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Degenerate Elliptic Equations with Measure Data and Nonlinear Potentials

TERO KILPELÄINEN - JAN MALÝ*

Introduction

Throughout this paper let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$. Let $\mathcal{M}^+(\Omega)$ be the set of all nonnegative finite Radon measures on Ω.

The problem

$$\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mu, \tag{1}$$

where $\mu \in \mathcal{M}^+(\Omega)$ and $1 < p \leq n$, is not, in general, solvable in $H^{1,p}_{\mathrm{loc}}(\Omega)$. If $p = 2$, we are in the case of the Laplacian and a generalized solution to (1) can be given by

$$u(x) = \int_\Omega G(x,y)\,\mathrm{d}\mu(y),$$

where G is the Green function. No corresponding integral representation is available when $p \neq 2$.

An existence result for equations

$$-\mathrm{div}\,\mathcal{A}(x, \nabla u) = \mu, \tag{2}$$

where $\mathcal{A}(x,\nabla u) \cdot \nabla u \approx |\nabla u|^p$ (see Section 1), was recently established by Boccardo and Gallouët [1] (for $p > 2 - 1/n$ and Ω bounded). In their paper, a function u in $H_0^{1,1}(\Omega)$ is found such that $\mathcal{A}(\cdot, \nabla u) \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega \varphi\,\mathrm{d}\mu$$

for all $\varphi \in C_0^\infty(\Omega)$. However such a solution is "very weak" and the uniqueness may fail [16]. It seems to be an important problem to find an appropriate class

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of functions in which both the existence and the uniqueness results for equation (2) are valid.

We pursue the existence problem and it turns out that the class of the so called $\mathcal{A}$-superharmonic functions (see Section 1) is wide enough to solve equation (2). Conversely, we show that each $\mathcal{A}$-superharmonic function solves equation (2) with some nonnegative (not necessarily finite) Radon measure μ on Ω. The uniqueness of $\mathcal{A}$-superharmonic solutions remains open; note that Serrin's non-unique solution [16] is not $\mathcal{A}$-superharmonic.

Our approach departs from [1] in three ways: We take care of the $\mathcal{A}$-superharmonicity of the solution. Our structural assumptions on the equation continue in the tradition of nonlinear potential theory (see e.g. [4-13]) and, therefore, are partially less, partially more, restrictive than the structural assumptions in [1][1]. Finally, we do not exclude the possibility $p \leq 2 - 1/n$. However, this requires us to interpret the derivative ∇u in a new way when u does not possess functions as distributional derivatives: we define the gradient $\mathrm{D}u$ of an $\mathcal{A}$-superharmonic function u by

$$\mathrm{D}u = \lim_{k\to\infty} \nabla(u \wedge k)$$

(see Section 1).

As to our method of proof, the main new feature is the use of $H^{1,p}$-estimates for truncated solutions rather than $H^{1,q}$-estimates for solutions.

In the second part of the paper, we establish pointwise estimates for an $\mathcal{A}$-superharmonic solution in terms of a nonlinear potential

$$\mathbf{W}^{\mu}_{1,p}(x;r) = \int_0^r (t^{p-n}\mu(B(x,t)))^{1/(p-1)}\,\frac{\mathrm{d}t}{t},$$

$x \in \mathbb{R}^n$, $r > 0$. The potential $\mathbf{W}^{\mu}_{1,p}$ is principal in the theory of nonlinear potentials. For example, a nonnegative Radon measure μ belongs to $H^{-1,p}(\mathbb{R}^n)$ if and only if

$$\int_{\mathbb{R}^n} \mathbf{W}^{\mu}_{1,p}(x;1)\,\mathrm{d}\mu(x) < \infty;$$

cf. [3]. Our main theorem, which gives a new link between the two nonlinear potential theories, reads as follows:

MAIN THEOREM. *Let u be an $\mathcal{A}$-superharmonic function in Ω and let μ be the nonnegative Radon measure*

$$\mu = -\operatorname{div} \mathcal{A}(x, \mathrm{D}u).$$

[1] Note added in August 1992: After the submission of this paper for publication, the existence result in [1] has been extended for more general classes of equations than ours by Rakotoson (*Quasilinear elliptic problems with measures as data*, Differential Integral Equations **4** (1991), 449-457) and by Boccardo and Gallouët (*Nonlinear elliptic equations with right hand side measures*, Comm. Partial Differential Equations **17** (1992), 641-655).

Suppose that $B(x,2r) \subset \Omega$. Then there is a constant c_1 depending only on n, p, and the structure of $\mathcal{A}$ such that

$$u(x) \geq \inf_{\Omega} u + c_1 \mathbf{W}^{\mu}_{1,p}(x;r).$$

If, in addition, $p > n-1$, then

$$u(x) \leq \sup_{\partial B(x,r)} u + c_2(\mathbf{W}^{\mu}_{1,p}(x;r) + (r^{p-n}\mu(\overline{B}(x,r)))^{1/(p-1)}),$$

where $c_2 = c_2(n,p) > 0$. In particular, if $\mathbf{W}^{\mu}_{1,p}(x;r)$ is finite, then $u(x)$ is finite.

That the potential $\mathbf{W}^{\mu}_{1,p}$ is bounded if the solution of (2) is bounded seems to be known to specialists. Nevertheless, we have not been able to locate the result in the literature. Moreover, our theorem gives a pointwise estimate for $\mathbf{W}^{\mu}_{1,p}$. Apparently, the reverse direction is not yet well understood. Our method breaks down if $p \leq n-1$ because it strongly employs the Sobolev embedding theorem on $(n-1)$-spheres. Loosely related to our upper estimate is a result of Rakotoson and Ziemer [15]; they show that if

$$\mu(B(x,r)) \leq cr^{n-p+\varepsilon}$$

for all small r, then a solution to (1) is not only bounded but locally Hölder continuous; see also [14].

Our paper is organized as follows. Section 1 provides the necessary preliminaries of the nonlinear potential theory of $\mathcal{A}$-superharmonic functions as well as precise definitions; some of the results in Section 1 may be of independent interest. Then existence problem is discussed in Section 2. In Section 3 we establish the first part of the main theorem and the concluding section contains the upper bound in the case when $p > n-1$.

NOTATION. Our notation is fairly standard and self-explanatory. Let us emphasize that Ω always stands for an open set in $\mathbb{R}^n$, $n \geq 2$, $\mathcal{M}^+(\Omega)$ for the set of all nonnegative finite Radon measures in Ω, and $C_0^\infty(\Omega)$ for the set of all infinitely many times differentiable functions with compact support in Ω. The minimum of functions u and v is denoted by $u \wedge v$, the maximum of u and v by $u \vee v$, and the positive part of u by $u^+ = u \vee 0$. The open ball $B(0,r)$ of radius r centered at origin is abbreviated by B_r and its sphere $\partial B(0,r)$ by S_r. The Lebesgue measure of a set E is written as $|E|$.

1. - $\mathcal{A}$-superharmonic functions

We assume throughout this paper that $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping which satisfies the following assumptions for some numbers $1 < p \leq n$ and

$\gamma \geq 1$:

$$\begin{aligned}&\text{the function } x \mapsto \mathcal{A}(x,h) \text{ is measurable for all } h \in \mathbb{R}^n, \text{ and}\\&\text{the function } h \mapsto \mathcal{A}(x,h) \text{ is continuous for a.e. } x \in \mathbb{R}^n;\end{aligned} \tag{1.1}$$

for all $h \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$

$$\mathcal{A}(x,h) \cdot h \geq |h|^p, \tag{1.2}$$

$$|\mathcal{A}(x,h)| \leq \gamma |h|^{p-1}, \tag{1.3}$$

$$(\mathcal{A}(x,h_1) - \mathcal{A}(x,h_2)) \cdot (h_1 - h_2) > 0 \tag{1.4}$$

whenever $h_1 \neq h_2$, and

$$\mathcal{A}(x,\lambda h) = \lambda|\lambda|^{p-2}\mathcal{A}(x,h) \tag{1.5}$$

for all $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

A solution u to equation

$$-\operatorname{div} \mathcal{A}(x,\nabla u) = 0 \tag{1.6}$$

always has a continuous version which we call *$\mathcal{A}$-harmonic*. Hence u is $\mathcal{A}$-harmonic in Ω if $u \in H^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ and

$$\int_\Omega \mathcal{A}(x,\nabla u) \cdot \nabla\varphi \, dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. A function u is called a *supersolution* of (1.6) if $u \in H^{1,p}_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega \mathcal{A}(x,\nabla u) \cdot \nabla\varphi \, dx \geq 0$$

for all nonnegative $\varphi \in C_0^\infty(\Omega)$.

A lower semicontinuous function $u: \Omega \to (-\infty,\infty]$ is called *$\mathcal{A}$-superharmonic* if u is not identically infinite in each component of Ω, and if for all open $D \subset\subset \Omega$ and all $h \in C(\overline{D})$, $\mathcal{A}$-harmonic in D, $h \leq u$ on ∂D implies $h \leq u$ in D.

To ease some formulations we say that u is *$\mathcal{A}$-hyperharmonic* in Ω if in each component of Ω u is either $\mathcal{A}$-superharmonic or identically ∞.

In this section we record some properties of $\mathcal{A}$-superharmonic functions. For more on their nonlinear potential theory see [4-6], [8-10], and [12].

Clearly, $u \wedge v$ and $\lambda u + \sigma$ are $\mathcal{A}$-superharmonic if u and v are, and σ, $\lambda \in \mathbb{R}$, $\lambda \geq 0$. Moreover we have the following relation between $\mathcal{A}$-superharmonic functions and supersolutions of (1.6).

PROPOSITION [4] 1.7. *(i) If u is a supersolution of* (1.6), *then there is an $\mathcal{A}$-superharmonic function v such that $u = v$ a.e. Moreover,*

$$v(x) = \operatorname{ess}\liminf_{y\to x} v(y) \qquad \textit{for all } x \in \Omega. \tag{1.8}$$

(ii) If v is $\mathcal{A}$-superharmonic, then (1.8) *holds. Moreover, v is a supersolution of* (1.6) *provided that $v \in H^{1,p}_{\mathrm{loc}}(\Omega)$.*

(iii) If v is $\mathcal{A}$-superharmonic and locally bounded, then $v \in H^{1,p}_{\mathrm{loc}}(\Omega)$ and therefore it is a supersolution.

It is worth mentioning that $\mathcal{A}$-superharmonic functions are $(1,p)$-finely continuous which in particular means that they are approximately continuous [8].

BALAYAGE 1.9. Let u be a nonnegative $\mathcal{A}$-superharmonic function in Ω and $E \subset \Omega$. If

$$R^u_E(x;\Omega) = \inf\{v(x) \colon v \text{ is } \mathcal{A}\text{-superharmonic, } v \ge 0, \text{ and } v \ge u \text{ on } E\},$$

then the lower semicontinuous regularization

$$\hat{R}^u_E(x;\Omega) = \hat{R}^u_E(x) = \lim_{r\to 0} \inf_{B(x,r)} R^u_E$$

of R^u_E is called the *balayage* of u on E. The function $\hat{R}^u_E$ is $\mathcal{A}$-superharmonic in Ω and $\mathcal{A}$-harmonic in $\Omega/\overline{E}$. Moreover, if Ω is bounded, $E \subset\subset \Omega$, and u is bounded, then $\hat{R}^u_E$ belongs to $H^{1,p}_0(\Omega)$ (see the proof of [6, 2.2]).

GRADIENT OF AN $\mathcal{A}$-SUPERHARMONIC FUNCTION 1.10. It may occur that an $\mathcal{A}$-superharmonic function is not locally integrable or that its distributional derivative is not a function. Thus we are led to the following concept of derivative which assigns a gradient (a function!) to each $\mathcal{A}$-superharmonic function and, more generally, to each function whose truncations are locally in a Sobolev space.

Suppose that u is a function in Ω such that the truncations $u \wedge k$, $k = 1, 2, \ldots$, belong to $H^{1,1}_{\mathrm{loc}}(\Omega)$. We let $\mathrm{D}u$ stand for the a.e. defined function

$$\mathrm{D}u = \lim_{k\to\infty} \nabla(u \wedge k).$$

If $u \in H^{1,1}_{\mathrm{loc}}(\Omega)$, the function $\mathrm{D}u$ is the distributional gradient of u. However, for $p \le 2 - 1/n$ there are $\mathcal{A}$-superharmonic functions u not in $H^{1,1}_{\mathrm{loc}}(\Omega)$ (see 1.16) and thus $\mathrm{D}u$ is not, in general, the distributional gradient of u.

We show that $|\mathrm{D}u|^{p-1}$ is locally integrable for an $\mathcal{A}$-superharmonic u. First we prove an auxiliary result.

LEMMA 1.11. *Let Ω be bounded and let u be a nonnegative, a.e. finite function on Ω. Suppose that for all $k \in \mathbf{N}$*

$$u \wedge k \in H_0^{1,p}(\Omega)$$

and

$$\int_\Omega |\nabla(u \wedge k)|^p \, dx \leq M k \tag{1.12}$$

for some constant M, independent of k. If $1 \leq q < \frac{n}{n-1}$, then

$$\int_\Omega |\nabla(u \wedge k)|^{q(p-1)} \, dx \leq c,$$

where $c = c(n, p, q, M, \operatorname{diam}\Omega)$.

PROOF. Let

$$\varkappa = \begin{cases} \dfrac{p}{n-p} & \text{if } p < n \\ \dfrac{2q}{n - q(n-1)} & \text{if } p = n. \end{cases}$$

Fix k and choose an integer m with $k < 2^m$. Writing $v = u \wedge 2^m$ we have by the Sobolev embedding theorem and (1.12) that

$$\begin{aligned} k^{\varkappa p} |\{k \leq u < 2k\}| &\leq \int_{\{k \leq u < 2k\}} v^{\varkappa p} \, dx \\ &\leq \int_\Omega (u \wedge 2k)^{\varkappa p} \, dx \\ &\leq c \left(\int_\Omega |\nabla(u \wedge 2k)|^p \, dx \right)^{\varkappa} \\ &\leq c (M k)^{\varkappa}. \end{aligned}$$

Using the Hölder inequality we obtain

$$\begin{aligned} &\int_{\{k \leq u < 2k\}} |\nabla v|^{q(p-1)} \, dx \\ &\qquad \leq |\{k \leq u < 2k\}|^{1 - q(p-1)/p} \left(\int_{\{k \leq u < 2k\}} |\nabla v|^p \, dx \right)^{q(p-1)/p} \\ &\qquad \leq c k^{\alpha}, \end{aligned}$$

where $c = c(n, p, q, M, \operatorname{diam}\Omega) > 0$ and

$$\begin{aligned}\alpha &= \varkappa(1-p)\left(1 - \frac{q(p-1)}{p}\right) + \frac{q(p-1)}{p}\\ &= \begin{cases} \dfrac{(p-1)(n-1)}{n-p}\left(q - \dfrac{n}{n-1}\right) & \text{if } p < n\\ -\dfrac{q(n-1)}{n} & \text{if } p = n\end{cases}\\ &< 0.\end{aligned}$$

Hence

$$\int_\Omega |\nabla v|^{q(p-1)}\,\mathrm{d}x = \int_{\{u<1\}} |\nabla v|^{q(p-1)}\,\mathrm{d}x + \sum_{j=1}^{m} \int_{\{2^{j-1}\le u<2^j\}} |\nabla v|^{q(p-1)}\,\mathrm{d}x \le c_1 + c\sum_{j=1}^{\infty} 2^{\alpha j},$$

and the lemma follows.

THEOREM 1.13. *Let u be a nonnegative $\mathcal{A}$-superharmonic function in a bounded open set Ω such that*

$$u \wedge k \in H_0^{1,p}(\Omega), \qquad k = 1, 2, \ldots. \tag{1.14}$$

Then $|\mathrm{D}u|^{p-1} \in L^q(\Omega)$ for each q with $1 \le q < \dfrac{n}{n-1}$.

PROOF. Assumption (1.2) implies

$$\int_{\{k-1\le u\le k\}} |\mathrm{D}u|^p\,\mathrm{d}x \le a_k,$$

where

$$a_k = \int_{\{k-1\le u\le k\}} \mathcal{A}(x, \mathrm{D}u)\cdot \mathrm{D}u\,\mathrm{d}x.$$

We show that the sequence a_k is decreasing, and hence

$$\int_\Omega |\nabla(u\wedge k)|^p\,\mathrm{d}x \le k a_1, \qquad k = 1, 2, \ldots$$

and the assertion follows from Lemma 1.11.

Now $u_k = u \wedge (k+1) \in H_0^{1,p}(\Omega)$ is a supersolution and

$$w_k = (1 - |u-k|)^+$$

an appropriate test function; we conclude

$$
\begin{aligned}
0 &\leq \int_{\Omega} \mathcal{A}(x, \nabla u_k) \cdot \nabla w_k \,\mathrm{d}x \\
&= \int_{\{k-1 \leq u < k\}} \mathcal{A}(x, \mathrm{D}u) \cdot \mathrm{D}u \,\mathrm{d}x - \int_{\{k \leq u < k+1\}} \mathcal{A}(x, \mathrm{D}u) \cdot \mathrm{D}u \,\mathrm{d}x \\
&= a_k - a_{k+1},
\end{aligned}
$$

as desired.

Lindqvist [11] showed that $|\mathrm{D}u|^{q(p-1)}$, $1 \leq q < \dfrac{n}{n-1}$, is locally integrable if u is p-superharmonic (i.e. $\mathcal{A}(x,h) = |h|^{p-2}h$). We extend his result for general $\mathcal{A}$-superharmonic functions by using a different technique.

THEOREM 1.15. *Suppose that u is $\mathcal{A}$-superharmonic in Ω and $1 \leq q < \dfrac{n}{n-1}$. Then both $|\mathrm{D}u|^{p-1}$ and $\mathcal{A}(\cdot, \mathrm{D}u)$ belong to $L^q_{\mathrm{loc}}(\Omega)$.*

Moreover, if $p > 2 - 1/n$, then $\mathrm{D}u$ is the distributional gradient of u.

PROOF. Fix open sets $G \subset\subset G' \subset\subset \Omega$. Then $m = \inf_{G'} u > -\infty$. Now the balayage $v(x) = \hat{R}^{u-m}_G(x; G')$ is a nonnegative $\mathcal{A}$- superharmonic function satisfying the weak zero boundary condition (1.14) in G'. Hence by Theorem 1.13, $|\mathrm{D}v|^{p-1} \in L^q(G')$. Since $u = v + m$ in G, we have that $|\mathrm{D}u|^{p-1} \in L^q(G)$ and therefore $\mathcal{A}(\cdot, \mathrm{D}u) \in L^q(G)$.

For $p > 2 - 1/n$ we can choose $q_0 < n/(n-1)$ such that $q_0(p-1) > 1$. Then the compactness properties of $H^{1,q_0(p-1)}(G)$ allow us to conclude that

$$u = \lim_{k \to \infty} u \wedge k \in H^{1,q_0(p-1)}(G)$$

and hence $\mathrm{D}u$ is the distributional gradient of u.

EXAMPLE 1.16. Let $p \leq 2 - 1/n$ and $\mathcal{A}(x,h) = |h|^{p-2}h$. Let u be the fundamental $\mathcal{A}$-superharmonic function,

$$u(x) = |x|^{\frac{p-n}{p-1}}.$$

Then

$$\mathrm{D}u(x) = \frac{p-n}{p-1}\, x|x|^{\frac{1-n}{p-1}-1}$$

is not the distributional derivative of u because $\dfrac{1-n}{p-1} \leq -n$.

THEOREM 1.17. *Suppose that u_j is a sequence of nonnegative $\mathcal{A}$-superharmonic functions in Ω. Then there is an $\mathcal{A}$-hyperharmonic function u*

in Ω *and a subsequence* u_{j_i} *of* u_j *such that* $u_{j_i} \to u$ *a.e. in* Ω *and* $\mathrm{D}u_{j_i} \to \mathrm{D}u$ *a.e. in* $\{u < \infty\}$.

PROOF. *Step 1.* Assume that $u_j \leq k$ for all j. Choose open sets $G \subset\subset G' \subset\subset \Omega$. By the Caccioppoli inequality [4, 2.16] we have

$$\int_{G'} |\nabla u_j|^p \, \mathrm{d}x \leq ck^p, \tag{1.18}$$

where the constant c is independent of j. Consequently, the sequence u_j is bounded in $H^{1,p}(G')$ and we may pick a subsequence u_{j_i} of u_j and a function $u \in H^{1,p}(G')$ such that $u_{j_i} \to u$ both weakly in $H^{1,p}(G')$ and pointwise a.e. We show that this subsequence, denoted from now on by u_j, has the desired properties in G.

If $v_i = \inf_{j \geq i} u_j$, then the lower semicontinuous regularization $\tilde{v}_i$ of v_i,

$$\tilde{v}_i(x) = \lim_{y \to x} \inf v_i(y),$$

is $\mathcal{A}$-superharmonic and $\tilde{v}_i = v_i$ a.e. in G' [5, Theorem 6.1]. Now the functions $\tilde{v}_i$ increase to an $\mathcal{A}$-superharmonic function v and $v = u$ a.e..

Next we show that

$$\nabla u_j \to \nabla u (= \nabla v) \qquad \text{a.e. in } G.$$

Fix $\varepsilon > 0$ and let

$$E_j = \{x \in G : (\mathcal{A}(x, \nabla u_j) - \mathcal{A}(x, \nabla u)) \cdot (\nabla u_j - \nabla u) > \varepsilon\}.$$

We estimate the measure of E_j. First we have that

$$\begin{aligned} |E_j| &\leq |E_j \cap \{|u_j - u| \geq \varepsilon^2\}| \\ &\quad + \frac{1}{\varepsilon} \int_{E_j \cap \{|u - u_j| < \varepsilon^2\}} (\mathcal{A}(x, \nabla u_j) - \mathcal{A}(x, \nabla u)) \cdot (\nabla u_j - \nabla u) \, \mathrm{d}x. \end{aligned} \tag{1.19}$$

Let $\eta \in C_0^\infty(G')$ be a cut-off function, $0 \leq \eta \leq 1$ and $\eta = 1$ in G. If

$$w_j = (u_j + \varepsilon^2 - u)^+ \wedge 2\varepsilon^2$$

and

$$\tilde{w}_j = (u + \varepsilon^2 - u_j)^+ \wedge 2\varepsilon^2,$$

then both $w_j \eta$ and $\tilde{w}_j \eta$ are nonnegative functions in $H_0^{1,p}(G')$, hence appropriate test functions. Since u and u_j are supersolutions and $u_j \to u$ weakly in $H^{1,p}(G')$, we obtain using (1.18) that

$$\int_{G'\cap\{|u_j-u|<\varepsilon^2\}} \mathcal{A}(x,\nabla u)\cdot(\nabla u_j-\nabla u)\eta\,\mathrm{d}x \le \int_{G'} \mathcal{A}(x,\nabla u)w_j\cdot\nabla\eta\,\mathrm{d}x$$

$$\le 2\gamma\varepsilon^2\int_{G'}|\nabla u|^{p-1}|\nabla\eta|\,\mathrm{d}x$$

$$\le c\varepsilon^2,$$

with c independent of j and ε, and similarly

$$\int_{G'\cap\{|u_j-u|<\varepsilon^2\}} \mathcal{A}(x,\nabla u_j)\cdot(\nabla u_j-\nabla u)\eta\,\mathrm{d}x \le \int_{G'} \mathcal{A}(x,\nabla u_j)\tilde{w}_j\cdot\nabla\eta\,\mathrm{d}x$$

$$\le c\varepsilon^2.$$

Adding these together we arrive at the estimate

$$\frac{1}{\varepsilon}\int_{G'\cap\{|u-u_j|<\varepsilon^2\}} (\mathcal{A}(x,\nabla u_j)-\mathcal{A}(x,\nabla u))\cdot(\nabla u_j-\nabla u)\eta\,\mathrm{d}x\le c\varepsilon,$$

and since the integrand is nonnegative we have by (1.19) that

$$|E_j|\le c\varepsilon+|E_j\cap\{|u_j-u|\ge\varepsilon^2\}| \tag{1.20}$$

where the constant c is independent of j and ε.

We infer from (1.20) that

$$\nabla u_j\to\nabla u \qquad \text{a.e. in } G. \tag{1.21}$$

Indeed, writing

$$A(h,h_0)=(\mathcal{A}(x,h)-\mathcal{A}(x,h_0))\cdot(h-h_0)$$

we have that $h\mapsto A(h,h_0)$ is continuous, $A(h,h_0)>0$ if $h\neq h_0$ and $A(h,h_0)\to\infty$ as $h\to\infty$. Using (1.20) we easily conclude (1.21).

The proof of the assertion under the assumption of Step 1 is now completed by using an exhaustion argument and choosing a diagonal subsequence.

Step 2. Now we treat the general case. In light of Step 1 we may select subsequences $u_j^{(k)}$ of u_j and find $\mathcal{A}$-superharmonic functions v_k such that for all $k=1,2,\dots$, $u_j^{(k+1)}$ is a subsequence of $u_j^{(k)}$, $u_j^{(k)}\wedge k\to v_k$ a.e., and $\nabla(u_j^{(k)}\wedge k)\to\nabla v_k$ a.e. It is easy to see that v_k increases to an $\mathcal{A}$-hyperharmonic function u and, moreover, $v_k=u\wedge k$ for each k. The diagonal sequence $u_j^{(j)}$ obviously has the desired properties.

2. - Existence of $\mathcal{A}$-superharmonic solutions

We introduce an operator $T_{\mathcal{A}} = T$ acting on the family of all $\mathcal{A}$-superharmonic functions u in Ω by

$$\langle Tu, \varphi \rangle = \int_\Omega \mathcal{A}(x, \mathrm{D}u) \cdot \nabla\varphi \,\mathrm{d}x$$

for all $\varphi \in C_0^\infty(\Omega)$. Note that the function $x \mapsto \mathcal{A}(x, \mathrm{D}u)$ is locally integrable in Ω (Theorem 1.15) and thus its divergence $\operatorname{div} \mathcal{A}(x, \mathrm{D}u)$ in the sense of distributions is $-Tu$, i.e.

$$T_{\mathcal{A}} u = Tu = -\operatorname{div} \mathcal{A}(x, \mathrm{D}u).$$

Obviously, Tu is a nonnegative Radon measure on Ω if u is a supersolution of (1.6). The following theorem shows that the same is true for each $\mathcal{A}$-superharmonic function u.

THEOREM 2.1. *Suppose that u is $\mathcal{A}$-superharmonic in Ω. Then there is a nonnegative Radon measure μ on Ω such that*

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega \varphi \,\mathrm{d}\mu$$

whenever $\varphi \in C_0^\infty(\Omega)$, that is,

$$Tu = \mu.$$

PROOF. Fix a nonnegative $\varphi \in C_0^\infty(\Omega)$. Since $\mathcal{A}(\cdot, \mathrm{D}u) \in L^1(\{\varphi \neq 0\})$ (Theorem 1.15), it follows from the Lebesgue dominated convergence theorem that

$$\begin{aligned}\langle Tu, \varphi \rangle &= \int_\Omega \mathcal{A}(x, \mathrm{D}u) \cdot \nabla\varphi \,\mathrm{d}x \\ &= \lim_{k\to\infty} \int_\Omega \mathcal{A}(x, \nabla(u \wedge k)) \cdot \nabla\varphi \,\mathrm{d}x \geq 0.\end{aligned}$$

Using the Riesz representation theorem we conclude that Tu is a nonnegative Radon measure.

REMARK 2.2. The proof of Theorem 2.1 shows: *if u is an $\mathcal{A}$-superharmonic function in Ω and $\mu = Tu$, then the measures $\mu_k = T(u \wedge k)$ converge weakly to μ on Ω.*

Next we show that the equation

$$-\operatorname{div} \mathcal{A}(x, \mathrm{D}u) = \mu, \tag{2.3}$$

where $\mu \in \mathcal{M}^{+}(\Omega)$, has an $\mathcal{A}$-superharmonic solution with weak zero boundary values.

THEOREM 2.4. *Suppose that Ω is bounded and $\mu \in \mathcal{M}^{+}(\Omega)$. Then there is an $\mathcal{A}$-superharmonic function u in Ω such that*

$$Tu = \mu$$

in Ω and

$$u \wedge k \in H_0^{1,p}(\Omega)$$

for all $k \in \mathbf{N}$.

PROOF. Let μ_j be a sequence of nonnegative measures associated with densities in $C_0^{\infty}(\Omega)$ such that μ_j converges weakly to μ. We may assume that

$$\mu_j(\Omega) \leq \mu(\Omega) + 1.$$

Let then u_j be the $\mathcal{A}$-superharmonic function in Ω such that $u_j \in H_0^{1,p}(\Omega)$ and

$$Tu_j = \mu_j$$

(see e.g. [13]). By Theorem 1.17 we may select a subsequence of u_j, denoted again by u_j, such that u_j converges to an $\mathcal{A}$-hyperharmonic function u a.e. on Ω. We have the estimate

$$\begin{aligned} \int_\Omega |\nabla(u_j \wedge k)|^p \,\mathrm{d}x &\leq \int_\Omega \mathcal{A}(x, \nabla u_j) \cdot \nabla(u_j \wedge k)\,\mathrm{d}x \\ &= \int_\Omega u_j \wedge k \,\mathrm{d}\mu_j \leq k\mu_j(\Omega) \leq k(\mu(\Omega) + 1). \end{aligned} \tag{2.5}$$

It now follows from the Poincaré inequality that

$$\int_\Omega (u_j \wedge k)^p \,\mathrm{d}x \leq ck,$$

with c independent of j and k. Hence

$$u = \lim_{j\to\infty} u_j < \infty \qquad \text{a.e.}$$

and u is thus $\mathcal{A}$-superharmonic. Moreover, $\nabla u_j \to Du$ a.e. in Ω (Theorem 1.17). These estimates also guarantee that

$$u \wedge k \in H_0^{1,p}(\Omega) \qquad \text{for all } k = 1, 2, \ldots .$$

Then let $\nu = Tu$. We show that $\nu = \mu$ which completes the proof. To this end, fix $1 < q < \dfrac{n}{n-1}$. From (2.5) and Lemma 1.11 we obtain

$$\int_\Omega |\mathcal{A}(x, \nabla u_j)|^q \, dx \leq c,$$

where c is independent of j. Therefore, $\mathcal{A}(x, \nabla u_j) \to \mathcal{A}(x, Du)$ weakly in $L^q(\Omega)$ because $\nabla u_j \to Du$ a.e. Thus we have for each $\varphi \in C_0^\infty(\Omega)$ that

$$\int_\Omega \varphi \, d\mu_j = \int_\Omega \mathcal{A}(x, \nabla u_j) \cdot \nabla \varphi \, dx \to \int_\Omega \mathcal{A}(x, Du) \cdot \nabla \varphi \, dx = \int_\Omega \varphi \, d\nu.$$

Since $\mu_j \to \mu$ weakly, we also have

$$\int_\Omega \varphi \, d\mu_j \to \int_\Omega \varphi \, d\mu.$$

Consequently, $\mu = \nu$ as desired.

REMARK 2.6. In light of Theorem 1.13 and the Poincaré inequality we have that, for $p > 2 - 1/n$, the solution u in Theorem 2.4 belongs to $H_0^{1,q}(\Omega)$ if $1 \leq q < \dfrac{n(p-1)}{n-1}$.

REMARK 2.7. Combining the methods of this paper and [1] we arrive at the following existence result: if μ is a finite *signed* Radon measure on Ω, then there is a "very weak" solution u to the problem

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu,$$
$$u = 0 \qquad \text{on } \partial\Omega.$$

That is, u has the properties that all its double-sided truncations $u_k = ((-k \vee u) \wedge u)$ belong to $H_0^{1,p}(\Omega)$ and $-\operatorname{div}\mathcal{A}(x, \nabla u_k) \to \mu$ weakly on Ω.

3. - Lower estimate

In this section we prove a lower estimate for an $\mathcal{A}$-superharmonic function in terms of a nonlinear potential.

THEOREM 3.1. *Let u be an $\mathcal{A}$-superharmonic function in Ω and $\mu = Tu$. If $B(x_0, 2r) \subset \Omega$, then*

$$u(x_0) \geq \inf_{\Omega} u + c\, \mathbf{W}^{\mu}_{1,p}(x_0; r)$$

where $c = c(n, p, \gamma) > 0$ and

$$\mathbf{W}^{\mu}_{1,p}(x_0; r) = \int_0^r (t^{p-n}\mu(B(x_0, t)))^{1/(p-1)} \frac{\mathrm{d}t}{t}.$$

For the proof we first record an appropriate form of Trudinger's weak Harnack inequality [17].

LEMMA 3.2. *Let u be a nonnegative supersolution of* (1.6) *in B_{4R}. If $q > 0$ is such that $q(n-p) < n(p-1)$, then*

$$R^{-n/q} \Bigg(\int_{B_{3R}} u^q \,\mathrm{d}x \Bigg)^{1/q} \leq c \inf_{B_{2R}} u,$$

where $c = c(n, p, q, \gamma) > 0$.

PROOF. By [17, Theorem 1.2], such a constant c exists if $u \leq 1$ in B_{4R}. However, as well known, the simpler structure of our equation allows us to obtain the inequality without boundedness restriction. Indeed, set $u_j = (u/j) \wedge 1$. Then

$$R^{-n/q} \Bigg(\int_{B_{3R}} u_j^q \,\mathrm{d}x \Bigg)^{1/q} \leq c \inf_{B_{2R}} u_j,$$

and hence

$$R^{-n/q} \Bigg(\int_{B_{3R}} (u \wedge j)^q \,\mathrm{d}x \Bigg)^{1/q} \leq c \inf_{B_{2R}} u.$$

Letting $j \to \infty$ we obtain the desired estimate.

Also we need the following well known estimate.

LEMMA 3.3. *Let u be a supersolution of* (1.6) *in an open set containing $\overline{B}_R$ such that $u > 0$ in B_R. Let $\eta \in C_0^\infty(B_R)$ be nonnegative. For all $\varepsilon \in (0, p-1)$ we have*

$$\int_{B_R} |\nabla u|^p u^{-1-\varepsilon} \eta^p \,\mathrm{d}x \leq c \int_{B_R} u^{p-1-\varepsilon} |\nabla \eta|^p \,\mathrm{d}x$$

where $c = (p\gamma/\varepsilon)^p$.

PROOF. For the sake of convenience we include a short proof. Fix $j \in \mathbf{N}$ and write $u_j = u + 1/j$. Set $v = u_j^{-\varepsilon}$ and $w = v\eta^p$. Then $w \in H_0^{1,p}(B_R)$ is nonnegative and hence

$$\begin{aligned} 0 &\le \int_{B_R} \mathcal{A}(x, \nabla u) \cdot \nabla w \, \mathrm{d}x \\ &= \int_{B_R} \mathcal{A}(x, \nabla u)\eta^p \cdot \nabla v \, \mathrm{d}x + p \int_{B_R} \mathcal{A}(x, \nabla u) v \eta^{p-1} \cdot \nabla \eta \, \mathrm{d}x. \end{aligned}$$

Using the structural assumptions and the Hölder inequality, it follows that

$$\begin{aligned} \varepsilon \int_{B_R} |\nabla u|^p u_j^{-1-\varepsilon} \eta^p \, \mathrm{d}x &\le \gamma p \int_{B_R} |\nabla u|^{p-1} u_j^{-\varepsilon} |\nabla \eta|^p \eta^{p-1} \, \mathrm{d}x \\ &\le \gamma p \left(\int_{B_R} |\nabla u|^p u_j^{-1-\varepsilon} \eta^p \, \mathrm{d}x \right)^{(p-1)/p} \left(\int_{B_R} u_j^{p-1-\varepsilon} |\nabla \eta|^p \, \mathrm{d}x \right)^{1/p}, \end{aligned}$$

which implies the reguired estimate when u is replaced by u_j. Letting $j \to \infty$ completes the proof.

The next estimate is a refined version of an estimate of Gariepy and Ziemer [2].

LEMMA 3.4. *Let u be a nonnegative supersolution of* (1.6) *in B_{4R}. Let $\eta \in C_0^\infty(B_{3R})$ be a cut-off function such that $0 \le \eta \le 1$, $\eta = 1$ on B_{2R}, and $|\nabla \eta| \le 10/R$. Then*

$$\int_{B_{3R}} |\nabla u|^{p-1} \eta^{p-1} |\nabla \eta| \, \mathrm{d}x \le c R^{n-p} \inf_{B_{2R}} u^{p-1},$$

where $c = c(n, p, \gamma) > 0$.

PROOF. We use the argument of the proof of Theorem 2.1 in [2]. Let $\varepsilon = \dfrac{1}{2} \min(p-1, p/(n-p)$ (if $p = n$, let $\varepsilon = (n-1)/2$)). Denote $q = p/(p-1)$, $\gamma_1 = p - 1 - \varepsilon$, and $\gamma_2 = (p-1)(1+\varepsilon)$. Using Lemmas 3.2 and 3.3, and the Hölder inequality we obtain

$$
\begin{aligned}
\int\limits_{B_{3R}} |\nabla u|^{p-1}\eta^{p-1}|\nabla\eta|\,\mathrm{d}x &= \int\limits_{B_{3R}} |\nabla u|^{p-1}\eta^{p-1}u^{-(1+\varepsilon)/q}u^{(1+\varepsilon)/q}|\nabla\eta|\,\mathrm{d}x \\
&\le \left(\int\limits_{B_{3R}} |\nabla u|^p u^{-1-\varepsilon}\eta^p\,\mathrm{d}x\right)^{1/q}\left(\int\limits_{B_{3R}} u^{(p-1)(1+\varepsilon)}|\nabla\eta|^p\,\mathrm{d}x\right)^{1/p} \\
&\le \left(c\int\limits_{B_{3R}} u^{\gamma_1}|\nabla\eta|^p\,\mathrm{d}x\right)^{1/q}\left(\int\limits_{B_{3R}} u^{\gamma_2}|\nabla\eta|^p\,\mathrm{d}x\right)^{1/p} \\
&\le cR^{-p}\left(\int\limits_{B_{3R}} u^{\gamma_1}\,\mathrm{d}x\right)^{1/q}\left(\int\limits_{B_{3R}} u^{\gamma_2}\,\mathrm{d}x\right)^{1/p} \\
&\le cR^{n-p}(\inf_{B_{2R}} u)^{\frac{\gamma_1}{q}+\frac{\gamma_2}{p}} = cR^{n-p}(\inf_{B_{2R}} u)^{p-1},
\end{aligned}
$$

and the lemma is proved.

The following estimate takes the measure data into account.

LEMMA 3.5. *Suppose that u is $\mathcal{A}$-superharmonic and $\mu = Tu$ in an open set containing $\overline{B}_R$. Then*

$$R^{p-n}\mu(B_{R/2}) \le c(\inf_{B_{R/2}} u - \inf_{B_R} u)^{p-1}$$

where $c = c(n,p,\gamma) > 0$.

PROOF. Write $a = \inf_{B_R} u$ and $b = \inf_{B_{R/2}} u$. Choose a positive integer $j \ge b$ and let $u_j = u \wedge j$. Let $\mu_j = Tu_j$ and let $\eta \in C_0^\infty(B_{3R/4})$, $0 \le \eta \le 1$, be a cut-off function such that $\eta = 1$ in $B_{R/2}$ and $|\nabla\eta| \le 10/R$. We use the test function $w_j = v_j\eta^p$ in B_R, where $v_j = (u_j \wedge b) - a$. Then $0 \le w_j \le b - a$ and $w_j = b - a$ on $B_{R/2}$. Using the Caccioppoli estimate for $b - a - v_j$ (cf. [4, Lemma 2.16]) and Lemma 3.4 for $u_j - a$, we obtain

$$\begin{aligned}
&\int\limits_{B_R} \mathcal{A}(x, \nabla u_j) \cdot \nabla w_j \,\mathrm{d}x \\
&= \int\limits_{B_{3R/4}} \mathcal{A}(x, \nabla u_j) \cdot \nabla v_j \eta^p \,\mathrm{d}x + p \int\limits_{B_{3R/4}} \mathcal{A}(x, \nabla u_j) v_j \eta^{p-1} \cdot \nabla \eta \,\mathrm{d}x \\
&\leq c \left(\int\limits_{B_{3R/4}} |b - a - v_j|^p |\nabla \eta|^p \,\mathrm{d}x + (b-a) \int\limits_{B_{3R/4}} |\nabla u_j|^{p-1} \eta^{p-1} |\nabla \eta| \,\mathrm{d}x \right) \\
&\leq c \left((b-a)^p \int\limits_{B_{3R/4}} |\nabla \eta|^p \,\mathrm{d}x + R^{n-p}(b-a)(b-a)^{p-1} \right) \\
&\leq c R^{n-p} (b-a)^p.
\end{aligned}$$

Now it follows that

$$(b-a)\mu_j(B_{R/2}) \leq \int\limits_{B_R} w_j \,\mathrm{d}\mu_j = \int\limits_{B_R} \mathcal{A}(x, \nabla u_j) \cdot \nabla w_j \,\mathrm{d}x \leq c R^{n-p}(b-a)^p.$$

Since $\mu_j \to \mu$ weakly (Remark 2.2), we have that

$$\mu(B_{R/2}) \leq \liminf_{j \to \infty} \mu_j(B_{R/2}) \leq c R^{n-p} (b-a)^{p-1};$$

this concludes the proof.

Now we are ready to prove Theorem 3.1.

PROOF OF THEOREM 3.1. We may assume that $x_0 = 0$. Choose a radius $r > 0$ such that $B_{2r} \subset \Omega$. Let $r_j = 2^{1-j} r$ and $a_j = \inf\limits_{B_{r_j}} u$. Then using the preceding lemma we have

$$\begin{aligned}
c \sum_{j=1}^{\infty} (r^{p-n} 2^{j(n-p)} \mu(B_{r_j}))^{1/(p-1)} &\leq \sum_{j=1}^{\infty} (a_j - a_{j-1}) \\
&= \lim_k (a_k - a_0) = u(0) - a_0 \leq u(0) - \inf_\Omega u.
\end{aligned}$$

The desired estimate follows, since

$$\mathbf{W}^{\mu}_{1,p}(0;r) = \int_0^r (t^{p-n}\mu(B_t))^{1/(p-1)}\,\frac{dt}{t}$$

$$\leq c\sum_{j=1}^{\infty}(r^{p-n}2^{j(n-p)}\mu(B_{r_j}))^{1/(p-1)}.$$

REMARK 3.6. The proof of Theorem 3.1 and the minimum principle imply: if u is $\mathcal{A}$-superharmonic and $\mu = Tu$ in a neighbourhood of $\overline{B}(x_0, 2r)$, then

$$u(x_0) \geq \inf_{\partial B(x_0,2r)} u + c\mathbf{W}^{\mu}_{1,p}(x_0;r),$$

where $c = c(n, p, \gamma) > 0$.

4. - Upper estimate when $p > n - 1$

In this section we establish the following upper estimate if $p > n - 1$.

THEOREM 4.1. *Suppose that $p > n - 1$. Let u be an $\mathcal{A}$-superharmonic function in Ω and $\mu = Tu$. If $\overline{B}(x_0, r_0) \subset \Omega$, then*

$$u(x_0) \leq \sup_{\partial B(x_0,r_0)} u + c\mathbf{W}^{\mu}_{1,p}(x_0;r_0) + cb_0;$$

here $c = c(n,p) > 0$,

$$\mathbf{W}^{\mu}_{1,p}(x_0;r_0) = \int_0^{r_0} (t^{p-n}\mu(B(x_0,t)))^{1/(p-1)}\,\frac{\mathrm{d}t}{t},$$

and

$$b_0 = (r_0^{p-n}\mu(\overline{B}(x_0,r_0)))^{1/(p-1)}.$$

COROLLARY 4.2. *Suppose that $p > n - 1$. Let u be $\mathcal{A}$-superharmonic in Ω and $\mu = Tu$. If for some $r > 0$*

$$\mathbf{W}^{\mu}_{1,p}(x_0;r) < \infty,$$

then $u(x_0)$ is finite.

We need the concept of p-capacity. If K is a compact subset of Ω, let

$${}_*\mathrm{cap}_p(K,\Omega) = \inf \int_\Omega |\nabla\varphi|^p\,\mathrm{d}x$$

where the infimum is taken over all $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 1$ on K. The *p-capacity* of a set $E \subset \Omega$ in Ω is the number

$$\operatorname{cap}_p(E, \Omega) = \inf_{\substack{U \subset \Omega \text{ open} \\ E \subset U}} \sup_{\substack{K \subset U \\ K \text{ compact}}} {}_*\operatorname{cap}_p(K, \Omega)$$

(see e.g. [9]).

The following lemma crucially relies on the fact that $p > n-1$. Its variants have constantly been used in nonlinear potential theory (see [12], [6], and [7]). For a proof see e.g. the proof of Lemma 5.3 in [7].

LEMMA 4.3. *For $p > n-1$ there is a constant $c = c(n,p) > 0$ such that if $E \subset B_R$ with*

$$\operatorname{cap}_p(E, B_R) \leq cR^{n-p},$$

then there is an $r \in \left(\frac{1}{4}R, \frac{1}{2}R\right)$ such that $S_r \cap E = \emptyset$.

LEMMA 4.4. *Let u be an $\mathcal{A}$-superharmonic function in an open set containing $\overline{B}_R$ and $\mu = Tu$. If $u \leq M$ on S_R, then for all $k > 0$*

$$\operatorname{cap}_p(\{u \geq M + k\},\ B_R) \leq k^{1-p}\mu(\overline{B}_R).$$

PROOF. Fix a positive integer j with $j \geq M + k$. Let $u_j = u \wedge j$ and $\mu_j = Tu_j$. Set

$$v_j = (u_j - M)^+ \wedge k.$$

Then v_j is an admissible test function for u_j in B_R and we obtain

$$\int_{B_R} \mathcal{A}(x, \nabla u_j) \cdot \nabla v_j \,\mathrm{d}x = \int_{B_R} v_j \,\mathrm{d}\mu_j \leq k\mu_j(\overline{B}_R).$$

Since $v_j = 0$ on S_R and $v_j = k$ on $\{u \geq M + k\}$, an approximation yields

$$\begin{aligned} k^p \operatorname{cap}_p(\{u \geq M + k\}, B_R) &\leq \int_{B_R} |\nabla v_j|^p \,\mathrm{d}x \leq \int_{B_R} \mathcal{A}(x, \nabla v_j) \cdot \nabla v_j \,\mathrm{d}x \\ &= \int_{B_R} \mathcal{A}(x, \nabla u_j) \cdot \nabla v_j \,\mathrm{d}x \leq k\mu_j(\overline{B}_R). \end{aligned}$$

The desired estimate follows since $\mu_j \to \mu$ weakly and hence

$$\limsup_{j \to \infty} \mu_j(\overline{B}_R) \leq \mu(\overline{B}_R).$$

The next lemma is the essential step in the proof of Theorem 4.1.

LEMMA 4.5. *Suppose that* $p > n - 1$. *Let* u *be* $\mathcal{A}$*-superharmonic in an open set containing* $\overline{B}_R$ *and* $\mu = Tu$. *Then there is a radius* $r \in \left(\frac{1}{4}R, \frac{1}{2}R\right)$ *such that if*

$$u \leq M \qquad \text{on } S_R,$$

then

$$u \leq M + c(R^{p-n}\mu(\overline{B}_R))^{1/(p-1)} \qquad \text{on } S_r;$$

here $c = c(n,p) > 0$.

PROOF. Set

$$k = \left(\frac{\mu(\overline{B}_R)}{c_0 R^{n-p}}\right)^{1/(p-1)},$$

where c_0 is the constant of Lemma 4.3. If

$$E = \{u \geq M + k\},$$

then Lemma 4.4 implies

$$\operatorname{cap}_p(E, B_R) \leq k^{1-p}\mu(\overline{B}_R) = c_0\, R^{n-p};$$

appealing to Lemma 4.3 we find a radius $r \in \left(\frac{1}{4}R, \frac{1}{2}R\right)$ such that $E \cap S_r = \emptyset$. Since

$$u \leq M + k \qquad \text{on } S_r,$$

the lemma follows.

LEMMA 4.6. *Suppose that* r_j *is a decreasing sequence of radii with*

$$c_1 \leq \frac{r_{j-1}}{r_j} \leq c_2$$

for some constants $c_2 \geq c_1 > 1$. *Then there are positive constants* $\tilde{c}_1 = \tilde{c}_1(n,p,c_1)$ *and* $\tilde{c}_2 = \tilde{c}_2(n,p,c_2)$ *such that*

$$\tilde{c}_1 \sum_{j=2}^{\infty}(r_j^{p-n}\mu(\overline{B}_{r_j}))^{1/(p-1)} \leq \mathbf{W}^{\mu}_{1,p}(0;r_1) \leq \tilde{c}_2 \sum_{j=1}^{\infty}(r_j^{p-n}\mu(B_{r_j}))^{1/(p-1)}.$$

PROOF. Obviously we have that

$$r_j^{\frac{p-n}{p-1}} \leq c \int_{r_j}^{r_{j-1}} s^{\frac{p-n}{p-1}} \frac{ds}{s}.$$

where $c = c(n, p, c_1)$. Hence

$$\sum_{j=2}^{\infty}(r_j^{p-n}\mu(\overline{B}_{r_j}))^{1/(p-1)} \le c\sum_{j=2}^{\infty}\int_{r_j}^{r_{j-1}}(s^{p-n}\mu(B_s))^{1/(p-1)}\,\frac{\mathrm{d}s}{s}$$
$$= c\int_0^{r_1}(s^{p-n}\mu(B_s))^{1/(p-1)}\,\frac{\mathrm{d}s}{s}.$$

Similarly,

$$r_{j-1}^{\frac{p-n}{p-1}} \ge c\int_{r_j}^{r_{j-1}} s^{\frac{p-n}{p-1}}\,\frac{\mathrm{d}s}{s}.$$

where $c = c(n, p, c_2)$ and we arrive at the second inequality of the claim.

Now we prove Theorem 4.1 and its corollary.

PROOF OF THEOREM 4.1. There is no loss of generality in assuming that $x_0 = 0$. Suppose that u is bounded on the sphere S_{r_0}. Using Lemma 4.5 we construct inductively a sequence r_j of radii such that for every $j = 0, 1, \dots$ we have

$$2 \le \frac{r_j}{r_{j+1}} \le 4$$

and

$$\sup_{S_{r_{j+1}}} u - \sup_{S_{r_j}} u \le c r_j^{p-n}\mu(\overline{B}_{r_j}))^{1/(p-1)}.$$

Since u is lower semicontinuous, we conclude in light of Lemma 4.6 that

$$u(0) \le \liminf_{x\to 0} u(x) \le \lim_j(\sup_{S_{r_j}} u)$$
$$\le \sup_{S_{r_0}} u + \sum_{j=0}^{\infty}(\sup_{S_{r_{j+1}}} u - \sup_{S_{r_j}} u)$$
$$\le \sup_{S_{r_0}} u + c\sum_{j=0}^{\infty} r_j^{p-n}(\mu(\overline{B}_{r_j}))^{1/(p-1)}$$
$$\le \sup_{S_{r_0}} u + c\mathbf{W}_{1,p}^{\mu}(0; r_0) + c(r_0^{p-n}\mu(\overline{B}_{r_0}))^{1/(p-1)},$$

and the theorem is proved.

PROOF OF THEOREM 4.2. We may assume that u is $\mathcal{A}$-superharmonic and nonnegative in a neighbourhood U of $\overline{B}(x_0, r_0)$ where $r_0 > r$. Let v be the balayage

$$v(x) = \hat{R}^u_{B(x_0,r)}(x; U).$$

Then $\mathbf{W}^{Tv}_{1,p}(x_0; r_0)$ is finite, since $\mathbf{W}^{\mu}_{1,p}(x_0; r)$ is. Hence because

$$\sup_{\partial B(x_0,r_0)} v < \infty,$$

Theorem 4.1 implies

$$u(x_0) = v(x_0) < \infty,$$

as desired.

REMARK 4.7. Our estimates (Theorems 3.1 and 4.1) can be used to give a partial answer to the following question posed by Peter Lindqvist: Given two equations of our type with mappings $\mathcal{A}$ and $\mathcal{A}^*$. Suppose that u is $\mathcal{A}$-superharmonic in Ω. Does there exist an $\mathcal{A}^*$-superharmonic function v such that

$$\{u = \infty\} = \{v = \infty\}?$$

The answer is affirmative at least if $p > n - 1$ and Ω is bounded. Indeed, let $\mu = Tu$ and let v be an $\mathcal{A}$-superharmonic solution to

$$-\operatorname{div} \mathcal{A}^*(x, \nabla v) = \mu$$

with weak zero boundary values. Then

$$\{u = \infty\} = \{x \colon \mathbf{W}^{\mu}_{1,p}(x; r_x) = \infty\} = \{v = \infty\}.$$

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