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Nonzero Time Periodic Solutions to an Equation of Petrovsky Type with Nonlinear Boundary Conditions: Slow Oscillations of Beams on Elastic Bearings

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In the linear theory of vibrating beams, the transversal displacement $u(x,t)$ at the instant t of the point with reference position x satisfies a linear second order (in time) equation

(E) $$u_{tt}(x,t) + u_{xxxx}(x,t) = 0, \qquad x \in (0,\ell), \qquad t \in R^1,$$

where all material parameters are supposed constant and have been scaled out.

If the beam rests on two identical bearings with purely elastic response characterized by a nonlinear function f, the boundary conditions read

(B_1) $$u_{xx}(0,t) = u_{xx}(\ell,t) = 0, \qquad t \in R^1,$$

(B_2) $$u_{xxx}(0,t) = -f(u(0,t)), \qquad u_{xxx}(\ell,t) = f(u(\ell,t)), \qquad t \in R^1.$$

Concerning the function f, we assume

(A_1) $f \in C^1(R^1)$, $\quad f(-v) = -f(v)$ for all $v \in R^1$,

(A_2) f is superlinear at infinity, i.e., for any $C > 0$, there is $K(C)$ such that

$$f(v) \geq Cv - K(C) \qquad \text{for all } v \geq 0,$$

(A_3) f satisfies the growth condition

$$\frac{1}{2} f(v)v - F(v) \geq c_1 |f(v)| - c_2 \qquad \text{for all } v \in R^1$$

where $F(v) = \int_0^v f(s)\,\mathrm{d}s$,

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(A_4) $f(0) = f'(0) = 0$,

where f' denotes the derivatives and the symbols c_i, $i = 1, 2, \dots$ stand for strictly positive constants.

We are interested in nonzero time periodic solutions to the above problem, i.e.

$$u(x, t+T) = u(x, t) \qquad \text{for all } x,\ t, \tag{P}$$

with a positive period T.

The problem has been proposed to me by Professor G. Capriz as a simple analogue of a more complicated shaft dynamics model where (E) is replaced by a pair of beam equations coupled by means of the nonlinearity f, which in that case depends also on the first time derivative of the unknown functions (see Capriz [3], [4]). The situation of interest is when the rotation speed of the shaft reaches a critical value, large self-excited oscillations occur, the amplitude of vibrations at the bearings tends to clearance while the amplitude at the center of the beam goes to infinity.

In the present case, however, we have a *conservative* system where the beam is at rest (does not rotate) and the response of the bearings is purely elastic, the dissipative effects neglected. As we will prove (Section 2, Theorem 1), free vibrations, i.e. non-zero time periodic solutions, exist at least for large periods T that are appropriately related to the lenght ℓ of the beam.

1. - Variational formulation

To solve the problem (E), (B), (P), we adopt the variational approach that has been, to our best knowledge, the only successful one to obtain analytical results.

The solution u is a critical point of the action functional

$$J(v) = \frac{1}{2} \int_0^T \int_0^\ell v_{xx}^2(x,t) - v_t^2(x,t)\, \mathrm{d}x\, \mathrm{d}t + \int_0^T F(v(0,t)) + F(v(\ell,t))\, \mathrm{d}t$$

which motivates the following definition.

DEFINITION 1. A function u is called a *weak solution* to the problem (E), (B_1), (B_2), (P), if

$$u = u(x,t) \text{ is continuous on } [0,\ell] \times R^1, \tag{1.1}$$

$$u \text{ satisfies (P)} \tag{1.2}$$

and the equality

$$
\int_0^T\int_0^\ell u(x,t)(\varphi_{xxxx}(x,t)+\varphi_{tt}(x,t))\,\mathrm{d}x\,\mathrm{d}t + \int_0^T f(u(0,t))\varphi(0,t)+f(u(\ell,t))\varphi(\ell,t)\,\mathrm{d}t = 0 \tag{1.3}
$$

holds for any test function $\varphi\in C^\infty([0,\ell]\times R^1)$ satisfying (P) along with the natural boundary conditions

$$
\varphi_{xx}(0,t)=\varphi_{xx}(\ell,t)=\varphi_{xxx}(0,t)=\varphi_{xxx}(\ell,t)=0, \qquad t\in R^1.
$$

However, the search for critical points of indefinite functionals like J is often a nontrivial task. The first successful attack has been carried out by Rabinowitz in the celebrated paper [6]. His truly pioneering work stimulated a rather vast amount of research, a complete list of which lies beyond the scope of the present paper. A nice survey may be found in Brézis [1].

Quite recently, the effort concentrated on certain qualitative properties of the critical points related to the minimality of the period T (see Salvatore [7], Tarantello [9]).

The essential stumbling block related to (E), (B), (P) is that the restoring force f is concentrated just at two boundary points in contrast with all the problems mentioned above where the elastic response is distributed continuously along the interval $(0,\ell)$. As a consequence, we encounter a lack of coercivity in certain sense calling for a refined analysis of critical points (cf. Section 4). In particular, the dual action principle used in Brézis-Coron-Nirenberg [2], Tanaka [8] and many others does not seem to work here.

2. - Main results

The aim of the present paper is to prove the following theorem.

THEOREM 1. *Let the function f satisfy the hypotheses* (A_1)–(A_4).

Then there exists a (sufficiently large) positive integer M such that the problem (E), (B_1), (B_2), (P) *possesses at least one nonzero time periodic solution u with the period*

$$
T = 2\pi\left(\frac{16\ell^2M^2}{\pi^2}\right).
$$

The method of the proof leans on the approximate method of Rayleigh-Ritz. We look for a sequence of approximate critical points of a slightly modified functional J_n restricted to Hilbert spaces of finite dimension (see Section 4).

The critical points are found by the help of a lemma of saddle point type. The proof of the lemma is standard. What is more technical is to verify its abstract hypotheses in the present context.

Finally, we have to pass to the limit in the sequence of approximate solutions (see Section 5). To this end, we need some information concerning the distribution of eigenvalues of the beam operator appearing in (E) (see Section 3).

3. - Spectral properties of the linear operator

To simplify the notation, we assume that $\ell = \frac{\pi}{4}$.

Next we denote

$$Q = \left\{(x,t)|x \in \left[0, \frac{\pi}{4}\right], \ t \in [0, T = 2\pi M^2]\right\}$$

and, finally, recall that the symbol $\|\cdot\|_p$ will denote the norm on the Lebesgue space $L_p(Q)$.

We start with the eigenvalue problem

$$(3.1) \qquad \begin{cases} w_{xxxx}(x) = \mu^4 w(x), & x \in \left(0, \frac{\pi}{4}\right), \\ w_{xx}(0) = w_{xx}\left(\frac{\pi}{4}\right) = w_{xxx}(0) = w_{xxx}\left(\frac{\pi}{4}\right) = 0. \end{cases}$$

which is known to possess a sequence of eigenvalues μ_k, $k = -1, 0, 1, \dots$ where

$$\mu_{-1} = \mu_0 = 0$$

and

$$(3.2) \qquad \cos\left(\mu_k \frac{\pi}{4}\right) \cosh\left(\mu_k \frac{\pi}{4}\right) = 1, \qquad k = 1, 2, \dots$$

(see Timoshenko-Young-Weaver [10]).

The corresponding orthonormal (in L_2) system of eigenfunctions reads

$$(3.3) \qquad \begin{cases} w_{-1}(x) = \dfrac{\sqrt{32}}{\pi} x, \qquad w_0(x) = \dfrac{2}{\sqrt{\pi}}, \\ w_k(x) = \dfrac{4}{\sqrt{\pi} W_k} \left[\cosh(\mu_k x) + \cos(\mu_k x) \right. \\ \left. - \dfrac{\cosh\left(\mu_k \frac{\pi}{4}\right) - \cos\left(\mu_k \frac{\pi}{4}\right)}{\sinh\left(\mu_k \frac{\pi}{4}\right) - \sin\left(\mu_k \frac{\pi}{4}\right)} (\sinh(\mu_k x) + \sin(\mu_k x))\right] \end{cases}$$

where the constants W_k are given by the formula

$$W_k = \cosh\left(\mu_k \frac{\pi}{4}\right) + \cos\left(\mu_k \frac{\pi}{4}\right) - \frac{\cosh\left(\mu_k \frac{\pi}{4}\right) - \cos\left(\mu_k \frac{\pi}{4}\right)}{\sinh\left(\mu_k \frac{\pi}{4}\right) - \sin\left(\mu_k \frac{\pi}{4}\right)} \left(\sinh\left(\mu_k \frac{\pi}{4}\right) + \sin\left(\mu_k \frac{\pi}{4}\right)\right).$$

From the relation (3.2) we deduce the asymptotic formula

$$\mu_k = 2(2k-1) + r(k) \qquad k = 1,\ 2,\ \ldots \tag{3.4}$$

where

$$|r(k)| \leq c_3 \exp(-c_4 k) \qquad \text{for all } k. \tag{3.5}$$

The proof of Theorem 1 is based, among other things, on the spectral analysis of the linear operator

$$\begin{cases} Lv = v_{tt} + v_{xxxx}, \ x \in \left(0, \frac{\pi}{4}\right), \ t \in (0, 2\pi M^2), \\ v_{xx}(0, \cdot) = v_{xx}\left(\frac{\pi}{4}, \cdot\right) = v_{xxx}(0, \cdot) = v_{xxx}\left(\frac{\pi}{4}, \cdot\right) = 0, \\ v(\cdot, t + 2\pi M^2) = v(\cdot, t). \end{cases} \tag{3.6}$$

In order to prevent accumulation of eigenvalues, we restrict ourselves to the class of functions having the symmetry properties

$$\begin{cases} v(x,t) = v(x, \pi M^2 - t) \\ v(x,t) = -v(x, \pi M^2 + t) \end{cases} \qquad \text{for all } x, t. \tag{3.7}$$

Note that, in view of (A_1), the symmetry is preserved by the superposition operator corresponding to f.

Consequently, the linear operator L determined for smooth functions by (3.6) and restricted to the class defined in (3.7) admits a spectral resolution in the form

$$Lv = \sum_{k=-1}^{\infty} \sum_{j=0}^{\infty} \lambda_{kj} a_{kj}(v) e_{kj} \tag{3.8}$$

where the eigenvalues are

$$\lambda_{kj} = \mu_k^4 - \frac{(2j+1)^2}{M^4} \tag{3.9}$$

and the symbols a_{kj} denote the Fourier coefficients

$$a_{kj}(v) = \int_0^{2\pi M^2} \int_0^{\frac{\pi}{4}} v(x,t) e_{kj}(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

evaluated with respect to the orthonormal system

$$e_{kj}(x,t) = \frac{1}{M\sqrt{\pi}}\, w_k(x)\, \sin\left(\frac{2j+1}{M^2}\, t\right).$$

PROPOSITION 1. *For any function* $v \in L_1(Q)$ *satisfying* (3.7) *for* a.e. x, t, *the following estimate*

$$\|v\|_\infty^2 \le c_5(\alpha, M)\left[\sum_{\substack{k,j\\ \lambda_{kj}\neq 0}} |\lambda_{kj}|^\alpha a_{kj}^2(v) + \sum_{\substack{k,j\\ \lambda_{kj}=0}} a_{kj}^2(v)\right] \tag{3.10}$$

holds for any $\alpha > \dfrac{3}{4}$.

PROOF. Since $2(2k-1)$ is even and $(2j+1)$ odd, we can use the asymptotic formula (3.4) along with the estimate (3.5) to obtain the inequality

$$|\lambda_{kj}| \ge \frac{1}{2M^4}\, |(M^4(2(2k-1))^4 - (2j+1)^2)|$$

which holds with a possible exception of a finite number of indices, i.e. for all

$$k \ge k_0(M), \qquad j \ge j_0(M).$$

Now we have

$$\|v\|_\infty \le \sum_{k=-1}^{\infty} \sum_{j=0}^{\infty} |a_{kj}|\, \|e_{kj}\|_\infty$$

$$\le \frac{c_6}{M}\left[\sum_{k=-1}^{k_0-1} \sum_{j=0}^{j_0-1} |a_{kj}| + \sum_{k=k_0}^{\infty} \sum_{j=j_0}^{\infty} |\lambda_{kj}|^{-\frac{\alpha}{2}} |\lambda_{kj}|^{\frac{\alpha}{2}} |a_{kj}|\right]$$

where the latter term is further estimated using the Hölder inequality

$$\sum_{k=k_0}^{\infty} \sum_{j=j_0}^{\infty} |\lambda_{kj}|^{-\frac{\alpha}{2}} |\lambda_{kj}|^{\frac{\alpha}{2}} |a_{kj}|$$

$$\le \left[\sum_{k=k_0}^{\infty} \sum_{j=j_0}^{\infty} |\lambda_{kj}|^{-\alpha}\right]^{\frac{1}{2}} \left[\sum_{k=k_0}^{\infty} \sum_{j=j_0}^{\infty} |\lambda_{kj}|^{\alpha} a_{kj}^2(v)\right]^{\frac{1}{2}}.$$

Finally, we estimate

$$\sum_{k=k_0}^{\infty}\sum_{j=j_0}^{\infty}|\lambda_{kj}|^{-\alpha} \leq (2M^4)^{\alpha}\sum_{k=k_0}^{\infty}\sum_{j=j_0}^{\infty}\frac{1}{|(M^4(2(2k-1))^4-(2j+1)^2|^{\alpha}}$$

$$\leq (2M^4)^{\alpha}\sum_{\substack{k,j\\ k^4-j^2\neq 0}}\frac{1}{(k^4-j^2)^{\alpha}}$$

where the last series is summable for any $\alpha > \frac{3}{4}$ (see [5]).

Proposition 1 has been proved.

Q.E.D.

4. - The Rayleigh-Ritz approximation

In this section, we will look for suitable critical points of a modified action functional

$$J_n(v) = \frac{1}{2}\int_0^{2\pi M^2}\int_0^{\frac{\pi}{4}} v_{xx}^2(x,t) - v_t^2(x,t) + \frac{1}{4n}v^4(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

$$+ \int_0^{2\pi M^2} F(v(0,t)) + F\left(v\left(\frac{\pi}{4},t\right)\right)\mathrm{d}t$$

restricted to the finite dimensional space

$$E_n = \operatorname{span}\{e_{kj}|k,\ j\leq n\}.$$

To find suitable critical points of J_n, we use the following assertion, the proof of which is standard and may be found in [5].

PROPOSITION 2. *Let $E = V_1 \oplus V_2 \oplus V_3$ be an orthogonal decomposition of a finite dimensional space E. Let the symbol S denote a sphere in E with the center at zero.*

Let $J \in C^1(E, R^1)$ be a functional satisfying the hypotheses

(4.1) $$\lim_{\|v\|_E\to\infty} J(v) = \infty,$$

(4.2) $$J \leq d \text{ on } S\cap(V_1\oplus V_2), \qquad V_2 \neq \{0\},$$

(4.3) $$J > d \text{ on } V_3,$$

and

$$J > c \text{ on } V_2 \oplus V_3. \tag{4.4}$$

Then there is a critical point $v_0 \in E$ *such that*

$$\nabla J(v_0) = 0, \qquad J(v_0) \in [c, d]. \tag{4.5}$$

The goal of the remaining part of this section is to verify the hypotheses of Proposition 2 for the choice

$$E = E_n, \qquad J = J_n,$$

$$V_1 = \operatorname{span}\{e_{kj} | \lambda_{kj} < 0,\ (k,j) \neq (0,0)\} \cap E_n,$$

$$V_2 = \operatorname{span}\{e_{00}\},$$

$$V_3 = \operatorname{span}\{e_{kj} | \lambda_{kj} \geq 0\} \cap E_n$$

with suitable constants $-\infty < c < d < 0$ independent of n.

To begin with, observe that the coercivity condition (4.1) is satisfied thanks to the additional term $\frac{1}{4n} \|v\|_4^4$.

Verification of the hypothesis (4.2).

Consider a sphere $S(r) = \left\{ v \in V_1 \oplus V_2, \sum_{k,j} |\lambda_{kj}| a_{kj}^2(v) = r^2 \right\}$. For $v \in S(r)$ we have

$$J_n(v) = \frac{1}{2} \sum_{k,j} \lambda_{kj} a_{kj}^2(v) + \frac{1}{4n} \|v\|_4^4 + \int_0^{2\pi M^2} F(v(0,t)) + F\left(v\left(\frac{\pi}{4}, t\right)\right) \mathrm{d}t$$

$$\leq -\frac{1}{2} r^2 + \frac{1}{4n} \|v\|_4^4 + \int_0^{2\pi M^2} F(v(0,t)) + F\left(v\left(\frac{\pi}{4}, t\right)\right) \mathrm{d}t.$$

As $\lambda_{kj} < 0$ on $V_1 \oplus V_2$, we can use Proposition 1 to obtain

$$c_5(1, M) \sum_{k,j} |\lambda_{kj}| a_{kj}^2(v) \geq \|v\|_\infty^2.$$

Thus the terms

$$\frac{1}{4n}\,\|v\|_4^4 + \int_0^{2\pi M^2} F(v(0,t)) + F\left(v\left(\frac{\pi}{4},t\right)\right)\,\mathrm{d}t$$

are of order $o(r^2)$ in accordance with (A_4).

Consequently there exist r small enough and $d < 0$ such that (4.2) holds independently of n.

As the next step, observe that $\lambda_{kj} \geq 0$ on V_3 and thus that (4.3) follows as $J|_{V_3} \geq 0$.

Verification of the hypotheses (4.4).

This is the most difficult step. Take $v \in V_2 \oplus V_3$, i.e.

$$v = \nu e_{00} + z, \quad \text{where} \quad z \in V_3, \qquad \nu \in R^1.$$

Evaluating J_n at v, we obtain

$$J_n(v) \geq \frac{1}{2}\left(\sum_{\substack{k,j\\ \lambda_{kj}\geq 0}} \lambda_{kj}a_{kj}^2(z) - \frac{\nu^2}{M^4}\right) + \int_0^{2\pi M^2} F(v(0,t))\,\mathrm{d}t.$$

In view of (A_2), we have the estimate

$$F(v) \geq Cv^2 - N(C)$$

for any $C > 0$ where $N(C) \to \infty$ as $C \to \infty$. Consequently, we obtain

$$\begin{aligned}
J_n(v) &\geq \frac{1}{2}\left(\sum_{\substack{k,j\\ \lambda_{kj}\geq 0}} \lambda_{kj}a_{kj}^2(z) - \frac{\nu^2}{M^4}\right)\\
&\quad + C\int_0^{2\pi M^2}\left(\nu\,\frac{2}{M\sqrt{\pi}}\,\sin\left(\frac{t}{M^2}\right) + z(0,t)\right)^2 \mathrm{d}t - 2\pi M^2 N(C)\\
&\geq \frac{1}{2}\left(\sum_{\substack{k,j\\ \lambda_{kj}\geq 0}} \lambda_{kj}a_{kj}^2(z) - \frac{\nu^2}{M^4}\right) + C\,\frac{4}{\pi}\,\nu^2 + C\int_0^{2\pi M^2} z^2(0,t)\,\mathrm{d}t\\
&\quad - C\left|\int_0^{2\pi M^2} \frac{4\nu}{M\sqrt{\pi}}\,\sin\left(\frac{t}{M^2}\right) z(0,t)\,\mathrm{d}t\right| - 2\pi M^2 N(C).
\end{aligned} \tag{4.6}$$

Making use of the orthogonality of the eigenfunctions we obtain

$$\int_0^{2\pi M^2} \frac{1}{M\sqrt{\pi}} \sin\left(\frac{t}{M^2}\right) z(0,t)\,\mathrm{d}t$$

$$= \int_0^{2\pi M^2} \frac{1}{M\sqrt{\pi}} \sin\left(\frac{t}{M^2}\right) \sum_{\substack{k,j \\ \lambda_{kj}\geq 0}} a_{kj}(z) w_k(0) \frac{1}{M\sqrt{\pi}} \sin\left(\frac{2j+1}{M^2}\, t\right) \mathrm{d}t \tag{4.7}$$

$$= \sum_{\substack{k \\ \lambda_{k0}\geq 0}} a_{k0}(z) w_k(0).$$

Further we have

$$\int_0^{2\pi M^2} z^2(0,t)\,\mathrm{d}t$$

$$= \int_0^{2\pi M^2} \left(\sum_{\substack{k,j \\ \lambda_{kj}\geq 0}} a_{kj}(z) w_k(0) \frac{1}{M\sqrt{\pi}} \sin\left(\frac{2j+1}{M^2}\, t\right)\right)^2 \mathrm{d}t \tag{4.8}$$

$$= \sum_j \left(\sum_{\substack{k \\ \lambda_{kj}\geq 0}} a_{kj}(z) w_k(0) \right)^2 \geq \left(\sum_k a_{k0}(z) w_k(0) \right)^2$$

and, finally,

$$\left| \sum_{\substack{k \\ \lambda_{k0}\geq 0}} a_{k0}(z) w_k(0) \right| \leq c_7 \sum_k \lambda_{k0}^{-\frac{1}{2}} \lambda_{k0}^{\frac{1}{2}} |a_{k0}(z)|$$

$$\leq c_7 \left[\sum_{\substack{k \\ \lambda_{k0}\geq 0}} \frac{1}{|\lambda_{k0}|} \right]^{\frac{1}{2}} \left[\sum_{\substack{k \\ \lambda_{k0}\geq 0}} \lambda_{k0} a_{k0}^2(z) \right]^{\frac{1}{2}} \tag{4.9}$$

$$\leq c_8 \left[\sum_k \lambda_{k0} a_{k0}^2(z) \right]^{\frac{1}{2}}$$

since $\lambda_{k0} = \mu_k^4 - \dfrac{1}{M^4}$.

Combining (4.6)–(4.9), we conclude that

$$J_n(v) \geq \left(\frac{4C}{\pi} - \frac{1}{2M^4}\right)\nu^2 + (C + c_9)\xi^2 - 2C\frac{2}{\sqrt{\pi}}\nu\xi - 2\pi M^2 N(C) \tag{4.10}$$

where $\xi = \left|\sum_{\substack{k \\ \lambda_{k0}\geq 0}} a_{k0}(z)w_k(0)\right|$. The constant c_9 is independent of M.

Consequently, if M is so large that

$$\left(\frac{4C}{\pi} - \frac{1}{2M^4}\right)(C + c_9) - C^2\frac{4}{\pi} > 0 \tag{4.11}$$

we can choose the constant $c = -2\pi M^2 N(C)$.

As we have verified all the assumptions of Proposition 2, we are allowed to use its conclusion to construct a sequence $\{u^n\}$ of approximate solutions such that

$$\begin{aligned} &\int_0^{2\pi M^2}\int_0^{\frac{\pi}{4}} u^n_{xx}(x,t)\varphi_{xx}(x,t) - u^n_t(x,t)\varphi_t(x,t) \\ &+ \frac{1}{n}(u^n)^3(x,t)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^{2\pi M^2} f(u^n(0,t))\varphi(0,t) + f\left(u^n\left(\frac{\pi}{4},t\right)\right)\varphi\left(\frac{\pi}{4},t\right)\,\mathrm{d}t = 0 \end{aligned} \tag{4.12}$$

for any test function $\varphi \in E_n$, and

$$\begin{aligned} &\frac{1}{2}\int_0^{2\pi M^2}\int_0^{\frac{\pi}{4}} (u^n_{xx})^2(x,t) - (u^n_t)^2(x,t) + \frac{1}{4n}(u^n)^4(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^{2\pi M^2} F(u^n(0,t)) + F\left(u^n\left(\frac{\pi}{4},t\right)\right)\,\mathrm{d}t \in [c,d] \end{aligned} \tag{4.13}$$

where $-\infty < c < d < 0$ are independent of n.

5. - Passing to the limit in the sequence of approximate solutions

We start with some estimates that may be deduced from (4.12), (4.13).

To begin with, we set $\varphi = u^n$ in (4.12), multiply by $1/2$ and substract (4.13) to obtain

$$\begin{aligned}\frac{1}{4n}\int_0^{2\pi M^2}\int_0^{\frac{\pi}{4}}(u^n)^4(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^{2\pi M^2}\frac{1}{2}f(u^n(0,t))u^n(0,t)\\ - F(u^n(0,t)) + \frac{1}{2}f\left(u^n\left(\frac{\pi}{4},t\right)\right)u^n\left(\frac{\pi}{4},t\right)\\ - F\left(u^n\left(\frac{\pi}{4},t\right)\right)\,\mathrm{d}t \in [-d,-c].\end{aligned} \tag{5.1}$$

The first consequence of (5.1) is that, by virtue of (A_3),

$$\int_0^{2\pi M^2}|f(u^n(0,t))| + \left|f\left(u^n\left(\frac{\pi}{4},t\right)\right)\right|\,\mathrm{d}t \leq c_{10} \tag{5.2}$$

and

$$\frac{1}{n}\|u^n\|_4^4 \leq c_{11}. \tag{5.3}$$

Next, we insert $\varphi^n = \sum_{k,j}\operatorname{sgn}(\lambda_{kj})a_{kj}(u^n)e_{kj}$ in (4.12):

$$\begin{aligned}\sum_{k,j}|\lambda_{kj}|a_{kj}^2(u^n) + \int_0^{2\pi M^2}\int_0^{\frac{\pi}{4}}\frac{1}{n}(u^n)^3(x,t)\varphi^n(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ + \int_0^{2\pi M^2}f(u^n(0,t))\varphi^n(0,t) + f\left(u^n\left(\frac{\pi}{4},t\right)\right)\varphi^n\left(\frac{\pi}{4},t\right)\,\mathrm{d}t = 0.\end{aligned}$$

According to Proposition 1, we have

$$\|\varphi^n\|_\infty \leq \left(\sum_{k,j}|\lambda_{kj}|a_{kj}^2(u^n)\right)^{\frac{1}{2}}$$

and, by virtue of (5.1), (5.3), we conclude that

$$\sum_{k=-1}^{\infty}\sum_{j=0}^{\infty}|\lambda_{kj}|a_{kj}^2(u^n) \leq c_{12}. \tag{5.4}$$

Our ultimate task is to estimate the part of u^n belonging to a possibly nonvoid kernel of the operator L. According to Proposition 1, the kernel is at most finite dimensional and, moreover, we have

$$\sum_{\substack{k,j \\ \lambda_{kj}=0}} a_{kj}^2(u^n) = \sum_{j=j_1,\dots,j_m} a_{k(j)j}^2(u^n) \tag{5.5}$$

where the index $k(j)$ is uniquely determined by the equation $\lambda_{kj} = 0$.

On the other hand, we have

$$\int_0^{2\pi M^2} v^2(0,t)\,\mathrm{d}t = \sum_{j=0}^{\infty} \left(\sum_{k=-1}^{\infty} a_{kj}(v) w_k(0) \right)^2 . \tag{5.6}$$

Combining (5.5) and (5.6) we get that for any $y \in \mathrm{Ker}(L)$ the estimate

$$\sum_{k,j} a_{kj}^2(y) \leq c_{13} \int_0^{2\pi M^2} y^2(0,t)\,\mathrm{d}t \tag{5.7}$$

i.e. that the L_2-norm on the boundary and the L_2-norm inside Q are equivalent on $\mathrm{Ker}(L)$.

Combining (5.2), (5.4)–(5.7) together with the growth condition (A_2), we obtain the estimate

$$\sum_{k=-1}^{\infty} \sum_{j=0}^{\infty} |\lambda_{kj}| a_{kj}^2(u^n) + \sum_{\substack{k,j \\ \lambda_{kj}=0}} a_{kj}^2(u^n) \leq c_{14}. \tag{5.8}$$

Note that the L_1- and L_2-norms on the boundary restricted to $\mathrm{Ker}(L)$ are equivalent as this is finite dimensional.

Using the conclusion of Proposition 1, we deduce the final result

$$\{u^n\} \text{ is precompact in } C(Q). \tag{5.9}$$

Now, it is matter of routine to pass to the limit in (4.12) as $n \to \infty$ to conclude that (1.3) holds for any accumulation point u of $\{u^n\}$. Moreover, the same procedure applied to (4.13) implies that the solution u is not identically zero.

Theorem 1 has been proved.

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