edge of $\mathcal{G}$ is defined as follows:

$$m_{\mathcal{G}}(B) = \begin{cases} \dfrac{m_{\mathcal{G}_\Gamma}(B)}{2} & \text{if } m_{\mathcal{G}_\Gamma}(B) \text{ is even,} \\[2ex] \dfrac{m_{\mathcal{G}_\Gamma}(B) - 1}{2} & \text{if } m_{\mathcal{G}_\Gamma}(B) \text{ is odd.} \end{cases}$$

Hence, if $m_{\mathcal{G}_\Gamma}(B) = 1$, then B is deleted. Note that $\mathcal{G}$ is not necessarily regular.

Let $q \in \mathcal{N}(\mathcal{G}_\Gamma)$. If $q \notin \partial E$, then all edges of $\mathcal{G}_\Gamma$ meeting at q have even multiplicity. Hence $\varrho^+_{\mathcal{G}_\Gamma}(q)$ and $\varrho^-_{\mathcal{G}_\Gamma}(q)$ are even. As $\mathcal{G}_\Gamma$ is regular, we have $\varrho^+_{\mathcal{G}_\Gamma}(q) = \varrho^-_{\mathcal{G}_\Gamma}(q)$. Therefore $\varrho_{\mathcal{G}}(q)$ is even.

If $q \in \partial E$ and ∂E is regular in a neighbourhood of q, then exactly two of the edges of $\mathcal{G}_\Gamma$ meeting at q have odd multiplicity and lie one on each side of q with respect to $\nu(q)$; let us denote by B_1 and B_2 these edges, and let $2k+1$ and $2k'+1$ be their multiplicities, respectively. All the other edges meeting at q have even multiplicity; let $2i$ (respectively $2j$) be the number (counted with multiplicities) of all these branches lying on the same (respectively on the opposite) side of B_1, B_2 with respect to $\nu(q)$. As $\mathcal{G}_\Gamma$ is regular, we have that $\varrho^-_{\mathcal{G}_\Gamma}(q) = \varrho^+_{\mathcal{G}_\Gamma}(q)$, i.e., $2k+1+2i = 2k'+1+2j$. Hence $\varrho_{\mathcal{G}}(q) = k+k'+i+j = 2(k+i)$ is even.

If $q \in \partial E$ is a simple cusp point of ∂E, then there exist exactly two edges B_1, $B_2 \in \mathcal{B}(\mathcal{G}_\Gamma)$ having odd multiplicity and lying both on one side of q with respect to $\nu(q)$. Let $2k+1$, $2k'+1$ be these multiplicities, and let $2i$ (respectively $2j$) be the number (counted with multiplicities) of all the other branches of $\mathcal{G}_\Gamma$ lying on the same (respectively on the opposite) side of B_1, B_2 with respect to $\nu(q)$ (see Figure 6.3). As $\mathcal{G}_\Gamma$ is regular, we have that $\varrho^-_{\mathcal{G}_\Gamma}(q) = \varrho^+_{\mathcal{G}_\Gamma}(q)$, i.e., $2j = 2i + 2k + 1 + 2k' + 1$.

Hence $\varrho_{\mathcal{G}}(q) = j + i + k + k' = 2(i + k + k') + 1$ is odd. It follows that the simple cusp points of ∂E are exactly the vertices q of $\mathcal{G}$ such that $\varrho_{\mathcal{G}}(q)$ is odd. Since in any finite graph $\mathcal{G}$ the number of the set of vertices q with $\varrho_{\mathcal{G}}(q)$ odd is even ([16, Theorem 1.2.1]), we deduce that n is even.

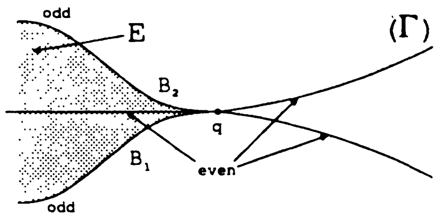


Fig. 6.3: *Computation of* $\varrho_{\mathcal{G}}^{-}(q)$ *and* $\varrho_{\mathcal{G}}^{+}(q)$ *when* $q \in \partial E$ *is a simple cusp point of* ∂E.

We still have to prove the theorem in the most delicate case, i.e., when the system of curves $\Gamma = \{\gamma^1, \dots, \gamma^m\}$ does not satisfy the finiteness property. We want to modify Γ into a system of curves Λ of class C^1, satisfying the finiteness property, such that $(\Lambda) \subseteq (\Gamma)$, and still

$$(6.9) \qquad |E \Delta \{z \in \mathbb{R}^2 \setminus (\Lambda) : I(\Lambda, z) \equiv 1 \pmod 2\}| = 0.$$

Then the argument of the previous part of the proof can be repeated for Λ (the C^1 regularity is enough for this purpose), so we can conclude as before that n is even.

Condition (6.9) will be achieved by requiring that each branch of Λ with even multiplicity does not meet ∂E and each branch with odd multiplicity is included in ∂E (recall the method for computing $I(\Lambda, z)$ mentioned before Lemma 6.2).

Let C be the set of those points $q \in (\Gamma)$ such that, for any neighbourhood U_q of q, $(\Gamma) \cap U_q$ cannot be written as a cartesian graph with respect to the tangent line $T_{(\Gamma)}(q)$ of (Γ) at q, and let K be the set of all accumulation points of C. Note that K is a compact set.

For any $q \in K$, let $R(q)$ be a small rectangle centered at q, having two sides parallel to $T_{(\Gamma)}(q)$ and such that each curve γ^i meeting $R(q)$ is a cartesian graph in $R(q)$ with respect to $T_{(\Gamma)}(q)$. Such a rectangle exists, since $\gamma^1, \dots, \gamma^m$ are regular curves of class $H^{2,p}$ parametrized with constant velocity. Furthermore, we can choose $R(q)$ in such a way that, if γ^i is a curve of Γ passing through $R(q)$, then γ^i meets the point q and does not meet the two sides of $R(q)$ which are parallel to $T_{(\Gamma)}(q)$. Finally, we may also assume that, if q is not a cusp point of ∂E, then $R(q)$ does not contain any cusp point of ∂E.

As K is a compact set, there exist $q_1, \dots, q_N$ points of K such that $U = \bigcup_{i=1}^{N} R(q_i)$ contains K. In order to construct the modified system of curves Λ, we argue by induction. Let $\Lambda_0 = \Gamma$, let $i \leq N$, and suppose that Λ_{i-1}

has been defined. Then Λ_i is obtained by modifying Λ_{i-1} only on $R(q_i)$, i.e., $(\Lambda_{i-1})\setminus R(q_i) = (\Lambda_i)\setminus R(q_i)$, in such a way that $\partial E \subseteq (\Lambda_i) \subseteq (\Lambda_{i-1})$, and that each branch of $(\Lambda_i) \cap R(q_i)$ with even multiplicity does not meet ∂E, while each branch of $(\Lambda_i) \cap R(q_i)$ with odd multiplicity is contained in ∂E. Moreover, by the construction of Λ_i, there exist a finite set F_i such that $(\Lambda_i) \cap R(q_i)\setminus F_i$ is a one dimensional C^1 submanifold. We define $\Lambda = \Lambda_N$. The previous remark implies that there exists a finite set F such that $(\Lambda) \cap U\setminus F$ is a one dimensional C^1 submanifold. By the definition of K, the set $C\setminus U$ has no accumulation points, so it is finite. It follows that Λ satisfies the finiteness property. As in the points of $(\Lambda)\setminus U$ we already have the correct multiplicity by Lemma 4.1, it is clear that $\Lambda = \Lambda_N$ satisfies the required properties.

Let us suppose first that $q_i \in K$ is such that $q_i \in (\Gamma)\setminus\overline{E}$. To simplify the notation, we assume that $q_i = 0$, that $T_{(\Lambda_i)}(0)$ coincides with the x-axis, and that $\nu(0) = \mathbf{e}_2$. Let $R(0) =]-a, a[\times]-b, b[$; we can assume that $R(0) \cap \overline{E} = \emptyset$. We shall work on the set $(\Lambda_{i-1}) \cap ([0, a[\times]-b, b[)$, since the modification of Λ_{i-1} on the set $(\Lambda_{i-1}) \cap]-a, 0]\times]-b, b[$ is analogous. Because of the assumptions on $R(0)$ and the inclusion $(\Lambda_{i-1}) \subseteq (\Gamma)$, the set $(\Lambda_{i-1}) \cap \{(x, y) \in R(0): x = a\}$ consists of a finite number of points $\{z_1, \ldots, z_h\}$, labelled by their y-coordinate. Let $\lambda^1, \ldots, \lambda^k$ be the curves of the system Λ_{i-1} which meet $R(0)$ and let $f_1, \ldots, f_k \in H^{2,p}(]0, a[)$ be the functions whose graphs are the traces of $\lambda^1, \ldots, \lambda^k$, i.e., $(\lambda^j) \cap R(0) = \mathrm{graph}(f_j)$. Note that $\{z_1, \ldots, z_h\} = \{f_1(a), \ldots, f_k(a)\}$. As (Λ_{i-1}) has a continuous unoriented tangent, it follows that $f_j'(x) = f_r'(x)$ whenever $f_j(x) = f_r(x)$.

For any $j \in \{1, \ldots, h\}$, let d_j be the number of elements of the set $\Lambda_{i-1}^{-1}(z_j)$. By the inductive property of Λ_{i-1} we have that d_j is even, for any $j \in \{1, \ldots, h\}$.

For any $j \in \{1, \ldots, h\}$, let S_j be the set of all continuous functions $g: [0, a] \to]-b, b[$ such that

$$\mathrm{graph}(g) \subseteq \bigcup_{r=1}^{h} \mathrm{graph}(f_r) \qquad \text{and } g(b) = z_j. \tag{6.10}$$

It is easy to see that S_j is non-empty and that each function $g \in S_j$ is of class C^1 and satisfies

$$\sup |g'| \leq \max_{1 \leq r \leq h} |f_r'| < +\infty \tag{6.11}$$

(recall that $f_j'(x) = f_r'(x)$ whenever $f_j(x) = f_r(x)$). Let $g_j: [0, a] \to [-b, b]$ be the function defined by

$$g_j(x) = \sup_{g \in S_j} g(x).$$

As all functions in S_j are continuous, by Lindelöf's Theorem there exists a sequence $\{g_j^k\}_k$ such that $g_j(x) = \sup_k g_j^k(x)$. As S_j is a lattice, we may assume that $g_j^k \leq g_j^{k+1}$ for every k. This implies that $g_j^k \to g_j$ pointwise as $k \to +\infty$, hence g_j satisfies (6.10). By (6.11) and by Ascoli-Arzelà Theorem it follows that g_j is continuous. Hence g_j is the greatest element of $\mathcal{S}_j$, and, consequently, $g_j \in \mathcal{C}^1([0,a])$.

By the maximality of g_j it follows that

$$g_j(c) = g_l(c) \Rightarrow g_j(x) = g_l(x) \quad \forall x \in [0,c]. \tag{6.12}$$

We are now in a position to replace the branches of (Λ_{i-1}) contained in $[0,a]\times[-b,b]$ by the curves given by the graphs of the functions g_j, $j = 1,\dots,h$, each curve counted with the multiplicity d_j equal to the multiplicity of z_j for the system of curves (Λ_{i-1}) The system obtained after the same operation on $[-a,a]\times[-b,b]$ will be Λ_i.

It is easy to see that (6.12) implies that there exists a finite set F_i such that $(\Lambda_i)\cap R(0)\backslash F_i$ is a one dimensional $\mathcal{C}^1$ submanifold.

This concludes the modification when $q_i \in (\Gamma)\backslash\overline{E}$. The previous arguments can be repeated exactly when $q_i \in \mathrm{int}(E)$, obviously taking $R(q_i)$ small enough such that $R(q_i) \subseteq \mathrm{int}(E)$.

We still have to consider the case in which $q_i \in \partial E$. We shall work in the most difficult case, i.e., when $q_i \in K$ is a simple cusp point of ∂E, the case in which ∂E is regular near q_i being easier.

We shall adopt the notation of the previous case, with the following modifications. We shall suppose that E lies on the right of 0, and that $R\cap\partial E = \mathrm{graph}(\phi_1)\cup\mathrm{graph}(\phi_2)$, where $\phi_1,\ \phi_2\colon]0,a[\to [-b,b]$ are functions of class $H^{2,p}$, with $\phi_1 \le \phi_2$. As ∂E is contained in (Λ_{i-1}), we have that $\phi_1(b)$, $\phi_2(b) \in \{z_1,\dots,z_h\}$, hence there exist $j_1 < j_2$ such that $\phi_1(b) = z_{j_1}$ and $\phi_2(b) = z_{j_2}$. Note that, in general, $\mathrm{graph}(\phi_1)$ is not a subset of the trace of a unique curve γ^i, and the same holds for $\mathrm{graph}(\phi_2)$. Observe also that there can be a curve γ^i which passes alternatively above $\mathrm{graph}(\phi_2)$ and below $\mathrm{graph}(\phi_1)$ an infinite number of times.

We will divide the set $\{1,\dots,h\}$ into three disjoint subsets, taking into account the part of (Λ_{i-1}) lying below the cusp, the part lying above the cusp, and the part lying inside the cusp. Let $S_1 = \{1,\dots,j_1\}$, $S_2 = \{j_2,\dots,h\}$, $S = \{j_1+1,\dots,j_2-1\}$.

For any $j \in \{1,\dots,h\}$ let d_j be the number of elements of the set $\Lambda_{i-1}^{-1}(z_j)$. By the inductive property of Λ_{i-1} we have that d_{j_1} and d_{j_2} are odd, while d_j is even, for any $j \neq j_1,\ j_2$. For any $j \in S_1$ let $\mathcal{S}_j$ be the set of all continuous functions $g\colon [0,a] \to]-b,b[$ such that

$$\mathrm{graph}(g) \subseteq \bigcup_{r=1}^{h} \mathrm{graph}(f_r \wedge \phi_1) \qquad \text{and } g(b) = z_j.$$

Then, as before, $\mathcal{S}_j$ is non-empty, if $g \in \mathcal{S}_j$, then $g \in \mathcal{C}^1$,

$$\sup|g'| \le \max\left\{\sup|\phi_1'|,\ \max_{1\le r\le h}|f_r'|\right\} < +\infty$$

(recall that $f_j'(x) = f_r'(x)$ whenever $f_j(x) = f_r(x)$), and $f_j'(x) = \phi_1'(x)$ whenever $f_j(x) = \phi_1(x)$).

Let, as before, $g_j: [0, a] \to [-b, b]$ be the function defined by

$$g_j(x) = \sup_{g \in S_j} g(x).$$

For what concerns the definition of g_j for $j \in S_2$, we define S_j as the set of all continuous functions $g: [0, a] \to]-b, b[$ such that

$$\text{graph}(g) \subseteq \bigcup_{r=1}^{h} \text{graph}(f_r \vee \phi_2) \qquad \text{and } g(b) = z_j,$$

and

$$g_j(x) = \inf_{g \in S_j} g(x).$$

Finally, inside the cusp we need a further attention, to treat the case in which a curve γ^i meets graph(ϕ_1) an infinite number of times. Hence, for any $j \in S$, let S_j be the set of all continuous functions $g: [0, a] \to]-b, b[$ such that

$$\text{graph}(g) \subseteq \bigcup_{r=1}^{h} \text{graph}((f_r \wedge \phi_1) \vee \phi_2) \qquad \text{and } g(b) = z_j.$$

Define $\tilde{g}_j(x) = \sup_{g \in S_j} g_j(x)$, and let

$$\overline{x} = \sup\{x \in [0, a]: \tilde{g}_j(x) = \phi_1(x)\}.$$

Define

$$g_j(x) = \begin{cases} \tilde{g}_j(x) & \text{if } \overline{x} < x \leq a, \\ \phi_1(x) & \text{if } 0 \leq x \leq \overline{x}. \end{cases}$$

The system of curves Λ_i is the system obtained from Λ_{i-1} by replacing the branches of (Λ_{i-1}) contained in $[0, a] \times [-b, b]$ with the curves given by the graphs of the functions g_j, for $j \in S_1 \cup S_2 \cup S$, each curve counted with the multiplicity d_j equal to the multiplicity of z_j for the system of curves (Λ_{i-1}). We conclude, as before, that there exists a finite set F_i such that $(\Lambda_i) \cap R(0) \setminus F_i$ is a one dimensional C^1 submanifold. □

As a particular case of Theorems 6.3 and 6.4, we get one of the main results of the paper, namely,

THEOREM 6.5. *Let $E \subseteq \mathbb{R}^2$ be a bounded open set with a piecewise $H^{2,p}$ boundary ∂E having a continuous unoriented tangent. Let $\mathcal{N}(\partial E) = \{q_1, \ldots, q_n\}$, and suppose that each q_i is a simple cusp point of ∂E. Then*

$$\overline{\mathcal{F}}(E) < +\infty \Longleftrightarrow n \text{ is even}.$$

7. - Localization

Some results of Sections 3, 4 can be localized. Let $\Omega \subseteq \mathbb{R}^2$ be an open set, and let $\mathcal{M}$ be the class of all measurable subsets of $\mathbb{R}^2$. The $L^1(\Omega)$-topology on $\mathcal{M}$ is the topology induced by the pseudo-distance $\mathrm{d}(E_1, E_2) = |\Omega \cap (E_1 \Delta E_2)|$. We say that $E \in \mathcal{M}$ is of class $C^2(\Omega)$, and we write $E \in C^2(\Omega)$, if E is bounded, open and $\Omega \cap \partial E$ is of class C^2, i.e., near a point $z \in \Omega \cap \partial E$ the set $\Omega \cap E$ is the subgraph of a function of class C^2 with respect to a suitable orthogonal coordinate system.

We define the map $\mathcal{F}(\cdot, \Omega): \mathcal{M} \to [0, +\infty]$ by

$$\mathcal{F}(E, \Omega) = \begin{cases} \displaystyle\int_{\Omega \cap \partial E} (1 + |\kappa(z)|^p) \, d\mathcal{H}^1(z) & \text{if } E \in C^2(\Omega), \\ +\infty & \text{elsewhere on } \mathcal{M}. \end{cases}$$

Note that

$$\mathcal{F}(E, \Omega) = \sup_{n \in \mathbf{N}} \mathcal{F}(E, \Omega_n) \tag{7.1}$$

for every sequence $\{\Omega_n\}_n$ of open sets invading Ω. Moreover $\mathcal{F}(E, \cdot)$ is increasing if considered as a set function, i.e., if Ω_1, Ω_2 are open,

$$\Omega_1 \subseteq \Omega_2 \Rightarrow \mathcal{F}(E, \Omega_1) \leq \mathcal{F}(E, \Omega_2). \tag{7.2}$$

By $\overline{\mathcal{F}}(\cdot, \Omega)$ we denote the lower semicontinuous envelope of $\mathcal{F}(\cdot, \Omega)$ with respect to the topology of $L^1(\Omega)$. It is known that, for every $E \in \mathcal{M}$, we have

$$\overline{\mathcal{F}}(E, \Omega) = \inf \left\{ \liminf_{h \to +\infty} \mathcal{F}(E_h, \Omega) : E_h \to E \text{ in } L^1(\Omega) \text{ as } h \to +\infty \right\}.$$

Note that $\overline{\mathcal{F}}(E, \Omega) < +\infty$ if and only if there exists a sequence $\{E_h\}_h$ of bounded open sets of class $C^2(\Omega)$ such that $E_h \to E$ in $L^1(\Omega)$ as $h \to +\infty$, and

$$\sup_h \mathcal{H}^1(\Omega \cap \partial E_h) < +\infty, \qquad \sup_h \int_{\Omega \cap \partial E_h} |\kappa_h(z)|^p \, d\mathcal{H}^1(z) < +\infty, \tag{7.3}$$

where κ_h denotes the curvature of $\Omega \cap \partial E_h$.

THEOREM 7.1. *Let E be a bounded open set of class $C^2(\Omega)$. Then*

$$\mathcal{F}(E, \Omega) \leq \liminf_{h \to +\infty} \mathcal{F}(E_h, \Omega) \tag{7.4}$$

for any sequence $\{E_h\}_h$ *of bounded open sets of class* $\mathcal{C}^2(\Omega)$ *such that* $E_h \to E$ *in* $L^1(\Omega)$ *as* $h \to +\infty$.

PROOF. Let $\{E_h\}_h$ be a sequence of bounded open sets of class $\mathcal{C}^2(\Omega)$ such that $E_h \to E$ in $L^1(\Omega)$ as $h \to +\infty$. We can suppose that the right hand side of (7.4) is finite, otherwise the result is trivial. Given a sequence $\{\Omega_n\}_n$ of relatively compact open sets invading Ω, it will be sufficient to prove that

$$\mathcal{F}(E, \Omega_n) \leq \liminf_{h \to +\infty} \mathcal{F}(E_h, \Omega_n), \tag{7.5}$$

for every $n \in \mathbf{N}$. In fact, (7.4) follows from (7.5), since, by (7.1) and (7.2),

$$\mathcal{F}(E, \Omega) = \sup_{n \in \mathbf{N}} \mathcal{F}(E, \Omega_n) \leq \sup_{n \in \mathbf{N}} \liminf_{h \to +\infty} \mathcal{F}(E_h, \Omega_n) \leq \liminf_{h \to +\infty} \mathcal{F}(E_h, \Omega).$$

Let us fix $n \in \mathbf{N}$. Let $\{E_{h_k}\}_k$ be a subsequence of $\{E_h\}_h$ with the property that

$$\lim_{k \to +\infty} \mathcal{F}(E_{h_k}, \Omega_n) = \liminf_{h \to +\infty} \mathcal{F}(E_h, \Omega_n) < +\infty.$$

For simplicity, this subsequence (and any further subsequence) will be denoted by $\{E_k\}_k$. Since $\{E_k\}_k$ satisfies (7.3), it follows that, for any k,

$$\Omega_n \cap \partial E_k \subseteq \bigcup_{i=1}^{m_k} (\gamma_k^i) \cup \bigcup_{j=1}^{r_k} (\beta_k^j),$$

where

$\{\gamma_k^1, \ldots, \gamma_k^{m_k}\}$ is a disjoint system of curves of class $\mathcal{C}^2$ such that $\overline{\Omega}_n \cap (\gamma_k^i) \neq \emptyset$ (see Lemma 3.1);

$\{\beta_k^1, \ldots, \beta_k^{r_k}\}$ is a finite family of simple regular curves of class $\mathcal{C}^2$ such that $\left|\frac{d\beta_k^j}{dt}\right|$ is constant, $\beta_k^j(0)$, $\beta_k^j(1) \in \partial\Omega$ and $(\beta_k^j) \cap \partial\Omega_n \neq \emptyset$ for any $j = 1, \ldots, r_k$;

all the sets $(\gamma_k^1), \ldots, (\gamma_k^{m_k})$, $(\beta_k^1), \ldots, (\beta_k^{r_k})$ are pairwise disjoint;

$$\sum_{i=1}^{m_k} l(\gamma_k^i) + \sum_{j=1}^{r_k} l(\beta_k^j) + \sum_{i=1}^{m_k} \|\kappa(\gamma_k^i)\|_{L^p}^p + \sum_{j=1}^{r_k} \|\kappa(\beta_k^j)\|_{L^p}^p < +\infty. \tag{7.6}$$

By Lemma 3.1 we have that $\{m_k\}_k$ is uniformly bounded with respect to k. Since $l(\beta_k^j) \geq 2\,\mathrm{dist}(\partial\Omega_n, \partial\Omega) > 0$ for any $j = 1, \ldots, r_k$, using (7.6) we get that $\{r_k\}_k$ is also uniformly bounded with respect to k. Passing to a suitable subsequence, we can assume that m_k and r_k are independent of k. These numbers will be denoted by m and r, respectively. Since all curves of the family $\{\gamma_k^1, \dots, \gamma_k^m, \beta_k^1, \dots, \beta_k^r\}$ meet $\overline{\Omega}_n$, by (7.6) their traces are contained in a bounded subset of $\mathbb{R}^2$ independent of k. Repeating the arguments of Theorem 3.1, by compactness there exists a family $\Gamma = \{\gamma^1, \dots, \gamma^m, \beta^1, \dots, \beta^r\}$ of regular curves of class $H^{2,p}$ such that $\gamma_k^i \rightharpoonup \gamma^i$ and $\beta_k^j \rightharpoonup \beta^j$ weakly in $H^{2,p}$ as $k \to +\infty$, for any $i = 1, \dots, m$ and any $j = 1, \dots, r$. Then, since by hypothesis $E \in \mathcal{C}^2(\Omega)$, it follows that $\Omega_n \cap (\Gamma) \supseteq \Omega_n \cap \partial E$. In fact, suppose by contradiction that there exists $z \in \Omega_n \cap (\partial E \setminus (\Gamma))$. Let $r > 0$ be such that $B_r(z) \subseteq \Omega_n$, $B_r(z) \cap (\Gamma) = \emptyset$, and

$$|B_{\frac{r}{2}}(z) \cap E| > 0, \qquad |B_{\frac{r}{2}}(z) \setminus E| > 0. \tag{7.7}$$

As $\gamma_k^i \to \gamma^i$ and $\beta_k^j \to \beta^j$ in $\mathcal{C}^1$ as $k \to +\infty$, there exists $k_0 \in \mathbf{N}$ such that $B_{\frac{r}{2}}(z) \cap \partial E_k = \emptyset$ for any $k \geq k_0$. This implies that either $B_{\frac{r}{2}}(z) \subseteq E_k$, or $B_{\frac{r}{2}}(z) \cap E_k = \emptyset$, which, together with (7.7), contradicts the fact that $E_k \to E$ in $L^1(\Omega)$ as $k \to +\infty$. Repeating the last part of the proof of Theorem 3.2 relatively to Ω_n we obtain (7.5). □

COROLLARY 7.1. *Let $E \subseteq \mathbb{R}^2$ be a bounded open set of class $H^{2,p}$ which is relatively compact in Ω. Then*

$$\overline{\mathcal{F}}(E, \Omega) = \int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z).$$

In particular, $\overline{\mathcal{F}}(E, \Omega) < +\infty$.

PROOF. Theorem 7.1 holds with the same proof if E is of class $H^{2,p}$, hence, passing to the infimum with respect to the approximating sequence $\{E_h\}_h$ in (7.4) we infer that $\int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z) \leq \overline{\mathcal{F}}(E, \Omega)$. The opposite inequality can be proved as in Corollary 3.2, using the hypothesis that E is relatively compact in Ω. □

The last part of this section is devoted to find an example of a set E in which we can calculate $\overline{\mathcal{F}}(E) = \overline{\mathcal{F}}(E, \mathbb{R}^2)$. Moreover, as we shall prove, for this set E one has that $\overline{\mathcal{F}}(E, \cdot)$ is not subadditive (see [4, Conjecture 5]). This shows that $\overline{\mathcal{F}}(E, \cdot)$ cannot be a measure and in particular that $\overline{\mathcal{F}}(E, \Omega)$ cannot be represented as an integral of the form (1.3) for a suitable choice of the function $\kappa(z)$.

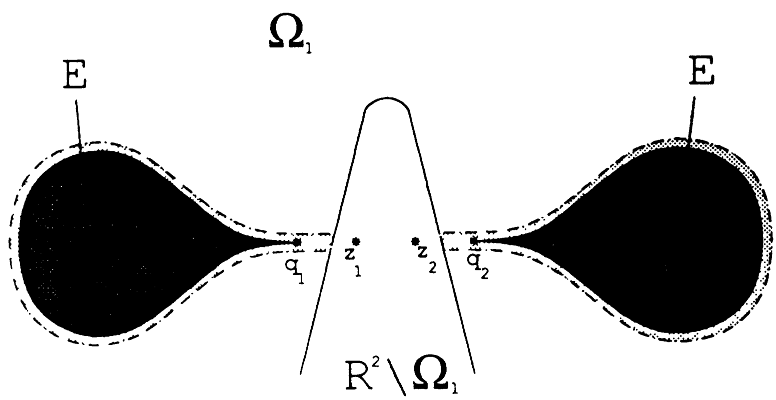


Fig. 7.1: *Each rectilinear segment of ∂E_h near the points q_1 and q_2 gives a contribution to the functional $\mathcal{F}(E_h)$ equal to its length, which is less than $\frac{1}{3}$.*

Let $E \subseteq \mathbb{R}^2$ be the set of Figure 7.1, where $q_1 = (0,0)$ and $q_2 = (1,0)$, and let C_1, C_2 be the two connected components of ∂E. Note that the unoriented tangent lines in q_1 and q_2 coincide with the x-axis. As in Theorem 4.1 (v), we denote by $\mathcal{A}(E)$ the set of all limit systems Γ of curves of class $H^{2,p}$ such that $(\Gamma) \supseteq \partial E$ and $E = \mathrm{int}(A_\Gamma \cup (\Gamma))$, where $A_\Gamma = \{z \in \mathbb{R}^2 \backslash (\Gamma) : I(\Gamma, z) = 1\}$.

LEMMA 7.1. *Let E be the set of Figure 7.1, and let $\Gamma = \{\gamma^1, \dots, \gamma^m\} \in \mathcal{A}(E)$. Then either there exist i, $j \in \{1, \dots, m\}$, $i \neq j$ and t_1, $t_2 \in [0,1]$ such that $\gamma^i(t_1) = \gamma^j(t_2) = q_1$, or there exist $i \in \{1, \dots, m\}$ and t_1, $t_2 \in [0,1]$ with $0 \leq t_1 < t_2 < 1$ such that $\gamma^i(t_1) = \gamma^i(t_2) = q_1$. The same holds for the point q_2.*

PROOF. Let $\{z_h^{(1)}\}_h$, $\{z_h^{(2)}\}_h$ be two sequences of points such that, for any h, $z_h^{(1)} = (x_h, y_h^{(1)})$, $z_h^{(2)} = (x_h, y_h^{(2)})$, $z_h^{(1)} \in C_1 \cap \{y \leq 0\}$, $z_h^{(2)} \in C_1 \cap \{y \geq 0\}$, and $z_h^{(1)} \to q_1$, $z_h^{(2)} \to q_1$ as $h \to +\infty$. Since $(\Gamma) \supseteq \partial E$, passing to suitable subsequences, there exist i, $j \in \{1, \dots, m\}$ and two sequences $\{s_h\}_h$, $\{t_h\}_h$ of points, with s_h, $t_h \in [0,1]$, such that $z_h^{(1)} = (x_h, y_h^{(1)}) = \gamma^i(s_h)$, $z_h^{(2)} = (x_h, y_h^{(2)}) = \gamma^j(t_h)$ for any h. By compactness, $s_h \to t_1$, $t_h \to t_2$ as $h \to +\infty$, so that $\gamma^i(t_1) = \gamma^j(t_2) = q_1$. If $i \neq j$ the proof is complete. Let us suppose that $i = j$, and assume by contradiction that $t_1 = t_2$. By the Mean Value Theorem, for any h there exists a point ξ_h between s_h and t_h such that the first component of the derivative of γ^i vanishes at ξ_h. As $\xi_h \to t_1 = t_2$, we conclude that the first component of the derivative of γ^i at t_1 vanishes. This is a contradiction, since any tangent unit vector of ∂E at q_1 is parallel to the x-axis. □

An important property of the set E is that C_1 lies on the left with respect to the y-axis, and that C_2 lies on the right with respect to the normal line of ∂E at q_2, which is the line $\{x = 1\}$. Let us denote by $\partial_r E$ the set of the regular points of ∂E, i.e., $\partial_r E = \partial E \setminus \{q_1, q_2\}$. Then the following result holds.

THEOREM 7.2. *Let E be the set of Figure* 7.1. *Then*

$$\overline{\mathcal{F}}(E) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} |\kappa(z)|^p \, d\mathcal{H}^1(z) + 2.$$

PROOF. We shall show that

$$\overline{\mathcal{F}}(E) = \inf\{\mathcal{F}(\Gamma) : \Gamma \in \mathcal{A}(E)\} = \inf\{\mathcal{F}(\Gamma) : \Gamma \in \mathcal{Q}(E)\}, \tag{7.8}$$

where $\mathcal{Q}(E)$ is the collection of those systems of curves Γ of class $H^{2,p}$ without crossings, satisfying the finiteness property and such that $\{z \in \mathbb{R}^2 : \exists r > 0 \ |B_r(z) \setminus E| = 0\} = \mathrm{int}(A_\Gamma \cup (\Gamma))$, with $A_\Gamma = \{z \in \mathbb{R}^2 \setminus (\Gamma) : I(\Gamma, z) \equiv 1 \pmod 2\}$ (see Theorem 6.2).

By Theorem 4.1 (v) and (6.7) we have

$$\inf\{\mathcal{F}(\Gamma) : \Gamma \in \mathcal{Q}(E)\} \geq \overline{\mathcal{F}}(E) \geq \inf\{\mathcal{F}(\Gamma) : \Gamma \in \mathcal{A}(E)\}.$$

Since

$$\inf\{\mathcal{F}(\Gamma) : \Gamma \in \mathcal{Q}(E)\} \leq \mathcal{H}^1(\partial E) + \int_{\partial_r E} |\kappa(z)|^p \, d\mathcal{H}^1(z) + 2,$$

to prove (7.8) it will be enough to show that, for any $\Gamma \in \mathcal{A}(E)$,

$$\mathcal{F}(\Gamma) \geq \mathcal{H}^1(\partial E) + \int_{\partial_r E} |\kappa(z)|^p \, d\mathcal{H}^1(z) + 2. \tag{7.9}$$

Let $\Gamma = \{\gamma^1, \ldots, \gamma^m\} \in \mathcal{A}(E)$. Let us use Lemma 7.1. Suppose that there exist i, $j \in \{1, \ldots, m\}$, $i \neq j$, and t_1, $t_2 \in [0,1]$ such that $\gamma^i(t_1) = \gamma^j(t_2) = q_1$. Up to a reparametrization of the curve γ^i, we can suppose that $t_1 = 0$, that the tangent unit vector of γ^i at $t = 0$ is $(1,0)$, and that γ^i is parametrized by the arc length s. Let $[0, l(\gamma^i)] = A \cup B$, where $A = \{s \in [0, l(\gamma^i)] : \gamma^i(s) \in \partial E\}$, and $B = \{s \in [0, l(\gamma^i)] : \gamma^i(s) \notin \partial E\}$. Clearly $B \neq \emptyset$. Let us prove that

$$\mathcal{H}^1(B) + \int_B |\ddot{\gamma}^i(s)|^p \, ds \geq 1. \tag{7.10}$$

There are two possibilities: either $(\gamma^i) \cap C_2 \neq \emptyset$, or $(\gamma^i) \cap C_2 = \emptyset$. If $(\gamma^i) \cap C_2 \neq \emptyset$, one obviously gets $\mathcal{H}^1(B) + \int_B |\ddot{\gamma}^i(s)|^p \, ds \geq |q_2 - q_1| = 1$, that is (7.10).

Suppose now that $(\gamma^i) \cap C_2 = \emptyset$, and denote by γ_1^i the x-component of the curve γ^i. By construction, we have that $\dot{\gamma}_1^i(0) = 1$. Since γ^i is closed, we can consider the smallest parameter $\sigma \in]0, l(\gamma^i)[$ such that $\gamma_1^i(\sigma) > 0$ and $\dot{\gamma}_1^i(\sigma) = 0$. It is easy to prove that $\gamma_1^i(s) > 0$ for any $s \in]0, \sigma]$. Since C_1 lies on the left with respect to the y-axis, we get $C_1 \cap \gamma^i(]0, \sigma]) = \emptyset$. Moreover, by assumption, we have also that $C_2 \cap \gamma^i([0, \sigma]) = \emptyset$. Hence $[0, \sigma] \subseteq B$, and using (3.5) we get

$$\mathcal{H}^1(B) + \int_B |\ddot{\gamma}^i(s)|^p \, ds \geq \sigma + \int_0^\sigma |\ddot{\gamma}^i(s)|^p \, ds \geq \frac{\pi}{2} \left[\left(\frac{p}{p'} \right)^{\frac{1}{p}} + \left(\frac{p'}{p} \right)^{\frac{1}{p'}} \right] > 1,$$

that gives (7.10).

Using the same arguments also for the curve γ^j, (7.9) follows from (7.10) applied to both γ^i and γ^j.

Let us suppose now that there exist $i \in \{1, \ldots, m\}$ and $t_1, t_2 \in [0, 1]$, $0 \leq t_1 < t_2 < 1$ such that $\gamma^i(t_1) = \gamma^j(t_2) = q_1$. If there exist $k \in \{1, \ldots, m\}$, $k \neq i$, and $t \in [0, 1]$ such that $\gamma^k(t) = q_2$, then (7.9) follows from the previous arguments. The last possibility (see Lemma 7.1) is whenever there exist t_3, $t_4 \in [0, 1]$, $0 \leq t_3 < t_4 < 1$, $t_3 \neq t_1$, $t_4 \neq t_2$ such that $\gamma^i(t_3) = \gamma^i(t_4) = q_2$, and this case can be treated with obvious modifications. □

THEOREM 7.3. *Let E be the set of Figure 7.1. Then there exist two open sets Ω_1, $\Omega_2 \subseteq \mathbb{R}^2$, with $\Omega_1 \cup \Omega_2 = \mathbb{R}^2$, such that*

$$\overline{\mathcal{F}}(E, \Omega_1) + \overline{\mathcal{F}}(E, \Omega_2) < \overline{\mathcal{F}}(E, \mathbb{R}^2) < +\infty.$$

PROOF. As we saw in Section 1, $\overline{\mathcal{F}}(E) = \overline{\mathcal{F}}(E, \mathbb{R}^2) < +\infty$.

Let $z_1 = \left(0, \frac{1}{3}\right)$ and $z_2 = \left(0, \frac{2}{3}\right)$. Let us fix two open subsets Ω_1, Ω_2 of $\mathbb{R}^2$ such that z_1, $z_2 \notin \Omega_1$, $\Omega_2 \cap E = \emptyset$, and $\Omega_1 \cup \Omega_2 = \mathbb{R}^2$. Then

$$\overline{\mathcal{F}}(E, \Omega_2) = 0, \tag{7.11}$$

and from Theorem 7.2 we have

$$\overline{\mathcal{F}}(E, \mathbb{R}^2) = \mathcal{H}^1(\partial E) + \int_{\partial_r E} |\kappa(z)|^p \, d\mathcal{H}^1(z) + 2. \tag{7.12}$$

Moreover one easily constructs a sequence $\{E_h\}_h$ of bounded open subsets of $\mathbb{R}^2$ of class $C^2(\Omega_1)$ with $E_h \to E$ in $L^1(\Omega_1)$ as $h \to +\infty$, such that (see Figure 7.1)

$$\mathcal{H}^1(\partial E) + \int_{\partial_\tau E} |\kappa(z)|^p \, d\mathcal{H}^1(z) + \frac{4}{3} = \lim_{h\to+\infty} \mathcal{F}(E_h, \Omega_1) \geq \overline{\mathcal{F}}(E, \Omega_1). \tag{7.13}$$

The conclusion of the theorem follows now from (7.11), (7.12), (7.13). □

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