ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

M. A. HERRERO
J. J. L. VELÁZQUEZ

Generic behaviour of one-dimensional blow up patterns

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 19, nº 3 (1992), p. 381-450
<http://www.numdam.org/item?id=ASNSP_1992_4_19_3_381_0>

© Scuola Normale Superiore, Pisa, 1992, tous droits réservés.

L'accès aux archives de la revue « Annali della Scuola Normale Superiore di Pisa, Classe di Scienze » (http://www.sns.it/it/edizioni/riviste/annaliscienze/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Generic Behaviour of One-Dimensional Blow up Patterns

M.A. HERRERO - J.J.L. VELÁZQUEZ

1. - Introduction

This paper is concerned with the Cauchy problem

$$u_t - u_{xx} = u^p \qquad \text{when } x \in \mathbb{R},\ t > 0 \tag{1.1}$$

$$u(x,0) = u_0(x) \qquad \text{when } x \in \mathbb{R}, \tag{1.2}$$

where $p > 1$ and $u_0(x)$ is a continuous, nonnegative and bounded function. By standard results, (1.1)-(1.2) has then a unique positive classical solution $u(x,t)$ (the solution, for short, in what follows), which exists at least for small times. A remarkable (and well known) fact is that the solution may develop singularities in finite time, no matter how smooth $u_0(x)$ is. More precisely, we say that $u(x,t)$ blows up in a finite time T if $u(x,t)$ satisfies (1.1), (1.2) in $S_T = \mathbb{R} \times [0,T)$ and

$$\limsup_{t \to T} \left(\sup_{x \in \mathbb{R}} u(x,t)\right) = +\infty.$$

Notice that this definition does not preclude the possibility of $u(x,t)$ remaining bounded at any fixed $x \in \mathbb{R}$ as $t \to T$. A point $x_0 \in \mathbb{R}$ is called a blow up point if there exist sequences $\{x_n\}$, $\{t_n\}$ such that $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} t_n = T$, and $\lim_{n\to\infty} u(x_n,t_n) = +\infty$. The set of blow up points (if any) is then termed as the blow up set.

Since the seminal paper by Fujita ([Fu]), great attention has been devoted to determining when do solutions of (1.1), (1.2) exhibit blow up, and in such case, what are the asymptotics of solutions as the blow up time is approached. To mention but a few results, conditions ensuring the existence of a single blow up point have been obtained in [W], [MW] and [FM] for various boundary value problems associated to (1.1). It was then shown that, if $u_0(x)$ is not constant, under our current hypothesis the blow up set is bounded and consists

Partially supported by CICYT Grant PB90-0235 and EEC Contract SC1-0019C.
Pervenuto alla Redazione il 31 Maggio 1991 e in forma definitiva il 10 Febbraio 1992.

of isolated points (cf. [CM] — where boundary value problems in finite intervals are considered — as well as [HV3]; see also [V] for a discussion of the relation between local and global — in space — properties of blowing up solutions).

In view of these facts, it appears that a basic problem in the study of blowing up solutions $u(x,t)$ of (1.1) consists in describing the asymptotics of $u(x,t)$ as $t \to T$ and x tends to any blow up point. To describe all the possible cases which may appear, we need to introduce some notation. Following [GP], [GK1], we define self similar variables as follows. Let $\overline{x}$ be a blow up point of u. We then set

$$\begin{aligned} &u(x,t) = (T-t)^{-\frac{1}{p-1}}\Phi(y,\tau) \\ &\text{where } x - \overline{x} = y(T-t)^{1/2}, \quad \tau = -\log(T-t) \end{aligned} \tag{1.3}$$

so that in the new variables, Φ satisfies

$$\Phi_\tau = \Phi_{yy} - \frac{1}{2}y\Phi_y + \Phi + f_1(\Phi), \qquad \text{where } f_1(\Phi) = \Phi^p - \frac{p}{p-1}\Phi. \tag{1.4}$$

Notice that, for any $T > 0$, the function

$$u_T(x,t) = \big((p-1)(T-t)\big)^{-\frac{1}{p-1}} \tag{1.5}$$

is an explicit solution of (1.1) for $t < T$, which corresponds to $\overline{\Phi}(y,\tau) = \overline{\Phi}(y) = (p-1)^{-\frac{1}{p-1}}$ in the new variables. Let us linearize about $\overline{\Phi}$ by setting

$$\Phi(y,\tau) = (p-1)^{-\frac{1}{p-1}} + \psi(y,\tau). \tag{1.6}$$

Then ψ solves

$$\psi_\tau = \psi_{yy} - \frac{1}{2}y\psi_y + \psi + f(\psi) \equiv A\psi + f(\psi) \tag{1.7}$$

where $f(s) = \Big((p-1)^{-\frac{1}{p-1}} + s\Big)^p - (p-1)^{-\frac{p}{p-1}} - \dfrac{ps}{p-1}$.

For $1 \le q < +\infty$ and any integer $k \ge 1$, we now consider the weighted spaces

$$L^q_w(\mathbb{R}) = \left\{ g \in L^q_{\text{loc}}(\mathbb{R}) : \int_{\mathbb{R}} |g(s)|^q e^{-s^2/4}\mathrm{d}s < +\infty \right\},$$

$$H^k_w(\mathbb{R}) = \left\{ g \in L^2_{\text{loc}}(\mathbb{R}) : \text{for any integer } j \in [0,k],\ g^{(j)} \in L^2_{\text{loc}}(\mathbb{R}) \right.$$

$$\left. \text{and} \int_{\mathbb{R}} |g^{(j)}(s)|^2 e^{-s^2/4}\mathrm{d}s < +\infty \right\}.$$

It is readily seen that $L^2_{\mathrm{loc}}(\mathbb{R})$ (respectively $L^q_w(\mathbb{R})$, $1 \le q < +\infty$, $q \neq 2$) is a Hilbert space (respectively a Banach space), when endowed with the norm

$$\begin{aligned} \|g\|^2_{2,w} &\equiv \langle g, g\rangle = \int_{\mathbf{R}} g(s)^2 e^{-s^2/4} \mathrm{d}s, \\ \|g\|^q_{q,w} &= \int_{\mathbf{R}} |g(s)|^q e^{-s^2/4} \mathrm{d}s. \end{aligned} \tag{1.8}$$

On the other hand, for $k \ge 1$, $H^k_w(\mathbb{R})$ can be given a structure of Hilbert space in a straightforward way. Since the L^2_w-norm will be repeatedly used henceforth, we shall drop the subscripts $(2, w)$ in (1.8) from now on. It is natural to consider (1.7) as a dynamical system in $L^2_w(\mathbb{R})$, since the operator A is self-adjoint in $L^2_w(\mathbb{R})$ and has eigenvalues $\lambda_n = 1 - \frac{n}{2}$; $n = 0, 1, 2, \ldots$ with eigenfunctions $H_n(y)$ given by

$$H_n(y) = c_n \tilde{H}_n\left(\frac{y}{2}\right), \qquad \text{where } c_n = \left(2^{n/2}(4\pi)^{1/4}(n!)^{1/2}\right)^{-1} \tag{1.9}$$

and $\tilde{H}_n(y)$ is the standard n-th Hermite polynomial, so that $\|H_n\| = 1$ for any n.

We are now in a position to state some basic results concerning possible blow up behaviours for (1.1), (1.2) which have been proved in [HV1] and [HV2].

THEOREM A. *Assume that the solution $u(x,t)$ of* (1.1), (1.2) *blows up at $x = \bar{x}$, $t = T$. Then one of the following cases occurs*

(1.10a) $\Phi(y,\tau) = (p-1)^{-\frac{1}{p-1}}$ *for any $\tau > 0$, or*

(1.10b) $$\Phi(y,\tau) = (p-1)^{-\frac{1}{p-1}} - \frac{(4\pi)^{1/4}(p-1)^{-\frac{1}{p-1}}}{\sqrt{2}p} \cdot \frac{H_2(y)}{\tau} + o\left(\frac{1}{\tau}\right)$$

as $\tau \to \infty$, or

(1.10c) *There exist an even integer $m \ge 4$, and a real constant $C < 0$ such that*

$$\Phi(y,\tau) = (p-1)^{-\frac{1}{p-1}} + Ce^{(1-\frac{m}{2})\tau} H_m(y) + o\left(e^{(1-\frac{m}{2})\tau}\right)$$

as $\tau \to \infty$,

where convergence takes place in H^1_w as well as in $C^{k,\gamma}_{\mathrm{loc}}$ for any integer $k \ge 0$ and any $\gamma \in (0,1)$.

We point out that a related result, namely the fact that (1.10b) holds unless $\Phi(y,\tau)$ has exponential-type decay as $\tau \to \infty$, has been proved simultaneously and independently in [FK]. As to the asymptotics in larger regions, we have

THEOREM B. *Let $u(x,t)$, $\overline{x}$ and T be as in Theorem* A. *Then one of the following cases occurs*

(1.11a) *If* (1.10a) *holds, then* $u(x,t) = \big((p-1)(T-t)\big)^{-\frac{1}{p-1}}$

(1.11b) *If* (1.10b) *holds, then*

$$\lim_{t\uparrow T}\Big((T-t)^{\frac{1}{p-1}}u\big(\overline{x}+\xi((T-t)|\log(T-t)|)^{1/2},t\big)\Big)$$
$$= (p-1)^{-\frac{1}{p-1}}\left(1+\left(\frac{p-1}{4p}\right)\xi^2\right)^{-\frac{1}{p-1}},$$

uniformly on compact sets of ξ,

(1.11c) *If* (1.10c) *holds, then*

$$\lim_{t\uparrow T}\Big((T-t)^{\frac{1}{p-1}}u\big(\overline{x}+\xi(T-t)^{\frac{1}{m}},t\big)\Big) = (p-1)^{-\frac{1}{p-1}}(1+C\xi^m)^{-\frac{1}{p-1}},$$

uniformly on compact sets of ξ, *where* m *is an even number and* $C>0$.

As to the precedings of these results, it was already known that, under fairly general assumptions on the initial values,

$$\lim_{t\uparrow T} u\left(\overline{x}+y(T-t)^{1/2},t\right) = (p-1)^{-\frac{1}{p-1}} \tag{1.12}$$
$$\text{uniformly on sets } |y|\le R \text{ with } R>0,$$

provided that $u(x,t)$ blows up at $x=\overline{x}$, $t=T$; cf. [GP] for the one-dimensional case, and [GK1] for any space dimension $N\ge 1$ and $p<\dfrac{N+2}{N-2}$. See also [GK2], [GK3] for a broad discussion of blow up properties for the N-dimensional version of (1.1). Then it follows that, in the similarity variables described in (1.3), the asymptotic behaviour of solutions at blow up is uniform in the first approximation (and coincides with that of the self-similar solution (1.5)), whereas different behaviours are possible when second-order asymptotics are considered. The existence of the blow up patterns (1.10b), (1.10c) was formally derived (without proofs) in [GHV], by means of the method of matched asymptotic expansions. The case described by (1.10b) was far from being unexpected; it has been derived — without proof — in [HSS] for the particular case $p=3$. On the other hand, formal analysis was previously available for the related equation

$$u_t - u_{xx} = e^u \tag{1.13}$$

which plays an important role in combustion theory (cf. for instance [BE]). In [D], higher order asymptotics near a blow up point were derived for (1.13) which correspond to (1.10b) for (1.1). Recently, rigorous results were obtained in [B1], where it was proved that there exist solutions (of suitable boundary value problems) which behave exactly as conjected in [D]. This remarkable result, however, was obtained in a non-constructive way. In particular the question of determining whether a given set of initial and side conditions eventually leads to such behaviour remained open. A classification result analogous to Theorem A for (1.13) has been also obtained in [HV1], where we also proved that (1.10b) actually occurs whenever $u_0(x)$ has a single maximum and is symmetric with respect to some point; this symmetry assumption was later dispensed with in [HV2].

The reader will notice that no mention has been made yet in these remarks to the flatter behaviours corresponding to (1.10c). As far as we know, these have not been conjected in the literature prior to [GHV]. However, the fact that solutions with such behaviour exist has been proved in [HV2] for problem (1.1), (1.2) and the particular case $m = 4$, and in [HV4] for the corresponding problem for the combustion model (1.13). While the proof of such result is rather technical, the basic ideas beneath it are simple enough. Suppose that $u(x,t)$ blows up at $x = \overline{x}$, $t = T$. Then the blow up profile of u will depend on the number of maxima which reach $x = \overline{x}$ at $t = T$. If only a single maximum arrives there, we shall have (1.10b), whereas if two maxima exist for $t < T$ which collapse exactly at $x = \overline{x}$, $t = T$, we will obtain (1.10c) with $m = 4$ (since $H_m(y)$ has $\left(\frac{m}{2}\right)$ extrema for m even). This strongly suggests that (1.10c) will hold for $m = 2k$, $k > 1$, if we can prove that there exist solutions for which k maxima exist for $t < T$ and collapse at the same point at $t = T$: a question which remains open as yet.

In view of the previous remarks it seems reasonable to expect that (1.10c) should correspond to unstable behaviours, since circumstances leading to such patterns look indeed easier to be disturbed that the single-maximum situation yielding (1.10b), which can be thought of as a stable behaviour. This idea is made precise in our main result, which we now proceed to state. Let us denote by $C_0^+(\mathbb{R})$ the set of continuous, nonnegative and compactly supported functions. We then have

THEOREM. *Let $\tilde{u}_0(x) \in C_0^+(\mathbb{R})$, and let $\tilde{u}(x,t)$ be the corresponding solution of* (1.1), (1.2). *Assume that $\tilde{u}$ blows up at a time $T < +\infty$, and let $\tilde{x}$ be a blow-up point of $\tilde{u}$. Then for any $\varepsilon > 0$ there exists $u_0(x) \in C_0^+(\mathbb{R})$ such that $\left(\max_{x\in\mathbb{R}} |u_0(x) - \tilde{u}_0(x)|\right) \leq \varepsilon$, and the solution of* (1.1) *with initial value $u_0(x)$ blows up at a single point $\overline{x}$ such that* (1.10b) *holds and $\lim_{\varepsilon\to 0} |\overline{x} - x| = 0$. Moreover, the behaviour corresponding to* (1.10b) *is stable under small perturbations in the L^∞-norm of the initial value $\tilde{u}_0$.*

In an informal way, this Theorem can be restated as saying that

If $u_0 \in C_0^+(\mathbb{R})$, blow up consists generically in a single point blow up with the behaviour described in (1.10b).

Actually, the assumption of $u_0(x)$ being compactly supported is unnecessarily restrictive. Our arguments apply provided that the blow up set is bounded, and by the results in [GK3] this holds whenever $u_0(x)$ decays fast enough as $|x| \to \infty$. Since blow up points are isolated by the results of [CM] and [HV3], it follows that, once the blow up set is bounded, there is a finite number of blow up points. As a matter of fact, it has been recently proved in [M] that there exist solutions of (1.1) which blow up at any given number of points $x_1, \dots, x_k$. Previously, in [B2] it was proved that there exist solutions of the higher-dimensional version of (1.13) which blow up at a given point with the behaviour corresponding to (1.10b) for such equation as in [B1], the author proves that his result is stable under small perturbations in the class of data he is considering. Both papers [M] and [B2] deal with boundary value problems. On the other hand, the fact that single-point blow up of type (1.10b) for equations (1.1), (1.13) is stable under small perturbations follows readily also by our results in [HV2]. Therefore, the main novelty herein consists in showing the instability of the behaviours corresponding to (1.10c), in the sense made precise in the statement of the Theorem.

In the course of proving the Theorem, however, a number of results of independent interest are obtained. We shall mention next a few of them. Here and henceforth, we shall freely use the customary asymptotic notations $o(\cdot)$, $O(\cdot)$, $\sim$, $\ll$, etc.

1) It is shown in Section 2 that solutions satisfying (1.10c) actually possess $\left(\frac{m}{2}\right)$ maxima for t close to T. Moreover, if $x(t)$ is a local maximum of $u(x,t)$, and $u(x(t),t) < ((p-1)(T-t))^{-\frac{1}{p-1}}$ at some $t < T$, then this maximum dissapears before blow up occurs. Furthermore, local maxima remaining until blow up are confined for $t \sim T$ in rather narrow parabolas near $x = \overline{x}$ (of type $|x - \overline{x}| \le C(T-t)^{1/2}$).

2) In Section 3 we improve the asymptotic estimates obtained for Φ in [HV1], [HV2]. Essentially, we improve the error bounds obtained in such papers, and in doing so we obtain optimal estimates, which coincide with those formally derived in [GHV]. We also analyze there an associated linear problem (cf. (3.19)), which is of the type

$$u_t = u_{xx} + V(x,t)u \tag{1.14}$$

where $V(x,t) \to \infty$ as $(x,t) \to (\overline{x},T)$ is a suitable way. Therefore the potential $V(x,t)$ becomes singular as blow up is approached.

3) In Section 4 perturbative expansions on the initial values are obtained which remain valid for times quite close to the blow up time. Results in this and the following Section are perhaps the more delicate ones in the paper.

4) Finally, in Section 5 we obtain perturbative estimates on the variation of the blow up time and the blow up region with respect to changes in the initial values. We then show that all behaviours can be altered by small perturbations on $u_0(x)$, except that corresponding to (1.10b).

We conclude this Introduction by sketching briefly the main ideas in the proof of the Theorem. We start from a solution $\tilde{u}(x,t)$ of (1.1) with initial value $\tilde{u}_0(x)$ satisfying the assumptions of the Theorem, which is assumed to blow up at $t = T < +\infty$ at points $x_1, \dots, x_k$. We then consider initial values

$$u_0(x) \equiv u_{0,\varepsilon}(x) = \tilde{u}_0(x) + \varepsilon \tilde{R}_0(x), \qquad 0 < \varepsilon < 1,$$

where $\tilde{R}_0(x)$ is some fixed function and ε is a small parameter. A formal asymptotic expansion yields then

$$u_\varepsilon(x,t) = \tilde{u}(x,t) + \varepsilon \tilde{R}(x,t) + \cdots \tag{1.15}$$

where

$$\begin{aligned} &\tilde{R}_t = \tilde{R}_{xx} + p\tilde{u}^{p-1}\tilde{R}, \qquad x \in \mathbb{R},\ t > 0, \\ &\tilde{R}(x,0) = \tilde{R}_0(x). \end{aligned}$$

This is the linear singular problem just mentioned above. We shall see that

$$\begin{aligned} &\tilde{R}(x,t) \sim \alpha_i (T-t)^{-\frac{p}{p-1}}, && \text{uniformly on sets,} \\ &|x - x_i| \le C(T-t)^{1/2}, && \text{for some real constants } \alpha_1, \dots, \alpha_k. \end{aligned} \tag{1.16}$$

Since $\tilde{u}(x,t) \sim ((p-1)(T-t))^{-\frac{1}{p-1}}$, one may then expect (1.15) to lose its validity when $\tilde{u}(x,t) \sim \varepsilon \tilde{R}(x,t)$, i.e., for times $T - t \sim \varepsilon$. As a matter of fact, we shall show (and this is one of the key points in the paper) that there is a common region of validity for the expansions (1.12), (1.15) where we have

$$u_\varepsilon(x,t) \sim ((p-1)(T_\varepsilon - t))^{-\frac{1}{p-1}}, \tag{1.17a}$$

$$u_\varepsilon(x,t) \sim ((p-1)(T-t))^{-\frac{1}{p-1}} + \varepsilon \alpha_i (T-t)^{-\frac{p}{p-1}} + \cdots, \tag{1.17b}$$

when $0 < \varepsilon \ll 1$, and this enables us to estimate the change in blow up time $\Delta T_\varepsilon = T_\varepsilon - T$. By selecting then α_i in a suitable way (which is shown to be done by means of an adequate choice of $\tilde{R}_0(x)$), we will then confine the blow up region to a small neighbourhood of any of the points $x_1, \dots, x_k$, say $x_1 = 0$. We just arrange things for blow up originated by points $x_2, \dots, x_k$ to occur later than T_ε, while at the same time reducing the number of local maxima arriving at blow up. A repeated application of the previous argument eventually leads to the situation where there is a single point blow-up, say at $x_1 = 0$, with perhaps many maxima collapsing there.

As a next step, we show that a further refinement in the choice of $\tilde{R}_0(x)$ leads to the expansion

$$\tilde{R}(x,t) \sim \alpha_1 x(T-t)^{-\frac{p}{p-1}} \qquad \text{for } |x| \le C(T-t)^{1/2}$$

and $t \sim T$. Then, if blow up of type (1.10c) occurs, we would have

$$\begin{aligned} u_\varepsilon(x,t) = \big((p-1)(T-t)\big)^{-\frac{1}{p-1}} - C(T-t)^{-\frac{1}{p-1}+\frac{m}{2}} H_m\left(\frac{x}{(T-t)^{1/2}}\right) \\ + \varepsilon\alpha_1 x(T-t)^{-\frac{p}{p-1}} + \cdots, \end{aligned} \tag{1.18}$$

in the common region of validity of expansions (1.17). Comparing the second and third terms on the right in (1.18), we see that, if $(T-t)^{\frac{m}{2}+1} \gg \varepsilon$, the structure corresponding to (1.10c) prevails, but if $(T-t)^{\frac{m}{2}+1} \gg \varepsilon$, the third term there dominates the second one.

Therefore, maxima located to the left of $x_1 = 0$ must fall below to level $\big((p-1)(T-t)\big)^{-\frac{1}{p-1}}$ as $\varepsilon \downarrow 0$, and they are thus unable to last until the blow up time. A repetition of this argument obliterates some maxima at any time, until we are left with the situation consisting of a single maximum arriving to $x = 0$, $t = T$. This is the stable case corresponding to (1.10b).

2. - The number of maxima near a blow up point

In this Section we shall prove the following result.

PROPOSITION 2.1. *Let $u_0(x)$ be continuous, nonnegative and bounded, and let $u(x,t)$ be the solution of* (1.1) - (1.2). *Assume that $u(x,t)$ blows up at $x = \overline{x}$, $t = T < +\infty$, and let $\Phi(y,\tau)$ be the function defined in* (1.3). *Then there exist $\delta > 0$ and $\eta > 0$ such that*

(2.1a) *If $\Phi(y,\tau)$ behaves as in* (1.10b), *$u(\cdot,t)$ has a single maximum in $[-\eta,\eta]$ for any $t \in (T-\delta, T)$,*

(2.1b) *If $\Phi(y,\tau)$ behaves as in* (1.10c), *$u(\cdot,t)$ has exactly $\left(\frac{m}{2}\right)$ maxima in $[-\eta,\eta]$ for any $t \in (T-\delta, T)$.*

We shall divide the proof of Proposition 2.1 into a number of steps. To begin with, we consider the following auxiliary functions (cf. [HV1], Section 6):

$$W(y,\tau) = \Phi^{-(p-1)}, \qquad G(y,\tau) = W(y,\tau) - (p-1) \tag{2.2}$$

so that G satisfies

$$G_\tau = G_{yy} - \frac{1}{2} yG_y + G - \left(\frac{p}{p-1}\right) \cdot \frac{G_y^2}{(p-1)+G} \equiv G_{yy} - \frac{1}{2} yG_y + G + L(y,\tau). \tag{2.3}$$

Assume first that (1.10b) holds, and let $\lambda(\tau) \in C([-\log T, +\infty))$ be such that

$$\lim_{\tau\to\infty} |\lambda(\tau)| = +\infty \tag{2.4a}$$

$$|\lambda(\tau)| \le \xi_0\sqrt{\tau} \qquad \text{for some } \xi_0 > 0 \text{ and } \tau > 0 \text{ large enough.} \tag{2.4b}$$

We then have

LEMMA 2.2. *Let $\lambda(\tau)$ be as in* (2.4). *Then there holds*

$$\lim_{\tau\to\infty} \left(\frac{\lambda(\tau)^2}{\tau}\right)^{-1} \left(G(\lambda(\tau),\tau) - \frac{(p-1)^2}{4p} \cdot \frac{\lambda(\tau)^2}{\tau}\right) = 0. \tag{2.5}$$

PROOF. As in [HV1], we use variation of constants formula in (2.3) to obtain that, whenever $-\log T < \tau_0 < \tau < +\infty$,

$$\begin{aligned} G(y,\tau) &= \frac{e^{\tau-\tau_0}}{\left(4\pi(1-e^{-(\tau-\tau_0)})\right)^{1/2}} \int_{\mathbb{R}} \exp\left(-\frac{\left(ye^{-(\tau-\tau_0)/2} - r\right)^2}{4\left(1-e^{-(\tau-\tau_0)}\right)}\right) G(r,\tau_0)\mathrm{d}r \\ &\quad - \int_{\tau_0}^{\tau} \frac{e^{\tau-s}\mathrm{d}s}{\left(4\pi(1-e^{-(\tau-s)})\right)^{1/2}} \int_{\mathbb{R}} \exp\left(-\frac{\left(ye^{-(\tau-s)/2} - r\right)^2}{\left(4(1-e^{-(\tau-s)})\right)}\right) L(r,s)\mathrm{d}r \\ &\equiv G_1(y,\tau) + G_2(y,\tau). \end{aligned} \tag{2.6}$$

For large enough τ, we then set

$$\tau_0 \equiv \tau_0(\tau) = \tau - \log(\lambda(\tau)^2). \tag{2.7}$$

By (2.4), we readily see that $\lim_{\tau\to\infty} \tau_0(\tau) = +\infty$ and $\tau_0 < \tau$ for sufficiently large τ. We now fix $R > 0$, and split $G_1(y,\tau)$ in (2.6) as follows

$$\begin{aligned} G_1(y,\tau) &= \frac{e^{\tau-\tau_0}}{\left(4\pi(1-e^{-(\tau-\tau_0)})\right)^{1/2}} \int_{|r|\le R} \exp\left(-\frac{\left(ye^{-(\tau-\tau_0)/2} - r\right)^2}{4(1-e^{-(\tau-\tau_0)})}\right) G(r,\tau_0)\mathrm{d}r \\ &\quad + \frac{e^{\tau-\tau_0}}{\left(4\pi(1-e^{-(\tau-\tau_0)})\right)^{1/2}} \int_{|r|\ge R} \exp\left(-\frac{\left(ye^{-(\tau-\tau_0)/2} - r\right)^2}{4(1-e^{-(\tau-\tau_0)})}\right) G(r,\tau_0)\mathrm{d}r \\ &\equiv G_{1,1}^R(y,\tau) + G_{1,2}^R(y,\tau). \end{aligned}$$

Let us denote henceforth by C a generic constant. Using Cauchy-Schwarz inequality, and recalling that

$$\|G(\cdot,\tau)\| \le \frac{C}{\tau} \qquad \text{as } C \to \infty$$

(cf. [HV1], Lemma 6.3), we see that

$$\begin{aligned}|G^R_{1,2}(\lambda(\tau),\tau)| &\le \frac{e^{\tau-\tau_0}\|G(\cdot,\tau_0)\|}{\left(4\pi(1-e^{-(\tau-\tau_0)})\right)^{1/2}}\left(\int_{|r|\ge R}\exp\left(\frac{r^2}{4}-\frac{(r-1)^2}{2(1-e^{-(\tau-\tau_0)})}\right)\mathrm{d}r\right)^{\frac{1}{2}}\\ &\le \frac{\lambda(\tau)^2}{\tau_0}\,h(R),\end{aligned}$$

where $h(R)\to 0$ as $R\to\infty$. Since $\dfrac{1}{\tau_0}\le\dfrac{C}{\tau}$ for large τ, we arrive at

$$|G^R_{1,2}(\lambda(\tau),\tau)| \le \frac{C}{\tau}\,\lambda(\tau)^2 h(R) \qquad \text{for large enough } \tau, \tag{2.8}$$

where $h(R)=o(1)$ as $R\to\infty$. On the other hand, it follows from (1.10b) and (2.2) that, if $|y|\le R$,

$$G(y,\tau)=\frac{b}{\tau}\,H_2(y)+g_R(y;\tau) \qquad \text{for } \tau \text{ large enough,} \tag{2.9}$$

where

$$b=\frac{(4\pi)^{1/4}(p-1)^{-\frac{1}{p-1}}}{\sqrt{2}p} \text{ and } g_R(y;\tau)=o\left(\frac{1}{\tau}\right) \text{ as } \tau\to\infty \text{ for any fixed } R.$$

We now substitute (2.9) into the expression for $G^R_{1,1}$ to obtain that $G^R_{1,1}(y,\tau)=Z_1(y,\tau)+Z_2(y,\tau)$, where $\lim\limits_{\tau\to\infty}\left(\dfrac{\lambda(\tau)^2}{\tau}\right)^{-1} Z_2(\lambda(\tau),\tau)=0$ and

$$\lim_{\tau\to\infty}\left(\frac{\lambda(\tau)^2}{\tau}\right)^{-1} G^R_{1,1}(\lambda(\tau),\tau)=\frac{b}{(4\pi)^{1/2}}\int_{|r|\le R} H_2(r)\exp\left(-\frac{(r-1)^2}{4}\right)\mathrm{d}r.$$

Since

$$\exp\left(\frac{2r-1}{4}\right)=\sum_{n=0}^{\infty}\frac{H_n(r)}{2^n\cdot n!\cdot c_n}, \qquad \text{with } c_n=\left(2^{n/2}(4\pi)^{1/4}(n^1)^{1/2}\right)^{-1},$$

we finally obtain, after letting $R\to\infty$,

$$\lim_{\tau\to\infty}\left(\frac{\lambda(\tau)^2}{\tau}\right)^{-1} G_{1,1}(\lambda(\tau),\tau)=\frac{(p-1)^2}{4p}. \tag{2.10}$$

We have yet to bound $G_2(\lambda(\tau),\tau)$. To this end, we proceed as in [HV1, Lemma 6.5]. Namely, we take $A>0$ such that $\tau_0+A<\tau$, $R>0$ arbitrary and δ large enough, to be selected presently. We then split G_2 as follows

$$
\begin{aligned}
G_2 = &\int_{\tau_0}^{\tau_0+A}\int_{R\le|r|\le\delta\sqrt{\tau_0}}(\cdots)+\int_{\tau_0}^{\tau_0+A}\int_{|r|\le R}(\cdots)+\int_{\tau_0+A}^{\tau}\int_{|r|\le\delta\sqrt{\tau_0}}(\cdots)\\
&+\int_{\tau_0}^{\tau}\int_{|r|\ge\delta\sqrt{\tau_0}}(\cdots)\equiv G_{2,1}+G_{2,2}+G_{2,3}+G_{2,4}.
\end{aligned}
\tag{2.11}
$$

We now recall that by the results in [HV1] (cfr. Lemmata 6.1 and 6.2 there) one has

$$
|L(y,\tau)|\le\frac{C}{\tau},\qquad \text{uniformly on sets } |y|\le C\sqrt{\tau} \text{ for large enough } \tau.
$$

whence

$$
|G_{2,3}(\lambda(\tau),\tau)|\le\frac{C}{\tau_0}\,e^{\tau-(\tau_0+A)}\le C\,\frac{\lambda(\tau)^2}{\tau}\,e^{-A}. \tag{2.12}
$$

On the other hand, $|\lambda(\tau)e^{-(\tau-s)/2}|\le e^{A/2}|\lambda(\tau)e^{-(\tau-\tau_0)/2}|=\xi_0e^{A/2}$, so that

$$
\begin{aligned}
&|G_{2,1}(\lambda(\tau),\tau)|\le C\,\frac{\lambda(\tau)^2}{\tau}\,g(A,R),\\
&\text{where } g(A,R)\to 0 \text{ as } R\to\infty \text{ for any fixed } A>0.
\end{aligned}
\tag{2.13}
$$

When $|r|\le R$, we make use of (1.10b) to obtain that $|L(y,\tau)|\le\dfrac{K_1}{\tau^2}$, where $K_1=K_1(R)$. It then follows that

$$
|G_{2,2}(\lambda(\tau),\tau)|\le\frac{K_1}{\tau_0^2}\int_{\tau_0}^{\tau_0+A}e^{\tau-s}ds\le K_2\,\frac{(\lambda(\tau))^2}{\tau^2}, \tag{2.14}
$$

where $K_2=K_2(A,R)$. Finally, to bound $G_{2,4}$ we argue as in [HV1, Lemma 6.5]. To this end, we observe that $L(y,\tau)=-\dfrac{p}{p-1}\,(u(x,t))^{-(1+p)}(u_x(x,t))^2$ and remark that

i) $u(x,t)$ is supercaloric, so that $u(x,t)\ge Ce^{-\theta x^2}$ for some $\theta>0$ whenever $x\in\mathbb{R}$ and $t\ge\delta>0$, and

ii) Proposition 1 in [GK1] yields the estimate $|u_x(x,t)|\le C(T-t)^{-\frac{1}{p-1}-\frac{1}{2}}$.

Altogether, these bounds give

$$
|L(y,\tau)|\le C\,\exp(\alpha\tau)\,\exp(\theta y^2e^{-\tau}),
$$

for some positive constants C, α and θ. We then readily see that

$$|G_{2,4}(\lambda(\tau),\tau)| \le C\lambda(\tau)^2 \exp(\alpha\tau - \beta\delta^2\tau_0)$$

and therefore, if $\delta > 0$ is large enough,

$$|G_{2,4}(\lambda(\tau),\tau)| = o\left(\frac{\lambda(\tau)^2}{\tau}\right) \qquad \text{as } \tau \to \infty. \tag{2.15}$$

Letting now $\tau \to \infty$, $R \to \infty$ and $A \to \infty$ (in this order), we obtain from (2.11)–(2.18) that

$$\lim_{\tau\to\infty} \left(\frac{\lambda(\tau)^2}{\tau}\right)^{-1} G_2(\lambda(\tau),\tau) = 0$$

and the proof is concluded. ∎

We next set out to derive the analogue of Lemma 2.2 for the case where (1.10c) holds. To this end, we first obtain the following gradient bound.

LEMMA 2.3. *Assume that* (1.10c) *holds, and let* $\xi_0 > 0$ *be fixed. We then have*

$$\begin{aligned} &|\Phi_y(y,\tau)| \le C|y|^{m-1}e^{(1-\frac{m}{2})\tau}, \\ &\textit{whenever } 2 \le |y| \le \xi_0 \exp\left(\left(\frac{1}{2}-\frac{1}{m}\right)\tau\right) \textit{ and } \tau \textit{ is large enough.} \end{aligned} \tag{2.16}$$

PROOF. Set $z = |\Phi_y|$. Arguing as in Lemma 2.6 in [HV2] we obtain that

$$z_\tau \le z_{yy} - \frac{yz_y}{2} + \frac{z}{2} + \frac{Cz}{\tau}, \qquad \text{for some } C > 0.$$

Using variation of constants formula in this inequality gives

$$z(y,\tau) \le \frac{\exp\left(\dfrac{\tau-\tau_0}{2} + C\log\dfrac{\tau}{\tau_0}\right)}{\left(4\pi(1-e^{-(\tau-\tau_0)})\right)^{1/2}} \int_{\mathbf{R}} \exp\left(-\frac{(ye^{-(\tau-\tau_0)/2}-r)^2}{4(1-e^{-(\tau-\tau_0)})}\right) z(r,\tau_0)\mathrm{d}r.$$

For any pair τ, y, in the set where $2 \le |y| \le \xi_0 \exp\left(\left(\frac{1}{2}-\frac{1}{m}\right)\tau\right)$, we now define

$$\tau_0 = \tau - 2\log|y|$$

so that

$$2\left(\frac{\tau}{m} - \log\xi_0\right) < \tau_0 < \tau.$$

For large enough τ, we have that $\tau_0 > -\log T$ and $\left(\frac{\tau}{\tau_0}\right)$ is uniformly bounded. Hence, recalling (1.10c)

$$z(y,\tau) \le \frac{C\exp\left(\frac{\tau-\tau_0}{2}\right)}{\left(4\pi(1-e^{-(\tau-\tau_0)})\right)^{1/2}} \left(\int_{\mathbb{R}} z(r,\tau_0)^2 e^{-r^2/4}\mathrm{d}r\right)^{1/2}$$

$$\cdot \left(\int_{\mathbb{R}} \exp\left(-\frac{(r-1)^2}{2(1-e^{-(\tau-\tau_0)})} + \frac{r^2}{4}\right)\mathrm{d}r\right)^{1/2}$$

$$\le C\exp\left(\frac{\tau-\tau_0}{2} + \left(1-\frac{m}{2}\right)\tau_0\right) \le C|y|^{m-1}e^{(1-\frac{m}{2})\tau}. \qquad \blacksquare$$

Let now $\mu(\tau)$ be such that $\mu(\tau) \in C([-\log T, \infty))$, and

(2.17a) $\quad \lim_{\tau\to\infty} |\mu(\tau)| = +\infty,$

(2.17b) $\quad |\mu(\tau)| \le \xi_0 e^{\left(\frac{1}{2}-\frac{1}{m}\right)\tau}$ for some $\xi_0 > 0$ and $\tau > 0$ large enough.

We prove

LEMMA 2.4. *Let $\mu(\tau)$ be as above, and assume that* (1.10c) *is satisfied. Then there holds*

$$G(\mu(\tau),\tau) = C(p-1)c_m(\mu(\tau))^m e^{(1-\frac{m}{2})\tau} + O\left((\mu(\tau))^m e^{(1-\frac{m}{2})\tau}\right) \tag{2.18}$$

as $\tau \to \infty$, where C, c_m are as in (1.10c).

PROOF. It proceeds along the lines of that in Lemma 2.2. We therefore shall sketch it, to stress only those points where some novelties appear. We start again from (2.6), and for large enough $\tau > 0$ we define τ_0 by means of the equation $\frac{\tau-\tau_0}{m} = \log\mu(\tau)$ there. We then split $G_1 = G_{1,1}^R + G_{1,2}^R$ just as before. The bound for $G_{1,2}^R$ reads now

$$|G_{1,2}^R(\mu(\tau),\tau)| \le C(\mu(\tau))^m e^{(1-\frac{m}{2})\tau} g(R), \qquad \text{with } g(R) = o(1) \text{ as } R \to \infty.$$

As a next step, we use (1.10c) to notice that, if $|y| \le R$,

$$\begin{aligned} &G(y,\tau) = C(p-1)e^{(1-\frac{m}{2})\tau} + g_R(y,\tau), \\ &\text{where } g_R(y,\tau) = o\left(e^{(1-\frac{m}{2})\tau}\right) \text{ as } \tau \to \infty \text{ for any fixed } R > 0. \end{aligned} \tag{2.19}$$

We take advantage of (2.19) to estimate $G_{1,1}^R$ as before, and we eventually obtain

$$G_1(\mu(\tau),\tau) = C(p-1)c_m\mu(\tau)^m e^{(1-\frac{m}{2})\tau} + o\big(\mu(\tau)^m e^{(1-\frac{m}{2})\tau}\big), \qquad \text{as } \tau \to \infty.$$

Let us split now G_2 as follow

$$G_2 = \int_{\tau_0}^{\tau}\int_{\Sigma_1}(\cdots) + \int_{\tau_0}^{\tau}\int_{\Sigma_2}(\cdots) \equiv G_{2,1} + G_{2,2},$$

where $\Sigma_1 = \left\{r \in \mathbb{R} : |r| \geq 2\xi_0 \exp\left(\left(\frac{1}{2} - \frac{1}{m}\right)\tau\right)\right\}$ and $\Sigma_2 = \mathbb{R}\backslash\Sigma_1$. To bound $G_{2,1}$, we use the fact that $|ye^{-(\tau-s)/2}| \leq \xi_0 e^{\left(\frac{1}{2}-\frac{1}{m}\right)s}$ to obtain that, for some positive constants θ and β,

$$|G_{2,1}(\mu(\tau),\tau)| \leq C\exp(-\theta e^{\beta\tau}) = o\big(\mu(\tau)^m e^{(1-\frac{m}{2})\tau}\big) \qquad \text{as } \tau \to \infty.$$

As to $G_{2,2}$, we notice that, by (2.16) and the lower bound for Φ derived in [HV2, Lemma 2.1],

$$|L(r,\tau)| \leq C(1 + r^{2(m-1)})e^{2(1-\frac{m}{2})\tau},$$

$$\text{whenever } |r| \leq 2\xi_0 \exp\left(\left(\frac{1}{2} - \frac{1}{m}\right)\tau\right).$$

On the other hand

$$\begin{aligned} I &\equiv \int_{\mathbb{R}} \left(1 + r^{2(m-1)}\right) \exp\left(-\frac{(r - \mu(\tau)e^{-(\tau-s)/2})^2}{4(1-e^{-(\tau-s)})}\right) \mathrm{d}r \\ &\leq C\int_{\mathbb{R}} \Big(1 + \left|r - \mu(r)e^{-(\tau-s)/2}\right|^{2(m-1)} \\ &\qquad + \left|\mu(\tau)e^{-(\tau-s)/2}\right|^{2(m-1)}\Big) \exp\left(-\frac{(r - \mu(\tau)e^{-(\tau-s)/2})^2}{4(1-e^{-(\tau-s)})}\right) \mathrm{d}r \end{aligned}$$

and, since $x^{2(m-1)}e^{-x^2/\alpha} \to 0$ as $x \to \infty$ for any fixed $\alpha > 0$, we arrive at

$$I \leq C\big(1 + \mu(\tau)^{2(m-1)}e^{-(m-1)(\tau-s)}\big),$$

whence

$$\begin{aligned} &|G_{2,2}(\mu(\tau),\tau)| \\ &\leq C\int_{\tau_0}^{\tau} \exp((\tau-s) + (2-m)s)(1 + \mu(\tau)^{2(m-1)}\exp(-(m-1)(\tau-s)))\mathrm{d}s \\ &\leq C\big(\mu(\tau)e^{-\tau}\big)^{m-2} \end{aligned}$$

and, since $(\mu(\tau)e^{-\tau})^{m-2} \le Ce^{\left(-\frac{m}{2}+\frac{2}{m}\right)\tau}$, we finally obtain

$$|G_{2,2}(\mu(\tau),\tau)| \le C\mu(\tau)^m e^{(1-\frac{m}{2})\tau}\tau e^{\left(1-\frac{2}{m}\right)\tau}$$

so that $|G_{2,2}(\mu(\tau),\tau)| = o\left(\mu(\tau)^m e^{(1-\frac{m}{2})\tau}\right)$ as $\tau \to \infty$, and the result follows. ∎

Let $\delta > 0$ and $\xi_0 > 0$, and let m be a positive integer, $m \ge 4$. We now set

$$Q(\xi_0) = \{(x,t) : |x| \ge \xi_0(T-t)^{1/2}|\log(T-t)|^{1/2},\ 0 < t < T\}, \tag{2.20a}$$

$$\begin{cases} L(\xi_0,t) = \{x : (x,t) \in Q(\xi_0)\}, \\ L_\delta \equiv L_\delta(\xi_0,t) = \{x : x \in L(\xi_0,t),\ |x| < \delta\}, \end{cases} \tag{2.20b}$$

$$Q_m(\xi_0) = \{(x,t) : |x| \ge \xi_0(T-t)^{\frac{1}{m}},\ 0 < t < T\}, \tag{2.20c}$$

$$\begin{cases} L_m(\xi_0,t) = \{x : (x,t) \in Q_m(\xi_0)\}, \\ L_{m,\delta} \equiv L_{n,\delta}(\xi_0,t) = \{x : x \in L_m(\xi_0,t),\ |x| < \delta\}. \end{cases} \tag{2.20d}$$

For simplicity, we shall normalize the blow up point by setting $\overline{x} = 0$ and prove

LEMMA 2.5. a) *Assume that* (1.10b) *holds, and let* $\xi_0 > 0$ *be fixed, but otherwise arbitrary. Then there exists* $\delta > 0$ *such that, if* t *is close enough to* T, *we have*

$$\sup_{L_\delta} u(x,t) < ((p-1)(T-t))^{-\frac{1}{p-1}}, \tag{2.21}$$

$$\lim_{\substack{(x,t)\to(0,T)\\(x,t)\in Q(\xi_0)}} u(x,t) = +\infty. \tag{2.22}$$

b) *Suppose now that* (1.10c) *is satisfied, and let* ξ_0, δ *and* t *be as in part* a). *Then* (2.21), (2.22) *hold true with* L_δ *and* $Q(\xi_0)$ *replaced by* $L_{m,\delta}$ *and* $Q_m(\xi_0)$ *respectively.*

PROOF. We begin by part a). As in [HV3], for $s \in (0,T)$ we consider the auxiliary function

$$v_s(x,t) = (T-s)^{\frac{1}{p-1}} u\Big(\lambda(s) + x(T-s)^{1/2},\ s + (T-s)t\Big),$$

where

$$\lambda(s) = \xi_0(T-s)^{1/2}|\log(T-s)|^{1/2}.$$

It has been shown in [HV3] that

$$v_s(x,t) = (p-1)^{-\frac{1}{p-1}}\left((1-t) + \left(\frac{p-1}{4p}\right)\xi_0^2\right)^{-\frac{1}{p-1}} + o(1) \tag{2.23}$$

as $s \uparrow T$, uniformly on compact sets of $\mathbb{R} \times [0,1]$.

Set now $x=0$, $\tilde{x}=\lambda(s)$, $\tilde{t}=s+t(T-s)$, and take $s_0\in(0,T)$. When (s,t) moves within the cylinder $[s_0,T]\times[0,1]$, $(\tilde{x},\tilde{t})$ varies over the set $Q(\xi_0)\cap\{t\in[s_0,T],$ $|x|\le\xi_0(T-s_0)^{1/2}|\log(T-s_0)|^{1/2}\}$. Furthermore, since

$$(p-1)^{-\frac{1}{p-1}}\left((1-t)+\left(\frac{p-1}{4p}\right)\xi_0^2\right)^{-\frac{1}{p-1}}<((p-1)(1-t))^{-\frac{1}{p-1}}+\eta_0,$$

for some $\eta_0=\eta_0(\xi_0)>0$, uniformly on $t\in[0,1]$, it follows from (2.23) that, for some $s_0=s_0(\xi_0)\in(0,T)$,

$$v_s(0,t)<((p-1)(1-t))^{-\frac{1}{p-1}},\qquad \text{for } s\ge s_0,$$

which in turn yields (2.21) with $\delta=\xi_0(T-s_0)^{1/2}|\log(T-s_0)|^{1/2}$. To obtain (2.22), we just remark that, by (2.23),

$$\begin{aligned}v_s(0,t)&\ge(p-1)^{-\frac{1}{p-1}}\left(\left(\frac{p-1}{4p}\right)\xi_0^2\right)^{-\frac{1}{p-1}}+o(1)\\&\ge\frac{(p-1)^{-\frac{1}{p-1}}}{2}\left(\left(\frac{p-1}{4p}\right)\xi_0^2\right)^{-\frac{1}{p-1}},\end{aligned}$$

for s close enough to T. Recalling the definition of $v_s(x,t)$, the result follows at once, since $\lambda(s)\to 0$ (whence $s\to T$) whenever $\tilde{x}\to 0$.

Finally, the proof of part b) is entirely similar. Instead of $v_s(x,t)$, we consider the auxiliary function

$$z_s(x,t)=(T-s)^{\frac{1}{p-1}}u\left(\xi_0(T-s)^{\frac{1}{m}}+x(T-s)^{\frac{1}{2}},\ s+t(T-s)\right)$$

and replace (2.23) by

$$z_s(x,t)=(p-1)^{-\frac{1}{p-1}}((1-t)+C\xi_0^m)^{-\frac{1}{p-1}}+o(1),$$

for some $C>0$ as $s\uparrow T$, uniformly on compact sets of $\mathbb{R}\times[0,1]$ (cf. [HV3]). We shall omit further details. ∎

End of the proof of Proposition 2.1.

Assume first that (1.10b) holds. Then the result is a consequence of the following fact

(2.24) There exists $R>0$ such that, for t close enough to T, every maximum of $u(x,t)$ lies in the interval

$$I=\{x:|x|\le R(T-t)^{1/2}\}.$$

To obtain (2.24), we argue as follows. By the analyticity of $u(\cdot, t)$ for any fixed t, there exists at most a countable number of local maxima of $u(\cdot, t)$, $\{\Gamma_j(t)\}$. Moreover, these maxima move smoothly except, perhaps, at points where they collapse. To avoid relabelling, we will assume that two branches are denoted by the same symbol after collapsing. Suppose now that (2.24) does not hold. Then there exist a curve of maxima $\Gamma(t)$ such that $\lim_{t\to T} \Gamma(t) = 0$, and a sequence of times $\{t_n\}$ such that $\lim_{n\to\infty} t_n = T$, and

$$|\Gamma(t_n)| \geq n(T - t_n)^{1/2}. \tag{2.25}$$

We then claim that

$$u(\Gamma(t_n), t_n) < ((p-1)(T - t_n))^{-\frac{1}{p-1}}. \tag{2.26}$$

Indeed, if (2.25) holds, for any fixed $\xi_0 > 0$ we may select a subsequence (also labelled by $\{t_n\}$) such that, either

$$|\Gamma(t_n)| \geq \xi_0 (T - t_n)^{1/2} |\log(T - t_n)|^{1/2} \tag{2.27a}$$

or

$$|\Gamma(t_n)| < \xi_0 (T - t_n)^{1/2} |\log(T - t_n)|^{1/2}. \tag{2.27b}$$

If (2.27a) holds, (2.26) follows at once from (2.21) in Lemma 2.5. Suppose now that (2.27b) is satisfied. We then define a sequence $\{\tau_n\}$ and a function $\lambda(\tau)$ as follows

$$\tau = -\log(T - t),$$

$$\lambda(\tau) = \Gamma(t)(T - t)^{-1/2}.$$

Notice that $\lim_{n\to\infty} |\lambda(\tau_n)| = \infty$ by (2.25). The desired inequality (2.26) is now a consequence of Lemma 2.2.

To conclude, we now set $M(t) = u(\Gamma(t), t)$, and notice that

$$\dot{M}(t) - M(t)^p \leq 0, \tag{2.28a}$$

whence

$$M(t) \leq \left(M(t_n)^{-(p-1)} - (p-1)(t - t_n)\right)^{-\frac{1}{p-1}}, \qquad \text{for } t > t_n. \tag{2.28b}$$

Since $M(t_n) \leq (\delta_1 + (p-1)(T - t_n))^{-\frac{1}{p-1}}$ for some $\delta_1 > 0$, (2.28b) yields that $M(t)$ stays uniformly bounded as $t \uparrow T$, and this provides a contradiction. The case where (1.10c) holds is similar. ∎

3. - Refined asymptotics as $\tau \to \infty$. The linearized problem

In this Section we shall derive some asymptotic estimates on the function $R(y,\tau)$ described in the Introduction, and which will be recalled below (cf. (3.20)). To this end, suitable improvements of the convergence results already obtained in [HV1], [HV2] are required. A first such step is contained in the following.

LEMMA 3.1. *Assume that* (1.10b) *holds. We then have*

$$\Phi(y,\tau) = (p-1)^{-\frac{1}{p-1}} - \frac{(4\pi)^{1/4}(p-1)^{-\frac{1}{p-1}}}{\sqrt{2}p} \cdot \frac{H_2(y)}{\tau} + O\left(\frac{\log \tau}{\tau^2}\right)$$

(3.1) *as* $\tau \to \infty$, *where* Φ *is given in* (1.3), *and convergence takes place in* $H^1_w(\mathbb{R})$ *as well as in* $C^{k,\gamma}_{\mathrm{loc}}(\mathbb{R})$ *for any* $k \geq 1$ *and* $\gamma \in (0,1)$.

PROOF. It consists in a suitable refinement of the arguments in Propositions 5.7 and 5.8 in [HV1]. Recalling (1.10b), we set

$$\theta(y,\tau) = \psi(y,\tau) - a_2(\tau)H_2(y) \tag{3.2}$$

so that θ satisfies

$$\theta_\tau = \theta_{yy} - \frac{1}{2}y\theta_y + \theta + (f(\psi) - \langle f(\psi), H_2\rangle H_2) \equiv A\theta + D(y,\tau), \tag{3.3}$$

where

$$f(\psi) = O(\psi^2) \qquad \text{as } \psi \to 0. \tag{3.4}$$

We can then represent $\theta(y,\tau)$ either as a Fourier series

$$\theta(y,\tau) = \sum_{k=0}^{\infty} \alpha_k(\tau)H_k(y) \tag{3.5}$$

or by means of variation of constants formula

$$\theta(y,\tau) = S_A(\tau-\tau_0)\theta(\cdot,\tau_0) + \int_{\tau_0}^{\tau} S_A(\tau-s)D(\cdot,s)\mathrm{d}s, \tag{3.6}$$

where S_A denotes the semigroup associated to the linear operator A given in (3.3). From (3.5) and (3.6), we deduce that

$$
\begin{aligned}
\theta(y,\tau) &= \sum_{k=0}^{\infty} \alpha_k(\tau_0) e^{\left(1-\frac{k}{2}\right)(\tau-\tau_0)} H_k(y) \\
&\qquad + \sum_{k=0}^{\infty} H_k(y) \int_{\tau_0}^{\tau} e^{\left(1-\frac{k}{2}\right)(\tau-s)} \left(\int_{\mathbb{R}} D(r,s) H_k(r) e^{-r^2/4} \mathrm{d}r \right) \mathrm{d}s \\
&\equiv \sum_{k=0}^{\infty} \alpha_k(\tau_0) e^{\left(1-\frac{k}{2}\right)(\tau-\tau_0)} H_k(y) + \sum_{k=0}^{\infty} H_k(y) I(\tau;\tau_0).
\end{aligned} \tag{3.7}
$$

We now remark that, if $k = 0, 1, 2$, $I(\tau;\tau_0)$ converges as $\tau \to \infty$. Indeed, by the delayed regularizing effect described in [HV1, Section 2], we have that

$$
\|D(\cdot,s)\| \le \|f(\psi(\cdot,s))\| \le C\|\psi(\cdot,s)^2\| = C\|\psi(\cdot,s)\|_{4,w}^2 \le C\|\psi(\cdot,s-a)\|^2,
$$

where $a > 0$, and from now on C will denote a generic constant, whose value may possibly change from line to line. Since we know already that $\|\psi(\cdot,\tau)\| = o\left(\frac{1}{\tau}\right)$ as $\tau \to \infty$, it turns out that

$$
|I(\tau,\tau_0)| \le \int_{\tau_0}^{\tau} e^{\left(1-\frac{k}{2}\right)(\tau-s)} \|D(\cdot,s)\| \mathrm{d}s \le C e^{\left(1-\frac{k}{2}\right)\tau} \int_{\tau_0}^{\tau} s^{-2} e^{-\left(1-\frac{k}{2}\right)s} \mathrm{d}s, \tag{3.8}
$$

whence the result. We now claim that

$$
\|\theta(\cdot,\tau)\|_{H_w^1} \le \frac{C}{\tau^2} \qquad \text{for large enough } \tau. \tag{3.9}
$$

To show (3.9), we rewrite (3.7) as follows

$$
\begin{aligned}
\theta(y,\tau) &= \sum_{k=0}^{2} e^{\left(1-\frac{k}{2}\right)\tau} H_k(y) \left(\alpha_k e^{-\left(1-\frac{k}{2}\right)\tau_0} + \int_{\tau_0}^{\infty} e^{-\left(1-\frac{k}{2}\right)s} \langle H_k, D(\cdot,s)\rangle \, \mathrm{d}s \right) \\
&\qquad + \sum_{k=3}^{\infty} \alpha_k e^{\left(1-\frac{k}{2}\right)(\tau-\tau_0)} H_k(y) \\
&\qquad - \sum_{k=0}^{2} H_k(y) e^{\left(1-\frac{k}{2}\right)\tau} \int_{\tau}^{\infty} e^{-\left(1-\frac{k}{2}\right)s} \langle H_k, D(\cdot,s)\rangle \, \mathrm{d}s \\
&\qquad + \sum_{k=3}^{\infty} H_k(y) e^{\left(1-\frac{k}{2}\right)\tau} \int_{\tau_0}^{\tau} e^{-\left(1-\frac{k}{2}\right)s} \langle H_k, D(\cdot,s)\rangle \, \mathrm{d}s \\
&\equiv T_1 + T_2 + T_3 + T_4.
\end{aligned} \tag{3.10}
$$

We next proceed to estimate the various terms in (3.10). To begin with, we have shown in [HV1, Proposition 5.8] that

(3.11a) $$\|T_2(\cdot,\tau)\|_{H^1_w} \le Ce^{-\tau/2}, \qquad \text{if } \tau > 0 \text{ is large enough,}$$

whereas (3.8) yields at once

(3.11b) $$\|T_3(\cdot,\tau)\|_{H^1_w} \le \frac{C}{\tau^2} \qquad \text{if } \tau > 0 \text{ is large enough.}$$

It remains to deal with T_4 and T_1. Since $\langle D(\cdot,s), H_k\rangle = \langle f(\psi(\cdot,s)), H_k\rangle$ for any $k \ge 3$, we may use (3.4) to obtain that, for large enough τ,

$$\begin{aligned}
\|T_4(\cdot,\tau)\| &\le \int_{\tau_0}^{\tau} \left(\sum_{k=3}^{\infty} e^{2\left(1-\frac{k}{2}\right)(\tau-s)}\right)^{1/2} \|f(\psi(\cdot,s)\|\mathrm{d}s \\
&\le C\int_{\tau_0}^{\tau} e^{-(\tau-s)/2}\left(1-e^{-(\tau-s)}\right)^{-1/2} s^{-2}\mathrm{d}s \\
&= C\int_{0}^{\tau-\tau_0} e^{-s/2}(1-e^{-s})^{-1/2}(\tau-s)^{-2}\mathrm{d}s \\
&\le \frac{C}{\tau^2}\int_0^{\tau/2}(1-e^{-s})^{-1/2}e^{-s/2}\mathrm{d}s + \frac{C}{\tau_0^2}e^{-\tau/4}\int_{\tau/2}^{\tau-\tau_0}(1-e^{-s})^{-1/2}\mathrm{d}s,
\end{aligned}$$

whence

(3.11c) $$\|T_4(\cdot,\tau)\| \le \frac{C}{\tau^2} \qquad \text{for } \tau > 0 \text{ sufficiently large.}$$

The H^1_w-bound for T_4 is now obtained by noting that T_4 solves

$$z_\tau = Az + \left(f(\psi) - \sum_{k=0}^{2}\langle (f(\psi), H_k\rangle\, H_k\right) \equiv Az + \delta(y,\tau),$$

where $\|\delta(\cdot,\tau)\| \le \|f(\psi(\cdot,\tau))\| \le \dfrac{C}{\tau^2}$ for large τ. We then use variation of constants formula in the equation above to deduce that

$$T_4(\cdot,\tau) = S_A T_4(\cdot,\tau-R) + \int_{\tau-R}^{\tau} S_A(\cdot,\tau-s)\delta(\cdot,s)\mathrm{d}s$$

and since both T_4 and δ are of order $O\left(\dfrac{1}{\tau^2}\right)$ for large τ, standard properties of evolution semigroups (cf. for instance Appendix A in [HV1]) yield at once that

(3.11d) $$\|T_4(\cdot,\tau)\|_{H^1_w} \le \frac{C}{\tau^2} \qquad \text{for large enough } \tau.$$

Since $\theta(\cdot,\tau) \to 0$ as $\tau \to \infty$, we then have that

$$T_1(y,\tau) \equiv 0. \tag{3.11e}$$

Taking into account (3.10), we readily see that (3.9) follows from (3.11). Consider now the equation satisfied by $a_2(\tau)$, namely

$$\begin{aligned} \dot{a}_2 &= \nu a_2^2 \langle H_2^2, H_2\rangle + 2\nu a_2 \langle \theta, H_2^2\rangle + \nu\langle \theta^2, H_2\rangle + \langle g(\psi), H_2\rangle \\ &= \nu a_2^2 \langle H_2^2, H_2\rangle + S(y,\tau), \end{aligned} \tag{3.12}$$

where $g(\psi) = O(\psi^3)$ as $\psi \to 0$ (cf. (5.25) in [HV1]). As a next step, we shall prove that

$$|S(y,\tau)| \le \frac{C}{\tau^3} \qquad \text{for large enough } \tau. \tag{3.13}$$

To derive (3.13), we first recall the following estimates obtained in [HV1, Proposition 5.8]

$$\begin{aligned} &|\langle g(\psi), H_2\rangle| \le C|a_2(\tau)|^3, \qquad |\langle \theta^2, H_2\rangle| \le \|\theta(\cdot,\tau)\|_{4,w}^2, \\ &|a_2\langle \theta, H_2^2\rangle| \le |a_2(\tau)|\, \|H_2^2\|_{4/3,w} \|\theta(\cdot,\tau)\|_{4,w}. \end{aligned} \tag{3.14}$$

For fixed $R > 0$, we then write

$$\theta(y,\tau) = S_A(R)\theta(\cdot,\tau - R) + \int_{\tau-R}^{\tau} S_A(\tau - s)D(\cdot,s)\mathrm{d}s.$$

The second term in the right above can be bounded by $Ca_2(\tau)^2$ (cf. (5.31) in [HV1]), whereas the results in [HV1, Section 2] yield that, if $R > 0$ is suitably chosen

$$\begin{aligned} \|\theta(\cdot,\tau)\|_{4,w} &\le \|S_A(R)\theta(\cdot,\tau - R)\|_{4,w} + Ca_2(\tau)^2 \\ &\le C(\|\theta(\cdot,\tau - R)\| + a_2(\tau)^2). \end{aligned} \tag{3.15}$$

Inequality (3.13) is now a consequence of (3.14) and (3.15), since we already know that $a_2(\tau) = O\left(\frac{1}{\tau}\right)$ as $\tau \to \infty$. We now set

$$a_2(\tau) = -\frac{K}{\tau} + w(\tau), \tag{3.16}$$

where K is the coefficient of $\left(\frac{H_2(y)}{\tau}\right)$ in (3.1). Plugging (3.16) into (3.12) gives

$$\begin{aligned} \frac{K}{\tau^2} + \dot{\omega}(\tau) = \dot{a}_2(\tau) &= \nu\left(-\frac{K}{\tau} + \omega(\tau)\right)^2 \langle H_2^2, H_2\rangle + S \\ &= \frac{K}{\tau^2} + \nu\omega(\tau)^2\langle H_2^2, H_2\rangle - \frac{2\omega(\tau)}{\tau} + S, \end{aligned}$$

where we have used the fact that $K\nu\langle H_2^2, H_2\rangle = 1$. We thus have

$$\dot{\omega}(\tau) = -\frac{2\omega(\tau)}{\tau} + \left(C\omega(\tau)^2 + O\left(\frac{1}{\tau^3}\right)\right) \qquad \text{for large } \tau. \tag{3.17}$$

Multiplying both sides in (3.17), by $(\operatorname{sgn}\omega(\tau))$ we obtain that, for large $\tau > 0$ and some $\varepsilon > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\tau^{2-\varepsilon}|\omega(\tau)|\right) \le C\tau^{-(1+\varepsilon)},$$

whence $|\omega(\tau)| \le C\tau^{-(2-\varepsilon)}$ in such case. We then may select $\varepsilon > 0$ such that $\omega(\tau)^2 \le C\tau^{-3}$ for large τ, whereupon (3.17) reads

$$\dot{\omega}(\tau) + \frac{2\omega(\tau)}{\tau} = O\left(\frac{1}{\tau^3}\right) \qquad \text{as } \tau \to \infty$$

which, after multiplication by $(\operatorname{sgn}\omega(\tau))$ and integration, yields

$$\omega(\tau) = O\left(\frac{\log\tau}{\tau^2}\right) \qquad \text{for large } \tau \tag{3.18}$$

and putting together (3.9) and (3.18), (3.1) holds in H_w^1, whence also in $C_{\mathrm{loc}}^{0,\gamma}$ for some $\gamma \in (0,1)$. Standard bootstrap arguments for parabolic equations yield then convergence in $C_{\mathrm{loc}}^{k,\gamma}$ for any $k \ge 1$ and any $\gamma \in (0,1)$. ∎

Suppose now that $\tilde{u}(x,t)$ solves (1.1), and let $\tilde{R}(x,t)$ be a solution of

$$\tilde{R}_t = \tilde{R}_{xx} + p\tilde{u}^{p-1}\tilde{R}, \qquad \text{when } x \in \mathbb{R},\ t > 0, \tag{3.19a}$$

$$\tilde{R}(x,0) = \tilde{R}_0(x), \qquad \text{when } x \in \mathbb{R}, \tag{3.19b}$$

where $\tilde{R}_0(x) \in L^\infty(\mathbb{R})$. Set now

$$R(y,\tau) = (T-t)^{\frac{1}{p-1}}\tilde{R}(x,t), \qquad y,\ \tau \text{ as in (1.3).} \tag{3.20}$$

We then have

PROPOSITION 3.2. *Let $R(y,\tau)$ be the function given in* (3.20). *Then there exists a real constant α such that*

$$R(y,\tau) = \alpha e^\tau + o(e^\tau) \qquad \text{as } \tau \to \infty, \tag{3.21}$$

where convergence takes place in $H_w^1(\mathbb{R})$, as well as in $C_{\mathrm{loc}}^{k,\gamma}$ for any $k \ge 1$ and $\gamma \in (0,1)$.

PROOF. To begin with, we readily see that R satisfies

$$R_\tau = R_{yy} - \frac{1}{2}yR_y + p\Phi^{p-1}R - \frac{R}{p-1}. \tag{3.22}$$

Since Φ is bounded, arguing as in [HV1, Section 2], we obtain that

$$\text{For any } r > 1,\ q > 1 \text{ and } L > 0 \text{ there exists } a = a(q,r) \text{ and } C = C(q,r,L) \text{ such that, if } a < \tau < a + L \quad \|R(\cdot,\tau)\|_{q,\omega} \le C\|R(\cdot,\tau - a)\|_{r,\omega}. \tag{3.23}$$

Assume now that (1.10b) holds, and let $K_1 = p(p-1)^{\frac{1}{p-1}}K$, where K is the positive coefficient of $\dfrac{H_2(y)}{\tau}$ in (3.1). We then rewrite (3.22) as follows

$$\begin{aligned} R_\tau &= R_{yy} - \frac{1}{2}yR_y + R - \frac{K_1}{\tau}H_2(y)R + \left(p\Phi^{p-1} - \frac{p}{p-1} + \frac{K_1}{\tau}H_2(y)\right)R \\ &\equiv AR - \frac{K_1}{\tau}H_2(y)R + E(y,\tau)R. \end{aligned} \tag{3.24}$$

We now claim that

$$\text{For any } q > 1, \text{ there exists } C = C(q) \text{ such that } \|E(\cdot,\tau)\|_{q,\omega} \le C\frac{\log\tau}{\tau^2} \quad \text{for large enough } \tau > 0. \tag{3.25}$$

To prove (3.25), we notice that if we define $\omega(y,\tau)$ by

$$\Phi(y,\tau) = (p-1)^{-\frac{1}{p-1}} - \frac{K}{\tau}H_2(y) + \omega(y,\tau)$$

we have that, by Taylor's expansion,

$$\left|p\Phi^{p-1} - \frac{p}{p-1} + K_1\frac{H_2(y)}{\tau}\right| \le C\left(|\omega(y,\tau)| + \left(\frac{KH_2(y)}{\tau} + \omega(y,\tau)\right)^2\right) \quad \text{for large enough } \tau, \text{ uniformly on compact sets in } |y|. \tag{3.26}$$

Then, if $\theta(y,\tau)$ is the function defined in (3.2), we see that

$$\omega(y,\tau) = \theta(y,\tau) + \left(a_2(\tau) + \frac{K}{\tau}\right)H_2(y). \tag{3.27}$$

Arguing as in Lemma 3.1 (cf. for instance (3.15) therein), it follows that, for any $q > 1$

$$\|\theta(\cdot,\tau)\|_{q,\omega} \le \frac{C}{\tau^2}, \qquad \text{if } \tau \text{ is large enough,} \tag{3.28a}$$

whereas (3.1) yields

$$\left|a_2(\tau)+\frac{K}{\tau}\right| \|H_2\|_{q,\omega} \le C\,\frac{\log\tau}{\tau^2} \tag{3.28b}$$

for any $q>1$ and $\tau>0$ large enough,

and (3.25) follows now at once from (3.26) and (3.29). We now set out to obtain

$$\|R(\cdot\,,\tau)\| \le C\tau^{\sigma}e^{\tau} \tag{3.30}$$

for some $\sigma>0$ and any $\tau>0$ sufficiently large. To this end, we multiply both sides of (3.24) by $R(y,\tau)e^{-y^2/4}$, integrate over the whole line, and use (3.25) and delayed L^q_ω-estimates ([HV1, Section 2]) to deduce that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|R(\cdot\,,\tau)\|^2 &\le \|R(\cdot\,,\tau)\|^2+\frac{C}{\tau}\|R(\cdot\,,\tau)\|^2_{4,w}+C\,\frac{\log\tau}{\tau^2}\|R(\cdot\,,\tau)\|^2_{4,w}\\ &\le \|R(\cdot\,,\tau)\|^2+\frac{C}{\tau}\|R(\cdot\,,\tau-a)\|^2\end{aligned} \tag{3.31}$$

for some $a>0$. Setting $y(\tau)=\|R(\cdot\,,\tau)\|^2$, we are thus led to

$$\frac{1}{2}y'(\tau)\le y(\tau)+\frac{C}{\tau}y(\tau-a),\qquad C>0,\ a>0, \tag{3.32a}$$

and (3.30) follows now from the differential inequality (3.32a). Indeed, for $\tau\ge\tau_0>a$, we have that, since $(s+a)^{-1}\le s^{-1}$

$$\begin{aligned}y(\tau)&\le Ce^{2\tau}+C\int_{\tau_0}^{\tau}s^{-1}e^{2(\tau-s)}y(s-a)\mathrm{d}s\\ &\le Ce^{2\tau}+C\int_{\tau_0-a}^{\tau-a}s^{-1}e^{2(\tau-s-a)}y(s)\mathrm{d}s\\ &\le Ce^{2\tau}+C\int_{\tau_0-a}^{\tau-a}s^{-1}e^{2(\tau-s)}y(s)\mathrm{d}s+Ce^{2\tau}+C\int_{\tau_0-a}^{\tau-a}s^{-1}e^{2(\tau-s)}y(s)\mathrm{d}s\\ &\le Ce^{2\tau}+C\int_{\tau_0}^{\tau}s^{-1}e^{2(\tau-s)}y(s)\mathrm{d}s.\end{aligned} \tag{3.32b}$$

Therefore, if we set $G(\tau)=\int_{\tau_0}^{\tau}s^{-1}e^{-2s}y(s)\mathrm{d}s$, G satisfies

$$\tau^{C}(\tau^{-C}G(\tau))'\le\frac{C}{\tau}$$

whence

$$G(\tau) \leq A\tau^C + B\tau^C \int_{\tau_0}^{\tau} s^{-(1+C)} \mathrm{d}s \leq M\tau^C,$$

for some $M > 0$ and $C > 0$. Substituting this in (3.32b) gives (3.30) with $\sigma = \frac{C}{2}$. The next step consists in improving the bound (3.30). To do this, we write

$$R(y,\tau) = \sum_{k=0}^{\infty} b_k(\tau) H_k(y) = b_0(\tau) H_0(y) + \chi(y,\tau) \tag{3.33}$$

and estimate separately $b_0(\tau)$ and $\chi(y,\tau)$ as follows. Let P be the orthogonal projection over the subspace spanned by $H_0(y)$, and let $Q = I - P$. Then, in view of (3.24), χ satisfies

$$\chi_\tau = \chi_{yy} - \frac{1}{2} y\chi_y + \chi - \frac{K_1}{\tau} Q(H_2 R) + Q(ER). \tag{3.34}$$

By (3.30) and delayed L^q_w-estimates,

$$\begin{aligned} |\langle Q(H_2R), \chi\rangle| &\leq \|H_2\|_{4,w} \|R(\cdot,\tau)\|_{4,w} \|\chi(\cdot,\tau)\| \leq C\|R(\cdot,\tau-a)\| \, \|\chi(\cdot,\tau)\| \\ &\leq C\tau^\sigma e^\tau \|\chi(\cdot,\tau)\| \qquad \text{if } \tau \gg 1, \end{aligned}$$

whereas, by (3.25) and (3.30)

$$\begin{aligned} |\langle ER, \chi\rangle &\leq \|E(\cdot,\tau)\|_{4,w} \|R(\cdot,\tau)\|_{4,w} \|\chi(\cdot,\tau)\| \\ &\leq C\tau^{\sigma-2} \log \tau \|\chi(\cdot,\tau)\|. \end{aligned}$$

We now multiply both sides of (3.34) by $\chi(y,\tau)e^{-y^2/4}$, integrate over the line, and use the fact that $\langle H_0, \chi\rangle = 0$ together with $\|\chi(\cdot,\tau)\| \leq \|R(\cdot,\tau)\|$ to arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau} \|\chi(\cdot,\tau)\|^2 \leq \frac{1}{2} \|\chi(\cdot,\tau)\|^2 + C\tau^{2\sigma-1} e^{2\tau}, \qquad \text{if } \tau \gg 1, \tag{3.35a}$$

which in turn yields

$$\|\chi(\cdot,\tau)\| \leq C\tau^{\sigma-1/2} e^\tau, \qquad \text{if } \tau \gg 1. \tag{3.35b}$$

On the other hand, by (3.22) and (3.33), we have that

$$\dot{b}_0(\tau) = b_0(\tau) - \frac{K_2}{\tau} \langle H_0, H_2 R\rangle + \langle ER, H_0\rangle.$$

We have shown in [GHV], [HV1] that, if we write

$$A_{n,m,\ell} = \int_{\mathbb{R}} H_n(y) H_m(y) H_\ell(y) e^{-y^2/4} \mathrm{d}y,$$

then $A_{n,m,\ell} \neq 0$ if and only if $(n+m+\ell)$ is even and $n \leq m+\ell$, $m \leq n+\ell$, $\ell \leq m+n$, in which case we have

$$A_{n,m,\ell} = (4\pi)^{-1/4}((n!)(m!)(\ell!))^{1/2} \cdot \left(\left(\frac{m+n-\ell}{2}\right)!\left(\frac{n+\ell-m}{2}\right)!\left(\frac{m+\ell-n}{2}\right)!\right)^{-1}$$

so that the differential equation for $b_0(\tau)$ actually reads

$$\dot{b}_0(\tau) = b_0(\tau) - \frac{K_1}{\tau} A_{2,2,0} b_2(\tau) + \langle ER, H_0 \rangle$$

whence, for $\tau \geq \tau_0$

$$\begin{aligned} b_0(\tau) &= Ce^{\tau} - K_1 A_{2,2,0} \int_{\tau_0}^{\tau} s^{-1} e^{\tau - s} b_2(s) \mathrm{d}s + \int_{\tau_0}^{\tau} \langle ER, H_0 \rangle e^{\tau - s} \mathrm{d}s \\ &\equiv Ce^{\tau} - I_1(\tau) + I_2(\tau). \end{aligned} \tag{3.36}$$

We now use (3.35b) to estimate $I_1(\tau)$ as follows

$$|I_1(\tau)| \leq C \int_{\tau_0}^{\tau} s^{-1} e^{\tau - s} \|\chi(\cdot, s)\| \mathrm{d}s \leq Ce^{\tau} \int_{\tau_0}^{\tau} s^{\sigma - 3/2} \mathrm{d}s.$$

Replacing σ by $\sigma + \varepsilon$ ($\varepsilon > 0$) if necessary, it may always be assumed that $\sigma \neq 1/2$, in which case the above inequality yields

$$\begin{aligned} &|I_1(\tau)| \leq C\tau^{\theta} e^{\tau}, \qquad \text{for } \tau \gg 1, \\ &\text{where } \theta = 0 \text{ if } \sigma < \frac{1}{2}, \text{ and } \theta = \sigma - \frac{1}{2} \text{ if } \sigma > \frac{1}{2}. \end{aligned} \tag{3.37a}$$

As to $I_2(\tau)$, we readily check that

$$\begin{aligned} |I_2(\tau)| &\leq \int_{\tau_0}^{\tau} \|E(\cdot, s)\|_{4,w} \|R(\cdot, s)\|_{4,w} e^{\tau - s} \mathrm{d}s \leq Ce^{\tau} \int_{\tau_0}^{\tau} s^{\sigma - 2} \log s \, \mathrm{d}s \\ &\leq Ce^{\tau} \int_{\tau_0}^{\tau} s^{\sigma + \delta - 2} \mathrm{d}s \qquad \text{for any } \delta \in \left(0, \frac{1}{2}\right), \end{aligned}$$

so that

$$\begin{aligned} &|I_2(\tau)| \leq C\tau^{\mu} e^{\tau}, \qquad \text{for } \tau \gg 1, \\ &\text{where } \mu = 0 \text{ if } \sigma < 1 - \delta \text{ and } \mu = \sigma + \delta - 1 \text{ if } \sigma > 1 - \delta. \end{aligned} \tag{3.37b}$$

From (3.36) and (3.37), it follows that

$$|b_0(\tau)| \le C(e^\tau + \tau^\theta e^\tau), \qquad \text{where } \theta \text{ is as in (3.36a).} \tag{3.38a}$$

We now substitute (3.35) and (3.38a) in (3.33) to obtain that, if $\sigma > \frac{1}{2}$,

$$\|R(\cdot,\tau)\| \le C\tau^{\sigma-1/2}e^\tau.$$

Iterating the previous argument, we then obtain after a finite number of steps that

$$\begin{aligned} &b_0(\tau) = Ce^\tau, && \text{for some } C > 0,\\ &\|\chi(\cdot,\tau)\| = o(e^\tau), && \text{as } \tau \to \infty. \end{aligned} \tag{3.38b}$$

Obviously, no iteration is required if $\sigma < \frac{1}{2}$. To conclude the proof in our case, we still have to show that (3.38b) holds also in H^1_w. This follows, however, by standard semigroup theory. Indeed, by (3.33), for any $\tau \ge \tau_0 > 0$, we may write

$$\chi(\cdot,\tau) = S_A(\tau_0)\chi(\cdot,\tau-\tau_0) + \int_{\tau-\tau_0}^{\tau} S_A(s-\tau_0)\left\{-\frac{K_1}{s}Q(H_2R) + Q(E)R\right\}\,\mathrm{d}s,$$

whence

$$\|\chi(\cdot,\tau)\|_{H^1_w} \le C\|\chi(\cdot,\tau-\tau_0)\| + \frac{C}{\tau}\left(\sup_{[\tau-\tau_0,\tau]}\Big(\|QH_2R(\cdot,s)\| + \|Q(E)R(\cdot,s)\|\Big)\right)$$

(cf. for instance [HV1, Appendix A]). It then follows at once that $\|\chi(\cdot,\tau)\|_{H^1_w} = o(e^\tau)$ as $\tau \to \infty$.

It remains to consider now the case where (1.10c) holds. Since the arguments are quite similar to those just explained before, we shall sketch briefly the main differences, and dispense with most of the details. To begin with, we replace (3.24) by

$$R_\tau = R_{yy} - \frac{1}{2}yR_y + R + \left(p\Phi^{p-1} - \frac{p}{p-1}\right) \equiv AR + E_1(y,\tau)R. \tag{3.39}$$

We now consider the auxiliary function $\sigma(y,\tau)$ given by

$$\Phi(y,\tau) = (p-1)^{-\frac{1}{p-1}} + \sigma(y,\tau)$$

so that $|E_1(y,\tau)| \le C|\sigma(y,\tau)|$ for $\tau \gg 1$. Arguing as for (3.28a), we then obtain that

$$\begin{aligned} &\text{For any } q > 1, \text{ there exists } C = C(q) \text{ such that}\\ &\|E_1(\cdot,\tau)\|_{q,w} \le Ce^{(1-\frac{m}{2})\tau} \qquad \text{for large enough } \tau. \end{aligned} \tag{3.40}$$

We use the particular case $q = 2$ of (3.40) (which follows from (1.10)) to deduce from (3.39) that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|R(\cdot,\tau)\|^2 &\le \|R(\cdot,\tau)\|^2 + \|E_1(\cdot,\tau)\|\,\|R(\cdot,\tau)\|_{4,w}^2\\ &\le \|R(\cdot,\tau)\|^2 + Ce^{(1-\frac{m}{2})\tau}\|R(\cdot,\tau-a)\|^2.\end{aligned}$$

Setting $z(\tau) = \|R(\cdot,\tau)\|^2$, we are thus led to the functional inequality

$$z'(\tau) \le 2z(y) + Ce^{(1-\frac{m}{2})\tau}z(\tau-a), \qquad C>0,\ a>0,$$

which can be dealt with as in the previous case to obtain

$$\|R(\cdot,\tau)\| \le Ce^{\tau} \quad \text{if} \quad \tau \gg 1. \tag{3.41}$$

We next write $R(y,\tau) = a_0(\tau)H_0(y) + \chi_1(y,\tau)$, where $\langle H_0,\chi_1\rangle = 0$. Keeping to our previous notations, and dropping the subscript 1 for convenience, we see that χ satisfies

$$\chi_\tau = \chi_{yy} - \frac{1}{2}y\chi_y + \chi + Q(ER).$$

Instead of (3.35a) we now derive

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|\chi(\cdot,\tau)\|^2 &\le \frac{1}{2}\|\chi(\cdot,\tau)\|^2 + |\langle Q(ER),\chi\rangle|\\ &\le \frac{1}{2}\|\chi(\cdot,\tau)\|^2 + \|E(\cdot,\tau)\|_{4,w}\|R(\cdot,\tau)\|_{4,w}\|\chi(\cdot,\tau)\|.\end{aligned}$$

Estimates (3.40) and (3.41) together with delayed L_w^q-bounds, repeatedly used so far, yield then

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\|\chi(\cdot,\tau)\|^2 \le \|\chi(\cdot,\tau)\|^2 + C\exp\left(\left(3-\frac{m}{2}\right)\tau\right).$$

It is an easy matter to obtain now that

$$\|\chi(\cdot,\tau)\|^2 \le f(\tau) \qquad \text{for } \tau \gg 1,$$

where $f(\tau) = \tau e^{\tau}$ if $\left(1-\frac{m}{2}\right) = -1$, and $f(\tau) = \exp\left(\left(3-\frac{m}{2}\right)\tau\right)$ if $\left(1-\frac{m}{2}\right) > -1$. In particular, $\|\chi(\cdot,\tau)\| = o(e^{\tau})$ as $\tau \to \infty$. On the other hand, (3.36) is to be replaced by

$$b_0(\tau) = Ce^{\tau} + \int_{\tau_0}^{\tau} e^{\tau-s}\langle E_1R, H_0\rangle\,\mathrm{d}s.$$

Using (3.40) (with $q = 2$) and (3.41), it follows that the integral above converges as $\tau \to \infty$, and

$$R(y,\tau) = \alpha e^{\tau} + o(e^{\tau}) \qquad \text{in } L^2_w \text{ as } \tau \to \infty.$$

Finally, convergence in H^1_w and $C^{k,\gamma}_{\text{loc}}$ are obtained as in the previous case. ∎

Our next Lemma provides an estimate on $\Phi(y,\tau)$ on regions $y = \xi\sqrt{\tau}$, which improves the results already obtained in [HV1], [HV2].

LEMMA 3.3. *For any $R > 0$, there exists $M = M(R) > 0$ such that*

a) *If* (1.10b) *holds,*

$$\left|\Phi(\xi(\sqrt{\tau},\tau)) - (p-1)^{-\frac{1}{p-1}}\left(1 + \left(\frac{p-1}{4p}\right)\xi^2\right)^{-\frac{1}{p-1}}\right| \le M\,\frac{\log\tau}{\tau}, \tag{3.42a}$$

uniformly on sets $|\xi| \le R$, provided that τ is large enough.

b) *If* (1.10c) *holds*

$$\left|\Phi\left(\xi e^{\left(\frac{1}{2}-\frac{1}{m}\right)\tau},\tau\right) - (p-1)^{-\frac{1}{p-1}}\left(1 + Cc_m\xi^m\right)^{-\frac{1}{p-1}}\right| \le Me^{-\tau/m} \tag{3.42b}$$

uniformly on sets $|\xi| \le R$, provided that τ is large enough.

PROOF. We know that Φ satisfies

$$\Phi_\tau = \Phi_{yy} - \frac{1}{2}y\Phi_y + \Phi^p - \frac{\Phi}{p-1}.$$

Assume that (1.10b) holds. We then define

$$\tilde{\Phi}(y,\tau) \equiv G\left(\frac{y}{\sqrt{\tau}}\right) + \frac{\beta}{\tau}, \tag{3.43a}$$

where

$$G(\xi) = (p-1)^{-\frac{1}{p-1}}\left(1 + \left(\frac{p-1}{4p}\right)\xi^2\right)^{-\frac{1}{p-1}}, \qquad \beta = \frac{(p-1)^{-\frac{1}{p-1}}}{2p}. \tag{3.43b}$$

Notice that

$$-\frac{\xi G'}{2} + G^p = \frac{G}{p-1} \tag{3.44}$$

and

$$G\left(\frac{y}{\sqrt{\tau}}\right) = (p-1)^{-\frac{1}{p-1}} - \frac{(p-1)^{-\frac{1}{p-1}}}{4p}\cdot\frac{y^2}{\tau} + O\left(\frac{1}{\tau^2}\right) \qquad \text{in } L^2_w,$$

whereas, by (1.10b),

$$\Phi(y,\tau) = (p-1)^{-\frac{1}{p-1}} - \frac{(p-1)^{-\frac{1}{p-1}}}{4p} \cdot \frac{y^2}{\tau} + \frac{(p-1)^{-\frac{1}{p-1}}}{2p\tau} + o\left(\frac{1}{\tau}\right) \qquad \text{as } \tau \to \infty \qquad \text{in } H^1_w.$$

Therefore, by Lemma 3.1,

$$\|\Phi(\cdot,\tau) - \tilde{\Phi}(\cdot,\tau)\| = O\left(\frac{\log \tau}{\tau^2}\right) \qquad \text{as } \tau \to \infty. \tag{3.45}$$

Actually, to derive (3.45) we split the integral in the left-hand side into two parts, on regions $|y| \le \sqrt{\tau}$ and $|y| > \sqrt{\tau}$ respectively. We then use Lemma 3.1 and Taylor's theorem to estimate the first part. The integral in the external region can be estimated in a straightforward way. On the other hand, taking into account (3.44), we readily check that

$$\begin{aligned}\tilde{\Phi}_\tau - \tilde{\Phi}_{yy} + \frac{1}{2} y\tilde{\Phi}_y - \tilde{\Phi}^p + \frac{\tilde{\Phi}}{p-1} \\ = -\frac{\xi G'(\xi)}{2\tau} - \frac{\beta}{\tau^2} - \frac{G''(\xi)}{2} + \left(G(\xi)^p - \left(G(\xi) + \frac{\beta}{\tau}\right)^p\right) + \frac{\beta}{(p-1)\tau},\end{aligned}$$

where $\xi = \dfrac{y}{\sqrt{\tau}}$. Set now

$$W(y,\tau) = \Phi(y,\tau) - \tilde{\Phi}(y,\tau),$$

$$B(W) = \frac{\Phi^p - \tilde{\Phi}^p}{\Phi - \tilde{\Phi}} \quad \text{if } \Phi \neq \tilde{\Phi}, \qquad B(W) = 0 \quad \text{if } W = 0.$$

We now subtract the differential equations satisfied by Φ and $\tilde{\Phi}$, and multiply both sides in the resulting equation by $(\operatorname{sgn} W)$. A routine computation reveals then that $Z = |W|$ satisfies

$$\begin{aligned} &Z_\tau - Z_{yy} + \frac{1}{2} y Z_y \\ &\le B(W)Z - \frac{Z}{p-1} + \frac{|\xi(G'(\xi))|}{\tau} \\ &\qquad + \frac{\beta}{\tau^2} + \left|\frac{G''(\xi)}{\tau} + \left(G(\xi) + \frac{\beta}{\tau}\right)^p - G(\xi)^p - \frac{\beta}{(p-1)\tau}\right| \\ &\equiv B(W)Z - \frac{Z}{p-1} + \frac{|\xi(G'(\xi))|}{\tau} + \frac{\beta}{\tau^2} + |L(y,\tau)|. \end{aligned} \tag{3.46}$$

We want to derive (3.42a) by means of application of variation of constants formula in the above inequality. To this end, some manipulations will prove useful. Firstly, we notice that

$$G''(0) + p\beta(G(0))^{p-1} - \frac{\beta}{p-1} = 0,$$

$$|\xi G'(\xi)| \le C|\xi|^2, \qquad |G''(\xi) - G''(0)| \le C|\xi|^2.$$

On the other hand, for any fixed $R > 0$, $G(\xi) \ge \theta_0 > 0$ when $|\xi| \le R$, and

$$\left| \left(G(\xi) + \frac{\beta}{\tau} \right)^p - G(\xi)^p - \frac{p\beta G(0)^{p-1}}{\tau} \right|$$
$$= \frac{p\beta}{\tau} \left| G(\xi)^{p-1} - G(0)^{p-1} \right| + O\left(\frac{1}{\tau^2}\right) \le \frac{C\xi^2}{\tau} + O\left(\frac{1}{\tau^2}\right).$$

Therefore, adding and subtracting $\dfrac{G''(0)}{\tau} + \dfrac{p\beta G(0)^{p-1}}{\tau}$ in $L(y,\tau)$ yields that, for any $R > 0$ fixed,

$$|L(y,\tau)| \le C\left(\frac{1}{\tau^2} + \chi_1\right),$$

where $\chi_1(y,\tau) = 1$ if $|y| \ge R\sqrt{\tau}$ and $\chi_1(y,\tau) = 0$ otherwise.

Set now $\varsigma = \max\{\Phi, \tilde{\Phi}\}$. Since $(a^p - b^p)(a-b)^{-1} \le p$ whenever $0 < a$, $b < 1$ and $a \ne b$, we have that if $\Phi \ne \tilde{\Phi}$

$$B(W) = \varsigma^{p-1} \left(\left(\frac{\Phi}{\varsigma}\right)^p - \left(\frac{\tilde{\Phi}}{\varsigma}\right)^p \right) \left(\left(\frac{\Phi}{\varsigma}\right) - \left(\frac{\tilde{\Phi}}{\varsigma}\right) \right)^{-1} \le p\varsigma^{p-1}.$$

We now recall that there exists $C > 0$ such that

$$\Phi(y,\tau) \le (p-1)^{-\frac{1}{p-1}} + \frac{C}{\tau} \tag{3.47}$$

(cf. estimate (2.20) in [HV2]). It is readily checked that such a bound also holds for $\tilde{\Phi}(y,\tau)$, whence

$$B(W)Z - \frac{Z}{p-1} \le \frac{pZ}{p-1} + \frac{CZ}{\tau} + O\left(\frac{1}{\tau^2}\right) - \frac{Z}{p-1}$$
$$= Z + \frac{CZ}{\tau} + O\left(\frac{1}{\tau^2}\right) \qquad \text{as } \tau \to \infty.$$

Substituting all these bounds in (3.46), and recalling that $y = \xi\sqrt{\tau}$, one is then led to

$$Z_\tau \le Z_{yy} - \frac{1}{2} y Z_y + Z + \frac{CZ}{\tau} + C\left(\frac{1}{\tau^2}(1+y^2) + \chi_1\right) \tag{3.48}$$

hence, for any $\tau > \tau_0 > 0$,

$$
\begin{aligned}
Z(y,\tau) \le{}& \frac{e^{(1+C/\tau_0)(\tau-\tau_0)}}{(4\pi(1-e^{-(\tau-\tau_0)}))^{1/2}} \int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(\tau-\tau_0)/2}-r)}{4(1-e^{-(\tau-\tau_0)})}\right)^2 Z(r,\tau_0)\mathrm{d}r \\
&+ C\int_{\tau_0}^{\tau} \frac{e^{(1+C/\tau_0)(\tau-s)}}{(4\pi(1-e^{-(\tau-s)}))^{1/2}} \\
&\int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(\tau-s)/2}-r)^2}{4(1-e^{-(\tau-s)})}\right)\left(\frac{(1+r^2)}{\tau^2}+\chi_1(r,s)\right)\mathrm{d}r\mathrm{d}s \\
\equiv{}& I_1(y,\tau)+I_2(y,\tau).
\end{aligned}
\tag{3.49}
$$

Following [HV1], we now relate τ and τ_0 by

$$e^{\tau-\tau_0} = \tau$$

so that $\tau = \tau_0 + \log\tau = \tau_0 + \log\tau_0 + \cdots$ as $\tau_0 \to \infty$, and there exist constants C_1, C_2 such that $C_1\tau_0 \le \tau \le C_2\tau_0$. By (3.45), we then have that, if $\tau_0 > 0$ is large enough

$$
\begin{aligned}
I_1(\xi\sqrt{\tau},\tau) &\le C\tau^{(1+C/\tau_0)}\|Z(\cdot,\tau_0)\| \left(\int_{\mathbb{R}} \exp\left(-\frac{(\xi-r)^2}{2(1-e^{-(\tau-\tau_0)})}+\frac{r^2}{4}\right)\mathrm{d}r\right)^{1/2} \\
&\le C\tau^{1+C/\tau_0}\frac{\log\tau_0}{\tau_0^2} \le C\frac{\log\tau}{\tau}.
\end{aligned}
\tag{3.50a}
$$

As to I_2, we see that

$$
\begin{aligned}
I_2(y,\tau) \le{}& C\int_{\tau_0}^{\tau} \frac{e^{\tau-s}}{(4\pi(1-e^{-(\tau-s)}))^{1/2}} \int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(\tau-s)/2}-r)^2}{4(1-e^{-(\tau-s)})}\right) \\
&\cdot\left(\frac{1}{s^2}\left(1+\left(r-ye^{-(\tau-s)/2}\right)^2+y^2e^{-(\tau-s)}\right)+\chi_1\right)\mathrm{d}y\mathrm{d}s.
\end{aligned}
$$

Setting $y = \xi\sqrt{\tau}$, we thus obtain

$$
\begin{aligned}
I_2(\xi\sqrt{\tau},\tau) \le{}& C\int_{\tau_0}^{\tau} e^{\tau-s}s^{-2}\left(1+\tau e^{-(\tau-s)}\right)\mathrm{d}s \\
&+ C\int_{\tau_0}^{\tau} \frac{e^{\tau-s}}{(4\pi(1-e^{-(\tau-s)}))^{1/2}} \int_{|r|\ge R\sqrt{s}} \exp\left(-\frac{(r-\xi\sqrt{\tau}e^{-(\tau-s)/2})^2}{4(1-e^{-(\tau-s)})}\right)\mathrm{d}r \\
\equiv{}& I_{2,1}+I_{2,2}.
\end{aligned}
$$

Notice that $|\xi\sqrt{\tau}e^{-(\tau-s)/2}| \le |\xi|\sqrt{s}$ for τ_0 large enough, so that if $|\xi| \le R/2$,

$$I_{2,2} \le C \int_{\tau_0}^{\tau} \frac{e^{\tau-s}}{(4\pi(1-e^{-(\tau-s)}))^{1/2}} \int_{|r|\ge(R\sqrt{s})/2} \exp\left(-\frac{r^2}{4(1-e^{-(\tau-s)})}\right) \mathrm{d}s$$

and since

$$I_{2,1}(\xi\sqrt{\tau},\tau) \le \frac{Ce^{\tau-\tau_0}}{\tau^2} + \frac{C}{\tau}\int_{\tau_0}^{\tau} \mathrm{d}s \le \frac{C\log\tau}{\tau}.$$

We finally obtain

$$I_2(\xi\sqrt{\tau},\tau) \le \frac{C\log\tau}{\tau} + o\left(\frac{\log\tau}{\tau}\right) \qquad \text{as } \tau\to\infty, \tag{3.50b}$$

uniformly on sets $|\xi| \le R$ and (3.42a) follows.

Assume now that (1.10c) holds. A careful examination of the proof of Proposition 5.8 in [HV1] reveals then that

$$\|\psi(\cdot,\tau) + Ce^{(1-\frac{m}{2})\tau}H_m\|_{H^1_w} \le Ce^{\left(1-\frac{m+1}{2}\right)\tau}. \tag{3.51}$$

We now adapt our previous argument as follows. First, we replace G defined in (3.43b) by

$$G(\xi) = (p-1)^{-\frac{1}{p-1}}\left(1 + Cc_m(p-1)^{\frac{p}{p-1}}\xi^m\right)^{-\frac{1}{p-1}}.$$

As in case a), $G(\xi)$ is even and we now have

$$-\frac{\xi G'}{m} + \frac{G}{p-1} = G^p \tag{3.52a}$$

and

$$\begin{aligned} &e^{-\left(1-\frac{2}{m}\right)\tau}G''\left(ye^{-\left(\frac{1}{2}-\frac{1}{m}\right)\tau}\right) \\ &= -Kc_m m(m-1)y^{m-2}e^{(1-\frac{m}{2})\tau} + O\left(e^{-(1-\frac{m}{2})\tau}\left(|y|e^{-\left(\frac{1}{2}-\frac{1}{m}\right)\tau}\right)^{2(m-1)}\right) \end{aligned} \tag{3.52b}$$

uniformly on sets $|y|e^{-\left(\frac{1}{2}-\frac{1}{m}\right)\tau} \le R$. Moreover, we have that

$$G\left(ye^{-\left(\frac{1}{2}-\frac{1}{m}\right)\tau}\right) = (p-1)^{-\frac{1}{p-1}} - C\left(ye^{-\left(\frac{1}{2}-\frac{1}{m}\right)\tau}\right)^m + o\left(e^{-\left(m-\frac{1}{2}\right)\tau}\right)$$

as $\tau\to\infty$, where equality is understood in L^2_w. Therefore, if we define

$$\tilde{\Phi}(y,\tau) = G\left(ye^{-\left(\frac{1}{2}-\frac{1}{m}\right)\tau}\right) + C(H_m(y) - c_m y^m)e^{(1-\frac{m}{2})\tau},$$

we obtain from (3.50) that

$$\|\Phi(\cdot,\tau)-\tilde{\Phi}(\cdot,\tau)\| = o\left(e^{\left(1-\frac{m+1}{2}\right)\tau}\right) \qquad \text{as } \tau\to\infty. \tag{3.53}$$

Set now $\xi = ye^{-\left(\frac{1}{2}-\frac{1}{m}\right)\tau}$, and let $\chi_2(y,\tau)=1$ if $|y|\geq Re^{\left(\frac{1}{2}-\frac{1}{m}\right)\tau}$, $\chi_2(y,\tau)=0$ otherwise. Taking advantage of (3.52) and using Hermite's equation, we obtain

$$\begin{aligned}
&\tilde{\Phi}_\tau - \tilde{\Phi}_{yy} + \frac{1}{2}y\tilde{\Phi}_y - \tilde{\Phi}^p + \frac{\tilde{\Phi}}{p-1}\\
&= G(\xi)^p - \left(G(\xi)+C(H_m(y)-c_my^m)e^{(1-\frac{m}{2})\tau}\right)^p\\
&\qquad + pG(0)^{p-1}C(H_m(y)-c_my^m)e^{(1-\frac{m}{2})\tau} + O\left(e^{-\left(1-\frac{2}{m}\right)\tau}|\xi|^{2m-2}\right)\\
&\leq C\left(|y|^{2(m-2)}e^{2(1-\frac{m}{2})\tau} + (1+|y|^{m-2})\,|\xi|^m e^{(1-\frac{m}{2})\tau}\right)\\
&\qquad + C\chi_2(y,\tau) \leq C(1+|y|^{2m-2})e^{2(1-\frac{m}{2})\tau}\\
&\qquad + C(1+|y|^{2(m-2)})e^{2(1-\frac{m}{2})\tau} + C\chi_2(y,\tau).
\end{aligned}$$

We now subtract the equations for Φ and $\tilde{\Phi}$, and multiply then by $\operatorname{sgn}(\Phi-\tilde{\Phi})$ throughout to obtain

$$Z_\tau \leq Z_{yy} - \frac{1}{2}yZ_y + Z + \frac{CZ}{\tau} + C(1+|y|^{2m-2})e^{2(1-\frac{m}{2})\tau} + C\chi_2(y,\tau). \tag{3.54}$$

Therefore, for $\tau>\tau_0>0$ and τ_0 large enough, there holds

$$\begin{aligned}
Z(y,\tau) &\leq \frac{Ce^{\tau-\tau_0}}{(4\pi(1-e^{-(\tau-\tau_0)}))^{1/2}}\int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(\tau-\tau_0)/2}-r)^2}{4(1-e^{-(\tau-\tau_0)})}\right) Z(r,\tau_0)\mathrm{d}r\\
&\qquad + C\int_{\tau_0}^{\tau} \frac{e^{\tau-s}}{(4\pi(1-e^{-(\tau-s)}))^{1/2}}\int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(\tau-s)/2}-r)^2}{4(1-e^{-(\tau-s)})}\right)\\
&\qquad\qquad \cdot\left(\left(1+|r|^{2m-2}\right)e^{2(1-\frac{m}{2})s} + \chi_2(r,s)\right)\,\mathrm{d}r\mathrm{d}s\\
&= I_1(y,\tau) + I_2(y,\tau)
\end{aligned}$$

We now set

$$\tau = \frac{m}{2}\tau_0 \tag{3.55}$$

Then, since $y=\xi e^{\left(\frac{1}{2}-\frac{1}{m}\right)\tau}$, it follows from (3.53) that

$$\left|I_1\left(\xi e^{\left(\frac{1}{2}-\frac{1}{m}\right)\tau},\tau\right)\right| \le Ce^{\tau-\tau_0}\|Z(\cdot,\tau_0)\| \cdot \left(\int_{\mathbb{R}} \exp\left(-\frac{(\xi-r)^2}{4(1-e^{-(\tau-\tau_0)})}\right)\mathrm{d}r\right)^{1/2} \le Ce^{-\tau/m}. \tag{3.56a}$$

To estimate I_2, we proceed as follows

$$\begin{aligned}\left|I_2\left(\xi e^{(1-\frac{m}{2})\tau},\tau\right)\right| &\le C\int_{\tau_0}^{\tau}\left(-\frac{(ye^{-(\tau-s)/2}-r)^2}{4(1-e^{-(\tau-s)})}\right)\\ &\quad\cdot\left(1+\left(r-ye^{-(\tau-s)/2}\right)^{2m-2}+y^{2m-2}e^{-(m-1)(\tau-s)}\right)e^{2(1-\frac{m}{2})s}\mathrm{d}r\mathrm{d}s\\ &\quad+C\int_{\tau_0}^{\tau}\frac{e^{\tau-s}}{(4\pi(1-e^{-(\tau-s)}))^{1/2}}\int_{\mathbb{R}}\exp\left(-\frac{(ye^{-(\tau-s)/2}-r)^2}{4(1-e^{-(\tau-s)})}\right)\chi_2(r,s)\mathrm{d}r\mathrm{d}s\\ &\equiv I_{2,1}+I_{2,2}.\end{aligned}$$

Recalling (3.55), we obtain

$$|I_{2,1}| \le Ce^{\left(\frac{2}{m}-1\right)\tau}(1+\tau) \le C\tau e^{\left(\frac{2}{m}-1\right)\tau},$$

whereas

$$|I_{2,2}| \le C\exp\left(-CR^2\exp\left(\left(1-\frac{2}{m}\right)\tau_0\right)\right)$$

so that

$$\left|I_2\left(\xi e^{\left(\frac{1}{2}-\frac{1}{m}\right)\tau},\tau\right)\right| \le C\tau e^{\left(\frac{2}{m}-1\right)\tau}, \qquad \text{as } \tau\to\infty, \tag{3.56b}$$

and (3.42b) follows from (3.56). ∎

We next improve the convergence result in Proposition 3.2 as follows

PROPOSITION 3.4. *Let $R(y,\tau)$ be the function given in* (3.20), *and let α be as in* (3.21). *Then*

a) *If* (1.10b) *holds, we have*

$$\lim_{\tau\to\infty}\left(e^{-\tau}R(\xi\sqrt{\tau},\tau)\left(1+\left(\frac{p-1}{4p}\right)\xi^2\right)^{\frac{p}{p-1}}\right)=\alpha \tag{3.57a}$$

uniformly on sets $|\xi|\le M$ with $M>0$.

b) *If* (1.10c) *holds, we have that for some real* C

$$\lim_{\tau\to\infty}\left(e^{-\tau}R\left(\xi e^{\left(\frac{1}{2}-\frac{1}{m}\right)\tau},\tau\right)(1+Cc_m\xi^m)^{\frac{p}{p-1}}\right)=\alpha \tag{3.57b}$$

uniformly on sets $|\xi|\le M$ *with* $M>0$.

PROOF. Assume first that (1.10b) holds. Consider the auxiliary function

$$H(\xi)=\alpha\left(1+\left(\frac{p-1}{4p}\right)\xi^2\right)^{-\frac{p}{p-1}}$$

so that H satisfies

$$\frac{\xi H'(\xi)}{2}=\frac{p}{p-1}\left(\left(1+\left(\frac{p-1}{4p}\right)\xi^2\right)^{-1}-1\right)H(\xi). \tag{3.58}$$

Define

$$\overline{R}(y,\tau)=e^{\tau}H\left(\frac{y}{\sqrt{\tau}}\right),\qquad \xi=\frac{y}{\sqrt{\tau}}.$$

Using (3.58), we see that

$$\begin{aligned}&\overline{R}_\tau-\overline{R}_{yy}+\frac{1}{2}y\overline{R}_y+p\Phi^{p-1}\overline{R}+\frac{\overline{R}}{p-1}\\&=-\frac{e^\tau}{\tau}\left(H''(\xi)+\frac{\xi H'(\xi)}{2}\right)\\&\quad+e^\tau H(\xi)\left(\frac{p}{p-1}\left(1+\left(\frac{p-1}{4p}\right)\xi^2\right)^{-1}-p\Phi^{p-1}(\xi)\right).\end{aligned}$$

Therefore, if we write $W(y,\tau)=R(y,\tau)-\overline{R}(y,\tau)$, W satisfies

$$\begin{aligned}&W_\tau-W_{yy}+\frac{1}{2}yW_y-p\Phi^{p-1}W+\frac{W}{p-1}\\&=-\frac{e^\tau}{\tau}\left(H''(\xi)+\frac{\xi H'(\xi)}{2}\right)\\&\quad-e^\tau H(\xi)\left(\frac{p}{p-1}\left(1+\left(\frac{p-1}{4p}\right)\xi^2\right)^{-1}-p\Phi^{p-1}(\xi)\right)\end{aligned}$$

so that $Z=|W|$ satisfies

$$\begin{aligned}Z_\tau\le Z_{yy}-\frac{1}{2}yZ_y-p\Phi^{p-1}Z-\frac{Z}{p-1}+\frac{Ce^\tau}{\tau}\\+Cpe^\tau\left|\frac{1}{p-1}\left(1+\frac{(p-1)}{4p}\xi^2\right)^{-1}-\Phi^{p-1}\right|.\end{aligned}$$

Let $K > 0$ be fixed, and let $\chi_3(y,\tau) = 1$ if $|y| \geq K\sqrt{\tau}$, $\chi_3(y,\tau) = 0$ otherwise. Recalling (3.42a), we finally arrive at

$$Z_\tau \leq Z_{yy} - \frac{1}{2} y Z_y + p\Phi^{p-1} Z - \frac{Z}{p-1} + \frac{Ce^\tau}{\tau} + C\frac{e^\tau \log \tau}{\tau} + Ce^\tau \chi_3,$$

whence, if $\tau > \tau_0 > 0$ and τ_0 is large enough

$$\begin{aligned} Z(y,\tau) \leq \frac{Ce^{\tau-\tau_0}}{(4\pi(1-e^{-(\tau-\tau_0)}))^{1/2}} \int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(\tau-\tau_0)/2} - r)^2}{4(1-e^{-(\tau-\tau_0)})}\right) Z(r,\tau_0)\mathrm{d}r \\ + C\int_{\tau_0}^{\tau} \frac{e^{\tau-s}}{(4\pi(1-e^{-(\tau-s)}))^{1/2}} \\ \int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(\tau-s)/2} - r)^2}{4(1-e^{-(\tau-s)})}\right)\left(\frac{e^s}{s}(1+\log s) + e^s\chi_3\right)\mathrm{d}r\mathrm{d}s \\ \equiv J_1(y,\tau) + J_2(y,\tau). \end{aligned}$$

As we have repeatedly done so far, we set $e^{\tau-\tau_0} = \tau$, $y = \xi\sqrt{\tau}$, and notice that by Proposition 3.2 (cf. the remarks following (3.45))

$$\|R(\cdot,\tau) - \overline{R}(\cdot,\tau)\| = o(e^\tau) \qquad \text{as } \tau \to \infty.$$

We then obtain that

$$|J_1(\xi\sqrt{\tau},\tau)| \leq Ce^{\tau-\tau_0}\|Z(\cdot,\tau_0)\| = o(e^\tau) \qquad \text{as } \tau \to \infty,$$

whereas

$$|J_2(\xi\sqrt{\tau},\tau)| \leq Ce^\tau \frac{(\log \tau)^2}{\tau} = o(e^\tau) \qquad \text{as } \tau \to \infty,$$

whence (3.57a). To derive (3.57b), we replace our former choice of H by

$$H(\xi) = \alpha(1 + Cc_m\xi^m)^{-\frac{p}{p-1}}$$

and proceed exactly as before. ∎

We shall require later a refinement of (3.57b). Assume that (1.10c) holds, $\alpha = 0$ in (3.21) and

$$\begin{aligned} &R(y,\tau) = \alpha_1 e^{\tau/2} H_1(y) + o(e^{\tau/2}) \qquad \text{as } \tau \to \infty, \text{ for some } \alpha_1, \\ &\text{where convergence takes place in } H^1_w \\ &\text{(and } C^{k,\gamma}_{\mathrm{loc}}(\mathbb{R}) \text{ for any } k \geq 1 \text{ and } \gamma \in (0,1)). \end{aligned} \tag{3.60}$$

Let $H_1(y) = c_1 y$. We shall prove

LEMMA 3.5. *Suppose that* (1.10c) *and* (3.60) *hold. Then*

$$\lim_{\tau\to\infty}\left(e^{-\left(1-\frac{1}{m}\right)\tau}R\left(\xi e^{\left(\frac{1}{2}-\frac{1}{m}\right)\tau},\tau\right)\right)=\alpha_1 c_1\xi(1+Cc_m\xi^m)^{-\frac{p}{p-1}} \tag{3.61}$$

for some real C uniformly on sets $|\xi|\le M$ with $M>0$.

PROOF. We merely sketch it, since it is similar to that of Proposition 3.4. We set $H(\xi)=\alpha_1 c_1\xi(1+Cc_m\xi^m)^{-\frac{p}{p-1}}$, and notice that

$$\frac{1}{m}(\xi H'(\xi)-H(\xi))=\frac{p}{p-1}H(\xi)\left((1+Cc_m\xi^m)^{-\frac{p}{p-1}}-1\right).$$

We then write

$$\overline{R}(y,\tau)=e^{\left(1-\frac{1}{m}\right)\tau}H(\xi),\qquad \xi=ye^{-\left(\frac{1}{2}-\frac{1}{m}\right)\tau}.$$

Since

$$\begin{aligned}&\overline{R}_\tau-\overline{R}_{yy}+\frac{1}{2}y\overline{R}_y-p\Phi^{p-1}\overline{R}+\frac{\overline{R}}{p-1}\\&=e^{\left(1-\frac{1}{m}\right)\tau}\left(-e^{-\left(1-\frac{1}{m}\right)\tau}H''(\xi)+\left(\frac{p}{p-1}(1+Cc_m\xi^m)^{-1}-p\phi^{p-1}\right)H(\xi)\right),\end{aligned}$$

standard computations show then that $Z=|R-\overline{R}|$ satisfies

$$Z_\tau\le Z_{yy}-\frac{1}{2}yZ_y+\frac{C}{\tau}Z+Ce^{\left(1-\frac{1}{m}\right)\tau}\left(e^{-\left(1-\frac{2}{m}\right)\tau}+e^{-\tau/m}+\chi_4\right),$$

where $\chi_4(y,\tau)=1$ if $|y|\ge Me^{\left(\frac{1}{2}-\frac{1}{m}\right)\tau}$ and $\chi_4(y,\tau)=0$ otherwise, $M>0$ being fixed but arbitrary. Arguing then as in Proposition 3.4, b), the result follows. ∎

At this stage, the reader might think that (3.60) is a rather artificial assumption. This is not the case, however, as shown by the following

LEMMA 3.6. *Assume that* (1.10c) *is satisfied and* $\alpha=0$ *in* (3.21). *Then* (3.60) *holds.*

PROOF. As in that of Proposition 3.2, the startpoint is the differential equation for R (cf. (3.22)), which we rewrite in a slightly different way, namely

$$R_\tau=R_{yy}-\frac{1}{2}yR_y+R+p\left(\Phi^{p-1}-\frac{1}{p-1}\right)R$$

so that, for $\tau>\tau_0\ge 0$, we can represent the solution as follows

$$
\begin{aligned}
R(\cdot,\tau) &= \langle R(\cdot,\tau_0), H_0\rangle\, e^{\tau-\tau_0} + \sum_{k=1}^{\infty} e^{\left(1-\frac{k}{2}\right)(\tau-\tau_0)} \langle R(\cdot,\tau_0), H_k(\cdot)\rangle\, H_k(\cdot) \\
&\quad + pH_0 \int_{\tau_0}^{\tau} e^{\tau-s} \langle H_0, \left(\phi^{p-1}(\cdot,s) - \frac{1}{p-1}\right) R(\cdot,s)\rangle\, \mathrm{d}s \\
&\quad + \sum_{k=1}^{\infty} pH_K(\cdot) \int_{\tau_0}^{\tau} e^{\left(1-\frac{k}{2}\right)(\tau-s)} \langle H_K, \left(\phi^{p-1}(\cdot,s) - \frac{1}{p-1}\right) R(\cdot,s)\rangle\, \mathrm{d}s \\
&\equiv T_1 + T_2 + T_3 + T_4.
\end{aligned}
\tag{3.62}
$$

Set $\|R(\cdot,\tau_0)\| = L$, and let us denote by C a generic positive constant depending on L. Arguing as in [HV1] (cf. Proposition 5.8 there), we readily see that

$$
\|T_2(\cdot,\tau)\| \le Ce^{(\tau-\tau_0)/2}. \tag{3.63a}
$$

Assume now that $\tau > \tau_0 + a^*$, where a^* is the time required for delayed regularizing effects to take place. There holds

$$
\begin{aligned}
\|T_4(\cdot,\tau)\| &\le p \int_{\tau_0}^{\tau_0+a^*} \frac{e^{(\tau-s)/2}}{4(1-e^{-(\tau-s)})^{1/2}} \left\| \left(\Phi^{p-1}(\cdot,s) - \frac{1}{p-1}\right) R(\cdot,s) \right\| \mathrm{d}s \\
&\quad + p \int_{\tau_0+a^*}^{\tau} \frac{e^{(\tau-s)/2}}{(1-e^{-(\tau-s)})^{1/2}} \left\| \left(\Phi^{p-1}(\cdot,s) - \frac{1}{p-1}\right) \right\|_{4,w} \|R(\cdot,s)\|_{4,w} \mathrm{d}s.
\end{aligned}
\tag{3.63b}
$$

As recalled in Corollary 2.2 of [HV1], $\|R(\cdot,s)\| \le C$ for $s \in [\tau_0, \tau_0 + a^*]$. Making use of our assumptions, we obtain that

$$
\begin{aligned}
\|T_4(\cdot,\tau)\| &\le Ce^{\tau/2} \left(\int_{\tau_0}^{\tau_0+a^*} \frac{e^{-s/2}\,\mathrm{d}s}{(1-e^{-(\tau-s)})^{1/2}} + \int_{\tau_0+a^*}^{\tau} \frac{e^{-s/2} e^{(1-\frac{m}{2})s} e^{s}}{(1-e^{-(\tau-s)})^{1/2}}\, \mathrm{d}s \right) \\
&\le Ce^{\tau/2} \int_{\tau_0}^{\tau} \frac{e^{-s/2}\,\mathrm{d}s}{(1-e^{-(\tau-s)})^{1/2}} \\
&= Ce^{(\tau-\tau_0)/2} \int_{\tau_0}^{\tau-\tau_0} \frac{e^{-s/2}\,\mathrm{d}s}{(1-e^{-(\tau-\tau_0)}e^{-s})^{1/2}} \\
&\le Ce^{(\tau-\tau_0)/2} \int_{0}^{\infty} \frac{e^{-s/2}\,\mathrm{d}s}{(1-e^{-s})^{1/2}},
\end{aligned}
\tag{3.63c}
$$

whence

$$\|T_4(\cdot,\tau)\| \le Ce^{(\tau-\tau_0)/2}. \tag{3.63d}$$

Since $R(\cdot,\tau) = o(e^\tau)$ in H^1_w as $\tau \to \infty$, it follows from (3.62) and (3.63) that

$$e^{-\tau_0}\langle R(\cdot,\tau_0), H_0\rangle + p\int_{\tau_0}^{\infty} e^{-s}\langle H_0, \left(\phi^{p-1}(\cdot,s) - \frac{1}{p-1}\right) R(\cdot,s)\rangle\, ds = 0 \quad \text{in } L^2_w$$

which, after substitution in (3.62), gives

$$\begin{aligned} R(\cdot,\tau) = {} & -pH_0e^\tau \int_{\tau_0}^{+\infty} e^{-s}\langle H_0\left(\phi^{p-1}(\cdot,s) - \frac{1}{p-1}\right), R(\cdot,s)\rangle\, ds \\ & + e^{(\tau-\tau_0)/2}\langle R(\cdot,\tau_0), H_1\rangle\, H_1 + \sum_{k=2}^{\infty} e^{\left(1-\frac{k}{2}\right)(\tau-\tau_0)}\langle R(\cdot,\tau_0), H_K\rangle\, H_K \\ & + pe^{(\tau-\tau_0)/2}H_1(\cdot)\int_{\tau_0}^{\tau} e^{-(s-\tau_0/2)}\langle H_1, \left(\phi^{p-1}(\cdot,s) - \frac{1}{p-1}\right) R(\cdot,s)\rangle\, ds \\ & + \sum_{k=2}^{\infty} pH_2(\cdot)\int_{\tau_0}^{\tau} e^{\left(1-\frac{k}{2}\right)(\tau-s)}\langle H_k, \left(\Phi^{p-1}(\cdot,s) - \frac{1}{p-1}\right) R(\cdot,s)\rangle\, ds \\ \equiv {} & J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned} \tag{3.64}$$

Using (1.10c), we readily check that

$$\|J_1(\cdot,\tau)\| \le C, \quad \|J_3(\cdot,\tau)\| \le C. \tag{3.65a}$$

On the other hand, arguing as for (3.63b), (3.63c), we now obtain

$$\|J_4(\cdot,\tau)\| \le C \tag{3.65b}$$

$$\|J_5(\cdot,\tau)\| \le Ce^{-\tau_0}(1 + (\tau - \tau_0)). \tag{3.65c}$$

Since obviously $\|J_2(\cdot,\tau)\| \le Ce^{(\tau-\tau_0)/2}$, (3.64) yields at once

$$\|R(\cdot,\tau)\| \le Ce^{(\tau-\tau_0)/2} \qquad (C \text{ depending on } L)$$

which in turn can be used to improve (3.65c) as follows. Let us split the integral in J_5 into two parts, performed over $(\tau_0, \tau_0 + a^*)$ and $(\tau_0 + a^*, \tau)$ respectively, where a^* is as before. Then

$$\|J_5(\cdot,\tau)\| \le C + p\int_{\tau_0+a^*}^{\tau} \left(1 - e^{-(\tau-s)}\right)^{-1/2} \left\| \Phi^{p-1}(\cdot,s) - \frac{1}{p-1} \right\|_{4,w} \|R(\cdot,s)\|_{4,w} \mathrm{d}s$$

$$\le C + C\int_{\tau_0+a^*}^{\tau} e^{(1-\frac{m}{2})s} e^{(s-\tau_0)/2} \left(1 - e^{-(\tau-s)}\right)^{-1/2} \mathrm{d}s \le C.$$

We have then obtained that

$$\left\| R(\cdot,\tau) - \alpha_1 H_1(\cdot) e^{(\tau-\tau_0)/2} \right\| \le C, \tag{3.66a}$$

where

$$\alpha_1 = \langle R(\cdot,\tau_0), H_0\rangle + p\int_{\tau_0}^{\infty} e^{-\frac{s}{2}} \langle H_1, \left(\phi^{p-1}(\cdot,s) - \frac{1}{p-1}\right) R(\cdot,s)\rangle \,\mathrm{d}s. \tag{3.66b}$$

Notice that in (3.66b), the integral is considered for any $s \ge \tau_0$. This is consistent, since it is readily seen that $\int_{\tau_0}^{\infty}(\cdots) = \int_{\tau_0}^{\tau}(\cdots) + \int_{\tau}^{\infty}(\cdots)$, and the second integral is bounded in the L^2_w-norm. Finally, convergence in H^1_w and $C^{k,\gamma}_{\mathrm{loc}}$ is obtained as in similar situations considered before. ∎

We conclude this Section with the following

LEMMA 3.7. *Let* $\mu(\tau_0) = \int_{\tau_0}^{+\infty} e^{-s/2} \langle H_1(\cdot), \left(\phi^{p-1}(\cdot,s) - \frac{1}{p-1}\right) R(\cdot,s)\rangle \,\mathrm{d}s$, *and assume that* $\|R(\cdot,\tau_0)\| \le L$. *Then*

$$\mu(\tau_0) \to 0 \quad \textit{as } \tau_0 \to \infty, \textit{ uniformly on } L. \tag{3.67}$$

PROOF. Let $a > 0$ be arbitrarily small. Then, for $\tau > \tau_0 + a$, a careful examination of the results in [HV1, Section 2] shows that

$$\|R(\cdot,\tau)\|_{2+\varepsilon,w} \le C_\varepsilon \|R(\cdot,\tau-a)\| \qquad \text{for some } \varepsilon > 0,\ C_\varepsilon > 0.$$

Let q be given by $\frac{1}{2+\varepsilon} = \frac{1}{2} - \frac{1}{q}$ $\left(\text{i.e., } q = \frac{2(2+\varepsilon)}{\varepsilon}\right)$. We can then bound $\mu(\tau_0)$ as follows

$$|\mu(\tau_0)| \le p \int_{\tau_0}^{\tau_0+a} e^{-(s-\tau_0)/2} \left\| \left(\Phi^{p-1}(\cdot\,,s) - \frac{1}{p-1} \right) R(\cdot\,,s) \right\| \mathrm{d}s$$

$$+ \int_{\tau_0+a}^{\infty} e^{-(s-\tau_0)/2} \left\| \Phi^{p-1}(\cdot\,,s) - \frac{1}{p-1} \right\|_{2(2+\varepsilon)/\varepsilon,w} \|R(\cdot\,,s)\|_{2+\varepsilon,w} \mathrm{d}s \equiv \mu_1 + \mu_2.$$

Since $\|R(\cdot\,,s)\| \le C_L$ for $s \in [\tau_0, \tau_0 + a] \subset [\tau_0, \tau_0 + 1]$, we see that

$$|\mu(\tau_0)| \le C_L \int_{\tau_0}^{\tau_0+a} e^{-(s-\tau_0)/2} \mathrm{d}s + C_\varepsilon \int_{\tau_0+a}^{\infty} e^{-(s-\tau_0)/2} e^{(1-\frac{m}{2})s} \|R(\cdot\,,s-a)\| \mathrm{d}s,$$

where we need to assume $\tau_0 \gg 1$, since the regularizing time for the $L_w^{2(1+\varepsilon)/\varepsilon}$ norm in μ_2 is very long indeed if $0 < \varepsilon \ll 1$. As $\|R(\cdot\,,s-a)\| \le C_L e^{(s-\tau_0)/2}$, we arrive at

$$|\mu(\tau_0)| \le C_L a + C_{\varepsilon,L} e^{-\tau_0}.$$

Taking then a $\ll 1$ and $\tau_0 \gg 1$, the result follows. ∎

4. - Approximation properties

Consider the auxiliary function $G(x,t)$ given by

$$u(x,t) = \tilde{u}(x,t) + \varepsilon \tilde{R}(x,t) + G(x,t), \tag{4.1}$$

where u, $\tilde{u}$ and $\tilde{R}$ are as in the previous Section. Our goal here consists in providing estimates of $\tilde{R}$ and G in terms of $\tilde{u}$. To this end, we shall assume that the following assumptions hold

(4.2a) There exists $\eta > 0$ such that $u_0(x) \ge \eta > 0$ for any $x \in \mathbb{R}$,

(4.2b) The blow up set of $\tilde{u}(x,t)$ remains in a compact subset of the real line.

We then show

LEMMA 4.1. *Let $\tilde{u}$, $\tilde{R}$ be as in* (4.1), *and let $T > 0$ be the blow up time of $\tilde{u}$. There exists $M > 0$ depending on $\tilde{u}_0, \tilde{R}_0$ and η such that*

$$|\tilde{R}(x,t)| \le M(\tilde{u}(x,t))^p, \qquad \textit{when } x \in \mathbb{R},\ \frac{T}{2} < t < T. \tag{4.3}$$

PROOF. As a starting point, we shall prove that

$$|\tilde{R}(x,t)| \le C(T-t)^{-\frac{p}{p-1}} \qquad \text{for some } C > 0. \tag{4.4}$$

Since we are assuming that (4.2b) holds, and we know that blow up points are isolated (cf. [CM], [HV3]), it turns out that there is a finite number of blow up points x_i, $1 \le i \le n$. For simplicity, we shall assume throughout that there is only one such point, located at $x = 0$. Clearly, it suffices to prove (4.3) for t close enough to T. We have already seen in Section 2 that there exists $C > 0$ such that

$$\tilde{u}(x,t) < ((p-1)(T-t))^{-\frac{1}{p-1}} \qquad \text{if } |x| > C(T-t)^{1/2}. \tag{4.5}$$

On the other hand, we have just shown in Section 3 that there exists $A > 0$ such that

$$|\tilde{R}(x,t)| \le A(T-t)^{-\frac{p}{p-1}} \qquad \text{if } |x| \le C(T-t)^{1/2}. \tag{4.6a}$$

Let $D_C = \{(x,t) : |x| > C(T-t)^{1/2}\}$. By (4.5), $Z = |\tilde{R}|$ satisfies

$$Z_t \le Z_{xx} + \frac{pZ}{(p-1)(T-t)} \qquad \text{in } D_C.$$

We now observe that

$$\text{For any } B > 0,\ \overline{Z}(x,t) \equiv \overline{Z}_B(x,t) = B(T-t)^{-\frac{1}{p-1}}$$

$$\text{is such that } \overline{Z}_t - \overline{Z}_{xx} - \frac{p\overline{Z}}{(p-1)(T-t)} = 0,$$

which by comparison implies that, if $B \ge A$ is large enough,

$$|\tilde{R}(x,t)| \le B(T-t)^{-\frac{p}{p-1}} \qquad \text{in } D_C. \tag{4.6b}$$

An immediate consequence of (4.6) is that

$$\begin{aligned} &|\tilde{R}(x,t)| \le C_1(T-t)^{-\frac{p}{p-1}} \qquad \text{for any } C_1 > 0 \text{ such that} \\ &C_1 \ge A \text{ (cf. (4.6a))} \qquad \text{and} \qquad C_1 T^{-\frac{p}{p-1}} \ge \|\tilde{R}_0\|_\infty \end{aligned} \tag{4.7}$$

so that (4.4) follows. We next proceed to prove (4.3). Assume first that (1.10b) holds. Then, by (1.11b) we have that

$$\begin{aligned} &\text{There exists } \alpha > 0 \text{ such that} \\ &\tilde{u}(x,t) \ge \alpha(T-t)^{-\frac{1}{p-1}} \qquad \text{for } |x| \le (T-t)^{1/2}|\log(T-t)|^{1/2} \\ &\text{and } t \text{ close enough to } T. \end{aligned} \tag{4.8}$$

From (4.7) and (4.8), we conclude that

$$\begin{aligned} &|\tilde{R}(x,t)| \le \beta\tilde{u}(x,t)^p \qquad \text{for some } \beta > 0, \\ &|x| \le (T-t)^{1/2}|\log(T-t)|^{1/2} \text{ and } t \text{ close enough to } T. \end{aligned} \tag{4.9}$$

Following [HV3], we next define

$$v_s(x,t) = (T-s)^{\frac{1}{p-1}}\tilde{u}(\lambda(s)+s(T-s)^{1/2},\ s+t(T-s)), \tag{4.10a}$$
$$\text{where } 0<s<T, \text{ and } \lambda(s)=(T-s)^{1/2}|\log(T-s)|^{1/2}.$$

As shown in [HV3], there holds

$$\lim_{s\uparrow T} v_s(x,t) = (p-1)^{-\frac{1}{p-1}}\left((1-t)+\frac{p-1}{4p}\right)^{-\frac{1}{p-1}}, \tag{4.10b}$$
$$\text{uniformly on sets } |x|\le K,\ 0\le t\le 1.$$

Consider now the auxiliary function

$$H_s(x,t) = (T-s)^{\frac{p}{p-1}}\tilde{R}\left(\lambda(s)+x(T-s)^{1/2},\ s+t(T-s)\right).$$

Clearly, H_s satisfies

$$(H_s)_t = (H_s)_{xx} + p(v_s(x,t))^{p-1}H_s$$

so that, using (4.10b) we see that $H(x,t)=|H_s(x,t)|$ is such that

$$H_t \le H_{xx} + K_1 H \qquad \text{for some } K_1>0, \tag{4.11a}$$
$$\text{whenever } |x|\le K \text{ and } 0\le t\le 1, \text{ uniformly as } s\to T.$$

On the other hand, by Proposition 3.4 and estimate (4.6a)

$$H(x,0) \text{ is uniformly bounded on sets } |x|\le K, \text{ and} \tag{4.6b}$$
$$H(x,t)\le A(1-t)^{-\frac{p}{p-1}} \text{ for } |x|\le K, \text{ uniformly as } s\to T.$$

Since $H_1(x,t)=e^{-K_1 t}H(x,t)$ is subcaloric on the set described in (4.11a) as $s\uparrow T$, we may use explicit representation formulae for solutions of the heat equation on bounded domains to obtain that

$$\text{There exists } C_1=C_1(K)>0 \text{ such that } |H_s(x,t)|\le C_1 \tag{4.12}$$
$$\text{when } |x|\le \frac{K}{2} \text{ and } 0\le t\le 1, \text{ uniformly as } s\to T.$$

In particular, setting $x=0$ and recalling that $\tilde{u}(x,t)\ge\eta>0$ by (4.2a), we deduce that

$$|H_s(0,t)|\le C_1\le C_2(v_s(0,t))^p \qquad \text{for } C_2=C_1\eta^{-p}.$$

Back to the original variables, this gives

$$|\tilde{R}(\lambda(s),s+t(T-s))|\le C_2\tilde{u}\,(\lambda(s),s+t(T-s))^p \tag{4.13}$$
$$\text{for any } t\in[0,1] \text{ and } s \text{ close enough to } T$$

and in turn, (4.13) can be restated as follows

$$|\tilde{R}(x,t)| \leq C_2\tilde{u}(x,t)^p \text{ whenever } T-\delta \leq t \leq T, \text{ and}$$

$$(T-t)^{1/2}|\log(T-t)|^{1/2} \leq |x| \leq \delta, \tag{4.14}$$

provided that $\delta > 0$ is small enough.

Putting together (4.9) and (4.14), we readily see that (4.3) holds indeed on some cylinder $Q_1 = \{(x,t) : |x| \leq \delta,\ T-\delta \leq t \leq T\}$ with $\delta > 0$ small enough. However, since $\tilde{u}(x,t)$ is bounded whenever $|x| \geq \delta$, we see at once that $W = |\tilde{R}|$ satisfies there

$$W_t \leq W_{xx} + p\tilde{u}^{p-1}W \leq W_{xx} + CW$$

for some $C > 0$. An argument analogous to that leading to (4.12b) yields then that $|\tilde{R}(x,t)| \leq C_2$ for some $C_2 > 0$ and any (x,t) outside of Q_1, and the result follows under our current assumptions. On the other hand, it is obvious that the previous argument carries over when blow up occurs at a finite number of points. Finally, the case where (1.10c) holds is similarly dealt with. ∎

We next obtain the main result in this Section, which yields an estimate for the error term G in (4.1).

PROPOSITION 4.2. *Let $u, \tilde{u}, \tilde{R}$ and G be as in* (4.1), *and assume that* (4.2) *holds. Then for any $\varepsilon > 0$ small enough, there exist positive constants C and θ, depending on $\tilde{u}_0$ but not on ε, such that*

$$|G(x,t)| \leq C\varepsilon^2\tilde{u}^{2p-1} \quad \textit{for any } x \in \mathbb{R} \textit{ and } 0 \leq t \leq T-\theta\varepsilon. \tag{4.15}$$

PROOF. As before, we examine first the case where blow up occurs at $x = 0$ only. We begin by remarking that the inequality in (4.15) can be recast as follows

$$\left|\frac{G}{\tilde{u}}\right| \leq C\varepsilon^2\tilde{u}^{2(p-1)}. \tag{4.16}$$

Since $G(x,0) = 0$ and $\tilde{u}(x,t) \geq \eta > 0$, it follows by continuity of blow up time with respect to the initial values (cf. for instance [HV2]) that for any given $\varepsilon > 0$ and $C > 0$, (4.16) holds for any $x \in \mathbb{R}$ and any $t > 0$ sufficiently small, say $0 < t < \delta$, where δ possibly depends on ε. On the other hand, G satisfies

$$G_t = G_{xx} + (\tilde{u} + \varepsilon\tilde{R} + G)^p - \tilde{u}^p - \varepsilon p\tilde{u}^{p-1}\tilde{R} \equiv G_{xx} + h(x,t). \tag{4.17a}$$

Since $\tilde{u}(x,t) \leq d(T-t)^{-\frac{1}{p-1}}$ for some $d > 0$ (cf. for instance (3.47)) and (4.3) holds, we now have that

$$\left|\frac{G}{\tilde{u}}\right| \leq Cd^{2(p-1)}\varepsilon^2(T-t)^{-2} \quad \text{if } t < \delta, \qquad \varepsilon\left|\frac{\tilde{R}}{\tilde{u}}\right| \leq Md^{p-1}(T-t)^{-1}\varepsilon.$$

Suppose now that $T - t \geq \theta\varepsilon$, where $\theta > 0$ is such that

$$Cd^{2(p-1)}\theta^{-2} \leq 1, \qquad Md^p\theta^{-1} \leq 1. \tag{4.17b}$$

We can then estimate the term $h(x,t)$ in (4.17a) as follows

$$\begin{aligned} |h(x,t)| &= \left|\tilde{u}^p\left(1+\varepsilon\frac{\tilde{R}}{\tilde{u}}+\frac{G}{\tilde{u}}\right)^p - \tilde{u}^p - \varepsilon p\tilde{u}^{p-1}\tilde{R}\right| \\ &\leq p\tilde{u}^{p-1}|G| + K_1\tilde{u}^p\left(\varepsilon^2\left(\frac{\tilde{R}}{\tilde{u}}\right)^2 + \left(\frac{G}{\tilde{u}}\right)^2\right) \end{aligned} \tag{4.17c}$$

for some $K_1 = K_1(p) > 0$, and any $t \leq \min\{\delta, T-\theta\varepsilon\} \equiv \tau(\delta,\varepsilon)$. We then have that $W = |G|$ satisfies

$$W_t \leq W_{xx} + p\tilde{u}^{p-1}W + K_1\tilde{u}^p\left(\varepsilon^2M^2\tilde{u}^{2(p-1)} + \left(C\varepsilon^2\tilde{u}^{2(p-1)}\right)^2\right)$$

in the strip $S_\tau = \mathbb{R}\times(0,\tau)$. Furthermore, recalling (4.17b) and selecting $K = 2K_1$, we obtain

$$W_t \leq W_{xx} + p\tilde{u}^{p-1}W + \varepsilon^2K\tilde{u}^{3p-2} \qquad \text{in } S_\tau. \tag{4.18}$$

Notice that C does not appear in (4.18). We now clain that

(4.19) There exists $C_1 > 0$ independent of C such that

$$\left|\frac{G}{\tilde{u}}\right| \leq C_1\varepsilon^2\tilde{u}^{2(p-1)} \text{ in any strip } S_\delta,$$

provided that (4.18) is satisfied there.

Once the claim has been established, the result follows from a typical continuation argument. Indeed, we select $C = 2C_1$, and denote by t^* the supremum of those times t for which (4.16) holds in S_t. Now, if $t^* < T - \theta\varepsilon$, we would obtain (4.19) in S_t, and by continuity (4.16) would be satisfied for some $t > t^*$, which contradicts the choice of t^*.

To prove (4.19), we proceed in several steps. First, we show that

(4.20) $$W(x,t) \leq \lambda\varepsilon^2(T-t)^{-\frac{p}{p-1}-1} \qquad \text{for some } \lambda > 0 \text{ large enough}$$

in any strip S_σ where (4.18) holds.

This is achieved as follows. First, for fixed $\sigma < T$, $\tilde{u}(x,t)$ is bounded in S_σ. Therefore, if (4.18) holds in S_σ we have that

$$W_t \leq W_{xx} + \mu W + \mu\varepsilon^2 \qquad \text{in } S_\sigma \text{ for some } \mu > 0.$$

Standard caloric estimates yield then

$$W(x,\sigma) \le \nu\varepsilon^2 \qquad \text{for some } \nu > 0 \tag{4.21}$$

which implies (4.20) if t is not too close to T. When $t \sim T$, we use (3.47) to remark that there exists then $\delta > 0$ (which can be supposed as small as we want) such that

$$\tilde{u}(x,t) \le \left((p-1)^{-\frac{1}{p-1}} + \delta\right)(T-t)^{-\frac{1}{p-1}}. \tag{4.22}$$

We then consider the auxiliary function $f(x,t) = \lambda_1\varepsilon^2(T-t)^{-\frac{p}{p-1}-1}$ with $\lambda_1 > 0$ to be selected presently, and take advantage of (4.22) to compute

$$\begin{aligned}\mathcal{L}f \equiv f_t - f_{xx} - p\tilde{u}^{p-1}f - K\varepsilon^2\tilde{u}^{3p-2} \ge \varepsilon^2(T-t)^{-\frac{p}{p-1}-2}\Bigg(\lambda_1\left(\frac{p}{p-1}+1\right)\\ - p\lambda_1\left((p-1)^{-\frac{1}{p-1}}+\delta\right)^{p-1} - K\left((p-1)^{-\frac{1}{p-1}}+\delta\right)^{3p-2}\Bigg)\end{aligned}$$

so that $\mathcal{L}f \ge 0$ for $t \sim T$ if $\lambda_1 \gg 1$. We then deduce (4.20) from (4.21) and (4.22).

Having shown (4.20), we continue with the proof of (4.19) by specializing to the case where (1.10b) occurs. We then take up an argument already used in the proof of our previous Lemma. Clearly, (4.19) holds in the set $|x| \le (T-t)^{1/2}|\log(T-t)|^{1/2}$, since $\tilde{u}(x,t) \ge \omega(T-t)^{-\frac{1}{p-1}}$ there for some $\omega > 0$, and (4.20) is satisfied. We then consider the function $v_s(x,t)$ given in (4.10a) and define

$$g(x,t) \equiv g_s(x,t) = (T-s)^{\frac{p}{p-1}+1}W\left(\lambda(s) + x(T-s)^{1/2},\ s + t(T-s)\right)$$

where $0 < s < T$, $W = |G|$ and $\lambda(s)$ is given in (4.10a). We readily check that

$$g_t \le g_{xx} + pv_s^{p-1}g + \varepsilon^2\Lambda(v_s)^{3p-2}$$

for some $\Lambda > 0$. By the results in [HV3], $v_s(x,t)$ is bounded in cylinders $Q_R = \{(x,t) : |x| \le R,\ 0 \le t \le 1\}$ as $s \to T$, whereas (4.20) implies that $g_s(x,t) \le \lambda\varepsilon^2(1-t)^{-\frac{p}{p-1}-1}$. We then repeat the steps leading to (4.12) to conclude that

$$0 \le g_s(x,t) \le \gamma\varepsilon^2 \qquad \text{in } Q_{R/2} \text{ for some } \gamma > 0, \text{ uniformly as } s \to T.$$

On the other hand, by (4.10b), $v_s(0,t) \ge \gamma_1 > 0$ as $s \to T$. It then follows that

$$g_s(0,t) \le \gamma_2\varepsilon^2 v_s(0,t)^{3p-2} \qquad \text{for some } \gamma_2 > 0.$$

Back to the original variables, this means that

$$W\big(\lambda(s), s + t(T-s)\big) \le \gamma_3 \varepsilon^2 \tilde{u}\big(\lambda(s), s + T(T-s)\big)^{3p-2}$$

for some $\gamma_3 > 0$, whence (4.19). In the region away from the blow up point $x = 0$, the bound is obtained as in the corresponding result in Lemma 4.1. Clearly, the argument extends to the case of a finite number of blow up points. The case where (1.10c) holds is similar. ∎

We next derive a refinement of (4.15) for the case where $R(y,\tau) = o(e^{\tau})$ in H^1_w as $\tau \to \infty$.

LEMMA 4.3. *Let the assumptions in Lemma* 3.5 *be satisfied, and assume that* ε, θ *are as in the statement of Proposition* 4.2. *Then for any fixed* $\mu > 0$, *there exists* $t_0 = t_0(\mu, \tilde{u}_0)$ *such that for* $\varepsilon > 0$ *small enough there holds*

$$|G(x,t)| \le \frac{C\varepsilon^2}{\mu^2}(T-t)^{-\frac{p}{p-1}-1} \tag{4.23}$$

whenever $|x| \le (T-t)^{\frac{1}{m}}$ *and* $t_0 \le t \le T - \mu\varepsilon$, *where the constant* C *does not depend on* ε *nor* μ.

PROOF. We have shown in the course of the proof of Proposition 4.2 that $Z = |G|$ satisfies

$$Z_t \le Z_{xx} + p\tilde{u}^{p-1}Z + C_1\left(\varepsilon^2\left(\frac{\tilde{R}}{\tilde{u}}\right)^2 + \left(\frac{G}{\tilde{u}}\right)^2\right)\tilde{u}^p \tag{4.24}$$

for $x \in \mathbb{R}$, $0 < t < T - \theta\varepsilon$, where C_1 is independent of ε (cf. (4.16), (4.17)). Recalling (3.57b), it follows that under our current assumptions

$$\frac{|\tilde{R}|}{\tilde{u}} \ll (T-t)^{-1} \qquad \text{uniformly on } |x| \le \xi_0(T-t)^{\frac{1}{m}} \text{ as } t \to T,$$

for any given $\xi_0 > 0$. Taking into account (4.15), we obtain from (4.24) that

$$Z_t \le Z_{xx} + p\tilde{u}^{p-1}Z + C(T-t)^{-\frac{p}{p-1}-2}\varepsilon^2\left(g(t) + \frac{1}{\mu^2}\right), \tag{4.25}$$

when $|x| \le \xi_0(T-t)^{\frac{1}{m}}$ and $0 \le t \le T - \mu\varepsilon$ with $\mu \ge \theta$, where $\lim_{t\to T} g(t) = 0$, and here and henceforth C will denote a generic constant which is independent of ε and μ. We now claim that it is possible to select constants λ and δ, $\delta < 1$, and a nonnegative function $\Lambda(\xi)$ such that

$$W(x,t) = \lambda\varepsilon^2(T-t)^{-\frac{p}{p-1}-\delta} + \varepsilon^2(T-t)^{-\frac{p}{p-1}-1}\Lambda\left(\frac{x}{(T-t)^{1/m}}\right)$$

satisfies

$$W(x,t) \geq Z(x,t) \quad \text{in} \quad Q_{0,\mu} = \left\{(x,t) : |x| < \xi_0(T-t)^{\frac{1}{m}},\ t_0 < t < T - \mu\varepsilon\right\} \tag{4.6}$$

for some $t_0 > 0$, where ξ_0 will be selected later. We shall derive (4.26) by means of a typical comparison argument in the cylinder $Q_{0,\mu}$. Recalling (4.15), we see that, in order to obtain

$$W(x,t_0) \geq Z(x,t_0), \qquad \text{when } |x| \leq (T-t)^{\frac{1}{m}},$$

it suffices to require that $\lambda = C(T-t_0)^{-1+\delta}$. With such a choice, our auxiliary function W reads

$$\begin{aligned} W(x,t) = {} & \varepsilon^2 C(T-t_0)^{-1+\delta}(T-t)^{-\frac{p}{p-1}-\delta} \\ & + \varepsilon^2 (T-t)^{-\frac{p}{p-1}-1}\Lambda\left(\frac{x}{(T-t)^{1/m}}\right). \end{aligned} \tag{4.27a}$$

On the other hand, the desired inequality at the parabolic boundary of $Q_{0,\mu}$ holds provided that

$$\Lambda(\xi_0) \geq C, \tag{4.27b}$$

where C is this time the constant in (4.15). Having obtained (4.27), (4.26) will follow from the maximum principle as soon as we have

$$\begin{aligned} PW \equiv {} & W_t - W_{xx} - p\tilde{u}^{p-1}W \\ & - C(T-t)^{-\frac{p}{p-1}-2}\varepsilon^2\left(g(t) + \frac{1}{\mu^2}\right) \geq 0 \qquad \text{in } Q_{0,\mu}. \end{aligned} \tag{4.28}$$

We now compute

$$\begin{aligned} PW = {} & \left(\frac{p}{p-1} + \delta\right)\varepsilon^2 C(T-t_0)^{-(1-\delta)}(T-t)^{-\frac{p}{p-1}-1-\delta} \\ & + \left(\frac{p}{p-1} + 1\right)\varepsilon^2(T-t)^{-\frac{p}{p-1}-2}\Lambda(\xi) \\ & + \frac{\varepsilon^2}{m}(T-t)^{-\frac{p}{p-1}-2}\xi\Lambda'(\xi) - \varepsilon^2(T-t)^{-\frac{p}{p-1}-1-\frac{2}{m}}\Lambda''(\xi) \\ & - C\varepsilon^2 p\tilde{u}^{p-1}(T-t_0)^{-1+\delta}(T-t)^{-\frac{p}{p-1}-\delta} - \varepsilon^2 p\tilde{u}^{p-1}(T-t)^{-\frac{p}{p-1}-1}\Lambda(\xi) \\ & - C\varepsilon^2(T-t)^{-\frac{p}{p-1}-2}\left(g(t) + \frac{1}{\mu^2}\right) \\ \equiv {} & S_1 + S_2 + S_3 - S_4 - S_5 - S_6 - S_7. \end{aligned} \tag{4.29}$$

Notice that we may assume without loss of generality that t_0 is close enough to T (this amounts in practice to restrict our attention to sufficiently small values of ε). Then there exists $\gamma > 0$ small enough such that

$$\tilde{u}^{p-1} \leq \left((p-1)^{-1} + \gamma\right) (T-t)^{-1}$$

in $Q_{0,\mu}$, and $S_1 \geq S_5$ in (4.29) if $p\gamma \leq \delta$. Furthermore, there exists $r \in (0,1)$ such that $S_2 - S_6 \geq rS_2$ in (4.29). As a matter of fact, this inequality holds provided that $1 - r \geq \dfrac{rp}{1-p} + p\gamma$. Since $(T - t_0)$ is small, $g(t) \leq \dfrac{2}{\mu^2}$, and we are thus led to

$$PW \geq \varepsilon^2 (T-t)^{-\frac{p}{p-1}-2} \left(r\Lambda(\xi) + \frac{\xi\Lambda'(\xi)}{m} - (T-t)^{1-\frac{2}{m}} \Lambda''(\xi) - \frac{2C}{\mu^2} \right). \tag{4.30}$$

We now select $\Lambda(\xi)$ in the form

$$\Lambda(\xi) = B(\xi) + \frac{4C}{r\mu^2}, \qquad \text{where } B(\xi) \geq 0, \ \xi B'(\xi) \geq 0,$$

$$|B''(\xi)| \text{ is bounded and } B(\xi_0) \geq C \qquad \text{(cf. (4.27b))},$$

for some large enough ξ_0. With such a choice, we obtain

$$r\Lambda(\xi) + \frac{\xi\Lambda'(\xi)}{m} - (T-t)^{1-\frac{2}{m}} \Lambda''(\xi) - \frac{2C}{\mu^2} \geq \frac{C}{\mu^2} > 0$$

so that (4.28) and (4.26) follow. Finally, in $Q_{0,\mu}$ we certainly have

$$\begin{aligned} W(x,t) &\leq C\varepsilon^2 \left(\frac{T-t}{T-t_0} \right)^{1-\delta} (T-t)^{-\frac{p}{p-1}-1} + \frac{C\varepsilon^2}{\mu^2} (T-t)^{-\frac{p}{p-1}-1} \\ &\leq \frac{C\varepsilon^2}{\mu^2} (T-t)^{-\frac{p}{p-1}-1} \end{aligned}$$

which yields (4.23). ∎

5. - Generic blow up behaviour

In this Section we shall proceed to complete the proof of our main result. This will be done in several steps, each of which is taken care of in the following paragraphs.

5.1. - *Perturbation theory for the linearized problem*

Let $\tilde{u}(x,t)$ be a solution of (1.1) which blows up at $t = T$, and assume that $\tilde{R}(x,t)$ solves (3.19). For any given point $x_i \in \mathbb{R}$, we set

$$R_i(y,\tau) = (T-t)^{\frac{1}{p-1}} \tilde{R}\left(\frac{x - x_i}{(T-t)^{1/2}},\ -\log(T-t)\right). \tag{5.1}$$

We then have

PROPOSITION 5.1. *Assume that $\tilde{u}(x,t) \geq \eta > 0$ and $\tilde{u}(x,t)$ blows up at points $x_1, \dots, x_j$. Let $R_i(y,\tau)$ $(i = 1, \dots, j)$ be as above, and let $\alpha_1, \dots, \alpha_j$ be given real constants. Then for any $\delta > 0$ there exists $\tilde{R}_0(x) \in L^\infty(\mathbb{R})$ with compact support such that, for $i = 1, \dots, j$,*

$$R_i(y,\tau) = \beta_i e^\tau + o(e^\tau) \tag{5.2}$$

as $\tau \to \infty$ in H^1_w and $C^{k,\gamma}_{\mathrm{loc}}$ for any $k \geq 1$ and $\gamma \in (0,1)$, where $|\beta_i - \alpha_i| < \delta$.

Before proving Proposition 5.1, two auxiliary results will be established. For $t > t_0 \geq 0$, let $E(t,t_0)$ be the evolution operator associated to (3.19a) in $L^2(\mathbb{R})$, i.e., for any $f(x) \in L^2(\mathbb{R})$, $E(t,t_0)f(x)$ is the solution $\tilde{R}(x,t)$ of (3.19a) such that $\tilde{R}(x,t_0) = f(x)$. Then there holds

LEMMA 5.2. *Let $\tilde{u}$ be as in the statement of Proposition 5.1, and let T be its blow up time. Then, for any $\delta < T$, the set $\Sigma = \{h(x)$: there exists $g(x) \in L^2(\mathbb{R})$ such that $h = E(T-\delta,0)g\}$ is dense in $L^2(\mathbb{R})$.*

PROOF. Let $G(x,\xi;t,\tau)$ be Green's function associated to (3.19), i.e., let G be a solution of

$$G_t = G_{xx} + p\tilde{u}^{p-1}G \qquad \text{when } x \in \mathbb{R},\ t > \tau, \tag{5.3a}$$

$$G(x,\xi;t,\tau) = \delta(x-\xi) \qquad \text{when } t = \tau, \tag{5.3b}$$

where $\delta(x-\xi)$ is Dirac's delta centered at ξ. Let H be a solution of the adjoint problem

$$-H_t = H_{xx} + p\tilde{u}^{p-1}H \qquad \text{when } x \in \mathbb{R},\ t < \mu \tag{5.4a}$$

$$H(x,\lambda;\mu,\mu) = \delta(x-\mu). \tag{5.4b}$$

Standard arguments (cf. for instance [F, Chapter I, Theorem 5]) yield then

$$G(\lambda,\xi;\mu,\tau) = H(\xi,\lambda;\tau,\mu) \qquad \text{for } \mu > \tau. \tag{5.5}$$

Suppose now that the result is false. Then there exists $t \in (0,T)$ and $h(x) \in L^2(\mathbb{R})$,

$h(x) \not\equiv 0$, such that

$$\int_{\mathbb{R}} h(x) \left(\int_{\mathbb{R}} G(x, \xi; t, 0)\varphi(\xi)\mathrm{d}\xi \right) \mathrm{d}x = 0 \qquad \text{for any } \varphi(x) \in L^2(\mathbb{R}).$$

Using Fubini's Theorem and (5.5), we then obtain

$$0 = \int_{\mathbb{R}} G(x, \xi; t, 0)h(x)\mathrm{d}x = \int_{\mathbb{R}} H(x, \xi; 0, t)h(x)\mathrm{d}x$$

whence

$$\int_{\mathbb{R}} H(x, \xi; 0, t)h(\xi)\mathrm{d}\xi = 0.$$

Set now

$$v(x, s) = \int_{\mathbb{R}} H(x, \xi; s, t)h(\xi)\mathrm{d}\xi; \qquad V(x, s) = v(x, t - s).$$

Then V solves

$$\begin{aligned} &V_s = V_{xx} + p\tilde{u}(x, t - s)^{p-1}V && \text{when } x \in \mathbb{R},\ 0 < s < t, \\ &V(x, 0) = h(x),\ V(x, t) = 0 && \text{if } x \in \mathbb{R}. \end{aligned}$$

Since $\tilde{u} \geq \eta > 0$, classical uniqueness results for backward evolution problems (cf. for instance [LM]) yield then that $h(x) \equiv 0$, and the proof is concluded. ∎

We next recall some bounds on the evolution in time of $R(y, \tau)$. More precisely, let R be a solution of

$$R_\tau = R_{yy} - \frac{1}{2} yR_y + p\Phi^{p-1}R - \frac{R}{p-1}, \qquad y \in \mathbb{R},\ \tau > \tau_0, \tag{5.6a}$$

$$R(\cdot, \tau_0) = R_0(\cdot) \qquad \text{at } \tau = \tau_0. \tag{5.6b}$$

Keeping track of the arguments in Proposition 3.2, we notice that the following estimates hold

(5.7) Assume that $\|R_0\| \leq L$ for some $L > 0$. Then there exist positive constants $A = A(L)$, α_0 and $\sigma \in (0, 1)$ such that, for $\tau > \tau_0$,

$$\|R(\cdot, \tau)\| \leq Ae^{\tau - \tau_0},$$

$$\|R(\cdot, \tau) - \alpha_0 e^{\tau - \tau_0}\| \leq Ae^{\tau - \tau_0}(1 + (\tau - \tau_0)^\sigma)^{-1}.$$

Consider now the linear problem

$$G_\tau = G_{yy} - \frac{1}{2} y G_y + G \qquad \text{when } y \in \mathbb{R},\ \tau > \tau_0, \tag{5.8a}$$

$$G(\cdot\,, \tau_0) = G_0(\cdot) \qquad \text{when } y \in \mathbb{R}. \tag{5.8b}$$

We then have

LEMMA 5.3. *Assume that* $\|R_0\| \le L$ *and* $R(\cdot\,,\tau)$ *solves* (5.6). *Then there exists* $C = C(L)$ *independent of* τ_0, *such that*

a) *If* (1.10b) *holds, then*

$$\|(R-G)(\cdot\,,\tau)\| \le e^{\tau-\tau_0}\Bigg(\|(R-G)(\cdot\,,\tau)\| + C\int_0^\infty ((1+s^\sigma)(\tau_0+s))^{-1}\mathrm{d}s + \frac{C}{\tau}\Bigg) \tag{5.9a}$$

where σ *is as in* (5.7).

b) *If* (1.10c) *holds, then*

$$\|(R-G)(\cdot\,,\tau)\| \le e^{\tau-\tau_0}\left(\|(R-G)(\cdot\,,\tau_0)\| + Ce^{(1-\frac{m}{2})\tau_0}\right). \tag{5.9b}$$

PROOF. Suppose that (1.10b) holds, and set $f = R - G$. Then $f(y,\tau)$ solves

$$\begin{aligned} f_\tau &= f_{yy} - \frac{1}{2} y f_y + f + \left(p\Phi^{p-1} - \frac{p}{p-1} + \frac{KH_2}{\tau}\right) \\ &\quad - \frac{KH_2}{\tau}\left(R - \alpha_0 e^{\tau-\tau_0} H_0\right) - \frac{K\alpha_0}{\tau} e^{\tau-\tau_0} H_2 H_0 \\ &\equiv Af + a_1(y,\tau)R + a_2(y,\tau) + a_3(y,\tau) \end{aligned}$$

where K is the coefficient of $\dfrac{H_2}{\tau}$ in (3.1). Since $\langle H_2, H_0\rangle = 0$, we obtain that

$$\begin{aligned} \|f(\cdot\,,\tau)\| &\le e^{\tau-\tau_0}\|f(\cdot\,,\tau_0)\| + \int_{\tau_0}^{\tau} e^{\tau-s}\|a_1(\cdot\,,s)\|_{4,w}\|R(\cdot\,,s)\|_{4,w}\mathrm{d}s \\ &\quad + \int_{\tau_0}^{\tau} Ks^{-1}e^{\tau-s}\|R - \alpha_0 e^{s-\tau_0}H_0\|_{4,w}\mathrm{d}s + \frac{Ce^{\tau-\tau_0}}{\tau}. \end{aligned} \tag{5.10}$$

We have already shown in (3.25) that

$$\|a_1(\cdot\,,s)\|_{4,w} \le C\frac{\log s}{s^2} \qquad \text{if } s > 0 \text{ is large enough.}$$

On the other hand, using (5.7) and delayed estimates (cf. [HV1, Section 2]) we see that

$$\|R(\cdot,s)\|_{4,w} \le C\|R(\cdot,s-a^*)\| \le Ce^{s-a^*-\tau_0} \qquad \text{if } s > a^*+\tau_0.$$

We now claim that

$$\|R-\alpha_0 e^{\tau-\tau_0}H_0\|_{4,w} \le \frac{Ce^{\tau-\tau_0}}{1+(\tau-\tau_0)^\sigma} \qquad \text{if } \tau > \tau_0. \tag{5.11}$$

Assume that (5.11) holds. Then substitution of the previous inequalities in (5.10) yields

$$\begin{aligned}
\|f(\cdot,\tau)\| &\le e^{\tau-\tau_0}\|f(\cdot,\tau_0)\| + C\int_{\tau_0}^{\tau} e^{\tau-s}s^{-2}(\log s)\ e^{s-a^*-\tau_0}\mathrm{d}s \\
&\qquad + Ce^{\tau-\tau_0}\int_{\tau_0}^{\tau} s^{-1}(1+(s-\tau_0)^\sigma)^{-1}\mathrm{d}s \\
&\le e^{\tau-\tau_0}\|f(\cdot,\tau)\| + Ce^{\tau-\tau_0}\int_{\tau_0}^{\tau} s^{-1}(1+(s-\tau_0)^\sigma)^{-1}\mathrm{d}s \\
&\le e^{\tau-\tau_0}\|f(\cdot,\tau)\| + Ce^{\tau-\tau_0}\left(\int_{\tau_0}^{\tau}(1+r)^\sigma(r+\tau_0)^{-1}\mathrm{d}r + \frac{1}{\tau}\right)
\end{aligned}$$

whence (5.9a) follows. To obtain (5.11), we set $z(y,\tau) = |R(y,\tau)-\alpha_0 e^{\tau-\tau_0}H_0|$ and notice that

$$z_\tau \le z_{yy} - \frac{1}{2}yz_y + Cz + |\alpha_0|e^{\tau-\tau_0}\left|p\Phi^{p-1} - \frac{p}{p-1}\right|H_0$$

which implies

$$\begin{aligned}
z(y,\tau) &\le e^{(C-1)(\tau-\overline{\tau})}S_0(\tau-\overline{\tau})z(y,\overline{\tau}) \\
&\qquad + C\int_{\overline{\tau}}^{\tau} e^{s-\tau_0}S(s-\overline{\tau})\left|p\Phi^{p-1} - \frac{p}{p-1}\right|\mathrm{d}s
\end{aligned} \tag{5.12}$$

where $t > \overline{\tau}$, and S_0 is the semigroup associated to $-Z_{yy} + \frac{1}{2}yZ_y$. We now set $\tau = \overline{\tau} + a^*$, where a^* is as in (3.23), and take advantage of the following estimate

$$\|S_0(\tau)g\|_{p,w} \le \frac{Ce^\tau}{(1-e^{-\tau})^{1/q}}\|g\|_{q,w} \qquad \text{for any } p > q \text{ and } g \in L^q_w \tag{5.13a}$$

(cf. (5.29) in [HV1]). On the other hand, an argument similar to the one leading to (3.25), gives

$$\left\| p\Phi^{p-1}(\cdot\,,s) - \frac{p}{p-1} \right\|_{r,w} \le \frac{C}{s} \qquad \text{for any } r > 1 \text{ and } s > 0. \tag{5.13b}$$

Using (5.7) and (5.13) in (5.12), we arrive at

$$\begin{aligned} z(y,\tau) &\le C\|S(a^*)z(\cdot\,,\overline{\tau})\|_{4,w} + C\int_{\overline{\tau}}^{\tau} e^{s-\tau_0}\left\| S(s-\overline{\tau})\left(p\Phi^{p-1}(\cdot\,,s) - \frac{p}{p-1}\right)\right\|_{4,w} \mathrm{d}s \\ &\le C\|z(\cdot\,,\overline{\tau})\| + C\int_{\overline{\tau}}^{\tau} e^{s-\tau_0}\left\| p\Phi^{p-1}(\cdot\,,s) - \frac{p}{p-1}\right\|_{5,w} (1-e^{-(\tau-s)})^{-1/5}\mathrm{d}s \\ &\le Ce^{\overline{\tau}-\tau_0}(1+(\overline{\tau}-\tau_0)^{\sigma})^{-1} + \frac{Ce^{\tau-\tau_0}}{\tau-\tau_0}\int_{\overline{\tau}}^{\tau}(1-e^{-(\tau-s)})^{-1/5}\mathrm{d}s \\ &\le Ce^{\tau-\tau_0}(1+(\tau-\tau_0)^{\sigma})^{-1} \end{aligned}$$

whence (5.11). The proof of (5.9b) is similar, and will be omitted. ∎

End of the proof of Proposition 5.1

Let $\varsigma \in C_0^\infty(\mathbb{R})$ be a standard nonnegative cutoff function such that $0 \le \varsigma \le 1$, $\varsigma = 1$ if $|x| \le 1$ and $\varsigma = 0$ if $|x| \ge 2$. Let ρ, θ be positive constants to be selected later, and consider the function

$$g(x) = \theta^{-\frac{1}{p-1}} \sum_{j=1}^{k} \alpha_j \varsigma\left(\frac{x-x_j}{\rho\sqrt{\theta}}\right) H_0(\cdot\,). \tag{5.14}$$

Pick now $i \in [1,k]$, set $\tau_0 = -\log\theta$, $y = \dfrac{x-x_i}{\sqrt{\theta}}$, and define

$$\tilde{G}_i(y,\tau_0) = \sum_{j=1}^{k} \alpha_j \varsigma\left(\frac{y}{\rho} + \frac{x_i - x_j}{\rho\sqrt{\theta}}\right) H_0(y) \equiv \theta^{\frac{1}{p-1}} g(x). \tag{5.15}$$

We readily check that

$$\begin{aligned} &\|\tilde{G}_i(y,\tau_0) - \alpha_i H_0(y)\| \\ &\le |\alpha_i| \left\| H_0\left(\varsigma\left(\frac{y}{\rho}\right) - 1\right)\right\| + \sum_{j\ne 1}^{k} |\alpha_j| \left\| H_0\left(\varsigma\left(\frac{y}{\rho} + \frac{x_i - x_j}{\rho\sqrt{\theta}}\right)\right)\right\| \equiv A_1 + A_2. \end{aligned}$$

We now select $\rho > 0$ large enough so that $A_1 \le \dfrac{\delta}{6}$, and then $\theta > 0$ small enough

so that $A_2 \leq \frac{\delta}{6}$. We thus obtain that at the time $\tau_0 = -\log\theta$,

$$\|\tilde{G}_i(y,\tau_0) - \alpha_i H_0(y)\| \leq \frac{\delta}{3}. \tag{5.16}$$

Let $E(t,0)$ be the evolution operator associated to (3.19), which has been recalled in this paragraph right after the statement of Proposition 5.1. By Lemma 5.2 and the continuity properties of E, we see that there exists $h_0(x) \in C_0^\infty(\mathbb{R})$ such that $h(x) \equiv E(T-\theta,0)h_0(x)$ satisfies

$$\theta^{\frac{1}{p-1}} \left(\int_{\mathbb{R}} (g(x) - h(x))^2 \mathrm{d}x \right)^{1/2} \leq \frac{\delta}{3},$$

where $g(x)$ is given in (5.14). Notice that

$$\begin{aligned} &\|\tilde{G}_i(y,\tau_0) - \theta^{\frac{1}{p-1}} h(x_i + \theta^{1/2} y, \tau_0)\| \\ &= \theta^{\frac{1}{p-1}} \left(\int_{\mathbb{R}} (g(x) - h(x))^2 \mathrm{d}x \right)^{1/2} \leq \frac{\delta}{3}. \end{aligned} \tag{5.17}$$

For $\tau > \tau_0$, we now consider the following functions

i) $G_i(y,\tau)$ defined as the solution of (5.8) such that

$$G_i(y,\tau_0) = e^{\tau_0} \tilde{G}_i(y,\tau_0),$$

ii) $h(x,t) = E(t,0)h_0(x), \qquad R^*(x,t) = e^{\tau_0} h(x,t),$

iii) $R_i(y,\tau_0) = \theta^{\frac{1}{p-1}} R^* \left(\frac{x - x_i}{\sqrt{\theta}}, \ -\log\theta \right) = e^{\tau_0} \hat{R}_i(y,\tau_0).$

Suppose now that (1.10b) holds. Recalling (5.8a), we have

$$\begin{aligned} &\|R_i(y,\tau) - G_i(y,\tau)\| = e^{\tau_0} \|\hat{R}_i(y,\tau_0) - \tilde{G}_i(y,\tau_0)\| \\ &\leq e^{\tau} \left(\|\hat{R}_i(y,\tau_0) - \tilde{G}_i(y,\tau_0)\| + C \int_0^\infty ((1+s^\sigma)(s+\tau_0))^{-1} \mathrm{d}s + \frac{C}{\tau} \right). \end{aligned} \tag{5.18}$$

On the other hand, by (5.17), $\|\hat{R}_i(y,\tau_0) - \tilde{G}_i(y,\tau_0)\| \leq \frac{\delta}{3}$. Furthermore, we clearly have that $\|\tilde{G}_i(y,\tau_0)\| \leq C$ where C depends only on $|\alpha_1|, \ldots, |\alpha_k|$. Since $\|\hat{R}_i(y,\tau_0)\| \leq \|\tilde{G}_i(y,\tau_0)\| + \|\hat{R}_i(y,\tau_0) - \tilde{G}_i(y,\tau_0)\|$, we deduce that $\|\hat{R}_i(y,\tau_0)\|$ can be bounded by some constant L uniformly as $\tau_0 \to \infty$, and therefore we may assume that the two last terms in the right-hand side in (5.18) are

bounded above by $\frac{\delta}{3}$ if θ is small enough (i.e., τ_0 is large enough). It then follows that

$$\|R_i(y,\tau_0) - G_i(y,\tau_0)\| \leq \frac{2\delta e^\tau}{3}$$

whereas, by (5.16),

$$\|R_i(y,\tau_0) - \alpha_i e^\tau H_0(y)\| \leq \|R_i(y,\tau) - G_i(y,\tau)\| + \|G_i(y,\tau) - \alpha_0 e^\tau H_0(y)\| \leq \delta e^\tau$$

which gives (5.2) (with convergence taking place in L^2_w instead of H^1_w). The rest of the proof is obtained as in Proposition 3.1 in Section 3. Finally, the case where (1.10c) occurs is similar. ∎

5.2. - *How is blow up affected by small changes on the initial values*

The main result in this paragraph is

PROPOSITION 5.4. *Let $\tilde{u}(x,t)$ be a solution of* (1.1) *with initial value $\tilde{u}(x,0) = \tilde{u}_0(x)$, such that $\tilde{u}$ blows up at $t = T < +\infty$, and $\tilde{u}(x,t) \geq \eta > 0$ for any $t \in (0,T)$. Moreover, assume that the blow up set of $\tilde{u}$ consists of k points, $x_1,\dots,x_k$. Then, for any $\sigma > 0$ and any fixed $j \in [1,k]$, there exists $u_0(x) \in C(\mathbb{R})$ such that $\max_{x\in\mathbb{R}} |u_0(x) - \tilde{u}_0(x)| \leq \delta$, and the blow up set of the solution $u(x,t)$ of* (1.1) *with initial value $u_0(x)$ is contained in the ball $B_\delta(x_j) = \{x \in \mathbb{R} : |x - x_j| < \delta\}$.*

We shall obtain Proposition 5.4 after some elaboration. First, we notice that it may be assumed that $j = 1$ and $x_1 = 0$. We then use Proposition 5.1 with $\alpha_1 = 2$, $\alpha_n = -2$ for $n = 2,\dots,k$, to deduce that there exists a compactly supported function $\tilde{R}_0(x)$ such that the corresponding functions $R_i(y,\tau)$ given in (5.1) satisfy

$$\begin{aligned} &R_i(y,\tau) = \beta_i e^\tau H_0(y) + o(e^\tau) \qquad \text{as } \tau \to \infty, \text{ in } H^1_w \text{ and } C^{k,\gamma}_{\text{loc}} \\ &\text{(for any } k \geq 1 \text{ and any } \gamma \in (0,1)), \text{ where } \beta_1 \geq 1 \\ &\text{and } \beta_n \leq -1 \text{ for } n = 2,\dots,k. \end{aligned} \tag{5.19}$$

Consider now functions $u_{0,\varepsilon}(x) = \tilde{u}_0(x) + \varepsilon\tilde{R}_0(x)$, where $\varepsilon > 0$ is small enough, and let $u_\varepsilon(x,t)$ be the solution of (1.1) with initial value $u_{0,\varepsilon}(x)$. Pick now $\rho > 0$ such that the balls $B_\rho(x_i)$ are disjoint for $i = 1,\dots,k$. As shown in [HV3], $\tilde{u}(x,t)$ is bounded in $U_T = \left\{(x,t) : x \in \mathbb{R}\setminus\bigcup_{k=1}^{k} B_\rho(x_i),\ 0 < t < T\right\}$. Using Lemma 4.1, it then follows that the solution of (3.19), $\tilde{R}(x,t)$, is also bounded in U_T. Recalling (4.15), we then deduce that there exist positive constants C, M and θ, independent of ε, such that

$$u_\varepsilon(x,t) \leq M \qquad \text{in } U_{T,\varepsilon} = U_T \cap \{t : 0 < t < T - \theta\varepsilon\}. \tag{5.20}$$

Assume now that (1.10b) holds at $x_1, \dots, x_k$. By the results of [HV3], we have that

$$\tilde{u}(x,t) \le \left((p-1)^{-\frac{1}{p-1}} - \sigma\right)(T-t)^{-\frac{1}{p-1}}$$

(5.21) for some $\sigma > 0$, whenever

$$\left((T-t)|\log(T-t)|\right)^{1/2} < |x - x_i| < \rho, \ i = 1, \dots, k,$$

which in turns implies

$$u_\varepsilon(x,t) \le \left((p-1)^{-\frac{1}{p-1}} - \frac{\sigma}{2}\right)(T-t)^{-\frac{1}{p-1}} \tag{5.22}$$

at the intervals $\left((T-t)|\log(T-t)|\right)^{1/2} < |x - x_i| < \rho$, $t = T - \lambda\varepsilon$, where $\lambda \ge \theta > 0$ is large enough, θ is as in the statement of Proposition 4.2, and $i = 1, \dots, k$. Indeed, setting $t = T - \lambda\varepsilon$ and taking into account the results in Section 4, we readily check that

$$|\varepsilon\tilde{R}| \le C\varepsilon\tilde{u} \cdot \tilde{u}^{p-1} \le \frac{C}{\lambda}(T-t)^{-\frac{1}{p-1}}$$

$$\varepsilon^2\tilde{u}^{2(p-1)}\tilde{u} \le \frac{C}{\lambda^2}(T-t)^{-\frac{1}{p-1}}$$

and (5.22) follows then from (5.21) and (4.15). Consider now the regions where $|x - x_i| \le ((T-t)|\log(T-t)|)^{1/2}$. As it has been noticed several times before, $\tilde{u}(x,t) \ge C(T-t)^{-\frac{1}{p-1}}$ for some $C > 0$ in such regions. Therefore (4.3) and (4.15) yield at once

$$u_\varepsilon(x,t) \ge \frac{C}{2}(T-t)^{-\frac{1}{p-1}} \tag{5.23}$$

whenever $|x - x_i| \le ((T-t)|\log(T-t)|)^{1/2}$ and $t = T - \lambda\varepsilon$, where $\lambda \ge \theta > 0$ is large enough, θ is as in the statement of Proposition 4.2, and $i = 1, \dots, k$.

Let us denote now by T_ε the blow up time of $u_\varepsilon(x,t)$. By the continuity of the blow up time with respect to initial values (cf. for instance [HV2]), we have that

$$T_\varepsilon = T + \Delta T_\varepsilon, \qquad \text{where } \Delta T_\varepsilon = o(1) \text{ as } \varepsilon \to 0.$$

Actually, there holds

LEMMA 5.5. *Let θ be as in Proposition 4.2. Then under our previous assumptions we have that*

$$-\theta \le \frac{\Delta T_\varepsilon}{\varepsilon} \le -(p-1)^{\frac{p}{p-1}}\beta_1 H_0(y), \tag{5.24}$$

where β_1 is as in (5.19).

PROOF. The first inequality in (5.24) has been already obtained as a consequence of Proposition 4.2. As to the second one, let us begin by showing that there exists $\nu > 0$ such that

$$\Delta T_\varepsilon \leq \nu\varepsilon. \tag{5.25}$$

To this end, let $g(x,t)$ be the solution of the Cauchy problem

$$g_t = g_{xx} + g^p \qquad \text{when } x \in \mathbb{R},\ t > 0,$$
$$g(x,0) = \begin{cases} \dfrac{C}{2}(\lambda\varepsilon)^{-\frac{1}{p-1}} & \text{if } |x| \leq ((\lambda\varepsilon)|\log(\lambda\varepsilon)|)^{1/2} \\ 0 & \text{elsewhere,} \end{cases}$$

where C is as in (5.23). Set now

$$\tilde{g}(x,t) = (\lambda\varepsilon)^{\frac{1}{p-1}} g((\lambda\varepsilon)^{1/2}x, (\lambda\varepsilon)t).$$

Then $\tilde{g}(x,t)$ solves (1.1) with initial value $\tilde{g}(x,0) = \dfrac{C}{2}$ if $|x| \leq |\log(\lambda\varepsilon)|^{1/2}$ and $\tilde{g}(x,0) = 0$ otherwise. Take now $\lambda \geq \theta > 0$ large enough, so that (5.22) and (5.23) hold. For $\varepsilon > 0$ sufficiently small, $\tilde{g}(x,t)$ will then blow up in a time $\tilde{T} = O(1)$, whence so does g at $T_g = O(\lambda\varepsilon)$. Inequality (5.25) follows then by comparison. As a next step, we remark that by (3.57a) and (5.1)

$$\tilde{R}(x,t) = (T-t)^{-\frac{p}{p-1}} \left(\beta_i H_0(y) \left(1 + \left(\frac{p-1}{4p} \right) \cdot \frac{(x-x_i)^2}{(T-t)|\log(T-t)|} \right)^{-\frac{p}{p-1}} + o(1) \right), \tag{5.26}$$

uniformly on sets $|x - x_i| \leq ((T-t)|\log(T-t)|)^{1/2}$ as $t \to T$ and $i = 1, \ldots, k$. We now make use of (4.15), together with (1.10b) and (5.26), to obtain that

$$\begin{aligned} u_\varepsilon(x,t) &= (T-t)^{-\frac{1}{p-1}} \left[\left((p-1) + \left(\frac{p-1}{4p} \right) \frac{(x-x_i)^2}{(T-t)|\log(T-t)|} \right)^{-\frac{p}{p-1}} + o(1) \right] \\ &+ \varepsilon(T-t)^{-\frac{1}{p-1}} \left[\beta_i H_0(y) \left(1 + \left(\frac{p-1}{4p} \right) \frac{(x-x_i)^2}{(T-t)|\log(T-t)|} \right)^{-\frac{p}{p-1}} + o(1) \right] \\ &+ O\left(\frac{1}{\lambda^2} \right) (T-t)^{-\frac{1}{p-1}} \end{aligned}$$

uniformly on sets $|x - x_i| \leq ((T-t)|\log(T-t)|)^{1/2}$, $0 < t < T - \lambda\varepsilon$, $i = 1, \ldots, k$, where $o(1) \to 0$ as $\varepsilon \to 0$ (so that t is allowed to tend to T). Setting $t = T - \lambda\varepsilon$, the previous formula reads

$$
\begin{aligned}
u_\varepsilon(x, T-\lambda\varepsilon) = (\lambda\varepsilon)^{-\frac{1}{p-1}} \Bigg[& \left((p-1) + \left(\frac{p-1}{4p}\right) \frac{(x-x_i)^2}{\lambda\varepsilon |\log(\lambda\varepsilon)|} \right)^{-\frac{p}{p-1}} \\
& + \frac{\beta_i H_0}{\lambda} \left(1 + \left(\frac{p-1}{4p}\right) \frac{(x-x_i)^2}{\lambda\varepsilon |\log(\lambda\varepsilon)|} \right)^{-\frac{p}{p-1}} \\
& + \frac{o(1)}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \Bigg].
\end{aligned} \tag{5.27}
$$

Consider now the function

$$
\begin{aligned}
f(s) = & \left((p-1) \left(1 + \left(\frac{p-1}{4p}\right) s^2 \right) \right)^{-\frac{1}{p-1}} \\
& + \left(\frac{\beta_i H_0}{\lambda} \left(1 + \left(\frac{p-1}{4p}\right) s^2 \right) \right)^{-\frac{p}{p-1}}
\end{aligned} \tag{5.28}
$$

in the interval $|s| \leq 1$. Clearly, $f(s)$ has a maximum at $s = 0$ (recall that $\beta_1 \geq 1$; cf. (5.19)). Therefore (5.27) gives

$$
\|u_\varepsilon(x, T-\lambda\varepsilon)\|_\infty \geq \left((p-1)^{-\frac{1}{p-1}} + \frac{\beta_i H_0}{\lambda} + o(1) + O\left(\frac{1}{\lambda^2}\right) \right) (\lambda\varepsilon)^{-\frac{1}{p-1}} \tag{5.29}
$$

whereas, by (3.47),

$$
\begin{aligned}
\|u_\varepsilon(x, T-\lambda\varepsilon)\|_\infty \leq \Bigg(& (p-1)^{-\frac{1}{p-1}} \\
& + C \left| \log\left(\frac{\Delta T_\varepsilon + \lambda\varepsilon}{T_\varepsilon} \right) \right|^{-1} \Bigg) (\lambda\varepsilon + \Delta T_\varepsilon)^{-\frac{1}{p-1}}.
\end{aligned} \tag{5.30}
$$

Moreover, by (5.25), $O < \dfrac{\Delta T_\varepsilon + \lambda\varepsilon}{T_\varepsilon} < \gamma\lambda\varepsilon$ for some $\gamma > 0$, provided that $\varepsilon > 0$ is small enough. From (5.28) and (5.29) it then follows that

$$
\begin{aligned}
& \left((p-1)^{-\frac{1}{p-1}} + \frac{\beta_i H_0}{\lambda} + o(1) + o\left(\frac{1}{\lambda^2}\right) \right) \\
& \leq \left((p-1)^{-\frac{1}{p-1}} + \frac{C}{|\log(\gamma\lambda\varepsilon)|} \right) \left(1 - \frac{1}{p-1} \frac{\Delta T_\varepsilon}{\lambda\varepsilon} + O\left(\frac{\Delta T_\varepsilon}{\lambda\varepsilon}\right)^2 \right)
\end{aligned}
$$

where $O\left(\dfrac{\Delta T_\varepsilon}{\lambda\varepsilon}\right)^2 = O\left(\dfrac{1}{\lambda^2}\right)$. This gives

$$
\limsup_{\varepsilon \to 0} \left((p-1)^{-\frac{1}{p-1}} \frac{\Delta T_\varepsilon}{\lambda\varepsilon} \right) + \frac{\beta_i H_0}{\lambda} = O\left(\frac{1}{\lambda^2}\right)
$$

multiplying throughout by λ and letting $\lambda \to \infty$, we are done. The case where (1.10c) is satisfied is similar. ∎

End of the proof of Proposition 5.4

Let us prove now that there are no blow up points in the balls $B_\rho(x_i)$ for $i = 2, \dots, k$. By the results of Section 2 (cf. the argument following (2.28) there), it suffices to show that

$$\begin{aligned} &u_\varepsilon(x,t) < \big((p-1)(T_\varepsilon - t)\big)^{-\frac{1}{p-1}} \\ &\text{in } B_\rho(x_i)\ (i = 2, \dots, k) \qquad \text{as } t \to T_\varepsilon. \end{aligned} \tag{5.31}$$

This is achieved as follows. Consider the functions $f_i(s)$ $(i = 2, \dots, k)$ given by (5.28) with β_i there. For any such i, $f_i(s)$ is decreasing in $|s|$ for $|s| \le 1$ provided that λ is large enough. We thus see that

$$u_\varepsilon(x, T - \lambda\varepsilon) \le \left((p-1)^{-\frac{1}{p-1}} + \frac{\beta_i H_0}{\lambda} + o(1) + O\left(\frac{1}{\lambda^2}\right)\right)(\lambda\varepsilon)^{-\frac{1}{p-1}}, \tag{5.32}$$

uniformly for $|x - x_i| \le (\lambda\varepsilon|\log(\lambda\varepsilon)|)^{1/2}$, where $o(1) \to 0$ as $(\lambda\varepsilon) \to 0$. We now select first $\lambda \gg 1$ so that $\frac{\beta_i H_0}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \le \frac{\beta_i H_0}{2\lambda} < 0$, and take then $\varepsilon > 0$ such that $(\lambda\varepsilon)$ is small, and $\frac{\beta_i H_0}{2\lambda} + o(1) < 0$. By the upper bound in (5.24), (5.32) yields now

$$u_\varepsilon(x, T - \lambda\varepsilon) \le ((\lambda\varepsilon)(p-1))^{-\frac{1}{p-1}} \le ((T_\varepsilon - T + \lambda\varepsilon)(p-1))^{-\frac{1}{p-1}},$$

when $|x - x_i| \le (\lambda\varepsilon|\log(\lambda\varepsilon)|)^{1/2}$. This estimate together with (5.20) and (5.22) yields (5.31) at $t = T - \lambda\varepsilon$. Finally, the case where not all points $x_i, \dots, x_k$ are of type (1.10b) is similar. ∎

We shall require later a refined version of (5.24) for the case where (1.10c) takes place

LEMMA 5.6. *Assume that* (1.10c) *is satisfied,* $\alpha \neq 0$ *in* (3.21), *and let* ΔT_ε *be as in Lemma* 5.5. *Then*

PROOF. Let us denote by $\Phi_\varepsilon, \tilde{\Phi}$ the auxiliary functions defined in (1.3) corresponding to u_ε, $\tilde{u}$. By (4.15), we then have

$$\begin{aligned} \Phi_\varepsilon(y,\tau) &= \tilde{\Phi}(y,\tau) + \varepsilon R(y,\tau) + o(\varepsilon^2 \tilde{u}^{2p-2}\tilde{\Phi}) \\ &= (p-1)^{-\frac{1}{p-1}} + Ce^{(1-\frac{m}{2})\tau} H_m(y) + \varepsilon\alpha_1 e^{\tau/2} H_1(y) + O(\varepsilon^2 e^{2\tau}) \end{aligned} \tag{5.33}$$

in H^1_w and $C^{k,\gamma}_{\text{loc}}$ for $0 < t \le T - \theta\varepsilon$. By (3.61), there holds

$$|\tilde{R}(x,t)| \le C(T-t)^{-\frac{1}{p-1}-\frac{1}{2}} \qquad \text{for } |x| \le M(T-t)^{\frac{1}{m}-\frac{1}{2}}. \tag{5.34a}$$

It then follows that

$$|R(y,\tau)| \le Ce^{\tau/2} \qquad \text{if } |y| \le Me^{\left(\frac{1}{2}-\frac{1}{m}\right)\tau},$$

$$|R(y,\tau)| \le Ce^{\tau} \qquad \text{if } |y| \ge Me^{\left(\frac{1}{2}-\frac{1}{m}\right)\tau},$$

so that, denoting by $A = \{y : |y| \le Me^{\left(\frac{1}{2}-\frac{1}{m}\right)\tau}\}$ and $B = \mathbb{R}\backslash A$, we readily see that

$$\|R(\cdot,\tau)\|^2 = Ce^{\tau}\int_A e^{-y^2/4}\mathrm{d}y + Ce^{2\tau}\int_B e^{-y^2/4}\mathrm{d}y \le Ce^{\tau} \tag{5.35}$$

for some $C > 0$, which gives

$$\|R(\cdot,\tau)\| \le Ce^{\tau/2} \qquad \text{as } \tau \to \infty. \tag{5.36}$$

We now define

$$t_\varepsilon = T - \varepsilon^{2/3}, \qquad \tau_\varepsilon = -\log(T - t_\varepsilon), \tag{5.37a}$$

$$v_\varepsilon(x,t) = (T-t_\varepsilon)^{\frac{1}{p-1}} u_\varepsilon\big((T-t_\varepsilon)^{1/2}x,\ t_\varepsilon + t(T-t_\varepsilon)\big). \tag{5.37b}$$

Notice that $T - t_\varepsilon = \varepsilon^{2/3} \gg \theta\varepsilon$ as $\varepsilon \downarrow 0$. Since, by (4.15),

$$|G(y,\tau)| \le C\varepsilon^2(T-t)^{-\frac{p}{p-1}-1} \text{ for } t \le T - \theta\varepsilon, \text{ and}$$

$$|v_\varepsilon(x,0) - \Phi(x,\tau_\varepsilon)| \le (T-t_\varepsilon)^{\frac{1}{p-1}}\left|\varepsilon\tilde{R}((T-t_\varepsilon)^{1/2}x, t_\varepsilon) + G((T-t_\varepsilon)^{1/2}x, t_\varepsilon)\right|,$$

standard estimates yield then

$$\|v_\varepsilon(x,0) - \Phi(x,\tau_\varepsilon)\| \le \varepsilon\|R(\cdot,\tau_\varepsilon)\| + \|G(\cdot,\tau_\varepsilon)\| \le C\varepsilon^{2/3} \tag{5.38}$$

whereas, by the basic estimate (1.10c),

$$\|\Phi_\varepsilon(\cdot,\tau_\varepsilon) - (p-1)^{-\frac{1}{p-1}}\| \le Ce^{\left(1-\frac{m}{2}\right)\tau_\varepsilon}. \tag{5.39}$$

Taking into account (5.33), we arrive at

$$\|v_\varepsilon(\cdot,0) - (p-1)^{-\frac{1}{p-1}}\| \le C\left(\varepsilon^{2/3} + \varepsilon^{\frac{m-2}{3}}\right).$$

Suppose now that $m \ge 6$. Then $\dfrac{m-2}{3} > \dfrac{2}{3}$, and the previous inequality yields

$$\|v_\varepsilon(\cdot,0) - (p-1)^{-\frac{1}{p-1}}\| \le C\varepsilon^{2/3}. \tag{5.40}$$

Consider now the auxiliary function

$$w_\varepsilon(x,t) = \left((S(t)(v_\varepsilon(x,0))^{-(p-1)} - (p-1)t\right)^{-\frac{1}{p-1}}.$$

As recalled in [HV1] (and readily checked by inspection), $w_\varepsilon(x,t)$ is a subsolution of (1.1). On the other hand,

$$|S(t)(v_\varepsilon(x,0) - (p-1)^{-\frac{1}{p-1}})| \le C\|v_\varepsilon(x,0) - (p-1)^{-\frac{1}{p-1}}\| \le C\varepsilon^{2/3}$$

uniformly when $t \in \left[\frac{1}{2}, 1\right]$. As

$$S(t)v_\varepsilon(x,0) = (p-1)^{-\frac{1}{p-1}} + S(t)\left(v_\varepsilon(x,0) - (p-1)^{-\frac{1}{p-1}}\right),$$

we obtain that

$$\begin{aligned} w_\varepsilon(x,t) &\ge \left(\left((p-1)^{-\frac{1}{p-1}} + O(\varepsilon^{2/3})\right)^{-(p-1)} - (p-1)t\right)^{-\frac{1}{p-1}} \\ &= \left((p-1)(1-t) + O(\varepsilon^{2/3})\right)^{-\frac{1}{p-1}} \end{aligned}$$

so that $w_\varepsilon(x,t)$ (and hence $v_\varepsilon(x,t)$) blows up in a time $t^* \le 1 + O(\varepsilon^{2/3})$. Back to the original variables, this means that

$$\Delta T_\varepsilon \le k\varepsilon^{4/3} \qquad \text{for some } k. \tag{5.41}$$

To proceed further, we use (4.23) together with (4.15) (this last one in the region where $|x| \ge M(T-t)^{1/m}$) to obtain

$$\begin{aligned} &\|G(\cdot,\tau)\| \le \frac{C\varepsilon^2}{\mu^2} e^{2\tau} + E(\varepsilon,\tau) \qquad \text{if } \varepsilon\mu e^\tau \le 1,\ \mu \text{ large}, \\ &\text{where } E(\varepsilon,\tau) = C\varepsilon^2 e^{2\tau} \exp\left(-Ce^{\left(1-\frac{2}{m}\right)\tau}\right). \end{aligned} \tag{5.42}$$

We now define t_ε by $T - t_\varepsilon = \lambda\varepsilon^{2/3}$, where $\lambda > 0$ will be selected presently, and take v_ε as in (5.37). Using (5.42) we now obtain

$$\|v_\varepsilon(\cdot,0) - (p-1)^{-\frac{1}{p-1}}\| \le C\left(\varepsilon^{2/3}\lambda^{-\frac{1}{2}} + \varepsilon^{2/3}(\mu\lambda)^{-2} + \lambda^{\frac{m}{2}-1}\varepsilon^{\frac{m-2}{3}} + o(\varepsilon^{2/3})\right)$$

for any fixed $\lambda > 0$. As in the previous case, we now estimate the blow up time of v_ε, $T^*(v_\varepsilon)$, by

$$T^*(v_\varepsilon) \le 1 + C\left(\lambda^{-1/2} + (\mu\lambda)^{-2} + \lambda^{\frac{m}{2}-1}\right)\varepsilon^{2/3}$$

since $T_\varepsilon = t_\varepsilon + (T - t_\varepsilon)T^*(v_\varepsilon) = T + \Delta T_\varepsilon$, this implies that

$$\Delta T_\varepsilon \leq C\left(\lambda^{1/2} + \lambda^{-1}\mu^{-2} + \lambda^{\frac{m}{2}}\right)\varepsilon^{4/3}$$

so that, letting first $\mu \to \infty$ and then $\lambda \to 0$, we deduce that

$$\limsup_{\varepsilon \to 0} \frac{\Delta T\varepsilon}{\varepsilon^{4/3}} \leq 0$$

and the proof is concluded. ∎

We next show

PROPOSITION 5.7. *Let $\tilde{u}(x,t)$ and T be as in Proposition* 5.4, *and assume that $\tilde{u}$ has a single blow up point at $x = 0$ such that* (1.10c) *holds for some $m \geq 4$. Then, for any $\delta > 0$, there exists an initial function $u_0(x)$ such that $(u_0(x) - \tilde{u}_0(x))$ has compact support, $\max_{x\in\mathbb{R}} |u_0(x) - \tilde{u}_0(x)| \leq \delta$, and the corresponding solution $u(x,t)$ has a single point blow up where the solution behaves as indicated by* (1.10b).

In the course of proving the Proposition, we shall use the following auxiliary result.

LEMMA 5.8. *Let $\tilde{u}$ be as in the statement of Proposition* 5.7. *Then there exists a compactly supported function $\tilde{R}_0(x)$ such that*

$$\begin{aligned} &R(y,\tau) = \alpha_1 e^{\tau/2} H_1(y) + o(e^{\tau/2}) \\ &\text{in } H^1_w \text{ as } \tau \to \infty, \text{ for some } \alpha_1 \neq 0, \end{aligned} \tag{5.43}$$

where $R(y,\tau)$ is given by (3.19), (3.20).

PROOF. It consists in a suitable modification of that of Proposition 5.1. Keeping to the notations used therein, we take a nonnegative cut-off function $\varsigma(x)$ just as before, as well as positive parameters ρ, θ to be selected later, and consider the function

$$g(x) = \theta^{-\frac{1}{p-1}} \varsigma\left(\frac{x}{\rho\sqrt{\theta}}\right) H_1\left(\frac{x}{\sqrt{\theta}}\right).$$

We now define

$$\tilde{G}(y,\tau_0) = \theta^{\frac{1}{p-1}} g(x), \qquad y = \frac{x}{\sqrt{\theta}}, \ \tau_0 = -\log\theta.$$

For any given $\delta > 0$, we can select $\rho > 0$ large enough so that

$$\|\tilde{G}(\cdot,\tau_0) - H_1(\cdot)\| \leq \frac{\delta}{6}. \tag{5.44}$$

By Proposition 5.1, there exists an initial value $\tilde{R}_{0,1}(x)$ with compact support such that the corresponding function $R_1(y,\tau)$ satisfies

$$R_1(y,\tau) = e^{\tau-\tau_0}H_0(y) + o(e^{\tau}) \qquad \text{in } H^1_w \text{ as } \tau \to \infty. \tag{5.45}$$

On the other hand, exactly as in the case of Proposition 5.1, we have that there exists $h_0(x) \in C_0^\infty(\mathbb{R})$ such that $h(x) = E(T-\theta,0)h_0(x)$ satisfies

$$\theta^{\frac{1}{p-1}}\left(\int_{\mathbb{R}} (g(x)-h(x))^2 \mathrm{d}x\right)^{1/2} \le \frac{\delta}{3}$$

and

$$\|\tilde{G}(\cdot,\tau_0) - \theta^{\frac{1}{p-1}}h(\theta^{1/2}y,\tau_0)\| \le \frac{\delta}{3}.$$

Denoting by $\tilde{R}(x,t) = E(t,0)h_0(x)$, we readily see that

$$\begin{aligned}\|\tilde{R}(\cdot,\tau_0)\| &\le \|\tilde{R}(\cdot,\tau_0) - \tilde{G}(\cdot,\tau_0)\| + \|\tilde{G}(\cdot,\tau_0)\| \\ &\le \|\tilde{G}(\cdot,\tau_0)\| + \frac{\delta}{3}.\end{aligned}$$

Take now $L = 2\|\tilde{G}(\cdot,\tau_0)\|$. Then, if δ is small enough we have that $\|\tilde{R}(\cdot,\tau_0)\| \le L$. By (5.44) and (5.9b) (this last one with G replaced by $H_1(y)e^{(\tau-\tau_0)/2}$), we obtain that

$$\|\tilde{R}(\cdot,\tau_0) - H_1(\cdot)e^{(\tau-\tau_0)/2}\| \le \frac{2\delta}{3}e^{\tau-\tau_0}. \tag{5.46}$$

Furthermore, by Proposition 3.2 we also have that

$$\|\tilde{R}(\cdot,\tau)\| = \beta e^{\tau-\tau_0} + o(e^{\tau}), \qquad \text{in } H^1_w \text{ as } \tau \to \infty, \tag{5.47}$$

for some real β $\left(\text{actually, (5.46) yields at once the bound } |\beta| \le \frac{2\delta}{3}\right)$. Consider now the initial value

$$\overline{R}_0(x) = \tilde{R}_0(x) - \beta\tilde{R}_{0,1}(x).$$

Clearly, $\overline{R}_0(x)$ has compact support, and (5.45) together with (5.47) yield $\overline{R}(\cdot,\tau) = o(e^{\tau-\tau_0})$ as $\tau \to \infty$. It then follows from Lemma 3.6 that

$$\overline{R}(\cdot,\tau) = \alpha_1 e^{(\tau-\tau_0)/2}H_1(\cdot) + o(e^{(\tau-\tau_0)/2}), \qquad \text{in } H^1_w \text{ as } \tau \to \infty, \tag{5.48}$$

where α_1 is given by (3.66b). To conclude with the proof, we just need to show that $\alpha_1 \neq 0$ in (5.48). To this end, we notice that

$$\begin{aligned}\langle \overline{R}(\cdot,\tau_0), H_1\rangle &= \langle \tilde{R}_0(\cdot,\tau_0), H_1\rangle - \beta\langle \tilde{R}_0(\cdot,\tau_0), H_1\rangle \\ &= \langle \tilde{R}_0(\cdot,\tau_0) - H_1, H_1\rangle - \beta\langle \tilde{R}_1(\cdot,\tau_0), H_1\rangle + 1\end{aligned}$$

whence

$$|\langle \overline{R}_0(\cdot,\tau_0), H_1\rangle - 1| \le \|\tilde{R}_0(\cdot,\tau_0) - H_1\| + |\beta|\,\|\tilde{R}_1(\cdot,\tau_0)\|.$$

Since $\|\tilde{R}_1(\cdot,\tau_0)\|$ is bounded (actually, $e^{\tau_0}\tilde{R}_1(\cdot,\tau) = h(\cdot,\tau)$ satisfies $\|h(\cdot,\tau)\| \le Ce^{\tau}$, cf. (5.7)), we obtain

$$|\langle \overline{R}_0(\cdot,\tau_0), H_1\rangle - 1| \le \frac{\delta}{3} + \frac{C\delta}{3}.$$

On the other hand, $\alpha_1 = \langle \overline{R}(\cdot,\tau_0), H_1\rangle + \mu(\tau_0)$ (cf. (Lemma 3.7)), where for τ_0 large enough $|\mu(\tau_0)| < \frac{\delta}{6}$, and the result follows. ∎

Proof of Proposition 5.7

We select

$$u_0(x) = \tilde{u}_0(x) + \varepsilon\tilde{R}_0(x),$$

where $\tilde{R}_0(x)$ is as in Lemma 5.8. If $\varepsilon > 0$ is small enough, the change in blow up time $(T_\varepsilon - T)$ is small, and by standard continuous dependence results, the number of maxima at $t = \frac{T}{2}$ does not change at all. Moreover, if at such a time $\tilde{u}(x,t) < ((p-1)(T-t))^{-\frac{1}{p-1}}$, we also have that $u_\varepsilon(x,t) < ((p-1)(T_\varepsilon - t))^{-\frac{1}{p-1}}$. In particular, any maximum where such inequality holds will never move towards a blow up point.

To proceed further, we make use of the basic asymptotic formula (1.10c). Together with the results in Section 4, this yields

$$\begin{aligned}\Phi_\varepsilon(y,\tau) = (p-1)^{-\frac{1}{p-1}} - Ce^{(1-\frac{m}{2})\tau}H_m(y) + o(e^{(1-\frac{m}{2})\tau}) \\ + \varepsilon R(y,\tau) + G(y,\tau) \qquad \text{as } \tau \to \infty,\end{aligned} \tag{5.49}$$

uniformly one sets $|y| \le M$ and $T - t \ge \mu\varepsilon$ for $\varepsilon > 0$ small and $\mu > 0$ arbitrarily large, so that in particular $T - t \ge \varepsilon^\beta$ for $0 < \beta < 1$. Making use of (4.23), we obtain

$$\begin{aligned}\Phi_\varepsilon(y,\tau) = (p-1)^{-\frac{1}{p-1}} - Ce^{(1-\frac{m}{2})\tau}H_m(y) + o(e^{(1-\frac{m}{2})\tau}) \\ + \varepsilon\alpha_1 e^{\tau/2}H_1(y) + \varepsilon o(e^{\tau/2}) + o(\varepsilon^{2(1-\beta)}) \qquad \text{as } \tau \to \infty,\end{aligned} \tag{5.50}$$

for y, τ as in (1.3).

We now select

$$T - t = \lambda\varepsilon^{\frac{2}{m-1}}, \qquad \lambda > 0 \text{ to be precised later.} \tag{5.51}$$

The motivation for this choice is that then (5.50) reads as follows

$$\Phi_\varepsilon(y,\tau) = (p-1)^{-\frac{1}{p-1}} + \varepsilon^{\frac{m-2}{m-1}}(C\lambda^{\frac{m}{2}-1}H_m(y) + \alpha_1\lambda^{-\frac{1}{2}}H_1(y)) + o(\varepsilon^{\frac{m-2}{m-1}}), \tag{5.52}$$

as $\tau \to \infty$, with $e^{-\tau} = \lambda\varepsilon^{\frac{2}{m-1}}$, uniformly on sets $|y| \le M$. Let us write

$$H(y,\lambda) = C\lambda^{\frac{m}{2}-1}H_m(y) + \alpha_1\lambda^{-\frac{1}{2}}H_1(y) \tag{5.53}$$

As a matter of fact, the proof under consideration will be finished as soon as we can show that

(5.54) Only one maximum of $u_\varepsilon(x,t)$ arrives to the blow up point x_ε as $t \to T_\varepsilon$.

(cf. (1.10b)). Notice that $x_\varepsilon \sim 0$ as $\varepsilon \to 0$ by the continuous dependence of blow up regions on initial values (cf. [HV2]). To show (5.54), we first remark that

(5.55) Either some maxima of $u_\varepsilon(x,t)$ collapse at a time $t < T_\varepsilon$, or there is at least one maximum $\overline{x}$ such that $\Phi_\varepsilon(\overline{y},\tau) < (p-1)^{-\frac{1}{p-1}} - \gamma\varepsilon^{\frac{m-2}{m-1}}$, for some $\gamma > 0$, where $\overline{y} = (\overline{x} - \overline{x})(T-t)^{-1/2}$ and $e^{-\tau} = \lambda\varepsilon^{\frac{2}{m-1}}$ with $\lambda \in [\lambda_0, \lambda_1]$ $(0 < \lambda_0 < \lambda_1 < +\infty)$ and $\varepsilon > 0$ is small enough.

Actually, this fact follows from inspection of (5.52). Assume without loss of generality that $\alpha_1 > 0$. Let $\overline{y} < 0$ be such that $\Phi_\varepsilon(\cdot,\tau)$ has a maximum located at $\overline{y}$ at some time $\tau \gg 1$. Such maximum may indeed collapse as τ increases. If this does not happens, though, it follows from (5.52) and (5.53) that, taking $\lambda > 0$ small enough, (5.55) should be satisfied, and we are done.

We then claim that, for $\overline{t} \sim \overline{T}_\varepsilon$,

$$u_\varepsilon(\overline{x},\overline{t}) < ((p-1)(T_\varepsilon - \overline{t}))^{-\frac{1}{p-1}}. \tag{5.56}$$

To this end, we notice that

$$T_\varepsilon - \overline{t} = T_\varepsilon - T + T - \overline{t} = \lambda\varepsilon^{\frac{2}{m-1}} + \Delta T_\varepsilon,$$

therefore, by convexity,

$$(T_\varepsilon - \overline{t})^{-\frac{1}{p-1}} \ge \left(\lambda\varepsilon^{\frac{2}{m+1}}\right)^{-\frac{1}{p-1}}\left(1 - \frac{1}{p-1}\frac{\Delta T\varepsilon}{(\lambda\varepsilon^{\frac{2}{m-1}})}\right)$$

whence

$$u_\varepsilon(\overline{x},\overline{t}) = (T-\overline{t})^{-\frac{1}{p-1}}\Phi_\varepsilon(\overline{y},\overline{\tau}) < \left(\lambda\varepsilon^{\frac{2}{m-1}}\right)^{-\frac{1}{p-1}}\left((p-1)^{-\frac{1}{p-1}} - \gamma\varepsilon^{\frac{m-2}{m-1}}\right)$$

$$\leq (T_\varepsilon-\overline{t})^{-\frac{1}{p-1}}\left((p-1)^{-\frac{1}{p-1}} - \gamma\varepsilon^{\frac{m-2}{m-1}}\right)\left(1-\frac{1}{p-1}(\Delta T\varepsilon)\left(\lambda\varepsilon^{\frac{2}{m-1}}\right)\right)^{-1}$$

since $\dfrac{1}{1-x} \leq 1+2x$ for $0<x\ll 1$, we finally obtain

$$u_\varepsilon(\overline{x},\overline{t}) \leq (T_\varepsilon-\overline{t})^{-\frac{1}{p-1}}\left((p-1)^{-\frac{1}{p-1}} - \gamma\varepsilon^{\frac{m-2}{m-1}}\right) \cdot \left(1+\frac{2}{p-1}(\Delta T\varepsilon)\left(\lambda\varepsilon^{\frac{2}{m-1}}\right)^{-1}\right). \tag{5.57}$$

Recalling the bounds for ΔT_ε obtained in Lemma 5.6, (5.56) follows at once from (5.57). Putting together (5.55) and (5.56), we have reduced the number of maxima which collapse at x_ε, at least by one. It is clear that we can repeat now the previous argument as many times as required, so that we eventually obtain that a single maximum of Φ_ε arrives to x_ε as $\tau\to\infty$, in which case (1.10b) necessarily holds. ∎

End of the proof of the Theorem

Assume that $\tilde{u}_0(x)$ has compact support, and the corresponding solution $\tilde{u}(x,t)$ blows up at n different points $x_1,\dots,x_n$. Take $\eta>0$, and replace $\tilde{u}_0(x)$ by $\tilde{u}_0(x)+\eta$. If we denote the corresponding solution of (1.1) with such initial value by $\tilde{u}_\eta(x,t)$, we readily see that if $\eta>0$ is small enough, the blow up set of $\tilde{u}_\eta$ lies in a compact subset of $\mathbb{R}$ (cf. for instance [GK3], [HV2]), the new blow up time T_η remains close to T ([HV2]), and for any $T^*<\min(T_\eta,T)$, the number of maxima of $\tilde{u}_\eta(\cdot,T^*)$ does not change with respect to that of $\tilde{u}(\cdot,T^*)$. We then make use of Proposition 5.4 to show that a slight perturbation of $(\tilde{u}_0(x)+\eta)$ localizes the blow up set in a small neighbourhood of one of the new blow up points, say x_1. By iterated application of Propositions 5.4 and 5.7, we can then change blow up behaviours (and if necessary, eliminate blow up points) until a single blow up point, where (1.10b) holds, is obtained. We finally truncate the initial value far enough from the origin to conclude. The fact that this last can be actually done follows from the arguments recalled in the proof of our next result, where stability under small perturbations of the patterns (1.10b) is proved.

PROPOSITION 5.9. *Let $\tilde{u}_0(x)$, $\tilde{x}$ and T be as in the statement of the Theorem, and suppose that there is a single point blow up at $\tilde{x}$ and* (1.10b) *holds there. Then there exists $\varepsilon>0$ small enough (depending on $u_0(x)$) such that, for any $u_0(x)\in C_0^+(\mathbb{R})$ satisfying $\max_{x\in\mathbb{R}}|\tilde{u}_0(x)-u_0(x)|<\varepsilon$, the corresponding solution*

of (1.1) *with initial value* $u_0(x)$ *has single point blow up* $\overline{x}$, *where* (1.10b) *holds. Moreover,* $\lim_{\varepsilon\to 0}(\overline{x}-\tilde{x})=0$.

PROOF. We shall merely sketch it, since the corresponding arguments have been already explained in detail. Keeping to our previous notations, we see that by assumption there exists $\delta>0$ small enough such that

$$\text{For } t\in(T-\delta,T),\ u(\cdot,t) \text{ has only one maximum } x(t) \text{ such that } u(x(t),t)>((p-1)(T-t))^{-\frac{1}{p-1}}. \tag{5.58}$$

Indeed, if we replace $>$ by $\geq$ there, (5.58) follows at once from (2.28) and (2.29). Strict inequality is then obtained by noting that, should it not hold, then the corresponding blow up time of $u(x,t)$ would be strictly less than T. This follows at once from the separation properties obtained in [GP] (cf. also Appendix in [HV2]).

Using now standard continuous dependence results and regularizing effects for parabolic equations, together with the continuity of blow up time with respect to initial data (cf. [HV2] for this last point), we readily see that for $u_0(x)\in C_0^+(\mathbb{R})$, close enough to $\tilde{u}_0(x)$, the corresponding solution $u(x,t)$ blows up at some $\overline{T}<+\infty$, and for $(\overline{T}-t)$ small enough, it has only one maximum $y(t)$, where

$$\overline{u}(y(t),t)>((p-1)(\overline{T}-t))^{-\frac{1}{p-1}},$$

whence the result. ∎

REFERENCES

[B1] A. BRESSAN, *On the asymptotic shape of blow up*, Indiana Univ. Math. J., **39, 4** (1990), 947-960.

[B2] A. BRESSAN, *Stable blow up patterns*, J. Differential Equations, **98** (1992), 57-75.

[BE] J. BEBERNES - D. EBERLY, *Mathematical problems from combustion theory*, Appl. Math. Sci. **83**, Springer (1989).

[CM] X.Y. CHEN - H. MATANO, *Convergence, asymptotic periodicity and finite point blow up in one-dimensional semilinear heat equations*, J. Differential Equations, **78** (1989), 160-190.

[D] J.W. DOLD, *Analysis of the early stage of thermal runaway*, Quart. J. Mech. Appl. Math. **38** (1985), 361-387.

[F] A. FRIEDMAN, *Partial Differential Equations of parabolic type*, Robert E. Krieger Co (1983).

[FK] S. FILIPPAS - R.V. KOHN, *Refined asymptotics for the blow up of* $u_t-\Delta u=u^p$, Comm. Pure. Appl. Math., XLV (1992), 821-869.

[FM] A. FRIEDMAN - J.B. MCLEOD, *Blow up of positive solutions of semilinear heat equations*, Indiana Univ. Math. J., **34** (1985), 425-447.

[Fu] H. FUJITA, *On the blowing up of solutions of the Cauchy problem for* $u_t=\Delta u+u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. I, **13** (1966), 109-124.

[GHV] V.A. GALAKTIONOV - M.A. HERRERO - J.J.L. VELÁZQUEZ, *The space structure near a blow up point for semilinear heat equations*, Soviet J. Comput. Math. and Math. Physics, **31** (1991), 399-411.

[GK1] Y. GIGA - R.V. KOHN, *Asymptotically self-similar blow up of semilinear heat equations*, Comm. Pure Appl. Math., **38** (1985), 297-319.

[GK2] Y. GIGA - R.V. KOHN, *Characterizing blow up using similarity variables*, Indiana Univ. Math. J., **36** (1987), 1-40.

[GK3] Y. GIGA - R.V. KOHN, *Nondegeneracy of blow up for semilinear heat equations*, Comm. Pure Appl. Math., **42** (1989), 845-884.

[GP] V.A. GALAKTIONOV - S.A. POSASHKOV, *Application of new comparison theorems in the investigation of unbounded solutions of nonlinear parabolic equations*, Differential Equations, **22**, **7** (1986), 116-1183.

[HV1] M.A. HERRERO - J.J.L. VELÁZQUEZ, *Blow up behaviour of one-dimensional semilinear parabolic equations*, Ann. Inst. H. Poincaré, to appear.

[HV2] M.A. HERRERO - J.J.L. VELÁZQUEZ, *Flat blow up in one-dimensional semilinear heat equations*, Differential Integral Equations, Vol. 5, n° 5 (1992), 973-997.

[HV3] M.A. HERRERO - J.J.L. VELÁZQUEZ, *Blow up profiles in one-dimensional, semilinear parabolic problems*, Comm. Partial Differential Equations, **17, 182**, (1992), 205-219.

[HV4] M.A. HERRERO - J.J.L. VELÁZQUEZ, *Plane structures in thermal runaway*, Israel J. Math., to appear.

[HSS] L.M. HOCKING - K. STEWARTSON - J.T. STUART, *A nonlinear instability burst in plane parallel flow*, J. Fluid Mech. **J1** (1972), 705-735.

[LM] J.L. LIONS - B. MALGRANGE, *Sur l'unicité retrograde*, Math. Scand. **8** (1960), 277-286.

[M] F. MERLE, *Solution of a nonlinear heat equation with arbitrarily given blow up points*, Comm. Pure. Appl. Math., to appear.

[MW] C.E. MÜLLER - F.B. WEISSLER, *Single point blow up for a general semilinear heat equation*, Indiana Univ. Math. J., **34** (1983), 881-913.

[V] J.J.L. VELÁZQUEZ, *Local behaviour near blow up points for semilinear parabolic equations*, J. Differential Equations, to appear.

[W] F.B. WEISSLER, *Single point blow up of semilinear initial value problems*, J. Differential Equations, **55** (1984), 204-224.

Departamento de Matemática Aplicada
Facultad de Ciencias Matemáticas
Universidad Complutense
28040 Madrid
Spain