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The Inverse of a Local Operator Preserves the Markov Property

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0. - Introduction

The present work was originally motivated by the analysis of a degenerate multicomponent Gaussian generalized random field which arises from the simplest example of quantum gauge field theory, i.e. the electromagnetic field. Let $\{X(\varphi); \varphi \in C_0^\infty(E^4 \to \mathbb{R}^4)\}$ be a generalized random field with an index set $\mathcal{D} = C_0^\infty(E^4 \to \mathbb{R}^4)$, which is identified with the space of smooth differential 1-forms over the 4-dimensional Euclidean space E^4 with compact supports, such that

$$E\left[e^{\sqrt{-1}X(\varphi)}\right] = \exp\left(-\frac{1}{2}(\mathrm{d}\varphi, (-\Delta)^{-2}\mathrm{d}\varphi)_{L^2}\right), \quad \varphi \in \mathcal{D}, \tag{0.1}$$

where d is the exterior derivative operator and Δ is the Laplacian acting componentwise. We note that the bilinear form $(\mathrm{d}\varphi, (-\Delta)^{-2}\mathrm{d}\varphi)_{L^2}$ is degenerate and invariant under the replacement by $\varphi + \mathrm{d}f$, where $f \in C_0^\infty(E^4)$. Speaking in connection with physics, $\{X(\varphi)\}$ describes the electromagnetic gauge potential field in Landau's gauge through the analytic continuation procedure to the Minkowski space. Because of its degeneracy we can not apply known general results to discuss the Markov property of $\{X(\varphi)\}$. (See e.g. [16], [30] and also [11]. In [8], [15], [28] and [29] the relation between locality and Markov property is discussed in the framework of potential theory). On the other hand the following idea is known among physicists (see e.g. [10], [20], [21] and [42]) as restriction to transversal test forms. We set $\mathcal{D}_t = \{\varphi \in \mathcal{D}; \delta\varphi = 0\}$, where δ is the coderivative operator and consider its Hilbert space completion $\mathcal{H}$ with respect to $(\mathrm{d}\varphi, (-\Delta)^{-2}\mathrm{d}\varphi)_{L^2}$. Since this bilinear form is non-degenerate on $\mathcal{D}_t$, $\mathcal{H}$ may be embedded in the space of tempered currents $\mathcal{S}'(E^4 \to \mathbb{R}^4)$ and

$$\mathcal{H} = \{\xi \in \mathcal{S}'(E^4 \to \mathbb{R}^4); \exists C < \infty \text{ s.t. } |\langle \xi, \varphi \rangle| \le C \|d\varphi\|_{L^2}\ \forall \varphi \in \mathcal{D}\}.$$

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Now the point is that the operator δd is no longer degenerate on $\mathcal{H}$. Thus applying Nelson's arguments [23] for the proof of the Markov property of the free Euclidean field, one can show a Markov property of $\{X(\varphi); \varphi \in \mathcal{D}_t\}$ (note the change of index sets). However the filtration subordinate to the latter generalized random field has a strange character which is caused by the lack of C^∞-module structure for $\mathcal{D}_t$. Because of this we aim here to discuss the Markov property of $\{X(\varphi); \varphi \in \mathcal{D}\}$ and not that of $\{X(\varphi); \varphi \in \mathcal{D}_t\}$.

Before describing how the question above is solved, we mention very briefly about the definition of Markov properties. We agree that a simple-minded definition of Markov property for random fields $\{X_t; t \in E^d\}$ is the following: $\sigma\{X_t; t \in D\}$, the σ-field generated by $\{X_t; t \in D\}$, and $\sigma\{X_t; t \in E^d \backslash \overline{D}\}$ are conditionally independent given $\sigma\{X_t; t \in \partial D\}$ for any open subset D with boundary ∂D. However this definition excludes a lot of interesting examples. In fact it is known that any translation homogeneous Gaussian Markov random field in the above sense must be a single Gaussian random variable if the dimension d is greater than 1 (see e.g. [17], [37] and [38]). McKean [22] proposed a general prescription by considering the germ σ-fields. His definition of Markov property, roughly speaking, involves all normal derivatives of the random field at ∂D. Then one is naturally led to the question how precisely one should know about the normal derivatives, for instance, whether it is sufficient to know the normal derivatives up to certain fixed order or not. Wong and Zakai [39] introduced a notion of localizable random currents and gave a geometrical formulation of normal derivatives of random currents. Moreover they presented interesting examples of Gaussian random currents with Markov property in their sense. Amont the recent developments we particularly mention the work by Albeverio, Høegh-Krohn and Surgailis [3] on integer valued random fields arising from grand canonical Gibbs fields, where the basic idea is that the Markov property must be preserved under local manipulations and is similar to ours. As for the subject related to Euclidean quantum fields we share in the benefits of the survey by Albeverio and Zegarlinski [6].

Concerning notions of Markov property there exists another important one besides the one in terms of germ σ-fields: Given an openn covering $\{D_+, D_-\}$ of E^d, one can ask whether $\sigma\{X_t; t \in D_+\}$ and $\sigma\{X_t; t \in D_-\}$ are conditionally independent given $\sigma\{X_t; t \in D_+ \cap D_-\}$ or not. Preiss and Kotecky [27] pointed out that if the above condition holds for any open coverings $\{D_+, D_-\}$ of E^d, then actually the random field $\{X_t\}$ is Markov in McKean's sense but not the other way around if one is concerned with *generalized* random fields. Moreover they clarified the difference between these two notions.

Our method for the analysis of (0.1) is the factorization of the covariance bilinear form. This is possible when one embeds a differential form of order say r, $r = 0, 1, \ldots, d$, as one of the homogeneous components of a section of the alternating algebra bundle generated by the cotangent bundle. We note that if we can make the embedding in such way that each homogeneous component is mutually independent, then the Markov property of total system implies that of the original random current. Then one can ask whether the extended random current is Markov or not. We consider the system determined by the following elliptic system:

$$(d+\delta)X_1 = X_2, \tag{0.2}$$

where X_2 is, for instance, a random current deriving from Gaussian white noises. Then we can transfer the degeneracy concerning the de Rahm-Hodge-Kodaira decomposition, which is not local, into that relative to the order of differential forms, which is local in contrast. Now the question is whether X_1 inherits the Markov property of X_2 related by $(d+\delta)X_1 = X_2$. Actually if X_2 is Gaussian distributed, this set up works very well as Rozanov showed [31], [32]. This type of problems with $d+\delta$ replaced by local operators with analytic symbols were discussed extensively in one-dimensional settings by Doob [7], Levinson and McKean [19], and Okabe [24], [25]. On the other hand multidimensional elliptic cases were discussed by Surgailis [34], [35] and Osipov [26]. In 1979 Kusuoka [18] gave a comprehensive answer to the question by showing that the inverse maps of invertible local linear operators preserve the Markov property relative to open coverings (not in the sense of McKean) in general multidimensional cases. However we find that the kernel of the linear operator $d+\delta$ is not small enough to apply Kusuoka's result in discussing our very question. Thus not a small part of the present work is precisely devoted to removing the harmful effect of the nontrivial kernel of $d+\delta$.

Our formulation of the problem is as follows: Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be real vector bundles over a paracompact manifold M and L a linear differential operator mapping C^∞-sections of $\mathbb{E}_2$ to those of $\mathbb{E}_1$. $\Gamma(\mathbb{E}_1)$ stands for the space of all C^∞-sections of $\mathbb{E}_1$ and $\mathcal{D}_1$ for the space of all C^∞-sections of $\mathbb{E}_1$ with compact supports. We use the analogous notations for the corresponding spaces for $\mathbb{E}_2$. Suppose we are given two generalized random fields $\{X_1(\xi); \xi \in \mathcal{D}_1\}$ and $\{X_2(\eta); \eta \in \mathcal{D}_2\}$ related by

$$X_1(L\eta) = X_2(\eta) \text{ a.s. for each } \eta \in \mathcal{D}_2.$$

The heart of the argument is an appropriate choice of a tolopogy on $\mathcal{D}_1$ with respect to which $\{X_1(\xi)\}$ is continuous in probability. Let $\mathcal{H}_2$ be a Fréchet space consisting of smooth sections of $\mathbb{E}_2$ satisfying the following postulates:

(0.3) $\mathcal{H}_2$ contains $\mathcal{D}_2$ and the inclusion $\mathcal{D}_2 \hookrightarrow \mathcal{H}_2$ is dense.

(0.4) $L : \mathcal{H}_2 \to \Gamma(\mathbb{E}_1)$ is injective and there exists a linear map $G : \mathcal{D}_1 \to \mathcal{H}_2$ such that the following diagram holds

$$\begin{array}{ccccc} \mathcal{D}_2 & \hookrightarrow & \mathcal{H}_2 & \hookrightarrow & \Gamma(\mathbb{E}_2) \\ L\downarrow & G\nearrow & & & \downarrow L \\ \mathcal{D}_1 & & \hookrightarrow & & \Gamma(\mathbb{E}_1). \end{array}$$

(0.5) If $\{\xi_n\} \subset \mathcal{D}_1$ and $G\xi_n \to 0$ in $\mathcal{H}_2$ then $X_1(\xi_n) \to 0$ in probability.

(0.6) If $\varphi \in \mathcal{H}_2$ satisfies supp $\varphi \subset D$ for some open subset D of M, then there exists a sequence $\{\eta_n\} \subset \mathcal{D}_2$ with supp $\eta_n \subset D$ such that $\eta_n \to \varphi$ in $\mathcal{H}_2$.

Let $\{D_+, D_-\}$ be an open covering of M. We shall say that $\{D_+, D_-\}$ is $\{L, \mathcal{H}_2\}$-admissible, for each $\xi \in \mathcal{D}_1$ with supp $\xi \subset D_+$ there exists a partition of unity $\{\chi_+, \chi_-\}$ subordinate to $\{D_+, D_-\}$ such that $\chi_- G\xi \in \mathcal{H}_2$ and there exists a sequence $\{\xi_n\} \subset \mathcal{D}_1$ with supp $\xi_n \subset D_+ \cap D_-$ such that $G\xi_n \to \chi_- G\xi$ in $\mathcal{H}_2$.

Under the above setting our main theorem in the present work asserts the following.

THEOREM 0.7. *Let $\{D_+, D_-\}$ be an $\{L, \mathcal{H}_2\}$-admissible open covering of M. If $\sigma\{X_2(\eta);$ supp $\eta \subset D_+\}$ and $\sigma\{X_2(\eta);$ supp $\eta \subset D_-\}$ are conditionally independent given $\sigma\{X_2(\varphi);$ supp $\varphi \subset D_+ \cap D_-\}$, then the corresponding relation also holds for $\{X_1(\xi)\}$.*

In order to discuss the system $(d+\delta)X_1 = X_2$ with a Gaussian white noise X_2, all we have to do is to put,

$$\mathcal{H}_2 = \{\xi \in C^\infty(E^d \to \wedge_* \mathbb{R}^d);\ (\xi, \xi)_{L^2} < \infty\},$$

where $\wedge_* \mathbb{R}^d$ is the exterior algebra generated by $\mathbb{R}^d$. Then we can verify that all the postulates (0.3)–(0.6) are satisfied and moreover any open covering $\{D_+, D_-\}$ of E^d is admissible relative to $\{d+\delta, \mathcal{H}_2\}$. Thus we succeed in proving the Markov property for the random current $\{X(\varphi); \varphi \in \mathcal{D}\}$ characterized by (0.1).

We also discuss the so called sharp Markov property for a first order elliptic system $\not{\partial} X_1 = X_2$ with a white noise X_2. Let V be a representation space of the Clifford algebra over $\mathbb{R}^d$, then the operator $\not{\partial}$, usually called the Dirac operator, acts on V-valued smooth functions on E^d. Moreover X_1 and X_2 are defined as generalized random fields with a common index space $\mathcal{D} = C_0^\infty(E^d \to V)$. We localize the generalized random field $\{X_1(\varphi); \varphi \in \mathcal{D}\}$ to hyperplanes in the spirit of Wong and Zakai. Let $\{j_\delta\}$ be a sequence of mollifiers tending to δ_0 the Dirac mass at $0 \in \mathbb{R}$. For each $\eta \in C_0^\infty(\mathbb{R}^{d-1} \to V)$ the sequence $\{X_1(j_\delta \otimes \eta)\}$ of random variables converges in probability as $\delta \downarrow 0$. We shall denote the limit by $Y_1(\delta_0 \otimes \eta)$. (We identify a hyperplane in E^d with $\mathbb{R}^{d-1}$ and hence the tensor product $j_\delta \otimes \eta$ is understood under this identification). What we mean by the sharp Markov property and what we shall prove is the following:

THEOREM 0.8. *$\sigma\{X_1(\varphi);$ supp $\varphi \subset (-\infty, 0) \times \mathbb{R}^{d-1}\}$ and $\sigma\{X_1(\varphi);$ supp $\varphi \subset (0, \infty) \times \mathbb{R}^{d-1}\}$ are conditionally independent given $\sigma\{Y_1(\delta_0 \otimes \eta);$ $\eta \in C_0^\infty(\mathbb{R}^{d-1} \to V)\}$.*

On the other hand we shall give an example showing that the splitting σ-field may spread to the whole σ-field contrary to what is intuitively suggested by the terminology 'sharp Markov property'.

Before briefly describing the contents, let us mention that these discussions can be applied to prove the Markov property of the generalized random field investigated in our previous works [2] (collaborated with S. Albeverio and R. Høegh-Krohn) and [4] (with S. Albeverio and T. Kolsrud). In particular the use of adjective 'markovian' which appeared in the title of [2] is justified by the present work.

In Section 1 we mention two mutually non-equivalent notions of Markov property and clarify the difference following Preiss and Kotecky. In Section 2 after some comments on Kusuoka's result we shall prove our main theorem and illustrate how (0.7) is applied. Finally in Section 3 we examine the sharp Markov property for the first order elliptic system.

1. - Two nonequivalent notions of Markov property

To begin with we want to clarify what we mean by Markov properties of generalized random fields.

Throughout this paper $(\Omega, \mathcal{F}, P)$ denotes the underlying probability space which we assume to be complete. $\mathcal{N}$ denotes the trivial sub σ-field $\{A \in \mathcal{F}; P(A)^2 = P(A)\}$. Let M be a *paracompact* C^∞-differentiable manifold. We consider a family of random variables $\{X(\varphi); \varphi \in C_0^\infty(M)\}$ indexed by the space $C_0^\infty(M)$ of all smooth functions on M with compact supports. (If one likes, $C_0^\infty(M)$ can be replaced by a certain space consisting of smooth sections of some vector bundle. Even the linear structure is dispensable except for Theorem (1.21)).

DEFINITION 1.1. If the family $\{X(\varphi)\}$ is linear in the following sense, we call it *a generalized random field*:

$$X(c_1\varphi_1 + c_2\varphi_2) = c_1X(\varphi_1) + c_2X(\varphi_2) \text{ a.s. } c_1, c_2 \in \mathbb{R}, \varphi_1, \varphi_2 \in C_0^\infty(M).$$

We identify $\{X(\varphi)\}$ and $\{X'(\varphi)\}$ if $X(\varphi) = X'(\varphi)$ a.s. for all $\varphi \in C_0^\infty(M)$.

Given a generalized random field $\{X(\varphi)\}$, the σ-field $\mathcal{F}$ is naturally filtered. That is, we introduce a family of sub σ-fields $\{\mathcal{F}_D; D \subset M, \text{ open}\}$, where each of $\mathcal{F}_D$ is defined as follows:

$$\mathcal{F}_D := \sigma\{X(\varphi); \text{ supp } \varphi \subset D\} \vee \mathcal{N} \quad (\mathcal{F}_\emptyset := \mathcal{N}).$$

We shall call the family of σ-fields $\{\mathcal{F}_D\}$ the *canonical filtration* subordinate to the generalized random field $\{X(\varphi)\}$. In addition to the usual property, $\{\mathcal{F}_D\}$ satisfies the following:

(1.2) *if* $\{U_\alpha\}$ *is an open covering of* D, *then* $\mathcal{F}_D = \vee \mathcal{F}_{U_\alpha}$.

Of course this is due to the paracompactness of M and the $C^\infty(M)$-module structure of $C_0^\infty(M)$.

NOTATION 1.3. $\mathcal{F}_+ \perp \mathcal{F}_- | \mathcal{F}_0$ means that two sub σ-fields $\mathcal{F}_+$ and $\mathcal{F}_-$ are conditionally independent given $\mathcal{F}_0$.

The σ-field $\mathcal{F}_0$ is often called a splitting σ-field for $\mathcal{F}_+$ and $\mathcal{F}_-$. We note that

LEMMA 1.4. *Suppose* $\mathcal{F}_0 \subset \mathcal{F}_-$. *Then* $\mathcal{F}_+ \perp \mathcal{F}_- | \mathcal{F}_0$ *is equivalent to that* $E[f|\mathcal{F}_-] = E[f|\mathcal{F}_0]$ *a.s. for all bounded* $\mathcal{F}_+$*-measurable functions* f.

In this paper we are concerned with several notions of Markov properties.

DEFINITION 1.5. We say that the filtration $\{\mathcal{F}_D\}$, and hence $\{X(\varphi)\}$, is *MI-Markov* if $\mathcal{F}_{D_+} \perp \mathcal{F}_{D_-} | \mathcal{F}_{D_+ \cap D_-}$ for any open convering $\{D_+, D_-\}$ of M.

REMARK 1.6. It is often practical to add some condition on open coverings. For example we require that $\inf\{\rho(x,y); x \in M \backslash D_+, y \in M \backslash D_-\} > 0$, where ρ is a fixed compatible metric. This weaker form is called *the* 0*-Markov property*.

In order to define the second notion of Markov properties, which is simply called Markov property in much of the literature on the subject, we need to introduce a family of sub σ-fields indexed by closed subsets of M. Let C be a closed subset. We consider the following sub σ-field:

$$\mathcal{F}_C := \cap\{\mathcal{F}_D; D \text{ open and } \supset C\}.$$

DEFINITION 1.7. We say that the filtration $\{\mathcal{F}_D\}$ is *MII-Markov*, if $\mathcal{F}_D \perp \mathcal{F}_{M\backslash\overline{D}} | \mathcal{F}_{\partial D}$ for any open subsets.

As an illustration we consider a white noise. Let G be a locally compact Hausdorff group with a countable base. We assume that M is a G-homogeneous space, i.e. G acts on M transitively from the left.

DEFINITION 1.8. By *a white noise* on M we mean a system of random variables $\{W(A)\}$ indexed by relatively compact Borel subsets of M which satisfies the following postulates.

(i) If $A_1, \dots, A_n$ are mutually disjoint, then $W(A_1), \dots, W(A_n)$ are mutually independent and

$$W\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n W(A_i) \quad \text{a.s.}.$$

(ii) If $A_n \downarrow \emptyset$, then $W(A_n) \to 0$ in probability.

(iii) The system $\{W(A)\}$ is homogeneous relative to the G-action, i.e.,

$$\{W(gA_1), \dots, W(gA_n)\} \overset{d}{=} \{W(A_1), \dots, W(A_n)\}, \quad \text{for all } g \in G,$$

whence $\overset{d}{=}$ stands for the equivalence in distribution.

It is known that the distributions of such random variables $\{W(A)\}$ must be infinitely divisible provided M is uncountable. (See e.g. [1]. Note that a white noise over an uncountable space must be non-atomic). In fact there exists a unique continuous conditionally positive definite function ψ on $\mathbb{R}$ such that

$$(1.9) \qquad E\left[e^{\sqrt{-1}\xi E(A)}\right] = e^{\psi(\xi)\mu(A)} \quad \xi \in \mathbb{R},\ A \text{ relatively compact,}$$

where μ is the G-invariant Radon measure on M (unique up to multiplicative constants). ψ is called the Lévy characteristic of the white noise. In particular if $\{W(A)\}$ is mean 0 Gaussian distributed, i.e., $\psi(\xi) = -\frac{1}{2}c\xi^2$ for some $c > 0$, $\{W(A)\}$ is called a Gaussian white noise.

The following procedure is standard. We set

$$\mathcal{M} := \{f : M \to \mathbb{R};\ \text{bounded measurable, vanishing outside a compact set}\}.$$

It is easy to see that there exists a unique family of random variables $\{Y(f); f \in \mathcal{M}\}$ such that

(1.10 i) $Y(c_1f_1 + c_2f_2) = c_1Y(f_1) + c_2Y(f_2)$ a.s. $c_1, c_2 \in \mathbb{R}$ and $f_1, f_2 \in \mathcal{M}$.

(1.10 ii) If $f_n \downarrow 0$, then $Y(f_n) \to 0$ in probability.

(1.10 iii) $Y(1_A) = W(A)$ a.s. for A relatively compact,

where 1_A is the indicator function of the set A. The random variable $Y(f)$ is called the stochastic integral of f with respect to the white noise $\{W(A)\}$. It immediately follows that

$$(1.11) \qquad E\left[e^{\sqrt{-1}Y(f)}\right] = \exp\left\{\int_{\mathbb{R}^d} \psi(f(x))\mu(\mathrm{d}x)\right\}, \quad f \in \mathcal{M}.$$

We define a generalized random field $\{X(\varphi)\}$ by restricting the index set to $C_0^\infty(M)$. The family $\{X(\varphi)\}$ shall be called a *white noise generalized random field* (or simply a white noise).

LEMMA 1.12. If D is an open subset of M, then $\mathcal{F}_D = \sigma\{W(A); A \subset D\} \vee \mathcal{N}$.

PROOF. The proof is done by showing that $W(K)$ is $\mathcal{F}_D$-measurable for each compact subset K of D. ∎

By virtue of this Lemma we see that *the white noise* $\{X(\varphi)\}$ *is MI-Markov* with respect to any open covering.

LEMMA 1.13. *$\mathcal{F}_C = \sigma\{W(A); A \subset C\} \vee \mathcal{N}$ for all closed subsets.*

PROOF. Let A be a relatively compact Borel subset. Then by (1.12) we have

$$E\left[e^{\sqrt{-1}\xi W(A)}\middle| \mathcal{F}_{C_\epsilon}\right] = e^{\sqrt{-1}\xi W(A\cap C_\epsilon)} e^{\psi(\xi)\mu(A\setminus C_\epsilon)} \quad \text{a.s.,}$$

where C_ϵ is the ϵ-neighbourhood of C with respect to a compatible metric. Since $C_\epsilon \downarrow C$ as $\epsilon \downarrow 0$, the martingale convergence theorem implies that

$$E\left[e^{\sqrt{-1}\xi W(A)}\middle|\bigcap_{\epsilon>0}\mathcal{F}_{C_\epsilon}\right] = e^{\sqrt{-1}\xi W(A\cap C)}e^{\psi(\xi)\mu(A\setminus C)} \quad \text{a.s..}$$

This shows that $\cap\mathcal{F}_{C_\epsilon} \subset \sigma\{W(A); A \subset C\} \vee \mathcal{N}$. On the other hand we see by (1.12) that the r.h.s. $\subset \mathcal{F}_C(\subset \mathcal{F}_{C_\epsilon})$. Thus we get the desired relation. ∎

The Lemmas (1.12) and (1.13) tell us that the canonical filtration subordinate to a white noise is particulary good natured, i.e.,

$$\mathcal{F}_{D_+} = \mathcal{F}_{M\setminus D_-} \vee \mathcal{F}_{D_+\cap D_-} \text{ for any open covering } \{D_+, D_-\} \text{ of } M,$$

$$\mathcal{F}_{C_+} = \mathcal{F}_{M\setminus C_-} \vee \mathcal{F}_{C_+\cap C_-} \text{ for any closed covering } \{C_+, C_-\} \text{ of } M.$$

We now return to the general situation again.

PROPOSITION 1.14. *Let $\{\mathcal{F}_D\}$ be an MI-Markov filtration. Then*

(1.15) $\mathcal{F}_{C_+} \perp \mathcal{F}_{C_-} | \mathcal{F}_{C_+\cap C_-}$ *for any closed covering $\{C_+, C_-\}$ of M.*

Conversely, if $\{\mathcal{F}_D\}$ has the property (1.2), *then* (1.15) *implies the MI-Markov property.*

The following version of the martingale convergence theorem is used in the proof of (1.14).

LEMMA 1.16. *Let H be a Hilbert space and $\mathcal{P}$ be a set of orthogonal projectors in H. Suppose $\mathcal{P}$ is directed in the following sense: for any pair $P_1, P_2 \in \mathcal{P}$ there exists $P_3 \in \mathcal{P}$ with $P_3 < P_1$ and $P_3 < P_2$. Then given $x \in H$ there is a sequence $\{P_n\} \subset \mathcal{P}$ such that $P_n x \to Qx$, where Q is the orthogonal projector onto $\bigcap_{P\in\mathcal{P}}$ Image P.* ∎

PROOF OF 1.14. Let f_+, f_- and f_0 be bounded $\mathcal{F}_{C_+}$ -, $\mathcal{F}_{C_-}$ - and $\mathcal{F}_{C_+\cap C_-}$-measurable functions respectively. If D is an open set containing $C_+ \cap C_-$, then $\{D \cup C_+, D \cup C_-\}$ is an open covering and $(D \cup C_+) \cap (D \cup C_-) = D$. Therefore we see that

$$E[f_+ f_- f_0] = E[E[f_+|\mathcal{F}_D]f_- f_0]$$

by using the MI-Markov property. Applying (1.16) we have

$$E[f_+ f_- f_0] = E[E[f_+|\mathcal{F}_{C_+\cap C_-}]f_- f_0],$$

which shows (1.15).

Next we prove the converse statement. Given an open covering $\{D_+, D_-\}$ of M, choose another open covering $\{D_{+,0}, D_{-,0}\}$ so that $\overline{D_{+,0}} \subset D_+$ and $\overline{D_{-,0}} \subset D_-$. We consider two families $\{D_{+,A} := D_{+,0} \cup A\}_{A\in\mathcal{A}}$ and $\{D_{-,A} := D_{-,0} \cup A\}_{A\in\mathcal{A}}$, where $\mathcal{A}$ is the collection of all open subsets A with

$\overline{A} \subset D_+ \cap D_-$. Clearly it follows that $\overline{D_{+,A}} \subset D_+$, $\overline{D_{-,A}} \subset D_-$ for $A \in \mathcal{A}$, and $\{D_{+,A}\}$ covers D_+, $\{D_{-,A}\}$ covers D_- and $\{D_{+,A} \cap D_{-,A}\}$ covers $D_+ \cap D_-$. Therefore, *thanks to the property* (1.2), we get

$$\mathcal{F}_{D_+} = \bigvee \mathcal{F}_{D_{+,A}} \subset \bigvee \mathcal{F}_{\overline{D_{+,A}}} \subset \mathcal{F}_{D_+}, \tag{1.17}$$
$$\bigvee \mathcal{F}_{\overline{D_{-,A}}} = \mathcal{F}_{D_-} \text{ and } \bigvee \mathcal{F}_{\overline{D_{+,A}} \cap \overline{D_{-,A}}} = \mathcal{F}_{D_+ \cap D_-}.$$

Let f be a bounded $\mathcal{F}_{D_-}$-measurable function, then, if $A \supset B$, we see *by* (1.15) that

$$E[E[f[\mathcal{F}_{\overline{D_{-,B}}}]|\mathcal{F}_{\overline{D_{+,A}}}] = E[E[f|\mathcal{F}_{\overline{D_{-,B}}}]|\mathcal{F}_{\overline{D_{+,A}} \cap \overline{D_{-,A}}}] \quad \text{a.s..}$$

Hence by (1.16) and (1.17) we obtain $E[f|\mathcal{F}_{D_+}] = E[f|\mathcal{F}_{D_+ \cap D_-}]$ a.s.. According to (1.4) this implies the MI-Markov property. ∎

REMARK 1.18. Concerning the topology of M, it is the *T_4-separation axiom* that we actually exploited in the proof of the converse relation. This remark also applies to the following Theorem (1.23).

This proposition has the following immediate corollary.

THEOREM 1.19. *Every MI-Markov filtration possesses the MII-Markov property as well.*

REMARK 1.20. In general the MI-Markov property is *strictly stronger* than the MII-Markov property. As an example we consider a translation homogeneous Gaussian generalized random field on E^d with independent values at every point (cf. [9]). It is of course MII-Markov. However it is known that the MI-Markov property is valid solely for the white noise among such a class (see e.g. [18] and [27]).

In the following discussion we want to single out the difference between the MI- and MII-Markov properties. The next Theorem and (1.22) are due to Preiss and Kotecky [27].

THEOREM 1.21. *Let $\{X(\varphi)\}$ be an MI-Markov generalized random field. Then it follows that*

(i) $\mathcal{F}_{D_+} = \mathcal{F}_{M \setminus D_-} \vee \mathcal{F}_{D_+ \cap D_-}$ *for any open covering* $\{D_+, D_-\}$ *of* M,

(ii) $\mathcal{F}_{C_+} = \mathcal{F}_{M \setminus C_-} \vee \mathcal{F}_{C_+ \cap C_-}$ *for any open covering* $\{C_+, C_-\}$ *of* M.

PROOF. (i) The relation $\mathcal{F}_{D_+} \subset \mathcal{F}_{M \setminus D_-} \vee \mathcal{F}_{D_+ \cap D_-}$ must be proved. Let φ be an element of $C_0^\infty(M)$ with supp $\varphi \subset D_+$. Given an open set $D \supset M \setminus D_-$, choose $\varphi_+ \in C_0^\infty(M)$ so that supp $\varphi_+ \subset D \cap D_+$ and supp $(\varphi - \varphi_+) \subset D_+ \cap D_-$

(using a partition of unity). Then we have

$$e^{\sqrt{-1}tX(\varphi)}E\left[e^{-\sqrt{-1}tX(\varphi)}\middle|\mathcal{F}_{D_+\cap D_-}\right] =$$
$$e^{\sqrt{-1}tX(\varphi_+)}E\left[e^{-\sqrt{-1}tX(\varphi_+)}\middle|\mathcal{F}_{D_+\cap D_-}\right] \quad \text{a.s. } t\in\mathbb{R}.$$

Because of the MI-Markov property,

$$E\left[e^{-\sqrt{-1}tX(\varphi_+)}\middle|\mathcal{F}_{D_+\cap D_-}\right] = E\left[e^{-\sqrt{-1}tX(\varphi_+)}\middle|\mathcal{F}_{D_-}\right] =$$
$$E\left[e^{-\sqrt{-1}tX(\varphi_+)}\middle|\mathcal{F}_{D\cap D_-}\right] \quad \text{a.s..}$$

Therefore $e^{\sqrt{-1}tX(\varphi)}E\left[e^{-\sqrt{-1}tX(\varphi)}\middle|\mathcal{F}_{D_+\cap D_-}\right]$ is $\mathcal{F}_D$- measurable if $D\supset M\backslash D_-$, i.e., it is $\mathcal{F}_{M\backslash D_-}$-measurable.

On the other hand we can find a family of probability measures $\{\mu_\omega\}_{\omega\in\Omega}$ on $\mathbb{R}$ such that

$$E\left[e^{-\sqrt{-1}tX(\varphi)}\middle|\mathcal{F}_{D_+\cap D_-}\right] = \int_{\mathbb{R}} e^{-\sqrt{-1}tx}\mu_{\cdot}(dx) \quad \text{a.s. } t\in\mathbb{R}.$$

Consider the following sets:

$$\Omega_n := \left\{\omega\in\Omega; \mathrm{Re}\int_{\mathbb{R}} e^{-\sqrt{-1}tx}\mu_\omega(dx)\geq\frac{1}{2} \text{ for } t\in\left[-\frac{1}{n},\frac{1}{n}\right]\right\},\quad n=1,2,\ldots.$$

Then $\Omega_n\in\mathcal{F}_{D_+\cap D_-}$ and $\Omega_n\uparrow\Omega$ as $n\to\infty$. We now fix n and suppose $|t|\leq\dfrac{1}{n}$. Then we see that

$$e^{\sqrt{-1}tX(\varphi)}1_{\Omega_n} =$$
$$e^{\sqrt{-1}tX(\varphi)}E\left[e^{-\sqrt{-1}tX(\varphi)}\middle|\mathcal{F}_{D_+\cap D_-}\right]\left(\int_{\mathbb{R}} e^{-\sqrt{-1}tx(\varphi)}\mu_{\cdot}(dx)\right)^{-1} 1_{\Omega_n} \quad \text{a.s..}$$

Hence the l.h.s. is $\mathcal{F}_{M\backslash D_-}\vee\mathcal{F}_{D_+\cap D_-}$-measurable. Since

$$X(\varphi) = \lim_{n\to\infty}\lim_{t\to 0}\frac{1}{\sqrt{-1}t}\left(e^{\sqrt{-1}tX(\varphi)}-1\right)1_{\Omega_n},$$

$X(\varphi)$ must be $\mathcal{F}_{M\backslash D_-}\vee\mathcal{F}_{D_+\cap D_-}$-measurable.

(ii) Let f and g be bounded $\mathcal{F}_{C_-}$ - and $\mathcal{F}_{M\backslash C_-}$-measurable functions respectively. According to (1.14) we have

$$E[fg|\mathcal{F}_{C_+}] = gE[f|\mathcal{F}_{C_+}] = gE[f|\mathcal{F}_{C_+\cap C_-}] \quad \text{a.s..}$$

Since $\mathcal{F}_M = \mathcal{F}_{C_-} \vee \mathcal{F}_{M\backslash C_-}$ by the already proved relation (i), we get $\mathcal{F}_{C_+} \subset \mathcal{F}_{M\backslash C_-} \vee \mathcal{F}_{C_+\cap C_-}$. This completes the proof. ∎

REMARK 1.22. In the previous Remark (1.20) we mentioned an example of MII-Markov generalized random fields *without* MI-Markov property. Actually the Proposition (1.21) suggests an alternative way to show the failure of MI-Markov property by proving the existence of open coverings $\{D_+, D_-\}$ of $\mathbb{R}^d$ which do not satisfy the condition (i) in (1.21) (see [27]). An outline of the argument is as follows: We consider a mean 0 translation homogeneous Gaussian generalized random field with independent values at every point. Then its covariance bilinear form determines a unique differential operator P. Provided P is not a constant, one can show that if D is a non-empty bounded open set with smooth boundary, then there exists a non-trivial solution f for $Pf = 0$ on $\mathbb{R}^d\backslash\partial D$ with finite norm $(f, f) < \infty$. This will imply that $\mathcal{F}_{\mathbb{R}^d} \neq \mathcal{F}_{\mathbb{R}^d\backslash\partial D} \vee \mathcal{F}_{\partial D}$.

Theorem (1.21) and Remark (1.22) tell us that the difference between the MI-Markov property and the MII-Markov property lies in the validity of condition (i) in (1.21). In fact we can prove the following theorem.

THEOREM 1.23. *Let $\{\mathcal{F}_D\}$ be an MII-Markov filtration with the property (1.2). Then the following implies the MI-Markov property:*

$$\mathcal{F}_{D_-} = \mathcal{F}_{D_-\backslash\partial D_+} \vee \mathcal{F}_{\partial D_+} \text{ for any open covering } \{D_+, D_-\} \text{ of } M.$$

Before going into the proof we mention a simple fact.

LEMMA 1.24. *Let $\mathcal{F}_+$, $\mathcal{F}_-$ and $\mathcal{F}_0$ be sub σ-fields of $\mathcal{F}$. If $\mathcal{F}_+ \perp \mathcal{F}_- | \mathcal{F}_0$, then $(\mathcal{F}_+ \vee \mathcal{F}_0) \perp (\mathcal{F}_- \vee \mathcal{F}_0) | \mathcal{F}_0$.*

PROOF OF (1.23). Given an open covering $\{D_+, D_-\}$, choose an open subset D_0 with $M\backslash D_- \subset D_0 \subset \overline{D_0} \subset D_+$. We consider a family $\{D_A := D_0 \cup A\}_{A\in\mathcal{A}}$, where $\mathcal{A}$ is the collection of all open subsets A with $\overline{A} \subset D_+ \cap D_-$. Since $(M\backslash\overline{D_A}) \cup (D_A \cap D_-) = D_-\backslash\partial D_A$, we see *by the assumption and the property* (1.2) that for each A

$$\mathcal{F}_{D_-} = \mathcal{F}_{D_-\backslash\partial D_A} \vee \mathcal{F}_{\partial D_A} = \mathcal{F}_{M\backslash\overline{D_A}} \vee \mathcal{F}_{D_A\cap D_-} \vee \mathcal{F}_{\partial D_A}.$$

Let f_1, f_2 and f_3 be bounded $\mathcal{F}_{M\backslash\overline{D_A}}$-, $\mathcal{F}_{D_A\cap D_-}$ - and $\mathcal{F}_{\partial D_A}$-measurable functions respectively. Then *by the MII-Markov property* and (1.24) we have

$$E[f_1f_2f_3|\mathcal{F}_{D_A} \vee \mathcal{F}_{\partial D_A}] = E[f_1|\mathcal{F}_{D_A} \vee \mathcal{F}_{\partial D_A}]f_2f_3 = E[f_1|\mathcal{F}_{\partial D_A}]f_2f_3 \quad \text{a.s..}$$

Therefore $E[f|\mathcal{F}_{D_A} \vee \mathcal{F}_{\partial D_A}]$ is $\mathcal{F}_{D_+\cap D_-}$-measurable, if f is $\mathcal{F}_{D_-}$-measurable. On the other hand a similar argument to (1.17) shows that $\mathcal{F}_{D_+} = \vee(\mathcal{F}_{D_A} \vee \mathcal{F}_{\partial D_A})$. Thus, according to (1.16), $E[f|\mathcal{F}_{D_+}]$ is $\mathcal{F}_{D_+\cap D_-}$-measurable. This completes the proof.■

REMARK 1.25. The canonical filtration subordinate to a random field satisfies the condition in (1.23) and hence the MI- and MII-Markov property are equivalent for *random fields*.

Finally we discuss the question whether the MI-Markov property is equivalent for two generalized random fields with the same finite dimensional distributions or not. We shall give an affirmative answer under some mild continuity assumption. We agree that $C_0^\infty(M)$ is equipped with the Schwartz topology, i.e., the strongest locally convex topology which makes all natural maps $C_0^\infty(D) \to C_0^\infty(M)$ continuous, where D runs through a covering consisting of relatively compact open subsets and the topology on each $C_0^\infty(D)$ is induced by the compact open topology on $C^\infty(M)$. (This topology is independent of the choice of open coverings). The resulting Hausdorff topological vector space shall be denote by $\mathcal{D}(M)$. Moreover $\mathcal{D}'(M)$ stands for the topological dual space of $\mathcal{D}(M)$ and $\langle \cdot, \cdot \rangle$ for the canonical pairing in the dual pair $\{\mathcal{D}'(M), \mathcal{D}(M)\}$.

DEFINITION 1.26. We say that a generalized random field $\{X(\varphi)\}$ is *stochastically continuous*, if $\varphi_n \to \varphi$ implies $X(\varphi_n) \to X(\varphi)$ in probability.

We dare to omit the word *sequentially*, since it turns out that this weaker form is sufficient in most interesting cases. (See e.g. [13] on related subjects). Suppose M is second countable. Then the Schwartz topology is a strict inductive limit of nuclear Fréchet topologies. Therefore given a stochastically continuous generalized random field, thanks to Minlos' theorem, we obtain a consistent system of probability measures on duals of nuclear Fréchet spaces. According to Kolmogorov the obtained countable projective limit is σ-additive. However, even if we do *not* assume the second countability for M, the projective limit is σ-additive. To prove this we recall that each connected component of a paracompact manifold is second countable. Since $\mathcal{D}'(M) \simeq \prod \mathcal{D}'(N)$, where N runs through all connected components of M, the σ-additivity follows (cf. [41]).

In the following discussion we assume that M is *second countable* till this assumption is canceled. What is favorable, when M is second countable, is that the topological σ-field for $\mathcal{D}'(M)$ equipped with the weak $*$ topology is *standard Borel* and moreover all Borel probability measures are *Radon*. In particular we have the following

LEMMA 1.27. *If a generalized random field $\{X(\varphi)\}$ on $(\Omega, \mathcal{F}, P)$ is stochastically continuous, then there exists a $\mathcal{D}'(M)$-valued random variable $\tilde{X}$ on $(\Omega, \mathcal{F}, P)$ such that $X(\varphi) = \langle \tilde{X}, \varphi \rangle$ a.s. for each $\varphi \in C_0^\infty(M)$.*

We express the canonical filtration of stochastically continuous generalized random fields in terms of $\mathcal{D}'(M)$-valued random variables.

NOTATIONS 1.28. $\pi_D : \mathcal{D}'(M) \to \mathcal{D}'(D)$ denotes the canonical restriction map for each open subset.

Let $\{X(\varphi)\}$ and $\tilde{X}$ be as in (1.27). Then concerning the canonical filtration we have the following

LEMMA 1.29. *$\mathcal{F}_D = \sigma\{\pi_D \circ \tilde{X}\} \vee \mathcal{N}$ for all open subsets.*

PROOF. Since the Borel structure for ($\mathcal{D}'(D)$, weak $*$) is standard and $\mathcal{D}(D)$ is separable, we obtain an into Borel isomorphism $\mathcal{D}'(D) \to \mathbb{R}^\infty$ by collecting a countable family of maps $w \mapsto \langle w, \varphi \rangle$. Therefore it follows that

$$\sigma\{\langle \tilde{X}, \varphi \rangle;\ \text{supp}\ \varphi \subset D\} = \sigma\{\langle \pi_D \circ \tilde{X}, \varphi \rangle; \varphi \in \mathcal{D}(D)\} = \sigma\{\pi_D \circ \tilde{X}\}.$$

This completes the proof. ∎

We now prove the following theorem *without* assuming that M is second countable.

THEOREM 1.30. *Let $\{X(\varphi)\}$ be a stochastically continuous generalized random field with the MI-Markov property. If $\{Y(\varphi)\}$ is a generalized random field with the same characteristic functional as $\{X(\varphi)\}$, then $\{Y(\varphi)\}$ is MI-Markov.*

PROOF. We denote the canonical filtration subordinate to $\{X(\varphi)\}$ by $\{\mathcal{F}_D^X\}$ and that subordinate to $\{Y(\varphi)\}$ by $\{\mathcal{F}_D^Y\}$. Note that if N is an open and closed subset of M and $\{U_+, U_-\}$ is an open covering of N, then $\mathcal{F}_{U_+}^X \perp \mathcal{F}_{U_- \cup (M \setminus N)}^X | \mathcal{F}_{U_+ \cap U_-}^X$. Suppose N is second countable in addition. Then, due to the existence of conditional probability kernels on standard Borel spaces, we deduce from (1.29) that $\mathcal{F}_{U_+}^Y \perp \mathcal{F}_{U_-}^Y | \mathcal{F}_{U_+ \cap U_-}^Y$. That is, if $\{D_+, D_-\}$ is an open covering of M, then $\mathcal{F}_{D_+ \cap N}^Y \perp \mathcal{F}_{D_- \cap N}^Y | \mathcal{F}_{D_+ \cap D_- \cap N}^Y$. Since each component of M is second countable, we see by (1.2) and (1.16) that $\{Y(\varphi)\}$ is MI-Markov. ∎

2. - Linear maps preserving the Markov property

The main purpose in the present work is to discuss the question when the inverse map of a local operator preserves Markov properties. The situation is as follows: Let $\mathbf{E}_1 \to M$ and $\mathbf{E}_2 \to M$ be vector bundles. We use the notation $\Gamma(\mathbf{E}_1)$ for the totality of C^∞-sections of $\mathbf{E}_1$ and $\mathcal{D}_1$ for the totality of C^∞-sections of $\mathbf{E}_1$ with compact supports. The same rule is applied to $\mathbf{E}_2$. Suppose a generalized random field $\{X_1(\xi)\}$ indexed by $\mathcal{D}_1$ and a local linear map $L : \Gamma(\mathbf{E}_2) \to \Gamma(\mathbf{E}_1)$ are given. Consider another generalized random field $\{X_2(\eta) := X_1(L\eta)\}$ indexed by $\mathcal{D}_2$. Since L is local, we may well expect that some local structures of $\{X_1(\xi)\}$ is deduced from that of $\{X_2(\eta)\}$. That is, we set up the question whether $\{X_1(\xi)\}$ inherits the Markov property of $\{X_2(\eta)\}$ or not.

Kusuoka [18] discussed the case when both of the generalized random fields are realized by random variables taking values in subspaces E_1 respectively E_2 of the Schwartz distributions and there exists a bijective map $L' : E_1 \to E_2$, which is dual to L. He stressed the importance of topologies on E_1 and E_2 which make L' homeomorphic and are compatible with the sheaf structure of the Schwartz distributions. From our point of view his result reads as follows:

Let E_i be a Hausdorff topological vector space with a continuous monomorphism $\iota_i : E_i \to \mathcal{D}_i'$ for $i = 1, 2$, and let $L' : E_1 \to E_2$ be a homeomorphic linear isomorphism. Suppose L is local, i.e. $\ker \pi_D^1 \circ \iota_1 \subset \ker \pi_D^2 \circ \iota_2 \circ L'$ for any open subset D of M.

We say that an open covering $\{D_+, D_-\}$ of M is admissible relative to $\{\iota_1 : E_1 \hookrightarrow \mathcal{D}_1', \iota_2 : E_2 \hookrightarrow \mathcal{D}_2'\}$, iff

$$\ker \pi^i_{D_+\cap D_-} \circ \iota_i = \ker \pi^i_{D_+} \circ \iota_i \oplus \ker \pi^i_{D_-} \circ \iota_i \quad \text{for } i = 1, 2 \tag{2.1}$$

and the canonical projection

$$\ker \pi^2_{D_+\cap D_-} \circ \iota_2 \to \ker \pi^2_{D_+} \circ \iota_2 \tag{2.2}$$

is continuous.

THEOREM 2.3. *Let X_2 be an E_2-valued random variable inducing a Radon probability measure. If the generalized random field $\{\langle \iota_2 X_2, \varphi\rangle; \varphi \in \mathcal{D}_2\}$ is MI-Markov with respect to an admissible open covering, then so is $\{\langle \iota_1 L'^{-1} X_2, \varphi\rangle; \varphi \in \mathcal{D}_1\}$.*

REMARKS 2.4. (i) By using a partition of unity subordinate to the open covering $\{D_+, D_-\}$, one sees that $\ker \pi_{D_+} \oplus \ker \pi_{D_-} = \ker \pi_{D_+\cap D_-}$. Therefore, if the partition of unity is compatible with $\iota_1 : E_1 \hookrightarrow \mathcal{D}_1'$, then we will get (2.1). On the other hand, if the topology for E_2 is, for example, Fréchet, then by virtue of the closed graph theorem the projection (2.2) is automatically continuous.

(ii) It is probable that the linearity is dispensable for the preservation of the Markov property in spite of the technical difficulty without it.

Let $\mathcal{S}'$ be the space of tempered distributions on E^d and $\iota : \mathcal{S}' \hookrightarrow \mathcal{D}'$ be the natural injection. Then it is not difficult to see that open coverings $\{D_+, D_-\}$ of E^d with $\inf\{|x-y|; x \in E^d \backslash D_+, y \in E^d \backslash D_-\} > 0$ are admissible relative to $\{\iota : \mathcal{S}' \hookrightarrow \mathcal{D}', \iota : \mathcal{S}' \hookrightarrow \mathcal{D}'\}$. This means that if L' is a differential operator mapping $\mathcal{S}'$ into $\mathcal{S}'$ bijectively and X_2 is an $\mathcal{S}'$-valued random variable which induces a 0-Markov generalized random field, then $L'^{-1} X_2$ induces a 0-Markov generalized random field as well.

REMARK 2.5. Concerning the conservation of the MII-Markov property we mention the following example. Let $\{X_2(\varphi)\}$ be a mean 0 Gaussian generalized random field with covariance functional

$$E[X_2(\varphi)^2] = \int_{\mathbb{R}^d} (1 + |k|^2)|\hat{\varphi}(k)|^2 \mathrm{d}k, \quad \varphi \in \mathcal{D}.$$

We recall that such a generalized random field $\{X_2(\varphi)\}$ is MII-Markov but *not* MI-Markov (see Remark (1.20)). Because of stochastic continuity, there exists a unique $\mathcal{S}'$-valued random variable such that $\langle \tilde{X}_2, \varphi \rangle = X_2(\varphi)$ a.s. for $\varphi \in \mathcal{D}$. We define another $\mathcal{S}'$-valued random variable $\tilde{X}_1$ by $\tilde{X}_1 = (2 - \Delta)^{-n} \tilde{X}_2$, where n is an integer greater than $(d+3)/4$. Then we have

$$E[\langle \tilde{X}_1, \varphi \rangle^2] = \int_{\mathbb{R}^d} (2 + |k|^2)^{-2n} (1 + |k|^2)|\hat{\varphi}(k)|^2 \mathrm{d}k, \quad \varphi \in \mathcal{D}.$$

We can easily see that if $n > (d+3)/4$, then

$$\left| \int_{\mathbb{R}^d} (2 + |k|^2)^{-2n} (1 + |k|^2) \left(e^{\sqrt{-1} k \cdot x} - 1 \right) \mathrm{d}k \right| \le C|x|, \quad x \in \mathbb{R}^d,$$

for some positive constant C. By vertue of this estimate and the Gaussian property of $\tilde{X}_1$, we can apply multiparameter versions of Kolmogorov's criterion (see e.g. [36]) and hence we may regard the $\mathcal{S}'$-valued random variable $\tilde{X}_1$ as a continuous random field. According to [16], the random field $\tilde{X}_1$ is not MII-Markov, for its spectral density is not the reciprocal of an entire function of minimal exponential type. (Note that the MI- and MII-Markov properties are equivalent as far as continuous random fields are concerned. See Remark (1.25)). Thus $(2 - \Delta)^{-n}$, $n > (d+3)/4$, does *not* necessarily preserve the MII-Markov property, while it preserves the MI-Markov property.

Somewhat unsatisfactory is that it is not so clear whether we can apply (2.3) to $L = \Delta$ or not, since Δ^{-1} maps $\mathcal{S} = \mathcal{S}(E^d)$, $d \ge 3$, into a truely larger space. Actually $\mathcal{S}$ is *not* dense in $\Delta^{-1}\mathcal{S}$, which is equipped with the topology transferred by Δ^{-1} from $\mathcal{S}$, and therefore the natural map $\iota_2 : E_2 = (\Delta^{-1}\mathcal{S})' \to \mathcal{D}'$ is *no longer* injective. Thus our effort is devoted to seeking an answer which covers the case $L = \Delta$ or which is flexible for nontrivial kernels of L'.

Our discussion starts from the suitable choice of a subspace $\mathcal{H}_2$ of $\Gamma(\mathbf{E}_2)$ which satisfies the following postulates.

ASSUMPTION 2.6. (i) $\mathcal{H}_2$ contains $\mathcal{D}_2$ and is equipped with a Fréchet topology such that $\mathcal{D}_2$ is dense.

(ii) L maps $\mathcal{H}_2$ into $\Gamma(\mathbf{E}_1)$ injectively. There exists a linear map $G : \mathcal{D}_1 \to \mathcal{H}_2$ such that $GL\eta = \eta$ for $\eta \in \mathcal{D}_2$ and $LG\xi = \xi$ for $\xi \in \mathcal{D}_1$.

Let $\mathcal{H}_1$ denote the image of $\mathcal{H}_2$ under L. Since $L : \mathcal{H}_2 \to \Gamma(\mathbf{E}_1)$ is injective, L induces a Fréchet topology on $\mathcal{H}_1$. Then it immediately follows that $\mathcal{H}_1$ contains

$\mathcal{D}_1(= LG\mathcal{D}_1)$ and the inclusion $\mathcal{D}_1 \hookrightarrow \mathcal{H}_1$ is also dense

$$\begin{array}{ccccc} \mathcal{D}_2 & \hookrightarrow & \mathcal{H}_2 & \hookrightarrow & \Gamma(\mathbf{E}_2) \\ L\downarrow & G\nearrow & L\downarrow\sim & & \downarrow L \\ \mathcal{D}_1 & \hookrightarrow & \mathcal{H}_1 & \hookrightarrow & \Gamma(\mathbf{E}_1). \end{array}$$

Although we do not specify the topology such that the dual L' behaves well in the sense of Theorem (2.3), nevertheless the kernel of L' cannot be too large or the image of L must be large. The following is the corresponding assumption.

ASSUMPTION 2.7. $\{X_1(\xi)\}$ is stochastically continuous with respect to the $\mathcal{H}_1$-topology.

We note that (2.7) implies the stochastic continuity of $\{X_2(\eta)\}$ with respect to the $\mathcal{H}_2$-topology. By a standard procedure we will get a unique family of random variables $\{Y_1(\varphi); \varphi \in \mathcal{H}_1\}$ such that the followings hold:

(2.8i) $\quad Y_1(a\varphi + b\phi) = aY_1(\varphi) + bY_1(\phi)$ a.s. for $a,\ b \in \mathbb{R}$ and $\varphi,\ \phi \in \mathcal{H}_1$;

(2.8ii) $\quad$ If $\varphi_n \to \xi$ in $\mathcal{H}_1$, then $Y_1(\varphi_n) \to Y_1(\varphi)$ in probability;

(2.8iii) $\quad Y_1(\xi) = X_1(\xi)$ a.s. for $\xi \in \mathcal{D}_1$.

The family $\{Y_1(\varphi); \varphi \in \mathcal{H}_1\}$ shall be called *the stochastic continuous extention* of $\{X_1(\xi)\}$. The same procedure also applies to $\{X_2(\eta)\}$. We use the analogous notation for the stochastic continuous extension of $\{X_2(\eta)\}$. Clearly it follows that

(2.9) $\quad Y_1(L\varphi) = Y_2(\varphi)$ a.s. for $\varphi \in \mathcal{H}_2$, $X_1(\xi) = Y_2(G\xi)$ a.s. for $\xi \in \mathcal{D}_1$.

In order to describe the canonical filtration $\{\mathcal{F}_D^2\}$ subordinate to $\{X_2(\eta)\}$, in terms of the random variables $\{Y_2(\varphi); \varphi \in \mathcal{H}_2\}$ we shall assume that

ASSUMPTION 2.10. If supp $\varphi \subset D$ for some $\varphi \in \mathcal{H}_2$ and some open subset D of M, then there exists a sequence $\{\eta_n\} \subset \mathcal{D}_2$ such that supp $\eta_n \subset D$ and $\eta_n \to \varphi$ in $\mathcal{H}_2$.

Then we immediately obtain the following description.

(2.11) $\quad \mathcal{F}_D^2 = \sigma\{Y_2(\varphi);\ \text{supp}\ \varphi \subset D\} \vee \mathcal{N}$ for all open subsets.

We still need one notion to state our theorem.

DEFINITION 2.12. An open covering $\{D_+, D_-\}$ of M is $\{L, \mathcal{H}_2\}$*-admissible*, if for each $\xi \in \mathcal{D}_1$ with supp $\xi \subset D_+$ there exists a partition of unity $\{\chi_+, \chi_-\}$ subordinate to $\{D_+, D_-\}$ such that $\chi_- G\xi \in \mathcal{H}_2$ and there exists a sequence $\{\xi_n\} \subset \mathcal{D}_1$ such that supp $\xi_n \subset D_+ \cap D_-$ and $G\xi_n \to \chi_- G\xi$ in $\mathcal{H}_2$.

REMARK 2.13. Since L is a linear differential operator, it follows that

$$L(\chi_- G\xi) = \chi_- \xi + \text{ terms containing derived functions of } \chi_-.$$

Together with the fact that supp $\chi_- \subset D_-$ and supp $(1-\chi_-) \subset D_+$ we see that supp $L(\chi_- G\xi) \subset D_+ \cap D_-$, if supp $\xi \subset D_+$. In particular, if either D_+ or D_- is bounded, then $L(\chi_- G\xi) \in \mathcal{D}_1$.

Under the assumptions (2.6), (2.7) and (2.10) we prove our main theorem.

THEOREM 2.14. *If* $\{X_2(\eta)\}$ *is MI-Markov with respect to an* $\{L, \mathcal{H}_2\}$-*admissible covering* $\{D_+, D_-\}$ *of* M, *then so is* $\{X_1(\xi)\}$.

Before going into the proof we mention the following usefull lemma.

LEMMA 2.15. *Suppose* $\mathcal{F}_+ \perp \mathcal{F}_- | \mathcal{F}_0$. *If* $\mathcal{F}_*$ *is a sub* σ-*field with* $\mathcal{F}_0 \subset \mathcal{F}_* \subset \mathcal{F}_+ \vee (\mathcal{F}_* \cap (\mathcal{F}_- \vee \mathcal{F}_0))$, *then* $\mathcal{F}_+ \perp \mathcal{F}_- | \mathcal{F}_*$.

PROOF OF 2.15. The argument breakes into twice repeat of the following algorithm: $\mathcal{F}_+ \perp \mathcal{F}_- | \mathcal{F}_0$ and $\mathcal{F}_0 \subset \mathcal{F}_* \subset \mathcal{F}_- \vee \mathcal{F}_0$ imply $\mathcal{F}_+ \perp \mathcal{F}_- | \mathcal{F}_*$. To prove this suppose f is bounded and $\mathcal{F}_+$-measurable. Then it follows from (1.4) and (1.24) that

$$\begin{aligned} E[f|\mathcal{F}_*] &= E[E[f|\mathcal{F}_- \vee \mathcal{F}_0]|\mathcal{F}_*] \\ &= E[E[f|\mathcal{F}_0]|\mathcal{F}_*] \\ &= E[f|\mathcal{F}_0] = E[f|\mathcal{F}_- \vee \mathcal{F}_0] \text{ a.s..} \end{aligned}$$

This means that $\mathcal{F}_+ \perp (\mathcal{F}_- \vee \mathcal{F}_0) | \mathcal{F}_*$ and hence the proof is complete. ∎

PROOF OF 2.14. Given an admissible open covering $\{D_+, D_-\}$ of M and $\xi \in \mathcal{D}_1$ with supp $\xi \subset D_+$, we choose a partition of unity $\{\chi_+, \chi_-\}$ which satisfies the condition in Definition (2.12). Then it follows from (2.9) that

$$\begin{aligned} X_1(\xi) &= Y_2(G\xi) \\ &= Y_2(\chi_+ G\xi) + Y_2(\chi_- G\xi) \\ &= Y_2(\chi_+ G\xi) + Y_1(L(\chi_- G\xi)) \text{ a.s..} \end{aligned}$$

(2.11) implies that $Y_2(\chi_+ G\xi)$ is $\mathcal{F}^2_{\bar{D}_+}$-measurable and $Y_2(\chi_- G\xi)$ is $\mathcal{F}^2_{\bar{D}_-}$-measurable. On the other hand we see from Definition (3.12) that $L(\chi_- G\xi)$ can be approximated in $\mathcal{H}_1$ by a sequence consisting of members in $\mathcal{D}_1$ with supports in $D_+ \cap D_-$. Due to the $\mathcal{H}_1$-stochastic continuity, this means that $Y_1(L(\chi_- G\xi))$ is $\mathcal{F}^1_{D_+ \cap D_-} \cap \mathcal{F}^2_{\bar{D}_-}$-measurable. Thus $X_1(\xi)$ is $\mathcal{F}^2_{\bar{D}_+} \vee (\mathcal{F}^1_{D_+ \cap D_-} \cap \mathcal{F}^2_{\bar{D}_-})$-measurable and hence

$$\mathcal{F}^1_{D_+} = \mathcal{F}^2_{\bar{D}_+} \vee (\mathcal{F}^1_{D_+ \cap D_-} \cap \mathcal{F}^2_{\bar{D}_-}).$$

On the other hand $\mathcal{F}^2_D \subset \mathcal{F}^1_D$ in general by virtue of the locality of L. We now apply Lemma (2.15) and obtain that $\mathcal{F}^2_{\bar{D}_+} \perp \mathcal{F}^2_{\bar{D}_-} | \mathcal{F}^1_{D_+ \cap D_-}$ from the *MI-Markov property* of $\{X_2(\eta)\}$. Finally by (1.24) the statement follows. ∎

REMARK 2.16. If L induces an automorphis of $S(E^d)$ and $\{X_1(\xi)\}$ is S-stochastically continuous, then Theorem (2.14) as well as Kusuoka's covers the conservation of the 0-Markov property.

As an illustration we consider the case where $L = \Delta$ and X_2 is a white noise on E^d with Lévy characteristic ψ (see Section 1). If $d \geq 5$, the following assumption is compatible with the *ccpd* (the abbreviation for continuous conditionally positive definite) property of ψ

$$\psi(\varsigma) = O\left(|\varsigma|^{\frac{d}{d-2}+\varepsilon}\right) \text{ as } \varsigma \to 0 \text{ for some } \varepsilon > 0. \tag{2.17}$$

We set

$$G(x) := \Gamma\left(\frac{d}{2}\right)\left(2\pi^{\frac{d}{2}}(d-2)\right)^{-1}|x|^{-(d-2)}, \ x \in \mathbb{R}^d \backslash \{0\}.$$

Then G is a fundamental solution for $-\Delta$. Our guide line to the choice of $\mathcal{H}_2$ is the following Sobolev inequality.

$$||G * \varphi||_{L^p} \leq C_{p,d}||\varphi||_{L^q} \text{ for } \varphi \in C_0^\infty(E^d), \ \frac{1}{q} = \frac{1}{p} + \frac{2}{d} \text{ and } p > \frac{d}{d-2},$$

where $G*\varphi$ denotes the convolution in distributional sense and $C_{p,d}$ is a positive constant depending on p and d (see e.g. [12], [33]). We define

$$\mathcal{H}_2 := \left\{\phi \in C^\infty(E^d); \int_{E^d} |\phi(x)|^p \mathrm{d}x < \infty \text{ for all } p > \frac{d}{d-2}\right\},$$

and introduce a locally convex topology induced by the compact open topology and the L^p-norms, $p > \dfrac{d}{d-2}$. Obviously $\mathcal{H}_2$ contains $C_0^\infty(E^d)$ and the inclusion $C_0^\infty(E^d) \hookrightarrow \mathcal{H}_2$ is dense. Moreover $G * \varphi \in \mathcal{H}_2$ for $\varphi \in C_0^\infty(E^d)$ precisely by definition. Since an L^2-harmonic function must be 0, Δ maps $\mathcal{H}_2$ into $C^\infty(E^d)$ injectively. By (2.17) we see that

$$|\psi(\varsigma)| \leq \text{Const}\left(|\varsigma|^2 + |\varsigma|^{\frac{d}{d-2}+\varepsilon}\right) \quad \text{for all } \varsigma \in \mathbb{R}.$$

Therefore

$$C_0^\infty(E^d) \in \varphi \mapsto \exp\left\{\int_{E^d} \psi(\varphi(x))\mathrm{d}x\right\} \in \mathbb{C}$$

is continuous with respect to the $\mathcal{H}_2$-topology, in other words $\{X_2(\varphi)\}$ is stochastically continuous with respect to the $\mathcal{H}_2$-topology. Thus, having observed that the assumption (2.10) is clearly satisfied, we obtain the desired space $\mathcal{H}_2$.

Let $\{Y_2(\phi); \phi \in \mathcal{H}_2\}$ be the stochastic continuous extention of $\{X_2(\varphi)\}$. We define a generalized random field $\{X_1(\varphi)\}$ by

$$X_1(\varphi) = Y_2(G * \varphi) \text{ for } \varphi \in C_0^\infty(E^d).$$

Then the distribution of $\{X_1(\varphi)\}$ is characterized by

$$E\left[e^{\sqrt{-1}X_1(\varphi)}\right] = \exp\left\{ \int_{E^d} \psi(G * \varphi(x))\mathrm{d}x \right\}, \quad \varphi \in C_0^\infty(E^d). \tag{2.18}$$

Hence taking account of Remark (2.13) (i.e. an open covering with relatively compact overlappings is $\{L, \mathcal{H}_2\}$-admissible) and Theorem (2.14) we obtain the following result.

PROPOSITION 2.19. *Let* $d = 5, 6, \ldots$ *and* ψ *be a ccpd function on* $\mathbb{R}$ *satisfying* (2.17). *Then the generalized random field* $\{X(\varphi)\}$ *with the characteristic functional* (2.18) *is locally MI-Markov, i.e.,* $\mathcal{F}_{D_+} \perp \mathcal{F}_{D_-} | \mathcal{F}_{D_+ \cap D_-}$ *for any open convering* $\{D_+, D_-\}$ *of* E^d *with* $D_+ \cap D_-$ *bounded.*

If $\psi(\varsigma) = \varsigma^2$, then $\{X(\varphi)\}$ in (2.18) is mean 0 Gaussian distributed and its covariance bilinear form is given by

$$E[X(\varphi)^2] = \int_{\mathbb{R}^d} |\hat{\varphi}(k)|^2 |k|^{-4} \mathrm{d}k, \quad \varphi \in C_0^\infty(E^d),$$

where $\hat{\varphi}$ is the Fourier image of φ and $|k|$ is the Euclidean norm of $k \in \mathbb{R}^d$. In what follows we are going to discuss the MI-Markov property for a more general class of Gaussian generalized random fields by applying Theorem (2.14). We note that our method is completely different from the conventional ones in terms of the orthogonality in the Hilbert space defined by the covariance bilinear form of the Gaussian generalized random field in question. Because we do not want to set up the problem too generally and we believe symmetry is one of the most beautiful principles, we single out Euclidean homogeneous Gaussian generalized random fields as objects to study. Then their spectral measures must be rotation invariant. We additionally assume that the spectral measures are absolutely continuous and their densities are reciprocals of polynomials. Namely for each Gaussian generalized random field $\{X(\varphi)\}$ under consideration there exists a unique polynomial P in one variable such that

$$E[X(\varphi)^2] = \int_{\mathbb{R}^d} |\hat{\varphi}(k)|^2 P(|k|^2)^{-1} \mathrm{d}k, \quad \varphi \in C_0^\infty(E^d). \tag{2.20}$$

We see that all the coefficients of P are real, the coefficient of the leading term is positive (we may assume that P is monic, i.e. the leading coefficient is 1, without loss of generality) and the multiplicity of any positive real root is even. We now assume that positive real roots are absent. Then P factorizes into the following form:

$$P = \prod_i (t + \lambda_i)(t + \overline{\lambda}_i) \prod_j (t + \mu_j) t^m \tag{2.21}$$

for some complex numbers λ_i with nonzero imaginary parts, positive numbers μ_j and a nonnegative integer m.

We want to factorize the differerntial operator $P(-\Delta)$ into two mutually adjoint differential operators somehow. For his purpose we have to extend the real number field. We consider a complex $*$-representation of the Clifford algebra over $\mathbb{R}^d$, i.e. a real inner product space V together with a system $\{\gamma_1, \ldots, \gamma_d, J\}$ of endomorphisms such that

$$\begin{aligned} &\gamma_i\gamma_j + \gamma_j\gamma_i = -2\delta_{ij}I, \quad \gamma_i^* = -\gamma_i \\ &J\gamma_i = \gamma_i J, \quad J^2 = -I \text{ and } J^* = -J, \end{aligned} \tag{2.22}$$

where I is the identity endomorphism and γ_i^* is the adjoint with respect to the inner product of V. We note that J defines a complex structure on the real vector space V. We introduce a different operator (Dirac operator):

$$\partial\!\!\!/ : C^\infty(E^d) \otimes V \ni \xi \mapsto \sum_{i=1}^{d} \gamma_i \frac{\partial}{\partial x^i} \xi \in C^\infty(E^d) \otimes V.$$

Then $\partial\!\!\!/^2 = (-\Delta) \otimes I$. (Since there is little fear of confusion, we will write simply $-\Delta$ from now on). We consider the following operator

$$L = \prod_i (-\Delta + \mathrm{Re}\lambda_i I + \mathrm{Im}\,\lambda_i J) \prod_j (\partial\!\!\!/ + \sqrt{\mu_j} J)\, \partial\!\!\!/^m$$

and the formal adjoint L^* with respect to the $L^2(E^d) \otimes V$-Hilbertian structure. By factorization (2.21) we have $L^*L = P(-\Delta)$. Let $\{X^i(\varphi)\}$, $i = 1, \ldots, \dim\, V$, be independent copies of the generalized random field $\{X(\varphi)\}$ under investigation. By using a fixed isometric isomorphism $\mathbb{R}^{\dim V} \to V$ we have another generalized random field $\{X_1(\underline{\varphi}); \underline{\varphi} \in C_0^\infty(E^d) \otimes V\}$. Since

$$\sigma\{X_1(\underline{\varphi});\ \mathrm{supp}\ \underline{\varphi} \subset D\} = \bigvee_{i=1,\ldots,\dim V} \sigma\{X^i(\varphi);\ \mathrm{supp}\ \varphi \subset D\}$$

for all open subsets, the Markov property of $\{X_1(\underline{\varphi})\}$ implies that of $\{X(\varphi)\}$. Now our task is to discuss the generalized random field $\{X_1(\underline{\varphi})\}$. We define another generalized random field $\{X_2(\underline{\varphi}); \underline{\varphi} \in C_0^\infty(E^d) \otimes V\}$ as follows:

$$X_2(\underline{\varphi}) = X_1(L\underline{\varphi}), \quad \varphi \in C_0^\infty(E^d) \otimes V.$$

Then the collection of componenets of $\{X_2(\underline{\varphi})\}$ referring to the fixed isometric isomorphism $\mathbb{R}^{\dim V} \to V$ forms an idependent system of Gaussian white noises. Therefore $\{X_2(\underline{\varphi})\}$ is MI-Markov with respect to any open covering $\{D_+, D_-\}$ of E^d. We note that the factor except for $\partial\!\!\!/^m$ in L maps $\mathcal{S}(E^d) \otimes V$ into itself bijectively. By virtue of Kusuoka's result we may reduce the problem to the case $L = \not{\partial}^m$ (i.e. $P = t^m$). We now assume $0 < 2m < d$. Then it is almost obvious that the natural choice of $\mathcal{H}_2$ is as follows:

$$\mathcal{H}_2 = \left\{ \phi \in C^\infty(E^d);\ \int_{E^d} |\phi(x)|^2 \mathrm{d}x < \infty \right\} \otimes V,$$

and hence

$$\mathcal{H}_1 = \left\{ \phi \in C^\infty(E^d);\ \int_{\mathbb{R}^d} |\hat{\phi}(k)|^2 |k|^{-2m} \mathrm{d}k < \infty \right\} \otimes V.$$

REMARK 2.23. Since $0 < m$, a measurable function $\hat{\phi}(\cdot)$ with $\int |\hat{\phi}(k)|^2 |k|^{-2m} \mathrm{d}k < \infty$ defines a tempered distribution by integration with respect to the Lebesgue measure. Thus we read that $\phi \in C^\infty(E^d) \cap S'(E^d)$ and its Fourier image $\hat{\phi}$ can be represented by a measurable function with $\int |\hat{\phi}(k)|^2 |k|^{-2m} \mathrm{d}k < \infty$ in the definition of $\mathcal{H}_1$.

At this moment the following proposition is the only thing to be clarified.

LEMMA 2.24. *Let m be an integer with $0 < 2m < d$. If the support of $\phi \in \mathcal{H}_1$ is contained in an open subset D of E^d, then there exists a sequence $\{\underline{\varphi}_n\} \subset C_0^\infty(D) \otimes V$ such that* $\int |\hat{\phi}(k) - \hat{\underline{\varphi}}_n(k)|^2 |k|^{-2m} \mathrm{d}k \to 0$ *as* $n \to \infty$.

One can show (2.24) by using the duality between the Sobolev spaces with index m and $-m$ respectively. The Lemma above implies that every open covering $\{D_+, D_-\}$ of E^d is $\{L, \mathcal{H}_2\}$-admissible. Therefore, invoking Theorem (2.14), we obtain the following

PROPOSITION 2.25. *Let d be a positive integer and P a real coefficient polynomial such that the leading coefficient is positive, none of the roots is real positive and the multiplicity of root 0 is less than $\frac{d}{2}$. Then a mean 0 Gaussian generalized random field $\{X(\varphi)\}$ with covariance bilinear form* (2.20) *is MI-Markov with respect to* any *open covering $\{D_+, D_-\}$ of E^d.*

3. - A detailed description of splitting σ-fields — the sharp Markov property

In this section we are going to discuss a generalized random field defined as a family of pathwise solutions of a first order elliptic system with a white noise. We shall specify the Markov property of the generalized random field in question by showing that the knowledge of the behaviour of the generalized random field at boundaries sufficies for splitting (the knowledge of normal derivatives is not necessary).

Let d be an integer greater than 2 and let $\{V, \gamma_1, \dots, \gamma_d\}$ be a real $*$ representation of the Clifford algebra over $\mathbb{R}^d$ (see (2.22)). The Dirac operator $\partial\!\!\!/$ is defined by

$$\partial\!\!\!/ : C^\infty(E^d) \otimes V \ni \xi \mapsto \sum_{i=1}^{d} \gamma_i \frac{\partial}{\partial x^i} \xi \in C^\infty(E^d) \otimes V.$$

Consider the following End(V)-valued function

$$G(x) := -\mathrm{vol}(S^{d-1})^{-1} |x|^{-d} \sum_{i=1}^{d} x^i \gamma_i, \quad x \in \mathbb{R}^d \setminus \{0\}.$$

Then G is a fundamental solution for $\partial\!\!\!/$. Sobolev's inequality tells us that

$$\|G * \xi\|_{L^p} \le C_{p,d} \|\xi\|_{L^q} \text{ for } \xi \in C_0^\infty(E^d) \otimes V, \tag{3.1}$$

where $p > \dfrac{d}{d-1}$, $\dfrac{1}{q} = \dfrac{1}{p} + \dfrac{1}{d}$ and $C_{p,d}$ is a positive constant depending on p and d. Now let ψ be a *ccpd* (abbr. continuous conditionally positive definite) function on V satisfying

$$\psi(\varsigma) = O\left(|\varsigma|^{\frac{d}{d-1}+\epsilon}\right) \text{ as } \varsigma \to 0 \text{ for some } \epsilon > 0. \tag{3.2}$$

We note that this behaviour is consistent with the ccpd property, since $\dfrac{d}{d-1} < 2$ $(d > 2)$. Then

$$D := C_0^\infty(E^d) \otimes V \ni \xi \mapsto \exp\left(\int_{E^d} \psi(G * \xi(x)) dx \right) \in \mathbb{C} \tag{3.3}$$

is continuous and positive definite (see Section 2). It is our aim in this section to discuss the Markov property of $\{X_1(\xi)\}$ with characteristic functional (3.3). We set

$$X_2(\xi) = X_1(\partial\!\!\!/ \xi), \ \ \xi \in D.$$

Then $\{X_2(\xi)\}$ is a V-valued white noise on E^d with Lévy characteristic ψ. Since the differential operator $\partial\!\!\!/$ is of first order, we may well except that the Markov property of $\{X_1(\xi)\}$ is 'sharp'. In what follows we shall illustrate what we mean by the sharp Markov property.

Let H be a hyperplane in E^d and D be one of the two connected components of $E^d \backslash H$. We introduce the following space:

$$\mathcal{H}_2 = \left\{ \varphi \in C^\infty(E^d \backslash H) \otimes V; \int_{E^d \backslash H} |\varphi(x)|^p dx < \infty \text{ for all } p > \frac{d}{d-1} \right\},$$

which is equipped with a locally convex topology induced by the compact open topology and the L^p-norms $p > \frac{d}{d-1}$. We regard $\mathcal{H}_2$ as embedded in $\mathcal{S}'(E^d) \otimes V$ by using integration with respect to the Lebesgue measure. Then we can easily verify that the natural inclusion $\mathcal{D} \hookrightarrow \mathcal{H}_2$ is dense, $G * \xi \in \mathcal{H}_2$ for $\xi \in \mathcal{D}$, and $\not{\partial}$ maps $\mathcal{H}_2$ into $\mathcal{S}'(E^d) \otimes V$ injectively. Moreover we have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{D} & \hookrightarrow & \mathcal{H}_2 & \hookrightarrow & \mathcal{S}'(E^d) \otimes V \\ \not{\partial} \downarrow & G \nearrow & & & \downarrow \not{\partial} \\ \mathcal{D} & & \hookrightarrow & & \mathcal{S}'(E^d) \otimes V. \end{array}$$

Let $\{Y_2(\varphi); \varphi \in \mathcal{H}_2\}$ be the stochastic continuous extention of $\{X_2(\xi)\}$ (see (2.8)). Then

$$X_1(\xi) = Y_2(G * \xi) \text{ a.s. for } \xi \in \mathcal{D}. \tag{3.4}$$

Our key observation is the following inequality:

LEMMA 3.5. *Let d be an integer greater than* 1. *If η is a measurable function on $\mathbb{R}^{d-1}$, then*

$$\int_{\mathbb{R} \times \mathbb{R}^{d-1}} \left(\int_{\mathbb{R}^{d-1}} (t^2 + |x-y|^2)^{-\frac{(d-1)}{2}} |\eta(y)| dy \right)^p dt dx \leq C_{d,p} (\|\eta\|_{L_q})^p,$$

where $p > \frac{d}{d-1}$, $q = \frac{d-1}{d} p$ and $C_{d,p}$ is a positive constant depending on d and p.

PROOF. Let $(E, \mathcal{B}, \mu)$ be a measure space and n a positive integer. If f is a positive measurable function on $(E, \mathcal{B})$, then by using Hölder's inequality we get

$$\int_{\mathbb{R}^n}\left(\int_E (|t|^2+f(x)^2)^{-\frac{\alpha}{2}}\mu(\mathrm{d}x)\right)^p \mathrm{d}t$$

$$\leq \int_{\mathbb{R}^n}\left(\int_E (|t|^2+f(x)^2)^{-\frac{p\alpha}{2}} f(x)^{p\alpha-n} f(x)^{-\alpha+\frac{n}{p}}\mu(\mathrm{d}x)\right)\left(\int_E f(x)^{-\alpha+\frac{n}{p}}\mu(\mathrm{d}x)\right)^{p-1}\mathrm{d}t$$

$$= \int_{\mathbb{R}^n}(|t|^2+1)^{-\frac{p\alpha}{2}}\mathrm{d}t\left(\int_E f(x)^{-\alpha+\frac{n}{p}}\mu(\mathrm{d}x)\right)^p,$$

where $p \geq 1$ and $\alpha \in \mathbb{R}$. We note that if $p\alpha > n$ then $\int_{\mathbb{R}^n}(|t|^2+1)^{-\frac{p\alpha}{2}}\mathrm{d}t < \infty$. Suppose $0 < n < d$ and $0 < \alpha < d$. Then we get the following estimate first applying the inequality above and next Sobolev's inequality:

$$\int_{\mathbb{R}^n\times\mathbb{R}^{d-n}}\left(\int_{\mathbb{R}^{d-n}}(|t|^2+|x-y|^2)^{-\frac{\alpha}{2}}|\eta(y)|\mathrm{d}y\right)^p \mathrm{d}t\mathrm{d}x$$

$$\leq \int_{\mathbb{R}^n}(|t|^2+1)^{-\frac{p\alpha}{2}}dt\int_{\mathbb{R}^{d-n}}\left(\int_{\mathbb{R}^{d-n}}|x-y|^{-\left(\alpha-\frac{n}{p}\right)}|\eta(y)|\mathrm{d}y\right)^p \mathrm{d}x \leq C_{d,p,\alpha}(\|\eta\|_{L^q})^p,$$

where $\max\left(0,\dfrac{\alpha-d+n}{n}\right) < \dfrac{1}{p} < \dfrac{\alpha}{d}$ and $\dfrac{1}{q} = \dfrac{d-n-\alpha}{d-n} + \dfrac{1}{p}\cdot\dfrac{d}{d-n}$. In particular, by choosing $n = 1$ and $\alpha = d-1$, we have proved the desired inequality. ∎

We identify the hyperplane H with $\mathbb{R}^{d-1}$ and therefore we use the convention $y = (0, \vec{y}) = (0, y^2, \ldots, y^d)$ to indicate the points in H. Let δ_0 be the Dirac mass at $0 \in \mathbb{R}$ and $\eta : \mathbb{R}^{d-1} \to V$ be measurable. By virtue of Lemma (3.5),

$$G*(\delta_0\otimes\eta)(x) := \frac{-1}{\mathrm{vol}(S^{d-1})}\int_{\mathbb{R}^{d-1}}|x-y|^{-d}\sum_{i=1}^{d}(x^i-y^i)\gamma_i\eta(\vec{y})\mathrm{d}\vec{y},\ \ x \in \mathbb{E}^d\backslash H,$$

defines an element of $\mathcal{H}_2$, if $\eta \in L^q(\mathbb{R}^{d-1})$ for all $q > 1$. We introduce the following space

$$\tilde{\mathcal{D}} = \mathcal{D} \oplus \{\delta_0\otimes\eta; \eta \in C_0^\infty(\mathbb{R}^{d-1})\otimes V\},$$

where the direct sum is considered in $S'(E^d)\otimes V$. If we denote the image of $\mathcal{H}_2$ under $\not{\partial}$ by $\mathcal{H}_1$, then we see that $\tilde{\mathcal{D}} \subset \mathcal{H}_1$. The following diagram explains the relation among the spaces introduced so far.

$$\begin{array}{ccccccc} \mathcal{D} & & \hookrightarrow & & \mathcal{H}_2 & \hookrightarrow & S'(E^d)\otimes V \\ \not{\partial}\downarrow & G & \nearrow & & \not{\partial}\downarrow & & \downarrow\not{\partial} \\ \mathcal{D} & \hookrightarrow & \tilde{\mathcal{D}} & \hookrightarrow & \mathcal{H}_1 & \hookrightarrow & S'(E^d)\otimes V. \end{array}$$

We consider the stochastic continuous extension $\{Y_1(\phi); \phi \in \mathcal{H}_1\}$ of $\{X_1(\xi)\}$. Then it is clear from the discussion so far that

$$Y_1(\not{\partial}\varphi) = Y_2(\varphi) \text{ a.s. for } \varphi \in \mathcal{H}_2. \tag{3.6}$$

We define a σ-field by

$$\mathcal{F}_C^{1,\mathrm{sharp}} := \sigma\{Y_1(\phi); \phi \in \tilde{\mathcal{D}}, \ \operatorname{supp}\phi \subset H\} \vee \mathcal{N}.$$

The superscripts *sharp* may be explained by the following:

$$\mathcal{F}_H^{1,\mathrm{sharp}} \subset \mathcal{F}_H^1. \tag{3.7}$$

(This is easily proved by approximating $\delta_0 \otimes \eta$ for $\eta \in C_0^\infty(\mathbb{R}^{d-1}) \otimes V$). By *the sharp Markov property* we mean the following:

THEOREM 3.8. *$\{X_1(\xi)\}$ is sharp Markov relative to H, i.e., $\mathcal{F}_D^1 \perp \mathcal{F}_{E^d\setminus\overline{D}}^1 | \mathcal{F}_H^{1,\mathrm{sharp}}$.*

PROOF. The point of the argument consists in calculating the distributional derivative $\not{\partial}(1_{E^d\setminus\overline{D}} G * \phi)$ for $\phi \in \tilde{\mathcal{D}}$ with $\operatorname{supp}\phi \subset \overline{D}$. Let $\{x^1, \ldots, x^d\}$ be an affine coordinate of E^d such that $D = \{x \in E^d; x^1 > 0\}$. We introduce the following differential forms: $\omega = dx^1 \wedge \ldots \wedge dx^d$ and $\theta = \sum_{i=1}^d \gamma_i \otimes *(dx^i)$, where the Hodge duality map $*$ is defined with respect to the orientation ω. Let f and g be V-valued C^∞-functions on some open subset of E^d. Then by an elementary calculus we see that $d(f, \theta g) = \{-(\not{\partial} f, g) + (f, \not{\partial} g)\}\omega$. If $f \in C^\infty(E^d\setminus\overline{D}) \otimes V$ satisfies $\not{\partial} f = 0$, then by using Stokes' formula we get

$$\int_{D_\epsilon} (f, \not{\partial}\xi)\omega = \int_{\partial D_\epsilon} (f, \theta\xi) = \int_{\mathbb{R}^{d-1}} (f(\epsilon, \vec{y}), \gamma_1\xi(\epsilon, \vec{y}))d\vec{y}, \xi \in \mathcal{D}.$$

where $D_\epsilon = \{x \in E^d; x^1 < \epsilon\}$, $\epsilon < 0$. Therefore it follows that

$$\int_{E^d\setminus\overline{D}} (G * \phi(x), \not{\partial}\xi(x))dx = \lim_{\epsilon\uparrow 0} \int_{\mathbb{R}^{d-1}} (G * \phi(\epsilon, \vec{y}), \gamma_1\xi(\epsilon, \vec{y}))d\vec{y}\,. \tag{3.9}$$

According to the theory of H^p-spaces associated with first order elliptic systems (see e.g. [33]), $G * \phi$ has nontangential limits almost everywhere on H and the boundary value $G * \phi(0-, \cdot)$ belongs to $L^q(\mathbb{R}^{d-1})$ for all $q > 1$. Thus from (3.9) we get

$$\dot{\not{\partial}}(1_{E^d \setminus \overline{D}} G * \phi) = -\delta_0 \otimes (\gamma_1 G * \phi(0-, \cdot)). \tag{3.10}$$

We now apply the algorithm in the proof of (2.14) with the stochastic continuity, (3.4), (3.5) and (3.6) taking into account. The conclusion is that

$$\mathcal{F}_D^1 \vee \mathcal{F}_H^{1,\text{sharp}} = \mathcal{F}_D^2 \vee (\mathcal{F}_H^{1,\text{sharp}} \cap \mathcal{F}_{E^d \setminus \overline{D}}^2). \tag{3.11}$$

Hence, by the independency of white noises, the sharp Markov property follows. ∎

The next two propositions clarify the relation to the MI- and MII-Markov property.

LEMMA 3.12. $\mathcal{F}_D^1 = \mathcal{F}_{\overline{D}}^1 = \bigcap_{\epsilon > 0} \mathcal{F}_{D_\epsilon}^1$ *and* $\mathcal{F}_H^{1,\text{sharp}} = \mathcal{F}_H^1 = \bigcap_{\epsilon > 0} \mathcal{F}_{H_\epsilon}^1$.

Where in general A_ϵ denotes the ϵ-neibourhood of $A \subset E^d$.

PROOF. Since the generalized random field $\{X_1(\xi)\}$ is stochastically continuous in the sense of (1.26), by (1.27), we have a $\mathcal{D}'$-valued realization X. Let $f : \Omega \to \mathbb{R}$ be bounded and $\cap \mathcal{F}_{D_\epsilon}^1$-measurable. Then given $\epsilon > 0$, thanks to (1.29), we can choose a $\sigma\{\pi_{D_\epsilon}\}$-measurable function $g_\epsilon : \mathcal{D}' \to \mathbb{R}$ so that $f = g_\epsilon \circ X$ a.s.. We now make use of *the translation invariance*. Let $\tau_x : \mathcal{D}' \to \mathcal{D}'$, $x \in \mathbb{R}^d$, be the natural action indeced by the $\mathbb{R}^d$-action on E^d. Since the measure on $\mathcal{D}'$ induced by X is $\{\tau_x\}$-invariant, we obtain $g_1 \circ \tau_{\epsilon\nu} \circ X = g_\epsilon \circ \tau_{\epsilon\nu} \circ X$ a.s., where ν is the unit outward normal vector for D. Note that $g_\epsilon \circ \tau_{\epsilon\nu} \circ X$ is $\mathcal{F}_D^1$-measurable, since $D_\epsilon = D + \epsilon\nu$. Therefore $g_1 \circ \tau_{\epsilon\nu} \circ X$ is $\mathcal{F}_D^1$-measurable for all $\epsilon > 0$. On the other hand $\{\tau_x\}$-invariance implies the stochastic continuity of $\epsilon \mapsto g_1 \circ \tau_{\epsilon\nu} \circ X$ as well. Hence $f(= g_1 \circ X$ a.s.) must be $\mathcal{F}_D^1$-measurable and the first relation is proved.

We see by (3.7), the first relation and the corresponding for $E^d \setminus \overline{D}$ that $\mathcal{F}_H^{1,\text{sharp}} \subset \bigcap_{\epsilon > 0} \mathcal{F}_{H_\epsilon}^1 \subset \mathcal{F}_D^1 \cap \mathcal{F}_{E^d \setminus \overline{D}}^1$. Actually the equality holds because of the sharp Markov property (3.8) and Lemma (1.4). This completes the proof. ∎

THEOREM 3.13. *Let D_+ and D_- be open half spaces which cover E^d. Then $\{X_1(\xi)\}$ is MI-Markov relative to the covering $\{D_+, D_-\}$.*

PROOF. We see from (3.11) that $\mathcal{F}_{D_-}^1 \subset \mathcal{F}_{D_-}^2 \vee \mathcal{F}_{\partial D_-}^{1,\text{sharp}}$. We choose an affine coordinate $\{x^1, \ldots, x^d\}$ so that $D_+ = \{x \in E^d; X^1 > 0\}$ and $D_- = \{x \in E^d; x^1 < 1\}$. Consider a family of open half spaces $D_\epsilon := \{x \in E^d; x^1 < \epsilon\}$, $0 < \epsilon < 1$. Since ∂D_ϵ is of Lebesgue measure zero and $\{X_2(\xi)\}$ is a white noise, it follows that $\mathcal{F}_{D_-}^2 = \mathcal{F}_{D_- \setminus \partial D_\epsilon}^2 \subset \mathcal{F}_{D_- \setminus \partial D_\epsilon}^1$. On the other hand by using the method in the proof of (3.12) we can prove that $\mathcal{F}_{\partial D_-}^{1,\text{sharp}} \subset \mathcal{F}_{D_- \setminus \partial D_\epsilon}^1$. Therefore we obtain $\mathcal{F}_{D_-}^1 = \mathcal{F}_{D_- \setminus \partial D_\epsilon}^1$, $0 < \epsilon < 1$. We now apply the algorithm in the proof of (1.23). The conclusion is that $\mathcal{F}^1_{D_+} \perp \mathcal{F}^1_{D_-} | \mathcal{F}^1_{D_+ \cap D_-}$. This completes the proof. ∎

REMARK 3.14. If $d = 4$, by virtue of the existence of quaternionic number field $\mathbb{H}$ we can construct a representation of the Clifford algebra in terms of 2×2 matricies with $\mathbb{H}$-entries. By choosing oriented orthogonal frames in $\mathbb{R}^4$ and in $\mathbb{H}$ respectively, we have an isomorphism $\mathbb{R}^4 \simeq \mathbb{H}$. Then the following map

$$\mathbb{R}^4 \simeq \mathcal{H} \ni x \mapsto \begin{pmatrix} 0 & x \\ \overline{x} & 0 \end{pmatrix} \in M_2(\mathbb{H}),$$

where $\overline{x}$ is the quaternionic conjugation of $x \in \mathbb{H}$, determines a representation of the Clifford algebra over $\mathbb{R}^4$. In this representation the Dirac operator is the matrix $\begin{pmatrix} 0 & \overline{\partial} \\ \partial & 0 \end{pmatrix}$, where ∂ and $\overline{\partial}$ is the quaternionic analogues of Cauchy-Riemann operators. In the previous papers [2] and [4], we studied the random field $\{X_1(\varphi)\}$ in this particular case but we did not discuss its Markov property. Hence the present work may be regarded as the continuation of [2] and [4].

Although we did not mention the elliptic regularity of $\not{\partial}$, we have already made use of it implicitly. In the following discussions the elliptic regularity is essential.

We consider a particular class of *ccpd* functions so called Poisson type. Let ψ be a *ccpd* function on a vector space V. (In our context V is the representation space of the Clifford algebra over $\mathbb{R}^d$). Then it is well known that there exists a unique triplet $\{\beta, \sigma, \nu\}$, where $\beta \in \mathrm{Hom}(V, \mathbb{R})$, $\sigma \in \odot^2 V$ (symmetric bilinear forms on V), nonnegative definite, and ν a Radon measure on $V \setminus \{0\}$ satisfying $\int |\lambda|^2/(1+|\lambda|^2)\nu(\mathrm{d}\lambda) < \infty$ such that

$$\psi(\varsigma) = \sqrt{-1}\beta(\varsigma) - \frac{1}{2}\sigma(\varsigma,\varsigma) + \int\limits_{V\setminus\{0\}} \left(e^{\sqrt{-1}(\varsigma,\lambda)} - 1 - \sqrt{-1}(\varsigma,\lambda)1_{|\lambda|<1}\right)\nu(\mathrm{d}\lambda), \quad \varsigma \in V.$$

This representation is usually named after Lévy and Khinchin (see e.g. [1]). From now on we assume that $\sigma = 0$ and ν is of finite variation and ψ satisfies (3.2). Under this condition the σ-field $\mathcal{F}^{1,\mathrm{sharp}}_C$ spreads to the whole σ-field contrary to what is suggested by the adjective 'sharp'. That is,

PROPOSITION 3.15. $\mathcal{F}^{1,\mathrm{sharp}}_C = \sigma\{X_1(\xi); \xi \in \mathcal{D}\} \vee \mathcal{N}$.

PROOF. As in the proof of (3.12) let X be a $\mathcal{D}'$-valued realization of $\{X_1(\xi)\}$. We introduce a Poisson point process $\{(\lambda_i, x_i)\}_{i=1}^\infty$ on $(V\setminus\{0\}) \times E^d$ with intensity (Lévy measure) $\nu \otimes l$, where l is the Lebesgue measure. Since ν is of finite variation, the map

$$\{(\lambda_i, x_i)\}_{i=1}^\infty \ni (\lambda, x) \mapsto x \in \{x_i\}_{i=1}^\infty$$

is injective and $\{x_i\}_{i=1}^{\infty}$ has no cluster point in E^d with probability one. It is easy to see that

$$\langle X, \not{\partial}\, \xi \rangle \stackrel{d}{=} \sum_{i=1}^{\infty} (\lambda_i, \xi(x_i)) \quad \text{for each } \xi \in \mathcal{D}.$$

We can realize the point process $\{(\lambda_i, x_i)\}_{i=1}^{\infty}$ on the same probability space on which X is defined so that we may obtain the following relation (cf. Lemma (1.27)).

$$\langle X, \not{\partial}\, \xi \rangle = \sum_{i=1}^{\infty} (\lambda_i, \xi(x_i)) \quad \text{for each } \xi \in \mathcal{D} \text{ a.s..}$$

Because of the elliptic regularity for $\not{\partial}$, X is real analytic in the connected open set $E^d \backslash \{x_i\}_{i=1}^{\infty}$. (Recall that $d > 2$ and $\{x_i\}_{i=1}^{\infty}$ has no cluster point in E^d almost surely). Since the Lebesge measure of H is 0, we see that $H \cap \{x_i\}_{i=1}^{\infty} = \emptyset$ a.s.. Let y be a point in H. By virtue of its real analyticity X is completely determined (including the singular parts) by the derivatives at y. Moreover since

$$\frac{\partial}{\partial x^1} X(y) = -\gamma_1 \left(\not{\partial} X - \sum_{i=2}^{d} \gamma_i \frac{\partial}{\partial x^1} X \right)(y) = \sum_{i=1}^{d} \gamma_1 \gamma_i \frac{\partial}{\partial x^i} X(y),$$

all the derivatives at y are determined by the tangential derivatives. Thus X restricted on H completely determines X almost surely, i.e., $\mathcal{F}_H^{1,\text{sharp}} = \sigma\{X_1(\xi); \xi \in \mathcal{D}\} \vee \mathcal{N}$. ∎

REMARK 3.16. Analogous phenomena also occur in Gaussian cases. Consider a Euclidean homogeneous mean 0 Gaussian generalized random field with the sharp Markov property. Then, according to Wong [37], its covariance bilinear form must be either

$$c \int_{\mathbb{R}^d} |\hat{\varphi}(k)|^2 (|k|^2 + m^2)^{-1} \mathrm{d}k \text{ or } c \int_{S^{d-1}} |\hat{\varphi}(rk)|^2 \mathcal{H}^{d-1}(\mathrm{d}k),$$

where c, m and r are nonnegative constants and $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$. In the latter case the random field pathwise satisfies $(\Delta + r^2)X = 0$ and hence it is regarded as a real analytic function almost surely. However given a bounded open set D, the Dirichlet problem

$$(\Delta + r^2)X = 0, \quad X|_{\partial D} \text{ given},$$

is not uniquely solvable for particular r's (i.e. $-r^2$ coincides with one of the eigenvalues of Δ with Dirichlet boundary condition), while the boundary value problem

$$(\Delta + r^2)X = 0, \quad X|_{E^d \setminus \overline{D}} \text{ given},$$

has at most one solution. Thus unless $r = 0$ the random field in question is not sharp Markov. (This fact was mentioned in [38]).

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REFERENCES

[1] S. Albeverio - R. Høegh-Krohn - H. Holden - T. Kolsrud, *Representation and construction of multiplicative noise*, J. Funct. Anal. **87** (1989), pp. 250-272.

[2] S. Albeverio - R. Høegh-Krohn - K. Iwata, *Covariant markovian random fields in four space-time dimensions with nonlinear electromagnetic interaction*, Proc. Dubna Conference 1987. P. Exner, P. Seba ed. Lecture Notes in Phys. **324** Springer 1989.

[3] S. Albeverio - R. Høegh-Krohn - D. Surgailis, *Some Euclidean Integer-Valued Random Field with Markov Properties*, Ruhr-Universität preprint 1990 to appear in Memorial Volume for R. Høegh-Krohn, Cambridge University Press.

[4] S. Albeverio - K. Iwata - T. Kolsrud, *Random fields as solutions of the inhomogeneous quaternionic Cauchy-Riemann equation. I. Invariance and analytic continuation*, Comm. Math. Phys. **132** (1990), pp. 555-580.

[5] S. Albeverio - K. Iwata - T. Kolsrud, *Homogeneous Markov generalized vector fields and quantum fields over 4-dimensional space-time*, to appear in Proc. Trento. Da Prato, Tubaro Edts. Lecture Notes in Math. Springer.

[6] S. Albeverio - B. Zegarlinski, *Global Markov Property: results and open problems*, to appear in Memorial Volume for R. Høegh-Krohn, Cambridge University Press.

[7] J.L. Doob, *The elementary Gaussian processes*, Ann. Math. Statist. **15** (1944), pp. 229-282.

[8] E.B. DYNKIN, *Markov processes and random fields*, Bull. Amer. Math. Soc. **3** (1980), pp. 975-999.

[9] I.M. GEL'FAND - N. YA. VILENKIN, *Generalized functions*, Vol. 4 English translation, Academic Press 1964.

[10] V. GEORGESCU - R. PURICE, *On the Markov property for the free Euclidean electromagnetic field*, Lett. Math. Phys. **4** (1980), pp. 465-467.

[11] L. GROSS, *The free euclidean Proca and electromagnetic fields*, Proc. International Conference Cumberland Lodge. Functional integration and its applications. 1974 Clorendon Press 1975.

[12] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators I*, Springer 1983.

[13] K. ITÔ, *Foundations of stochastic differential equations in infinite dimensional spaces*, CBNS - NSF Regional Conf. Series in Appl. Math. Vol. **47** SIAM 1984.

[14] K. IWATA, *On Linear Maps Preserving Markov Properties and Applications to Multicomponent Generalized Random Fields*, Dissertation Ruhr-Universität Bochum 1990.

[15] T. KOLSRUD, *On the Markov property for certain Gaussian random fields*, Probab. Theory Related Fields **74** (1986), pp. 393-402.

[16] S. KOTANI, *On a Markov property for stationary Gaussian processes with a multidimensional parameter*, Proc. 2nd Japan USSR Symposium Probability, Lecture Notes in Math. **330** Springer 1977.

[17] H. KÜNSCH, *Gaussian Markov random fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **26** (1979), pp. 53-73.

[18] S. KUSUOKA, *Markov fields and local operators*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **26** (1979), pp. 199-212.

[19] N. LEVINSON - H.P. MCKEAN JR., *Weighted trigonometric approximation on $\mathbb{R}^1$ with application to the germ field of a stationary Gaussian noise*, Acta Math. **112** (1964), pp. 99-143.

[20] J. LÖFFELHOLZ, *Euclidean fields I, II and III*, Publication Joint Insitute for Nuclear Research Dubna E5 12779, 12780 and 12781 (1979).

[21] J. LÖFFELHOLZ, *The Markov property of the free Euclidean electromagnetic field*, Karl Marx Universität preprint 1982.

[22] H.P. MCKEAN, *Brownian motion with a several dimensional time*, Theory Probab. Appl. **8** (1963), pp. 335-354.

[23] E. NELSON, *The free Markov field*, J. Funct. Anal. **12** (1973), pp. 211-227.

[24] Y. OKABE, *Stationary Gaussian processes with Markovian property and M. Sato's hyperfunctions*, Japan. J. Math. **41** (1973), pp. 66-22.

[25] Y. OKABE, *On the germ field of stationary Gaussian processes with Markovian property*, J. Math. Soc. Japan **28** (1976), pp. 86-95.

[26] E. OSIPOV, *Markov properties of solutions of stochastic partial differential equations in a finite volume*, Novosibirsk preprint 1989.

[27] D. PREISS - R. KOTECKY, *Markov property of generalized random fields*, Seventh Winter School, Czechoslovakia 1979.

[28] M. RÖCKNER, *Markov property of generalized field and axiomatic potential theory*, Math. Ann. **264** (1983), pp. 153-177.

[29] M. Röckner, *Generalized Markov Fields and Dirichlet Forms*, Acta Appl. Math. **3** (1985), pp. 285-311.

[30] Yu A. Rozanov, *Markov Randon Fields*, Springer 1982.

[31] Yu A. Rozanov, *Boundary problems for stochastic partial differential equations*, Proc. BiBoS, S. Albeverio, Ph. Blanchard, L. Streit Edts., Lecture Notes in Math. **1250**, Springer 1987.

[32] Yu A. Rozanov, *Some Functional Models for Random Fields*, Chapel Hill preprint 1988.

[33] E.M. Stein - G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press 1971.

[34] D. Surgailis, *On the Markov property of a class of linear infinity divisible fields*, Z. Warsch. verw. Geb. **49** (1979), pp. 293-311.

[35] D. Surgailis, *On convariant stochastic differential equations and Markov property of their solutions*, Istituto Fisico Università di Roma preprint 1979.

[36] H. Totoki, *A method of construction of measures on function spaces and its applications to stochastic processes*, Mem. Fac. Sci. Kyushu Univ. Ser. A **15** (1961), pp. 178-190.

[37] E. Wong, *Homogeneous Gauss-Markov random fields*, Ann. Math. Stat. **40** (1969), pp. 1625-1634.

[38] E. Wong - B. Hajek, *Stochastics Processes in Engineering Systems*, Springer 1985.

[39] E. Wong - M. Zakai, *Isotropic Gauss Markov Currents*, Probab. Theory Related Fields **82** (1989), pp. 137-154.

[40] E. Wong - M. Zakai, *Spectral representation of isotropic random currents*, Séminaire de Probabilités XXIII. J. Azéma, P. Meyer and M. Yor Eds. Lecture Notes in Math. **1372** Springer 1989.

[41] Y. Yamasaki, *Measures on infinite dimensional spaces*, World Scientific 1985.

[42] Te Hai Yao, *Construction of quantum fields from Euclidean tensor field*, J. Math. Phys. **17** (1976), pp. 241-247.

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