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A Free Boundary Problem for a Nonlinear Degenerate Elliptic System Modeling a Thermistor

XINFU CHEN - AVNER FRIEDMAN

0. - Introduction

In this paper we study a nonlinear elliptic system for the electric potential φ and the temperature u. When the electric conductivity $\sigma(u)$ depends on the temperature u and Joule's heating is taken into account in the heat equation, the (time-independent) system has the form

$$\nabla(\sigma(u)\nabla\varphi) = 0, \qquad \nabla^2 u = -\sigma(u)|\nabla\varphi|^2.$$

We are interested in the case where $\sigma(u)$ becomes (approximately) zero for large u. The study of the system is motivated by a device called thermistor. A thermistor is an electric circuit device made of ceramic material (typically a cylinder of diameter $\sim 5\,\text{mm}$ and height $\sim 2\,\text{mm}$) where the electrical resistivity $1/\sigma(u)$ increases 5 orders of magnitude as the temperature increases beyond a critical level u^* (typically between 100°C and 200°C). If there is a current surge in the circuit, the thermistor will act as a circuit breaker. In comparison with a circuit breaker such as a fuse, the thermistor has the advantage that when the surge has fallen off the thermistor will cool down and the circuit will resume its normal function without needing replacement or resetting. For more details see, for instance, [10], [11], [13].

The case where $\sigma(u)$ is uniformly positive was considered by several authors, who proved existence of a solution, and under special boundary conditions, also uniqueness; see Cimatti and Prodi [5] and Cimatti [3] and the references therein. More recently Chen and Friedman [2] considered the case where

$$\begin{aligned} &\sigma(u) > 0 \quad \text{if } u < u^*, \qquad \sigma(u) = 0 \quad \text{if } u \geq u^*, \\ &\text{with } \sigma(u) \text{ continuous across } u = u^*. \end{aligned} \tag{0.1}$$

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They established existence of a weak solution. Under special boundary conditions (as in [4]) they also proved uniqueness, and showed that the set of infinite resistance

$$S = \{x;\ \sigma(u(x)) = 0\}$$

is a level surface of a harmonic function.

In the present paper we assume that

$$\sigma(u) = 1 \quad \text{if } u < u^*, \qquad \sigma(u) = 0 \quad \text{if } u \geq u^*. \tag{0.2}$$

Note that in contrast with (0.1), here $\sigma(u)$ is discontinuous at $u = u^*$. The methods of the present paper are entirely different from the method in [2]; the boundary conditions are also different.

We establish existence and uniqueness of a solution, and derive a characterization of the set S of infinite resistance, as a curve lying in a compact subset of the thermistor. All our results are for 2-dimensional domains only.

In Section 1 we state the thermistor problem and the main results. In Section 2 we transform the problem into a simpler problem for the electric potential and an auxiliary function ψ, $\psi = u + \frac{1}{2}\varphi^2$. Next, in Section 3, we use the conformal mapping $x + iy \to \tilde{\varphi} + i\varphi$ ($\tilde{\varphi}$ = harmonic conjugate of $-\varphi$) to transform the problem in (φ, ψ) into a problem for ψ only. (The use of such a conformal mapping was suggested by Howison [8]). In fact, ψ turns out to be a solution to a variational inequality whose properties are studied in Section 4. In Section 5 we go back from ψ as a function of $(\varphi, \tilde{\varphi})$ to recover ψ as well as φ as functions of (x, y). This step is quite delicate and requires a special choice of the additive constant in the definition of $\tilde{\varphi}$; it completes the proof of existence and uniqueness of a solution, and of the characterization of S. A more general uniqueness result is proved in Section 6.

All the above results are proved in case the thermistor is a rectangle. In Section 7 we extend the results to general 2-dimensional domains.

1. - The problem and the main results

Introduce rectangles

$$\begin{aligned} &R = \{(x, y);\ -a < x < a,\ -b < y < b\}, \\ &R^+ = R \cap \{y > 0\}, \\ &R^- = R \cap \{y < 0\}. \end{aligned} \tag{1.1}$$

The thermistor problem which will be considered here consists of the system of elliptic equations

$$\nabla(\sigma(u)\nabla\varphi) = 0 \qquad \text{in } R, \tag{1.2}$$

$$\nabla(\nabla u + \sigma(u)\varphi\nabla\varphi) = 0 \qquad \text{in } R \tag{1.3}$$

and the boundary conditions

$$\begin{aligned} &\varphi(x,b) = A, \qquad \varphi(x,-b) = -A \qquad &&\text{if } -a < x < a, \\ &\varphi_x(\pm a, y) = 0 &&\text{if } -b < y < b, \end{aligned} \tag{1.4}$$

$$u = 0 \qquad \text{on } \partial R, \tag{1.5}$$

where

$$\begin{aligned} \varphi &= \text{ electric potential}, \\ u &= \text{ temperature}, \\ \sigma(u) &= \text{ electric conductivity}. \end{aligned}$$

The boundary condition (1.4) means: the voltages at $y = +b$ and at $y = -b$ are fixed (and then normalized to $\pm A$ at $y = \pm b$) and no current flows out of the domain at $x = \pm a$. The boundary condition (1.5) means that the temperature on the thermistor boundary is kept constant (and then normalized to be 0). The voltage drop across the thermistor is $2A$, a quantity which, when large enough, should cause the temperature u to increase to the level $u = u^*$ somewhere within the device.

Most of the mathematical literature is concerned with the case where $\sigma(u)$ is uniformly positive; see [3], [4], [5], [7], [9]. In a recent paper Chen and Friedman [2] considered the case (0.1). By approximating $\sigma(u)$ by uniformly positive $\sigma_\varepsilon(u)$ ($\varepsilon > 0$), they show that the correponding solutions $(\varphi_\varepsilon, u_\varepsilon)$ converge to a weak solution (φ, u), and that $u \leq u^*$. Although the boundary conditions taken in [2] are different from (1.4), (1.5), their method applies as well to the present boundary conditions.

In this paper we assume that $\sigma(u)$ satisfies (0.2). If we approximate it by a sequence of positive functions $\sigma_\varepsilon(u)$, we get a limiting pair (φ, u) with

$$u \leq u^* \qquad \text{almost everywhere in } R; \tag{1.6}$$

however it is not clear whether (φ, u) forms a "weak solution" of the problem (1.2)–(1.5) in the distribution sense defined in [2]. The main interest is actually to determine the set of infinite resistance; this is formally the set where $\sigma(u) = 0$ or, in view of (0.2), (1.6), the set

$$S = \{(x,y) \in R;\ u(x,y) = u^*\}. \tag{1.7}$$

For the σ and the special boundary conditions considered in [2], S was found to be a level curve of a harmonic function; this curve connects one point on the boundary to another point on the boundary.

In this paper we adopt a more direct formulation of a solution to (1.2)–(1.5). We shall seek a solution for which S is either empty or an interval lying on the x-axis:

(1.8) $$S = \{(x,0);\ -a_* \leq x \leq a_*\} \qquad \text{where } 0 \leq a_* < a.$$

Set

$$S_0 = \{(x,0);\ -a_* < x < a_*\},$$

$$T = \{(x,0);\ -a < x < -a_* \text{ or } a_* < x < a\}.$$

The weak formulation of (1.2), (1.3) in a neighbourhood of S formally implies that

(1.9) $$\left[\frac{\partial \varphi}{\partial n}\right] = 0 \qquad \text{on } S_0,$$

(1.10) $$\left[\frac{\partial u}{\partial n} + \varphi \frac{\partial \varphi}{\partial n}\right] = 0 \qquad \text{on } S_0,$$

where n is the normal to S_0 $\left(\text{so that } \frac{\partial}{\partial n} \text{ is } \frac{\partial}{\partial y}\right)$ and $\left[...\right]$ means the jump across S_0. The choice of S as postulated in (1.8) is a somewhat intuitive guess: We expect the temperature to reach its maximum possible level u^* only on an interval $-a_* \leq x \leq a_*$ located on $\{y = 0\}$; since (by (1.5)) $u(\pm a, 0) = 0 < u^*$, if u is continuous in $\overline{R}$ then a_* must be smaller than a. One might, with equal justification, guess that S is a domain with nonempty interior, having some symmetry properties; however, it will be shown in Section 6 that there is no solution with such an S.

We shall require that

(1.11)
u is continuous in $\overline{R}$,
φ is continuous in $\overline{R}\backslash S$ and uniformly continuous in R^+ and in R^-,

and that

(1.12) φ and u are uniformly C^1 from each side of S_0;

φ may be discontinuous across S_0 (and in fact it will be).

Notice that in view of (1.12), the functions in (1.9), (1.10) are well defined.

REMARK 1.1. If $A = 0$ then $\varphi = 0$ and therefore also $u \equiv 0$. By a fixed-point argument (using elliptic estimates) one can show that, if A is small enough, then there exists a unique solution to (1.2)–(1.5) with $u < u^*$ in $\overline{R}$; hence $S = \emptyset$. On the other hand, if A is large enough then the set $\{u = u^*\}$ is nonempty. For, otherwise, we shall have $\sigma(u) \equiv 1$ and $\varphi = Ay/b$. It follows that

$$\Delta u = -\sigma |\nabla \varphi|^2 = -\frac{A^2}{b^2} \qquad \text{in } R,$$

and therefore,

$$u(x,y) = \frac{A^2}{b^2} \iint_R G(x,y;\xi,\eta)\mathrm{d}\xi\,\mathrm{d}\eta$$

where G is the Green function for Δ with zero boundary conditions on ∂R. Hence, $u(0,0) > u^*$ if $A \gg b$, a contradiction.

Because the boundary data for φ are symmetric in x and anti-symmetric in y, and the boundary data for u are symmetric in both x and y, it is natural to look for a solution which satisfies:

$$\varphi(x,y) = -\varphi(x,-y) = \varphi(-x,y) \qquad \text{in } R\backslash S, \tag{1.13}$$

$$u(x,y) = u(x,-y) = u(-x,y) \qquad \text{in } R. \tag{1.14}$$

We add two more conditions:

$$\varphi > 0 \qquad \text{in } R^+, \tag{1.15}$$

and

$$\varphi_y(x,0^+) \geq 0 \qquad \text{on } S_0. \tag{1.16}$$

If S_0 is empty then $\varphi = Ay/b$ so that $\varphi > 0$ in R^+ and $\varphi_y(x,0) > 0$. The conditions (1.15), (1.16) mean that, when S_0 is nonempty, φ is still positive in R^+ and the current is still moving downward at S_0.

DEFINITION 1.1. A pair of functions (φ, u) is called a solution of the thermistor problem (1.2)–(1.5) if it is a smooth solution of (1.2), (1.3) in $R\backslash S$ satisfying (1.4)–(1.16).

Observe that the relation (1.9) is a consequence of the first condition in (1.13). This condition also implies that

$$\varphi(x,0) = 0 \qquad \text{on } T. \tag{1.17}$$

THEOREM 1.1. *There exists a unique solution to the thermistor problem.*

The proof is given in Sections 2–5.

In Section 6 we shall extend the concept of a solution, allowing S to have nonempty interior. We shall prove that the only possible solution is the one with S as in (1.8). Theorem 1.1 asserts that the "hot" set S, where $u = u^*$, is an interval, defined in (1.8). This was conjectured by Howison [9]. Some numerical computations are reported in Westbrook [12]. In Section 7 we shall extend the results to the case where R is a general domain in $\mathbb{R}^2$.

2. - Reduction of the problem

In this section we reduce the problem for (φ, u) to a problem for (φ, ψ) where

$$\psi = u + \frac{1}{2}\varphi^2; \tag{2.1}$$

such a transformation was used already in [4] for other boundary conditions. Clearly ψ is continuous in R and

$$\psi(x,y) = \psi(x,-y) = \psi(-x,y) \qquad \text{in } R. \tag{2.2}$$

One can check that

$$\Delta\psi = 0 \qquad \text{in } R \backslash S. \tag{2.3}$$

From (2.2) it follows that $\psi_y(x,0) = 0$ on T (since $\psi \in C^1$ on T). The relation (1.10) gives $[\psi_y] = 0$ and, in view of the symmetry of ψ in y, $\psi_y = 0$ along S_0. We conclude that

$$\psi_y(x,0\pm) = 0 \qquad \text{if } -a < x < a, \quad x \neq \pm a_*. \tag{2.4}$$

Finally,

$$\psi = \frac{A^2}{2} \qquad \text{on } y = \pm b, \tag{2.5}$$

$$\psi = \frac{\varphi^2}{2} \qquad \text{on } x = \pm a \tag{2.6}$$

and

$$\psi < u^* + \frac{\varphi^2}{2} \qquad \text{on } \overline{R} \backslash S, \tag{2.7}$$

$$\psi = u^* + \frac{\varphi^2}{2} \qquad \text{on } S. \tag{2.8}$$

The thermistor problem can now be reformulated in terms of (φ, ψ):

(i) φ satisfies (1.2), (1.4), (1.13); it is continuously differentiable uniformly from each side of S except possibly at $(\pm a_*, 0)$, it is continuous in $R \backslash S_0$, and it satisfies (1.15), (1.16);

(ii) φ satisfies (2.1)–(2.8); it is continously differentiable uniformly from each side of S except possibly at $(\pm a_*, 0)$, and it is continuous in R.

3. - Conformal mapping

It will suffice to construct φ, ψ in R^+.

We shall use conformal mapping in order to transform the problem for (φ, ψ) into a problem for ψ alone. We introduce the harmonic conjugate $-\tilde{\varphi}$ of φ; it is uniquely determined up to an additive constant. We choose $\tilde{\varphi}$ such that $\tilde{\varphi}(a,0) = -\tilde{\varphi}(-a,0)$, which makes it symmetric in x, and then set

$$q = \tilde{\varphi}(a,0), \qquad q \text{ positive.} \tag{3.1}$$

Notice, by the maximum principle, that $\varphi < A$ in R^+.

Set

$$X = \tilde{\varphi}, \qquad Y = \varphi \tag{3.2}$$

and consider the conformal mapping

$$(x,y) \to \tilde{\varphi} + i\varphi = X + iY, \tag{3.3}$$

or

$$(x,y) \to (X(x,y), Y(x,y)). \tag{3.4}$$

It maps $\overline{R^+}$ into the closure of the rectangle R^*:

$$R^* = \{(X,Y);\ -q < X < q,\ 0 < Y < A\}; \tag{3.5}$$

here

$$\begin{aligned} &\partial R^+ \cap \{y = b\} && \text{is mapped into} && \partial R^* \cap \{Y = A\}, \\ &\partial R^+ \cap \{x = \pm a\} && \text{is mapped into} && \partial R^* \cap \{X = \pm q\} \end{aligned}$$

and

$$\partial R^+ \cap T \text{ is mapped onto the two segments of the set}$$

$$T' = \{(X,0);\ -q < X < -q_1 \text{ or } q_1 < X < q\} \qquad (0 < q_1 < q).$$

The image of S is a curve S' in $\overline{R^*}$ since $\varphi(x, 0_+) \geq 0$. From (1.16) we deduce that $\tilde{\varphi}(x, 0_+) \geq 0$. This implies that S' must be a graph

$$S' = \{(X,Y);\ Y = g(X),\ -q_1 \leq X \leq q_1,\ 0 \leq g(X) < A\}.$$

We denote by Ω^* the region bounded by S' and the X-axis, and set

$$D = R^* \backslash \overline{\Omega^*}.$$

LEMMA 3.1. *The mapping* (3.3) *is injective from* $\overline{R^+}$ *onto* $\overline{D}$.

PROOF. Using complex notation, let ς_0 be any point in D. We shall prove that

$$\int_{\partial D} \frac{dz}{f(z) - \varsigma_0} = 2\pi i \tag{3.7}$$

where $f(z) = \tilde{\varphi}(z) + i\varphi(z)$. Indeed, along the edge $y = b$ of R^+ the argument of $f(z) - \varsigma_0$ changes by 2 $\arctan \frac{A}{q}$; along $\{x = \pm a\}$ it changes by $\arctan \frac{A}{q}$; along S' it changes by π, and along T' it does not change. These observations clearly yield the assertion (3.7).

Equality (3.7) implies that $f(z)$ takes the value ς_0 at precisely one point of D. Thus f is conformal mapping from R^+ into D. Since both domains have piecewise C^1 boundary, we can appeal to a general result in conformal mappings [1; p. 369] to deduce that f can be extended continuously as an injective mapping from $\overline{R^+}$ onto $\overline{D}$.

Set

$$\Psi(X, Y) = \psi(x, y).$$

Then Ψ satisfies:

$$\Delta\Psi = 0 \qquad \text{in } D, \tag{3.8}$$

$$\Psi = \frac{Y^2}{2} \qquad \text{on } X = \pm q, \tag{3.9}$$

$$\Psi = \frac{A^2}{2} \qquad \text{on } Y = A, \tag{3.10}$$

$$\Psi_Y = 0 \qquad \text{on } T', \tag{3.11}$$

$$\frac{\partial\Psi}{\partial n} = 0 \qquad \text{on } S', \tag{3.12}$$

and

$$\Psi = u^* + \frac{Y^2}{2} \qquad \text{on } S', \tag{3.13}$$

$$\Psi < u^* + \frac{Y^2}{2} \qquad \text{on } R' \backslash S'. \tag{3.14}$$

We shall reduce the problem for Ψ into a variational inequality for a function W defined by

$$
(3.15)\qquad
\begin{aligned}
W(X,Y) &= \int_{g(X)}^{Y} \left[u^* + \frac{\tilde{Y}^2}{2} - \Psi(X,\tilde{Y})\right] \mathrm{d}\tilde{Y} \qquad \text{for } -q_1 < X < q_1,\\
W(X,Y) &= \int_{0}^{Y} \left[u^* + \frac{\tilde{Y}^2}{2} - \Psi(X,\tilde{Y})\right] \mathrm{d}\tilde{Y} \qquad \text{for } (X,0) \in T'.
\end{aligned}
$$

The condition (3.12) can be written in the form

$$
(3.16)\qquad g'(X)\Psi_X - \Psi_Y = 0 \qquad \text{on } Y = g(X).
$$

Using (3.16) one can verify that

$$
\Delta W = Y \qquad \text{in } D;
$$

the condition (3.11) is used in verifying this equation for (X,Y) above T'. Also, $W > 0$ in D, $W = 0$ on S' and

$$
\frac{\partial W}{\partial Y} = u^* + \frac{Y^2}{2} - \Psi = 0 \qquad \text{on } S'.
$$

This shows that if we extend W by 0 into Ω^* then W satisfies a variational inequality:

LEMMA 3.2. *The function W defined by* (3.15) *and extended by* 0 *into* Ω^* *satisfies the variational inequality*

$$
(3.17)\qquad -\Delta W \geq -Y, \qquad W \geq 0, \qquad W(-\Delta W + Y) = 0 \qquad \textit{a.e. in } R^*,
$$

with boundary conditions

$$
(3.18)\qquad
\begin{aligned}
&\frac{\partial W}{\partial Y} = u^* && \textit{on } Y = A,\\
&W = u^* Y && \textit{on } X = \pm q,\\
&W = 0 && \textit{on } Y = 0.
\end{aligned}
$$

4. - The variational inequality

In this section we shall assume that q is a fixed positive constant. It is well known (see, for instance, [6]) that the variational inequality (3.17), (3.18) has a unique solution. However, in order to go backward and recover the functions ψ and φ, we must first show that

$$\text{(4.1)}\qquad \begin{array}{l}\text{the set } \{(X,Y)\in\overline{R^*};\ W(X,Y)=0\} \text{ is a simply connected}\\ \text{domain of the form } \{(X,Y);\ -q\le X\le q,\ 0\le Y\le g(X)\}\end{array}$$

and

$$\text{(4.2)}\qquad \frac{\partial W}{\partial Y}>0 \qquad \text{in } \{W>0\},$$

i.e., $\psi < u^* + Y^2/2$ in $D\cap\{W>0\}$.

To do this we use the penalty method. We approximate W by the solution W_ε to

$$\text{(4.3)}\qquad -\Delta W_\varepsilon + \beta_\varepsilon(W_\varepsilon) = -Y \qquad \text{in } R^* \qquad (\varepsilon>0)$$

with the same boundary conditions (3.18) as for W; here the $\beta_\varepsilon(t)$ are C^∞ functions in t satisfying:

$$\text{(4.4)}\qquad \begin{array}{l}\beta_\varepsilon(t)=0 \qquad \text{if } t\ge 0,\\ \beta'_\varepsilon(t)\ge 0, \qquad \beta''_\varepsilon(t)\le 0\\ \text{and } \beta_\varepsilon(t)\to-\infty \qquad \text{as } \varepsilon\to 0 \text{ if } t<0.\end{array}$$

From the general theory of variational inequalities [6; Chapter 1] we know that there exists a unique solution W_ε and that $W_\varepsilon \to W$ uniformly in $W^{2,\infty}_{\text{loc}}(R^*)$ as $\varepsilon\to 0$, where W is the unique solution of the variational inequality (3.17) with boundary conditions (3.18).

We wish to derive additional properties of W. To do this we first establish these properties for W_ε and then let $\varepsilon\to 0$. We observe that the first compatibility condition holds at $(\pm q,0)$ for W_ε, since, upon using the boundary conditions, we get $-\Delta W_\varepsilon + \beta_\varepsilon(W_\varepsilon) = \beta_\varepsilon(0) = 0 = Y$ at $(\pm q, 0)$. It follows that W_ε is C^2 in a neighbourhood of $(\pm q,0)$. On the other hand, in a neighbourhood of $(\pm, A)$ the function $\beta_\varepsilon(W_\varepsilon)$ vanishes and (4.3) becomes $\Delta W_\varepsilon = Y$, so that $\Delta(W_{\varepsilon,Y}) = 1$. Since $W_{\varepsilon,Y} = u^*$, both on $Y=A$ and on $X=\pm q$, we can apply L^p boundary estimates to $W_{\varepsilon,Y}$ and conclude that, for any $p>1$,

$$W_{\varepsilon,Y}\in W^{2,p} \qquad \text{in a neighbourhood } N \text{ of } (\pm q, A).$$

This implies that $W_{\varepsilon,XXY}$, $W_{\varepsilon,XYY}$ and $W_{\varepsilon,YYY}$ are in $L^p(N)$ and, by Sobolev's imbedding, $W_{\varepsilon,XY}$ and $W_{\varepsilon,YY}$ are in $C^\alpha(\overline{N})$. From (4.3) we deduce that also $W_{\varepsilon,XX}\in C^\alpha(\overline{N})$.

LEMMA 4.1. *The solution W satisfies:*

$$\text{(4.5)}\qquad W_{XX}\ge 0, \qquad W_{YY}\ge 0 \qquad \textit{in } R^*.$$

PROOF. Consider the function $\varsigma = \partial^2 W_\varepsilon / \partial Y^2$. Differentiating (4.3) twice with respect to Y, we get

$$-\Delta\varsigma + \beta'_\varepsilon(W_\varepsilon)\varsigma = -\beta''_\varepsilon(W_\varepsilon)\left(\frac{\partial W_\varepsilon}{\partial Y}\right)^2 \geq 0.$$

On $\{-q < X < q,\ Y = A\}$

$$\varsigma_Y = (W_\varepsilon)_{YYY} = (-W_{\varepsilon,XX} + Y)_Y = 1.$$

On the sides $X = \pm q$, $\varsigma = 0$, and on $Y = 0$

$$\varsigma = \beta_\varepsilon(0) - W_{\varepsilon,XX} = 0.$$

Since, as noted above, ς is continuous in $\overline{R^*}$, the maximum principle yields $\varsigma > 0$ in R^*. Taking $\varepsilon \to 0$ the assertion $W_{YY} \geq 0$ follows.

The proof of $W_{XX} \geq 0$ is similar. Here

$$(W_{\varepsilon,XX})_Y = 0 \quad \text{on } Y = A, \qquad W_{\varepsilon,XX} = 0 \quad \text{on } Y = 0,$$

and

$$W_{\varepsilon,XX} = -W_{\varepsilon,YY} + \beta_\varepsilon(W_\varepsilon) + Y = Y > 0 \qquad \text{on } X = \pm q.$$

COROLLARY 4.2. *$W_Y(X,Y) \geq 0$ in R^*.*

PROOF. Since $W(X,0) = 0$ and $W(X,Y) \geq 0$, we have $W_Y(X,0) \geq 0$. Since also $W_{YY} \geq 0$, the assertion follows.

From Corollary 4.2 we deduce that there exists a function $g(X)$ defined for $X \in [-q,q]$ such that $g(X) \in [0,A]$, $W(X,Y) = 0$ for all $0 \leq Y \leq g(X)$ and $W(X,Y) > 0$ for all $g(X) < Y \leq A$. By symmetry,

$$W(X,Y) = W(-X,Y) \tag{4.6}$$

so that $W_X(0,Y) = 0$. Since $W_{XX} \geq 0$,

$$W_X(X,Y) \geq 0 \qquad \text{if } X \geq 0.$$

It follows that

$$g(X) = g(-X), \tag{4.7}$$

and $g(X)$ is non-increasing in X for $X > 0$. Define $q_1 = \inf\{X \in [0,q);\ g(X) = 0\}$. We can apply Theorems 6.1, 6.2 in [6; Chapter 2, Section 6] to deduce that

$$\begin{aligned} &g(X) \text{ is analytic for } -q_1 < X < q_1, \\ &g'(X) < 0 \qquad \text{if } 0 < X < q_1. \end{aligned} \tag{4.8}$$

We also have

$$g(q_1 - 0) = 0. \tag{4.9}$$

Indeed, if $g(q_1 - 0) > 0$ then along the segment $\ell = \{(q_1, Y);\ 0 < Y < g(q_1 - 0)\}$ the function W_{YY} has zero Cauchy data. Since this function is harmonic in D, it follows (by unique continuation) that $W_{YY} \equiv 0$, a contradiction.

We summarize:

THEOREM 4.3. *For any $q > 0$, there exists a unique solution W to the variational inequality* (3.17), (3.18), *satistying the properties* (4.2), (4.5), (4.6)*; furthermore, if the set $R^+ \cap \{W = 0\}$ is nonempty, then it has the form* (4.1), *and* (4.7), (4.8), (4.9) *hold for some $0 \leq q_1 < q$.*

5. - Proof of Theorem 1.1

In Section 4 we have constructed the function $W(X, Y)$. We now define a function $\Psi(X, Y)$ by

$$\Psi = u^* + \frac{Y^2}{2} - W_Y. \tag{5.1}$$

If (φ, ψ) is a solution of the thermistor problem then, as shown in Section 3, $\Psi(X, Y)$ must coincide with $\psi(x, y)$. There remains the problem of reconstructing the functions φ and ψ from Ψ. As we shall see, this can be done for one and only one choice of the parameter q.

For clarity we shall write

$$R^* = R^*_q = \{-q < X < q,\ 0 < Y < A\}$$

and denote the corresponding W and Ψ by W_q and Ψ_q. The free boundary will be denoted by

$$Y = g_q(X) \qquad -q_1 \leq X \leq q_1,$$

where q_1 is a function of q (given by Theorem 4.3). We also set

$$D_q = R^*_q \backslash \{Y \leq g_q(X)\}. \tag{5.2}$$

By Riemann's conformal mapping theorem, there exists a unique conformal mapping $h_q(Z)$ $(Z = X + iY)$ of the closure of D_q onto the closure of the rectangle in the $z = x + iy$ plane:

$$R^+ = \{-a < x < a,\ 0 < y < b\}$$

such that

$$\begin{aligned} &\pm q + iA \text{ is mapped into } \pm a + ib \\ &\text{and } ig(0) \text{ is mapped into } 0. \end{aligned} \tag{5.3}$$

From the uniqueness of $h_q(Z)$ and the symmetry about the imaginary axis of D_q and R^+ and of the points in (5.3), it follows that $h_q(Z)$ is symmetric with respect to the imaginary axis; in particular, $h_q(q) = -h_q(-q)$. Hence if

$$h_q(q) = a, \tag{5.4}$$

then also $h_q(-q) = -a$, and it follows that

$$\begin{aligned} &h_q \text{ maps the edge } Y = A \text{ of } D_q \text{ onto the edge } y = b \text{ of } R^+, \\ &\text{the edges } X = \pm q \text{ of } D_q \text{ onto the edges } x = \pm a \text{ of } R^+, \\ &\text{and the remaining boundary of } D_q \text{ onto the edge } y = 0 \text{ on } R^+. \end{aligned} \tag{5.5}$$

Denote the inverse of $h_q(Z)$ by f; i.e.

$$x + iy = h_q(X + iY) \quad \text{if and only if} \quad X + iY = f(x + iy). \tag{5.6}$$

Next define functions φ, $\tilde{\varphi}$ by

$$\tilde{\varphi}(x, y) = X(x, y) = \operatorname{Re} f, \qquad \varphi(x, y) = Y(x, y) = \operatorname{Im} f. \tag{5.7}$$

From (5.5) if follows that φ satisfies all the boundary conditions in R^+ required in the definition of a solution to the thermistor problem (including (1.15), (1.16)) where

$$S = \text{ image of } S' \text{ under the mapping } h. \tag{5.8}$$

Since φ is harmonic in R^+, it satisfies of course also (1.2). The function ψ also satisfies all the required conditions stated in Section 2. Observe that if $h_q(q) \neq a$ then φ, defined by (5.6), (5.7), does not satisfy some of the required boundary conditions.

LEMMA 5.1. *There exists a unique $q > 0$ such that $h_q(q) = a$.*

Once Lemma 5.1 is proved, the assertion of Theorem 1.1 follows. To prove Lemma 5.1 we need several auxiliary lemmas.

LEMMA 5.2. *If q is small enough, then $0 < x(q) < bq/A$.*

In the remainder of this section we denote $X + iY$ by (X, Y).

PROOF. We first show that the concidence set of W_q, i.e., the set $\{W_q = 0\} \cap R_q^*$, is empty if q is small enough. Consider the solution to

$$\Delta V = Y \qquad \text{in } R_q^*$$

with the same boundary conditions as W, and set

$$\tilde{V}_q(X,Y) = \frac{1}{q} V(qX, qY), \qquad 0 < Y < \frac{A}{q}, \quad -1 < X < 1.$$

Then

$$\Delta \tilde{V}_q = q^2 Y \to 0 \qquad \text{as } q \to 0$$

so that $\tilde{V}_q \to V^*$ where V^* is harmonic in $|X| < 1$, $0 < Y < \infty$ with boundary conditions

$$V^*(X,0) = 0, \qquad V^*(\pm 1, Y) = u^* Y,$$

$$V^* = O(Y) \qquad \text{as } Y \to \infty.$$

Since, for $\varepsilon > 0$, $\varepsilon(Y^2+1-X^2)$ is a superharmonic function in $\{|X| < 1,\ Y > 0\}$ which majorizes the harmonic function $\pm(V^* - u^*Y)$ on $\{Y = 0\}$, $\{|X| = 1\}$ and for $Y \to \infty$, it follows that $|V^* - u^*Y| < \varepsilon(Y^2 + 1 - X^2) \to 0$ if $\varepsilon \to 0$, i.e., $V^* \equiv u^*Y$. Consequently,

$$\frac{\partial}{\partial Y} \tilde{V}_q(X,0) \to \frac{\partial V^*}{\partial Y} = u^* \qquad \text{as } q \to 0.$$

The convergence is uniform for $-1 \le X \le 1$, and it implies that

$$V_Y(X,0) \sim u^* \qquad \text{uniformly in } X, \quad -1 \le X \le 1,$$

provided q is small.

Next we can apply the maximum principle to V_{YY} (cf. the proof of Lemma 4.1) and deduce that $V_{YY} > 0$. It follows that $V_Y(X,Y) > 0$ if q is small and therefore $V(X,Y) > 0$ in R^*. Hence V is the solution W_q of the variational inequality, and thus the coincidence set of W_q is empty, i.e.,

$$D_q = \{-q < X < q,\ 0 < Y < A\} = R_q^*$$

if q is small.

Consider the function $x = x(X,Y)$, the real part of h_q (cf. (5.6)). We need to show that

$$x(q,0) < bq/A. \tag{5.9}$$

Suppose (5.9) is not true. Then $x(q,0) \ge bq/A$ and there exists an $\alpha \in (0,q]$ such that $x(\alpha,0) = bq/A$.

We shall compare the harmonic function $x(X,Y)$ with the harmonic function bX/A in the rectangle $R' = \{0 < X < q,\ 0 < Y < A\}$. On the top edge $Y = A$, $y \equiv b$ so that

$$\frac{\partial x}{\partial Y} = -\frac{\partial y}{\partial X} = 0; \qquad \text{also } \frac{\partial X}{\partial Y} = 0.$$

On the left edge $X = 0$, $x = 0$ (by symmetry of h_q). On the bottom edge $Y = 0$, if $0 \le X \le \alpha$ then $y(X,0) = 0$ so that

$$\frac{\partial x}{\partial Y} = -\frac{\partial y}{\partial X} = 0; \qquad \text{also } \frac{\partial X}{\partial Y} = 0.$$

Finally, on the remaining boundary of R' which consists of the segments from $(\alpha,0)$ to $(q,0)$ and from $(q,0)$ to (q,A), $x(X,Y) \ge bq/A$ whereas $q \ge X$. It follows that

$$x(X,Y) \ge bX/A \qquad \text{in } D_q$$

and, furthermore,

$$\frac{\partial}{\partial X}(x - bX/A)|_{X=0} > 0.$$

This implies $\left(\text{since } \frac{\partial x}{\partial X} = \frac{\partial y}{\partial Y}\right)$ that

$$\frac{\partial}{\partial Y}(y - bY/A)|_{X=0} > 0$$

and, by integration,

$$0 = y(0,A) - b = \int_0^A \frac{\partial}{\partial Y}(y(0,Y) - bY/A) > 0,$$

a contradiction.

LEMMA 5.3. *If $q > aA/b$, then $y(q,0) > b - aA/q$.*

PROOF. Suppose this is not true. Then there is an $\alpha \in (0,A)$ such that

$$y(q,\alpha) = b - aA/q. \tag{5.10}$$

We can now prove by comparison that

$$y(X,Y) \le b - \frac{a}{q}(A - Y) \qquad \text{in } D_q. \tag{5.11}$$

Indeed, both sides of (5.11) are harmonic functions. On $\{Y = A\}$ they coincide; on $\{X = 0\}$ and on $\{X = q,\ \alpha < Y < A\}$

$$\frac{\partial y}{\partial X} = -\frac{\partial x}{\partial Y} = 0 \qquad \text{and} \qquad \frac{\partial}{\partial X}\left(b - \frac{a}{q}(A - Y)\right) = 0.$$

Finally, by (5.10), on the remaining boundary which is contained in $X > 0$,

$$y < y(q, \alpha) = b - aA/q \leq b \frac{a}{q}(A - Y) \qquad \text{since } Y \geq 0.$$

It follows that (5.11) holds and, furthermore,

$$\frac{\partial}{\partial Y}\left(y - b + \frac{a}{q}(A - Y)\right)\bigg|_{Y=A} > 0.$$

Hence

$$\frac{\partial}{\partial X} = \frac{\partial y}{\partial Y} > \frac{a}{q} \qquad \text{on } Y = A$$

and, by integration,

$$2a = x(q, A) - x(-q, A) = \int_{-q}^{q} \frac{\partial x}{\partial X}\, \mathrm{d}X > \frac{a}{q} \cdot 2q = 2a,$$

a contradiction.

We have proved that

$$h_q(q, 0) < bq/A \qquad \text{if } q \text{ is small,}$$

$$h_q(q, 0) = a + i\eta \qquad \text{for some } 0 < \eta < b, \text{ if } q \text{ is large.}$$

Let us extend the function $Y = g_q(X)$ by 0 to $|X| > q_1$. Then the mapping $q \to g_q$ is continuous in q (when g_q is endowed with the uniform topology), by [6; Chapter 2]. It then follows that also the mapping $q \to h_q^{-1}$ is continuous in q (when h_q^{-1} is endowed with the uniform topology); in fact, one can prove this by using the equicontinuity of the family h_q^{-1} defined on $\overline{R^+}$. We deduce that also the inverse conformal mappings vary continuously with q and, in particular, $q \to h_q(q, 0)$ is continuous. It follows that there exists a $q_0 > 0$ such that $h_{q_0}(q_0, 0) = a$. This proves the existence part of Lemma 5.1.

To prove that such a solution q_0 is unique, we begin by examining more closely the dependence of W_q and the free boundary

$$Y = g_q(X), \qquad -q_1 < X < q_1,$$

on q.

LEMMA 5.4. *If $q' > q$ then*

$$W_{q'}\left(\frac{q'}{q} X, Y\right) \leq W_q(X, Y) \qquad \textit{in } D_q \tag{5.12}$$

and

$$g_{q'}(q'X) \geq g_q(qX) \qquad \textit{for } -\frac{q_1}{q} < X < \frac{q_1}{q}. \tag{5.13}$$

PROOF. Set

$$\widetilde{W} = W_{q'}\left(\frac{q'}{q}X, Y\right).$$

Then, in $\{\widetilde{W} > 0\}$

$$\Delta\widetilde{W} = \frac{q'^2}{q^2}\widetilde{W}_{XX} + \widetilde{W}_{YY} \geq \widetilde{W}_{XX} + \widetilde{W}_{YY} = Y = \Delta W_q$$

since $q'/q > 1$ and $\widetilde{W}_{XX} \geq 0$ (by Lemma 4.1). Since the boundary conditions for $\widetilde{W}$ are identical to those for W_q in D_q, (5.12) follows by a comparison theorem for variational inequalities ([6; Chapter 1]). The inequality (5.13) is a consequence of (5.12).

Consider the conformal mappings h_q and $h_{q'}$ for $q < q'$, and suppose they satisfy:

$$h_q(\pm q, 0) = h_{q'}(\pm q', 0) = \pm a. \tag{5.14}$$

We wish to show that this leads to a contradiction. We begin by introducing the harmonic functions

$$\eta_1(X,Y) = y_q(qX, A - q(A-Y)),$$
$$\eta_2(X,Y) = y_{q'}(q'X, A - q'(A-Y)),$$

and the corresponding $\xi_1(X,Y)$, $\xi_2(X,Y)$ defined in the same way with y_q, $y_{q'}$ replaced by x_q, $x_{q'}$; here $h_q = x_q + iy_q$, $h_{q'} = x_{q'} + iy_{q'}$. Observe that η_1 is defined for $-1 < X < 1$ and for Y such that

$$A - q(A-Y) \geq g_q(qX);$$

for simplicity we extend $g_q(qX)$ by 0 to $-1 \leq X < -q_1/q$ and $q_1/q < X \leq 1$, and similarly extend $g_{q'}(q'X)$. Thus, for fixed X, the Y-interval in the domain of definition of η_1 is $\tilde{Y}_q \leq Y \leq A$, where

$$A - q(A - \tilde{Y}_q) = g_q(qX).$$

The length of this interval is $A - \tilde{Y}_q$. Similarly, for fixed X, the Y-interval in the domain of definition of η_2 is $\tilde{Y}_{q'} \leq Y \leq A$, where

$$A - q'(A - \tilde{Y}_{q'}) = q_{q'}(q'X)$$

and its length is $A - \tilde{Y}_{q'}$. Since $g_{q'}(q'X) \geq g_q(qX)$, for $-1 < X < 1$, we get

$$q'(A - \tilde{Y}_{q'}) \leq q(A - \tilde{Y}_q)$$

and, since $q' > q$,

$$A - \tilde{Y}_{q'} < A - \tilde{Y}_q.$$

This means that the domain of definition I_q of η_1 contains the domain of definition $I_{q'}$ of η_2.

We now compare η_1, η_2 in the domain $I_{q'}$. Clearly $\eta_1 \geq 0 = \eta_2$ on the free boundary. From (5.14) we deduce that the horizontal side and the vertical sides of the domains of definitions of η_1 and η_2 are mapped into the same horizontal and vertical sides of R^+. Therefore

$$\eta_1 = \eta_2 \qquad \text{on } Y = A$$

and

$$\frac{\partial \eta_i}{\partial X} = \frac{\partial \xi_i}{\partial Y} = 0 \qquad \text{on the vertical sides.}$$

Applying the maximum principle we conclude that $\eta_1 > \eta_2$ in $I_{q'}$ and, furthermore,

$$\left.\frac{\partial(\eta_1 - \eta_2)}{\partial Y}\right|_{Y=A} < 0.$$

It follows that

$$\left.\frac{\partial}{\partial X}(\xi_1 - \xi_2)\right|_{Y=A} < 0$$

and, by integration,

$$0 = (\xi_1 - \xi_2)(1, A) - (\xi_1 - \xi_2)(-1, A) = \int_{-1}^{1} \frac{\partial}{\partial X}(\xi_1 - \xi_2)(X, A)\, \mathrm{d}X < 0,$$

a contradiction.

We have thus proved that there cannot be more than one solution q to the equation $h_q(q, 0) = q$. This completes the proof of Theorem 1.1.

6. - A more general uniqueness theorem

Definition 1.1 of a solution (φ, u) presupposes that S consists of an interval lying on the x-axis. From the physical background of the problem one might equally well look for a solution where S has the form

(6.1) $$S = \{(x, y);\ |y| \leq k(x),\ -\gamma \leq x \leq \gamma\}$$

where

(6.2) $$\begin{aligned} &k(-x) = k(x), \\ &k(x) > 0 \qquad \text{if } 0 \leq |x| < \gamma_1, \\ &k(x) = 0 \qquad \text{if } \gamma_1 \leq |x| \leq \gamma. \end{aligned}$$

Let

$$S_0 = \{(x,0);\ |x| \le \gamma,\ k(x) = 0\}, \tag{6.3}$$
$$S_1 = S \backslash S_0.$$

Formally, the weak formulation of (1.2) gives

$$\frac{\partial \varphi}{\partial n} = 0 \qquad \text{on } \partial S_1 \tag{6.4}$$

where n is the outward normal to S_1. Since the function ψ introduced in Section 2 satisfies (see [3], [4])

$$\nabla(\sigma(u)\nabla(\psi) = 0, \tag{6.5}$$

the weak formulation of (1.3), written in terms of ψ, gives

$$\frac{\partial \psi}{\partial n} = 0 \qquad \text{on } \partial S_1. \tag{6.6}$$

We also need to replace (1.15), (1.16) by

$$\varphi > 0 \qquad \text{in } (R\backslash S) \cap \{y > 0\}, \tag{6.7}$$

$$\frac{\partial \varphi}{\partial n} \ge 0 \qquad \text{on } (S_0 \cup \partial S_1) \cap \{y \ge 0\} \text{ except at } (\pm\gamma, 0). \tag{6.8}$$

Finally we assume that (1.13), (1.14) hold in $R\backslash S$, u is continuous in $R\backslash(\operatorname{int} S_1)$, and continuously differentiable in $(R\backslash \operatorname{int} S_1) \cap \{y \ge 0\}$, and φ has the same smoothness properties as u except for a jump discontinuity that it might have across the interior of the interval S_0.

DEFINITION 6.1. A pair (φ, u) satisfying all the above properties as well as (1.2)–(1.5) with R replaced by $R\backslash S$ is called a solution to the thermistor problem.

THEOREM 6.1. *If* (φ, u) *is a solution to the thermistor problem (according to Definition* 6.1*) then* S_1 *is empty so that* (φ, u) *coincides with the solution asserted in Theorem* 1.1.

PROOF. Suppose (φ, u) is a solution and consider the mapping $\tilde{\varphi} + i\varphi$ introduced in Section 3. It maps S_1 onto an interval l along which $\tilde{\varphi} = 0$. Thus, the image of R is $D\backslash\bar{l}$ where D is defined as in Section 3 and $\bar{l}$ is the closure of an interval

$$l = \{(0, Y);\ g(0) < Y < \delta\}, \qquad \delta \ge g(0).$$

By properties of the solution, the function $\Psi(X,Y) = \psi(x,y)$ satisfies:

$$\Psi = u^* + \frac{Y^2}{2} \qquad \text{on } l$$

and (cf. (6.6))

$$\frac{\partial \psi}{\partial X} = 0 \qquad \text{from both sides of } l,$$

if l is nonempty. Since the Cauchy data of Ψ are analytic, Ψ can be extended as harmonic function across l. By uniqueness to the Cauchy problem,

$$\Psi(X,Y) \equiv u^* - \frac{X^2}{2} + \frac{Y^2}{2},$$

and this holds throughout D, a contradiction to the boundary condition for Ψ on $Y = A$. This proves that l is empty and then so is the set S_1, and the theorem follows.

7. - General domains

Let Ω be a general domain with piecewise C^1 boundary $\partial\Omega$ and choose four points B, C, D, E on $\partial\Omega$, arranged clockwise along $\partial\Omega$. Let R_p be a rectangle

$$R_p = \{(x,y);\ -p < x < p,\ -b < y < b\}$$

and set

$$B' = (-p,b), \qquad C' = (p,b), \qquad D'(p,-b), \qquad E' = (-p,-b).$$

Denote by F_p the conformal mapping which maps Ω onto R_p such that

$$F_p(B) = B', \qquad F_p(C) = C', \qquad F_p(E) = E'.$$

THEOREM 7.1. *There exists a unique value $p > 0$ such that $F_p(D) = D'$.*

Once this theorem is proved, we can immediately solve the thermistor problem which consists of (1.2), (1.3) in Ω with

$$\begin{aligned}
&\varphi = A && \text{on the arc } \overset{\frown}{BC},\\
&\varphi = -A && \text{on the arc } \overset{\frown}{DE},\\
&\frac{\partial \varphi}{\partial n} = 0 && \text{on the arcs } \overset{\frown}{CD} \text{ and } \overset{\frown}{EB},\\
&u = 0 && \text{on } \partial\Omega,
\end{aligned}$$

by first solving it in the rectangle R_p with a pair (φ, u), and then taking $(\varphi \circ F_p^{-1}, u \circ F_p^{-1})$ to be the solution in Ω (cf. [7]).

PROOF OF THEOREM 7.1. Denote by G_p the point on the arc CDE whose image under F_p is D'. We claim that

(7.1) if $p' > p$ then $G_{p'}$ lies between C and G_p.

Suppose this is not true, i.e. suppose that $G_{p'}$ lies between G_p and E or at G_p, and consider the imaginary parts Y_p and $Y_{p'}$ of F_p and $F_{p'}$. On $\widehat{BC}$ both are equal to b. On $\widehat{EB}$ and $\widehat{CG_p}$ both have zero normal derivatives. On $\widehat{G_{p'}E}$ both Y_p and $Y_{p'}$ are equal to $-b$. Finally, on the arc $\widehat{G_pG_{p'}}$ there holds $Y_{p'} \geq -b$ and $Y_p = -b$ so that $Y_{p'} \geq Y_p$. By the maximum principle it follows that $Y_{p'} \geq Y_p$ in Ω and

$$\frac{\partial}{\partial n}(Y_{p'} - Y_p) < 0 \qquad \text{on } \widehat{BC},$$

so that

$$\frac{\partial}{\partial s}(X_{p'} - X_p) < 0 \qquad \text{on } \widehat{BC}$$

where $\partial/\partial s$ denote the tangential derivative (counter-clockwise). Integrating on s we get

$$2p = X_p(C) - X_p(B) > X_{p'}(C) - X_{p'}(B) = 2p',$$

which is a contradiction.

The assertion (7.1) implies that there exists at most one value p for which $F_p(D) = D'$. Observe next that the function $p \to F_p(D)$ is continuous. Therefore, in order to complete the proof of Theorem 7.1 it suffices to show:

$$\text{if } \ p \to 0 \ \text{ then } G_p \to E, \tag{7.2}$$

$$\text{if } \ p \to \infty \text{ then } G_p \to C. \tag{7.3}$$

To prove (7.2) suppose the assertion is not true. Then

$$G_p \to E_* \neq E \qquad \text{as } p \to 0.$$

The harmonic function X_p satisfies

$$|X_p| \leq p \qquad \text{in } \Omega$$

and therefore

$$X_p \to 0, \qquad Y_p \to \text{const.}$$

uniformly in Ω, as $p \to 0$. This is a contradiction since $Y \equiv \lim Y_p$ is harmonic in Ω and

$$Y = b \quad \text{on } \widehat{BC}, \qquad Y = -b \quad \text{on } \widehat{E_*E}.$$

To prove (7.3) we work with $\frac{1}{p}X_p$, $\frac{1}{p}Y_p$. The image of Ω under $\frac{1}{p}(X_p + iY_p)$ is a rectangle with one side of lenght 1 and the other side of length b/p which goes to zero as $p \to \infty$. Thus we can apply the proof of (7.2) to deduce the assertion (7.3).

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