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THOMAS J. RICHARDSON

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# Limit Theorems for a Variational Problem Arising in Computer Vision<sup>1</sup>

THOMAS J. RICHARDSON

## PART I

### Introduction

#### 1. - Background

In this paper we present a result for a “free-discontinuity” [8] problem which has applications in computer vision. It is in the context of this application that the result finds relevance. We are interested in minimizers of the functional

$$E(f, \Gamma) = \beta \int_{\Omega} (f - g)^2 + \int_{\Omega \setminus \Gamma} |\nabla f|^2 + \alpha \text{length}(\Gamma),$$

where  $\Omega \subset \subset \mathbb{R}^2$  ( $A \subset \subset B$  means the closure of  $A$  is a compact subset of  $B$ ),  $\Gamma \subset \Omega$  is relatively closed,  $g \in L^\infty(\Omega)$ ,  $f \in W^{1,2}(\Omega)$ , and  $\alpha$  and  $\beta$  are constants. (For convenience, whenever we do not explicitly state with respect to which measure we are integrating, Lebesgue measure is to be understood). When the function  $g$  is interpreted as the intensity of an observed image, the function  $f$  is thought to represent a piecewise smooth approximation to  $g$  while  $\Gamma$  represents the set of edges in the image. This approach to the *segmentation problem* of computer vision is known as the *Variational Formulation* and was introduced by Mumford and Shah, [15, 16].

A related functional of interest arises by allowing  $\beta$  and  $\alpha$  to tend to zero while keeping their ratio fixed. In this limit the minimizers of  $E$  approach locally constant functions on  $\Omega \setminus \Gamma$ . Mumford and Shah were thus lead to also introduce the following functional

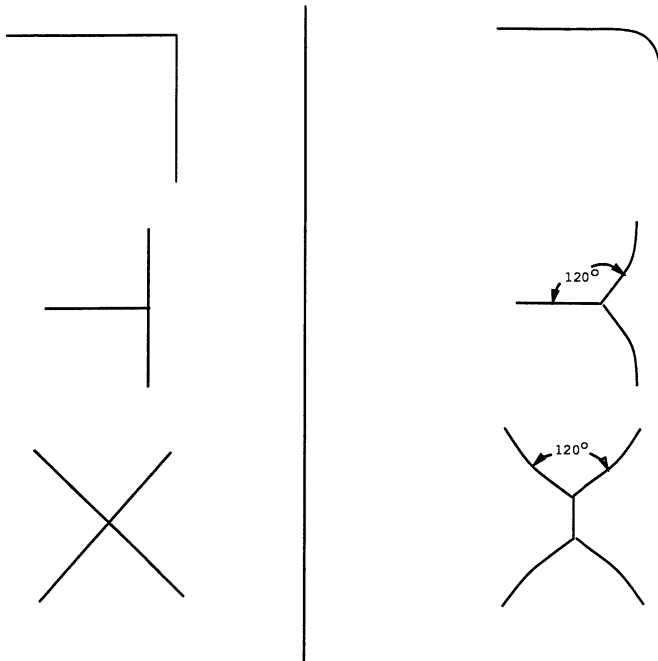
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$$E_0(f, \Gamma) = \beta \sum_i \int_{\Omega_i} (f_i - g)^2 + \alpha \text{length}(\Gamma),$$

where  $\Gamma = \Omega \setminus \bigcup_i \Omega_i$  and the  $f_i$  are constants. This functional, because of its greater simplicity, lends itself to more thorough analysis. We will be considering minimizers of this functional as well.

Mumford and Shah conjectured that there exist minimizers of  $E$  in which  $\Gamma$  is composed of a finite number of  $C^1$  curves. Such a result has in fact been shown for  $E_0$  [16, 23, 14]. In [16] the Authors studied the first variation of  $E$  positing the existence of such minimizers. The results of their analysis show that (regular)  $\Gamma$ , which minimize  $E$ , can possess only very restricted types of singularities. (Essentially the same results also apply to minimizers of  $E_0$ ). Within the context of computer vision, the constraints placed on possible segmentations by these results must be considered a drawback of the variational formulation. In particular, corners and  $T$ -junctions, long recognized as significant features in images, tend to be distorted. The following are some of the constraints on  $\Gamma$ 's, which minimize  $E$  and  $E_0$ , proved by Mumford and Shah in [16]. They are illustrated in Figure 1.



Non-minimal geometries

Corresponding minimal geometries

Figure 1 - Calculus of variations results

- (i) If  $\Gamma$  is composed of  $C^{1,1}$  arcs, then at most three arcs can meet at a single point, and they do so at  $120^\circ$ .
- (ii) If  $\Gamma$  is composed of  $C^{1,1}$  arcs, then they meet  $\partial\Omega$  only at an angle of  $90^\circ$ .
- (iii) If  $\Gamma$  is composed of  $C^{1,1}$  arcs, then it never occurs that two arcs meet at an angle other than  $180^\circ$ .
- (iv) If  $x \in \Gamma$  and, in a neighbourhood of  $x$ ,  $\Gamma$  is the graph of a  $C^2$  function, then  $(\beta(f - g)^2 + |\nabla f|^2)^+ - (\beta(f - g)^2 + |\nabla f|^2)^- + \alpha \text{curv}(\Gamma) = 0$ , where the superscripts  $+$  and  $-$  denote the upper and lower trace of the associated function on  $\Gamma$  at  $x$  and  $\text{curv}(\Gamma)$  denotes the curvature of  $\Gamma$  at  $x$ .

These results help to characterize the structure of  $\Gamma$  which minimize  $E$  (or  $E_0$ ) but they do so only in a local fashion. That is, they only say something about solutions on the level of microscopic detail. The results presented in this paper show, at least in an asymptotic sense, that the solutions found by minimizing  $E$  and  $E_0$  may be quite reasonable when viewed globally. The main theorem states that, as  $\beta \rightarrow \infty$ , the  $\Gamma$  which minimize  $E$  and  $E_0$  are “close to” (in the sense of Hausdorff metric) what is “appropriate”, where appropriate is defined in terms of the discontinuity set of the image.

The energy functional associated with the variational formulation is ad hoc. It seems necessary, for producing models for vision, to make ad hoc choices at some level unless one is specifically interested in reproducing human vision, in which case one can appeal to empirical evidence. The difficulty with the results stemming from the calculus of variations is that they do not support the use of the variational approach as an image segmenting scheme with respect to the goal of obtaining intuitively appealing segmentations. The imposed properties of minimal edges are a consequence of the particular ad hoc structure of the functional (e.g., the use of ‘length’ as opposed to some other penalty term on the edges) and do not reflect an intrinsic property of the problem at hand. How then can one improve upon such ad hoc models? One approach is developed in [17]. Consider the set of all possible minimizers of the functional  $E$ , over all possible values of the parameters. Each of these minimizers possess the properties which the model imposes. However, if we take the closure of these functions in an appropriate topology, we may widen the class of functions considerably. What we show in Part II is that particularly meaningful members of such a closure may be found by taking the parameters associated with the functional to certain limits. In fact, one can produce essentially any piecewise smooth function in this way and, hence, obtain a more general model for images and their edges. An idea which follows naturally from this one is to develop an algorithm in which the same limit is taken. Roughly speaking, this is what has been done in [17]. A complete description of the algorithm is beyond the scope of this paper; however, for the sake of completeness and to motivate the results of this paper, the following provides a short synopsis.

We begin with a theorem which states that, as  $\beta$  tends to  $\infty$ , the edges found by minimizing  $E(\beta)$  will converge to the discontinuity set of the data  $g$ , assuming it is piecewise smooth. A similar theorem holds for  $E_0$  when  $g$  is

piecewise constant. (The proofs of these theorems are presented in this paper). This implies, in particular, that one can recover  $T$ -junctions and corners, at least asymptotically, by the variational method. We also characterize the degree of corruption of the image which can be allowed before these results break down. The limit theorems are not enough to fulfill the requirements of a practical segmentation formulation because in effect they require the “scale” to tend to the microscopic. For a fixed value of  $\alpha$ , “scale” can be related to  $\beta^{-\frac{1}{2}}$ . This parameter is proportional to the range over which smoothing occurs. In general, the errors one obtains in the localization of boundaries (such as the rounding of corners) vary directly with the “scale” of the segmentation. Thus, the relative errors do not improve as one tends towards the microscopic scale. The goal of the algorithm, which is developed in [17], is to take the limit suggested by the limit theorems, retaining “coarse scale” boundaries, letting them tend to limit positions while preventing the segmenting of smaller scale features. We will not concern ourselves here with how this is accomplished; the purpose of this paper is to present the proof of the limit theorems.

## 2. - Mathematical preliminaries

This section provides an introduction to the relevant mathematical framework. In the first section, definitions of the Hausdorff and Minkowski measures and the Hausdorff metric are provided. In the next section, we define and present some basic properties of the space  $BV$  and a subspace  $SBV$ , the special functions of bounded variation, which was introduced by De Giorgi and Ambrosio in [7]. The space  $SBV(\Omega)$  plays an important role in the study of the fundamental mathematical questions associated with the variational formulation. It is in the  $SBV$  setting that the most general existence results have been achieved. Also, our asymptotic theorems for minimizers of  $E$  and  $E_0$  (see Section II) are proved in the  $SBV$  setting.

### 2.1. Metrics and measures

In this section we introduce a variety of ideas useful in dealing with the ‘edges’ of an image. The ‘image’ will be a real valued function defined on a bounded open rectangle  $\Omega \subset \mathbb{R}^2$ . However, for the definitions to follow,  $\Omega$  represents an arbitrary bounded open subset of  $\mathbb{R}^n$ . A set of *edges* generally refers to a closed subset of  $\Omega$ . The following concepts can be applied to such objects.

For  $A \subset \mathbb{R}^n$ , the  $\varepsilon$ -neighbourhood of  $A$  will be denoted by  $[A]_\varepsilon$  and is defined by

$$[A]_\varepsilon = \left\{ x \in \mathbb{R}^n : \inf_{y \in A} \|x - y\| < \varepsilon \right\}$$

where  $\|\cdot\|$  denotes the Euclidean norm. In the terminology of mathematical

morphology [21],  $[A]_\varepsilon$  is the dilation of  $A$  with the open ball of radius  $\varepsilon$ . A notion of distance between sets, which we will often use, is defined by the following,

$$d_H(A_1, A_2) = \inf\{\varepsilon : A_1 \subset [A_2]_\varepsilon \subset [A_1]_\varepsilon\}.$$

Known as the *Hausdorff metric*,  $d_H(\cdot, \cdot)$ , is in fact a metric on the space of all non-empty compact subsets of  $\mathbb{R}^n$ .

Rigorous results establishing the existence of regular (i.e., piecewise  $C^1$ )  $\Gamma$  as minimizers of  $E$  do not yet exist; a more general measure than ‘length’ is therefore required. A variety of measures for subsets of  $\mathbb{R}^n$  have been investigated (see [11] for many examples). Perhaps the most widely used and studied are the Hausdorff measures [10, 11, 18].

For a non-empty subset  $A$  of  $\mathbb{R}^n$ , the *diameter* of  $A$  is defined by  $\text{diam}(A) = \sup\{\|x - y\| : x, y \in A\}$ . Let

$$\omega_s = \frac{\Gamma\left(\frac{1}{2}\right)^2}{\Gamma\left(\frac{s}{2} + 1\right)}$$

where  $\Gamma(\cdot)$  is the usual Gamma function. For integer values of  $s$ ,  $\omega_s$  is the volume of the unit ball in  $\mathbb{R}^s$ . For  $s > 0$  and  $\delta > 0$  define

$$\mathcal{H}_\delta^s(A) = 2^{-s}\omega_s \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(U_i)^s : A \subset \bigcup_{i=1}^{\infty} U_i, \text{diam}(U_i) \leq \delta \right\}.$$

The *Hausdorff  $s$ -dimensional measure* of  $A$  is then given by

$$\mathcal{H}^s(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(A) = \sup_{\delta > 0} \mathcal{H}_\delta^s(A).$$

Note that the factor  $2^{-s}\omega_s$  in the definition of  $\mathcal{H}^s(\cdot)$  is included for proper normalization. For integer values of  $s$ , Hausdorff measure gives the desired value on sets where the usual notions of length, area, and volume apply.

Many properties of Hausdorff measure can be found in [10, 11, 18]. For the results of this paper we will require only  $\mathcal{H}^1$  and  $\mathcal{H}^0$ .

Another measure we will be needing is Minkowski content [11]. We will use  $|\cdot|$  to denote Lebesgue measure in  $\mathbb{R}^n$ . For any  $A \subset \mathbb{R}^n$ ,  $0 \leq s \leq n$ , and  $\varepsilon > 0$ , define,

$$\mathcal{M}_\varepsilon^s(A) = \frac{|[A]_\varepsilon|}{\varepsilon^{n-s}\omega_{n-s}}.$$

As in the definition of Hausdorff measure, the term  $\omega_{n-s}$  is included for proper normalization. In general,  $\lim_{\varepsilon \rightarrow 0} \mathcal{M}_\varepsilon^s(A)$  may not exist (for an example see [11], Section 3.2.40). However, *lower* and *upper Minkowski contents* can be defined

by

$$\mathcal{M}_*^s(A) = \liminf_{\varepsilon \rightarrow 0} \mathcal{M}_\varepsilon^s(A)$$

and

$$\mathcal{M}^{*s}(A) = \limsup_{\varepsilon \rightarrow 0} \mathcal{M}_\varepsilon^s(A)$$

respectively. If these two values agree, then one refers to the common value as the *s-dimensional Minkowski content* of  $A$ , and it is denoted simply as  $\mathcal{M}^s(A)$ .

The following theorem relates Minkowski content to Hausdorff measure. A subset  $\Gamma$  of  $\mathbb{R}^n$  is called *m-rectifiable* if there exists a Lipschitzian function mapping a bounded subset of  $\mathbb{R}^m$  onto  $\Gamma$ .

**THEOREM 1** [11, Theorem 3.2.39]. *If  $\Gamma$  is a closed m-rectifiable subset of  $\mathbb{R}^n$  then  $\mathcal{M}^m(A) = \mathcal{H}^m(\Gamma)$ .*

## 2.2. Essential boundaries

The problem of defining the perimeter of a set proved difficult from the point of view of the calculus of variations. The topological boundary does not in general possess sufficient mathematical properties to fulfill the usual requirements of the calculus. Federer [11] introduced a notion based on the idea of density. The *essential boundary* of a set is those points where the set has density other than zero or one. To be more precise, for a borel set  $A \subset \Omega$  we define,

$$A_t = \left\{ x \in \Omega : \lim_{\rho \rightarrow 0^+} \frac{|B \cap B_\rho(x)|}{\omega_n \rho^n} = t \right\} \quad t \in [0, 1],$$

where  $\omega_n$  is, as before, the volume of the unit ball in  $\mathbb{R}^n$ .  $A_t$  is the set where  $A$  has density  $t$ . The essential boundary  $\partial^* A$  is given by

$$\partial^* A = \Omega \setminus (A_0 \cup A_1).$$

The essential boundary possesses the following property,

$$(1) \quad \partial^* A \supset A_{\frac{1}{2}} \quad \text{and} \quad \mathcal{H}^{n-1} \left( \partial^* A \setminus A_{\frac{1}{2}} \right) = 0,$$

also, the set  $\partial^* A$  is countably rectifiable in the sense of Federer ([11], Chapter 3), i.e.,

$$\partial^* A \subset \bigcup_{n=1}^{\infty} \Gamma_n \cup N,$$

where the  $\Gamma_n$  are  $C^1$  hypersurfaces and  $\mathcal{H}^{n-1}(N) = 0$ .

A result which characterizes the essential boundary very nicely is the following. For bounded measurable sets  $A$ , if  $\mathcal{H}^{n-1}(\partial^*A) < \infty$  then

$$(2) \quad \mathcal{H}^{n-1}(\Omega \cap \partial^*A) = \inf \left\{ \liminf_{n \rightarrow \infty} \mathcal{H}^{n-1}(\Omega \cap \partial A_n) : \right. \\ \left. A_n \rightarrow A \text{ locally in measure, } A_n \text{ polyhedral} \right\}.$$

A measurable set  $A \subset \mathbb{R}^n$ , satisfying  $\mathcal{H}^{n-1}(K \cap \partial^*A) < \infty$  for all compact  $K \subset \mathbb{R}^n$ , is referred to as a *Caccioppoli set*.

This concept of boundary will be helpful in the formulation of the piecewise constant version of the limit theorem proved in Section 4. Also, it was used by Mumford and Shah in their proof of the existence of minimizers for  $E_0$ .

### 2.3. SBV functions

Let  $J$  be an open interval in  $\mathbb{R}$ , then  $u : J \rightarrow \mathbb{R}$  is a function of *bounded variation* in  $J$  if

$$(3) \quad V_J(u) = \sup \left\{ \sum_{i=1}^{k-1} |u(t_{i+1}) - u(t_i)| : \inf J < t_1 < \dots < t_k < \sup J \right\} < +\infty.$$

$V_J(u)$  is called the total variation of  $u$  in  $J$ . The space  $BV(J)$  is the space of Borel functions  $u : J \rightarrow \mathbb{R}$  such that

$$\text{ess-}V_J(u) = \inf \{V_J(v) : v = u \text{ almost everywhere}\} < +\infty.$$

In higher dimensions this definition can be generalized by slicing arguments [2]. The space  $BV(\Omega)$  can be characterized in other ways. In particular the functions in  $BV(\Omega)$  are those functions  $u \in L^1(\Omega)$  such that  $Du$ , the distributional derivative of  $u$ , is representable as a bounded Radon measure on  $\Omega$  with values in  $\mathbb{R}^2$  [11, 12].

For each  $x \in \Omega$ ,  $u \in BV(\Omega)$ , one can define the *approximate upper* (and *lower*) *limit of  $u$  at  $x$* . The upper limit is the greatest lower bound on all  $t \in [-\infty, \infty]$  such that  $\{x \in \Omega : u(x) > t\}$  has 0 density at  $x$ , i.e.,

$$(4) \quad u^+(x) = \inf \left\{ t \in [-\infty, \infty] : \lim_{\rho \rightarrow 0^+} \frac{|\{u > t\} \cap B_\rho(x)|}{\rho^n} = 0 \right\}.$$

Similarly, the approximate lower limit is

$$(5) \quad u^-(x) = \sup \left\{ t \in [-\infty, \infty] : \lim_{\rho \rightarrow 0^+} \frac{|\{u < t\} \cap B_\rho(x)|}{\rho^n} = 0 \right\}.$$

Points where  $u^+ = u^-$  are points of approximate continuity for  $x$ . The remainder, the *jump set* of  $u$ , those  $x$  for which  $u^-(x) < u^+(x)$ , is denoted  $\mathcal{S}_u$ . If  $u \in L^\infty(\Omega)$



then the points of approximate continuity are precisely the Lebesgue points of  $u$ , i.e.,

$$\Omega \setminus \mathcal{S}_u = \left\{ x \in \Omega : \exists z : \lim_{\rho \rightarrow 0^+} \rho^{-n} \int_{B_\rho(x)} |u - z| dx = 0 \right\}.$$

By the Lebesgue derivation theorem we conclude  $|\mathcal{S}_u| = 0$ .

It turns out that for  $\mathcal{H}^{n-1}$ -almost all  $x \in \mathcal{S}_u$  one can define an approximate tangent to  $\mathcal{S}_u$  while  $u^+$  and  $u^-$  provide one sided limits [12]. Given  $\nu$ , a unit vector in  $\mathbb{R}^n$ ,  $z \in \mathbb{R}$ , we say that  $z = u^+(x, \nu)$  if

$$\lim_{\rho \rightarrow 0^+} \frac{|\{y \in B_\rho(x) : \langle y - x, \nu \rangle > 0, |u(y) - z| > \varepsilon\}|}{\rho^n} = 0$$

for every  $\varepsilon > 0$ . Similarly, one defines  $u^-(x, \nu) = u^+(x, -\nu)$ . For  $\mathcal{H}^{n-1}$ -almost all  $x \in \mathcal{S}_u$  there is a unique  $\nu$  such that  $u^-(x) = u^-(x, \nu)$  and  $u^+(x) = u^+(x, \nu)$ . The vector  $\nu$  represents a normal to  $\mathcal{S}_u$ .

For any function  $u \in BV(\Omega)$ , the measure  $Du$  can be decomposed as

$$Du = \nabla u dx + Ju + Cu.$$

The first term,  $\nabla u dx$ , represents the part of  $Du$  which is absolutely continuous with respect to Lebesgue measure (which we denote by  $dx$ ), and  $\nabla u \in L^1(\Omega, \mathbb{R}^2)$  is thus the corresponding Radon-Nikodym derivative.  $Ju + Cu$  represents the part of  $Du$  which is singular with respect to Lebesgue measure. The measure  $Ju$  is defined on any Borel set  $B \in \Omega$  by

$$Ju(B) = \int_{B \cap \mathcal{S}_u} (u^+ - u^-) \nu_n d\mathcal{H}^{n-1},$$

where  $\nu_n(x)$  is the approximate normal to  $\mathcal{S}_u$  at  $x \in \mathcal{S}_u$ . The measure  $Cu(B)$  is a bounded Radon measure on  $\Omega$  with values in  $\mathbb{R}^2$ . It is a fact that, if  $\mathcal{H}^{n-1}(B) < +\infty$ , then  $Cu(B) = 0$  [2, 7]. It is clear that  $Ju$  captures the jump of discontinuity set of  $u$ , and  $\nabla u dx$  captures the smooth part. Thus, a reasonable formulation of the variational problem in this setting is: find minimizers of

$$(6) \quad E(u) = \beta \int_{\Omega} (u - g)^2 + \int_{\Omega} |\nabla u|^2 + \alpha \mathcal{H}^{n-1}(\mathcal{S}_u)$$

for  $u \in BV(\Omega)$  (where  $\Omega$  is  $n$  dimensional). The difficulty which arises is that the functional  $E$  gives no control over  $Cu$ . In fact the Cantor-Vitelli function in one dimension satisfies  $Du = Cu$  [3]. A consequence of this is that  $E$  is not coercive in  $BV(\Omega)$ , i.e.,  $E$  bounded sets are not compact.

We say  $u \in SBV(\Omega)$  if  $u \in BV(\Omega)$  and  $Cu = 0$ .  $SBV$  possesses some very useful properties. For example, as with  $BV(\Omega)$ , membership in  $SBV$  can be

determined by examining one-dimensional sections, and  $SBV$  is closed under  $L^1$  limits of  $BV$ -norm bounded sequences. Furthermore, the functional  $E$  is lower-semicontinuous in  $SBV$  with respect to the  $L^1$  topology. This issue will be discussed further in Section 2.5. The remainder of this section is devoted to stating some results on  $SBV$  functions which we will later require. Although all of the results hold in  $\mathbb{R}^n$  in general, we will require them only in  $\mathbb{R}^2$ . Since restricting to  $\mathbb{R}^2$  simplifies the statements of the results, we will confine ourselves to this case. In particular, we assume henceforth that  $\Omega \subset\subset \mathbb{R}^2$ .

Let  $B \subset\subset \Omega$  be an open set with Lipschitz boundary such that  $\mathcal{S}_u \cap \partial B$  has only a finite number of points. From the trace theorems for  $BV$  functions (see [12] Theorem 2.10) it follows that, for  $u \in BV(\Omega)$ ,

$$\int_B \phi \, Du = - \int_B u \operatorname{div} \phi \, dx + \int_{\partial B} u \phi \cdot \nu \, d\mathcal{H}^1$$

for every bounded Borel vector field  $\phi \in C^1(\overline{B}, \mathbb{R}^2)$ , where  $\nu$  is the outward normal to  $B$ . Thus, if it is also true that  $u \in SBV(\Omega)$  then

$$(7) \quad - \int_B u \operatorname{div} \phi \, dx + \int_{\partial B} u \phi \cdot \nu \, d\mathcal{H}^1 = \int_B \phi \cdot \nabla u \, dx + \int_{B \cap \mathcal{S}_u} (u^+ - u^-) \phi \cdot \nu_n \, d\mathcal{H}^1.$$

By some rather deep results due to De Giorgi-Carriero-Leaci [9], it is possible to characterize the condition  $x \in \mathcal{S}_u$  by examining the decay of certain functionals evaluated in balls centered at  $x$ .

**THEOREM 2** [9, Theorem 3.6]. *Let  $x \in \Omega$  and  $u \in SBV(\Omega)$ . If,*

$$\lim_{\rho \rightarrow 0} \rho^{-1} \left[ \int_{B_\rho(x)} |\nabla u|^2 \, dy + \mathcal{H}^1(\mathcal{S}_u \cap B_\rho(x)) \right] = 0$$

*then  $x \notin \mathcal{S}_u$ .*

The proof of this theorem is based on a generalization to  $SBV$  of the Poincaré-Wirtinger inequality.

To state the next result we need to introduce some more notation. Let  $u \in SBV(\Omega)$ . For every compact set  $K \subset \Omega$  we set,

$$F(u, K) = \int_K |\nabla u|^2 + \mathcal{H}^1(\mathcal{S}_u \cap K)$$

and

$$\Phi(u, K) = \inf\{F(v, K) : v \in SBV(\Omega), v = u \in \Omega \setminus K\}.$$

Obviously  $\Phi \leq F$ , and we define the deviation from minimality as

$$\Psi(u, K) = F(u, K) - \Phi(u, K).$$

THEOREM 3 [9, Theorem 4.13]. *There exist universal constants  $\xi, \gamma > 0$  such that, if  $u \in SBV(\Omega)$ ,  $B_\rho(x) \subset\subset \Omega$  for some  $\rho > 0$ , and each of the following three conditions holds,*

$$\begin{aligned} F(u, \overline{B}_\rho(x)) &\leq \xi\rho, \\ \lim_{t \rightarrow 0^+} t^{-1} \Psi(u, \overline{B}_t(x)) &= 0, \\ \Psi(u, \overline{B}_t(x)) &\leq \gamma t, \text{ for every } t \leq \rho, \end{aligned}$$

then  $\lim_{\rho \rightarrow 0} \rho^{-1} F(u, \overline{B}_\rho(x)) = 0$ , (and hence  $x \notin \mathcal{S}_u$ ).

PROOF. See [9] or [3]. □

The proof of this theorem is based on another theorem which we quote below. To state this theorem it is convenient to reintroduce  $\alpha$  into the notation. Thus temporarily we set,

$$F(u, \alpha, K) = \int_K |\nabla u|^2 + \alpha \mathcal{H}^1(\mathcal{S}_u \cap K)$$

and

$$\Phi(u, \alpha, K) = \inf\{F(v, \alpha, K) : v \in SBV(\Omega), v = u \in \Omega \setminus K\}.$$

THEOREM 4. *For any  $\delta \in (0, 1)$  there exist two universal constants  $\xi$  and  $\theta$  such that, if  $\rho > 0$ ,  $\overline{B}_\rho(x) \in \Omega$  and  $u \in SBV(\Omega)$  with*

$$\begin{aligned} F(u, \alpha, \overline{B}_\rho(x)) &\leq \xi\rho, \\ \Psi(u, \alpha, \overline{B}_\rho(x)) &\leq \theta F(u, \alpha, \overline{B}_\rho(x)), \end{aligned}$$

then

$$F(u, \alpha, \overline{B}_{\frac{\rho}{2}}(x)) \leq \left(\frac{1}{2}\right)^{2-\delta} F(u, \alpha, \overline{B}_\rho(x)).$$

The theorem is proved by contradiction. Assuming the theorem is false, it is possible to find  $\delta \in (0, 1)$ , an  $\alpha > 0$ , and sequences  $\xi_n, \theta_n, \rho_n, x_n, u_n$  such that  $\xi_n \downarrow 0$ ,  $\theta_n \downarrow 0$ ,  $\overline{B}_{\rho_n}(x_n) \subset \Omega$ ,

$$\begin{aligned} F(u_n, \alpha, \overline{B}_{\rho_n}(x_n)) &\leq \xi_n \rho_n, \\ \Psi(u_n, \alpha, \overline{B}_{\rho_n}(x_n)) &\leq \theta_n F(u_n, \alpha, \overline{B}_{\rho_n}(x_n)), \end{aligned}$$

and

$$F(u_n, \alpha, \overline{B}_{\frac{\rho_n}{2}}(x_n)) > \left(\frac{1}{2}\right)^{2-\delta} F(u_n, \alpha, \overline{B}_{\rho_n}(x_n)).$$

By rescaling and translating, one obtains a sequence  $v_n$  of functions in  $SBV(B_1)$  such that  $F\left(v_n, \frac{\alpha}{\xi_n}, \overline{B_1}\right) = 1$ ,  $\Psi\left(v_n, \frac{\alpha}{\xi_n}, \overline{B_1}\right) \leq \theta_n$  and

$$F\left(v_n, \frac{\alpha}{\xi_n}, \overline{B_{\frac{1}{2}}}\right) \geq \left(\frac{1}{2}\right)^{2-\delta}.$$

Since  $\frac{\alpha}{\xi_n} \uparrow \infty$  and the deviation from minimality in  $B_1$  tends to 0, in the limit the functions  $v_n$  should behave like harmonic functions. But if  $v$  is an harmonic function, then

$$\int_{B_{\frac{1}{2}}} |\nabla v|^2 dx \leq \left(\frac{1}{2}\right)^2 \int_{B_1} |\nabla v|^2 dx$$

and in this way a contradiction is found.

#### 2.4. Partitions in Caccioppoli sets

It will be convenient when treating the piecewise constant version of the problem to be able to translate back and forth between an  $SBV$  formulation and one based on partitions. In this section we state the facts which will be of use in this respect.

A sequence  $\{R_i\}$  is a *Borel partition* of  $\Omega$  if each  $R_i$  is a Borel set,  $R_i \cap R_j = \emptyset$  whenever  $i \neq j$ , and  $\bigcup_i R_i = \Omega$ . We will call a Borel partition a *Caccioppoli partition* if  $\sum_{i=1}^{\infty} \mathcal{H}^1(\Omega \cap \partial^* R_i) < \infty$ .

LEMMA 5. *Let  $\{R_i\}$  be a Caccioppoli partition of  $\Omega$ , and let  $a_i$  be a bounded sequence of real numbers such that  $a_i \neq a_j$  for  $i \neq j$ . Then,*

$$u = \sum_{i=1}^{\infty} a_i \chi_{R_i} \in SBV_{\text{loc}}(\Omega)$$

and

$$\begin{aligned} \nabla u &\equiv 0 \\ 2\mathcal{H}^1(\mathcal{S}_u) &= \sum_{i=1}^{\infty} \mathcal{H}^1(\Omega \cap \partial^* R_i). \end{aligned}$$

PROOF. See [5, Lemma 1.10]. □

LEMMA 6. *Let  $u \in SBV_{\text{loc}}(\Omega)$  be such that  $\nabla u \equiv 0$  and  $\mathcal{H}^1(\mathcal{S}_u) < \infty$ . Then there exists a Caccioppoli partition  $\{R_i\}$  and a sequence  $\{a_i\}$  satisfying  $a_i \neq a_j$*

for  $i \neq j$  such that

$$2\mathcal{H}^1(\mathcal{S}_u) = \sum_{i=1}^{\infty} \mathcal{H}^1(\Omega \cap \partial^* R_i)$$

$$u = \sum_{i=1}^{\infty} a_i \chi_{R_i} \text{ a.e. in } \Omega.$$

PROOF. See [5, Lemma 1.11]. □

### 2.5. Existence results

The following theorem established the existence of “regular” minimizers for  $E_0$ .

**THEOREM 7** [14]. *Let  $\Omega$  be an open rectangle and let  $g \in L^\infty(\Omega)$ . For all one-dimensional sets  $\Gamma \subset \Omega$  such that  $\Gamma \cup \partial\Omega$  is made up of a finite number of  $C^{1,1}$ -arcs, meeting each other at their end-points, and for all locally constant functions  $f$  on  $\Omega \setminus \Gamma$ , there exist an  $f$  and a  $\Gamma$  which minimize  $E_0$ .*

Mumford and Shah [16] proved a similar theorem with the restriction that  $g$  be continuous on  $\bar{\Omega}$ . In this case they showed that  $\Gamma$  is composed of a finite number of  $C^2$  curves. The theorem quoted above was proved by Morel and Solimini in [14] using direct, constructive methods. Finally, another proof, using  $\Gamma$  restricted to be unions of line segments and then taking limits as the segment lengths tend to zero, was achieved by Wang [23]. In the  $n$  dimensional case, existence results were obtained by Congedo and Tamanini [5] with  $\Gamma$  lying in the class of relatively closed sets.

Equivalent results for minimizers of  $E$  in the two dimensional case have not been achieved. However, for a “weak” version of the problem, to be detailed later, an existence result has been attained. This result was achieved by proving a regularity result on minimizers of a yet weaker version of the problem posed in the  $SBV$  setting. Formulating the problem in  $SBV$ , the functional appears as below

$$E_{SBV}(f) = \beta \int_{\Omega} (g - f)^2 + \int_{\Omega} \|\nabla f\|^2 + \alpha \mathcal{H}^1(\mathcal{S}_f),$$

where  $f \in SBV(\Omega)$ ,  $\nabla f dx$  is the part of  $Df$  which is absolutely continuous with respect to Lebesgue measure, and  $\mathcal{S}_f$  is the jump set of  $f$ . Ambrosio [2] proved a compactness theorem and a lower-semicontinuity theorem for the space  $SBV(\Omega)$  which allows the assertion of existence of minimizers to  $E_{SBV}$ . These theorems are required for the proofs of our asymptotic results in Part II.

**THEOREM 8** [2] [3, Theorem 3.1]. *Let  $\{u_n\} \subset SBV(\Omega) \cap L^\infty(\Omega)$  be a sequence satisfying*

$$\limsup_{n \rightarrow \infty} \left\{ \|u_n\|_\infty + \int_{\Omega} |\nabla u_n|^2 dx + \mathcal{H}^1(\mathcal{S}_{u_n}) \right\} < \infty.$$

Then, there exists a subsequence  $u_{n_k}$  converging in  $L^1_{\text{loc}}(\Omega)$  to  $u \in SBV(\Omega)$ . Moreover,

$$\begin{aligned} Ju_n &\rightarrow Ju && \text{weakly as radon measures} \\ \nabla u_n &\rightarrow \nabla u && \text{weakly in } L^1(\Omega; \mathbb{R}^2). \end{aligned}$$

We mention in passing that, if it can be shown that  $u_n \rightarrow u$  in  $L^1_{\text{loc}}(\Omega)$  by other means, then the weak convergence results apply to the original sequence.

To complete the proof of existence of  $SBV(\Omega)$  minimizers of  $E_{SBV}$ , the following has been proved.

**THEOREM 9** [2] [3, Theorem 4.2]. *If  $u_n \rightarrow u$  in  $L^1_{\text{loc}}(\Omega)$  with  $\|u_n\|_{\infty} \leq C < \infty$ , then*

$$E_{SBV}(u) \leq \liminf_{n \rightarrow \infty} E_{SBV}(u_n).$$

The “weak” formulation of the functional  $E$  takes the form

$$E(f, \Gamma) = \beta \int_{\Omega} (f - g)^2 + \int_{\Omega \setminus \Gamma} |\nabla f|^2 + \alpha \mathcal{H}^1(\Gamma),$$

where  $\Gamma$  is a relatively closed subset of  $\Omega$  and  $f \in W^{1,2}(\Omega)$ . To make the connection between this formulation and the  $SBV$  formulation, the first step is to note that the  $SBV$  formulation is more general. The following proposition makes this assertion.

**PROPOSITION 10** [3, Proposition 3.3]. *Let  $\Gamma \subset \Omega$  be a closed set such that  $\mathcal{H}^1(\Gamma) < \infty$ , and let  $u \in W^{1,1}(\Omega \setminus \Gamma) \cap L^{\infty}(\Omega)$ . Then,  $u \in SBV(\Omega)$  and  $S_u \subset \Gamma \cup N$  with  $\mathcal{H}^1(N) = 0$ .*

A consequence of this proposition is that the minimum achieved under the  $SBV$  formulation is less than or equal to the infimum of the “weak” formulation. However, an equivalence between this formulation and the weak formulation was achieved through a regularity theorem for  $SBV$  minimizers in [9].

**THEOREM 11** [9]. *Let  $u \in SBV(\Omega)$  be a minimizer of  $E_{SBV}$ . Then,*

- (i)  $u \in L^{\infty}(\Omega)$ ,  $\|u\|_{\infty} \leq \|g\|_{\infty}$ ;
- (ii)  $u \in W^{2,p}(\Omega \setminus \overline{S}_u) \forall p \in [1, \infty)$  and  $\Delta u = \beta(u - g)$  in  $\Omega \setminus \overline{S}_u$ ;
- (iii) the function  $\tilde{u}(x) = u^+(x) = u^-(x)$  belongs to  $C^1(\Omega \setminus \overline{S}_u)$ ;
- (iv)  $\mathcal{H}^1(\Omega \cap \overline{S}_u \setminus S_u) = 0$ .

The most difficult and interesting part of this theorem is the last statement. The proof uses the two theorems quoted in Section 2.3. The most important

result needed for the proof (beyond what has already been mentioned in Section 2.3) is that, if  $f$  minimizes  $E_{SBV}$ , then the set

$$\Omega_0 = \left\{ x \in \Omega : \limsup_{\rho \rightarrow 0} \rho^{-1} \left[ \int_{B_\rho(x)} |\nabla u|^2 \, dy + \mathcal{H}^1(\mathcal{S}_u \cap B_\rho(x)) \right] = 0 \right\}$$

is open. This is established by showing that if  $x \in \Omega_0$  then the conditions of Theorem 3 are satisfied at  $x$  and also in some neighbourhood of  $x$ . But all points where these conditions are satisfied are in  $\Omega_0$  by Theorem 3; thus,  $\Omega_0$  is open. To see why (iv) follows, let  $\Gamma = \Omega \setminus \Omega_0$ :

$$\Gamma = \bigcup_{\delta > 0} \Gamma_\delta$$

where

$$\Gamma_\delta = \left\{ x \in \Omega : \limsup_{\rho \rightarrow 0} \rho^{-1} \left[ \int_{B_\rho(x)} |\nabla u|^2 \, dy + \mathcal{H}^1(\mathcal{S}_u \cap B_\rho(x)) \right] \geq \delta \right\}.$$

$\Gamma_\delta$  has zero Lebesgue measure since

$$\Gamma_\delta \subset \left\{ x \in \Omega : \limsup_{\rho \rightarrow 0} \rho^{-2} \int_{B_\rho(x)} |\nabla u|^2 \, dy = \infty \right\}.$$

A general result for Hausdorff measures [9] implies

$$\int_B |\nabla u|^2 \, dy + \mathcal{H}^1(\mathcal{S}_u \cap B) \geq \delta \mathcal{H}^1(\Gamma_\delta \cap B)$$

for all  $\delta > 0$  and Borel sets  $B$ . Thus, by setting  $B = \Gamma_\delta \setminus \mathcal{S}_u$ , we obtain

$$\mathcal{H}^1(\Gamma_\delta \setminus \mathcal{S}_u) = 0$$

and, since  $\delta$  is arbitrary,  $\mathcal{H}^1(\Gamma \setminus \mathcal{S}_u) = 0$ . Finally, since  $\Gamma$  is relatively closed in  $\Omega$ , we get  $\Omega \cap \overline{\mathcal{S}_u} \setminus \mathcal{S}_u \subset \Gamma \setminus \mathcal{S}_u$  and (iv) is proved.

The following is thus established,

**THEOREM 12.** *Let  $\Omega$  be an open rectangle and let  $g \in L^\infty(\Omega)$ . For all relatively closed sets  $\Gamma \subset \Omega$  and for all functions  $f \in C^1(\Omega \setminus \Gamma)$ , there exists an  $f$  and a  $\Gamma$  which minimize  $E$ .*

An independent proof of the existence theorem for the  $\mathbb{R}^2$  case, due to Dal Maso-Morel-Solimini, was given in [6]. In the same reference the authors

also showed that minimizers of  $E$  possess a regularity property which they called the concentration property. For details we refer to [6]. Their results strengthen the existence theorem to allow  $\Gamma$  to be a closed subset of  $\overline{\Omega}$ , rather than  $\Omega$ , when  $\Omega$  is a rectangle. An important implication for the work presented in this paper is that  $\mathcal{H}^1(\overline{\mathcal{S}}_f \cap \partial\Omega) = 0$  (in the two dimensional case). Thus we have LEMMA 13. *In the  $\mathbb{R}^2$  case, statement (iv) of Theorem 11 can be strengthened to*

$$\mathcal{H}^1(\overline{\mathcal{S}}_u \setminus \mathcal{S}_u) = 0.$$

PROOF. See [6]. □

## PART II

### Asymptotic results

In this section we state and prove asymptotic theorems for minimizers of  $E_0$  and  $E$ , respectively. Since the results strongly depend on those found for minimizers of  $E_{SBV}$ , it is convenient to state and prove the results in that framework. Under some mild regularity assumptions on  $g$ , it is shown that as  $\beta$  tends to infinity,  $\mathcal{S}_f(\beta)$ , the jump set of a minimizer of  $E_{SBV}$ , will converge to  $\mathcal{S}_g$  in the Hausdorff metric. (Of course the same also holds for the  $\Gamma(\beta)$  which minimize  $E$ ). Furthermore, we show that the result still holds if the image  $g$  is corrupted, by smearing and additive noise say, provided the corruption decays sufficiently quickly as  $\beta$  tends to infinity.

### 3. - Minimizers of $E$

#### 3.1. Problem formulation

In general, by a minimizer of  $E$  we mean a minimizer such as described in Theorem 12; in particular,  $f \in C^1(\Omega \setminus \overline{\mathcal{S}}_f)$  and  $\Gamma$  is a relatively closed subset of  $\Omega$ . It will be convenient also to refer back to the formulation in the  $SBV$  setting. Thus, we will also consider that  $f \in SBV(\Omega)$  and  $\Gamma = \overline{\mathcal{S}}_f \cap \Omega$ . Without loss of generality, we will set the parameter  $\alpha = 1$  and subsequently drop it from our notation.

##### 3.1.1. Assumptions on the domain

We will be assuming that our domain  $\Omega$  is a rectangle. We do this primarily to allow a reflection argument, based on Lemma 13, which assures that the boundary of the domain does not cause the introduction of spurious boundaries (see Theorem 26).



### 3.1.2. Assumptions on the image

We will need some mild assumptions on the regularity of the image in order to achieve the desired result. We summarize them below.

ASSUMPTION 1.  $g_u \in L^\infty(\Omega) \cap SBV(\Omega)$ ,  $\int_{\Omega} |\nabla g_u|^2 + \mathcal{H}^1(\mathcal{S}_{g_u}) < \infty$  and  $\mathcal{S}_{g_u}$  has no isolated points, i.e., if  $x \in \mathcal{S}_{g_u}$  then  $\forall \rho > 0$ ,  $\mathcal{H}^1(\mathcal{S}_{g_u} \cap B_\rho(x)) > 0$ .

ASSUMPTION 2. If  $A \subset \Omega$  is an open set satisfying  $\text{dist}(A, \mathcal{S}_{g_u}) > 0$  then there exists an  $L < \infty$  such that, if  $x$  and  $y$  are the end points of a line segment lying in  $A$ , then  $|g_u(x) - g_u(y)| \leq L|x - y|$ . We refer to  $L$  as the Lipschitz constant associated with  $A$ .

Essentially we have assumed that  $g_u \in C^{0,1}(\Omega \setminus [\mathcal{S}_{g_u}]_\varepsilon)$  for any  $\varepsilon > 0$ .

### 3.1.3. The noise model

For each  $\beta \in \mathbb{R}$ , we define a class of functions, parametrized by  $\beta$ , which represents the set of images which might actually be observed. We will denote this class of functions by  $Y(\beta)$ , and we assume  $Y(\beta) \subset L^\infty(\Omega)$ . The following are the properties we require of  $Y(\beta)$ :

$$(8) \quad \lim_{\beta \rightarrow \infty} \sup_{g \in Y(\beta)} \beta \int_{\Omega} (g - g_u)^2 = 0,$$

and

$$(9) \quad \forall \varepsilon > 0, \lim_{\beta \rightarrow \infty} \sup_{g \in Y(\beta)} \|(g - g_u)(1 - \chi_{[\mathcal{S}_{g_u}]_\varepsilon})\|_\infty = 0.$$

For vision applications, a more natural model for the corruption might be based on smearing and additive noise. We show how we can cast our assumptions in a form that makes this explicit, provided we assume that the Lipschitz constants referred to above are uniformly bounded on  $\Omega \setminus \overline{\mathcal{S}_{g_u}}$  and that the Minkowski content of  $\mathcal{S}_{g_u}$  is finite. Let  $\mathcal{S}_r$  be the class of maps taking  $L^\infty(\Omega)$  to  $L^\infty(\Omega)$  having the property that the value of the image function at a point  $x \in \Omega$  lies within the range of essential values that the argument function takes in a ball of radius  $r$  around  $x$ . This models, in a quite general way, smearing of the image and, hence, distortion of the boundaries. More formally,  $\Phi \in \mathcal{S}_r$  iff  $\Phi$  has the property

$$\Phi(g)(x) \in [\text{ess inf}\{g(x) : x \in B_r\}, \text{ess sup}\{g(x) : x \in B_r\}].$$

An example of such a  $\Phi$  would be a smoothing operator defined using a mollifier with support lying inside the ball of radius  $r$ , but nonlinear perturbations are also allowed. Suppose there is a constant  $c_b > 0$  such that  $c_b r \geq [\mathcal{S}_{g_u}]_r$  for all

$r > 0$  and that the uniform Lipschitz constant for  $g_u$  on  $\Omega \setminus \overline{S_{g_u}}$  is  $L$ . Suppose further that  $g$  has a representation of the form

$$(10) \quad g = \Phi(g_u) + \vartheta w$$

for some  $\Phi \in \mathcal{S}_r$  and  $w \in L^\infty$  with  $\|w\|_\infty \leq 1$  and  $\vartheta$  a real scalar. Let  $h_r : (0, \infty) \rightarrow [0, \infty)$  and  $h_\vartheta : (0, \infty) \rightarrow [0, \infty)$  be any functions satisfying

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \beta h_r(\beta) &= 0 \\ \lim_{\beta \rightarrow \infty} \beta^{\frac{1}{2}} h_\vartheta(\beta) &= 0. \end{aligned}$$

Define  $Y(\beta)$  to be those functions  $g$  which can be written in the form (10), with  $\vartheta \leq h_\vartheta(\beta)$ , for some  $\Phi \in \mathcal{S}_r$ , with  $r \leq h_r(\beta)$ . With this definition of  $Y(\beta)$ , equations (8) and (9) are satisfied since

$$\beta \int_{\Omega} (g - g_u)^2 \leq \beta [|\mathcal{S}_{g_u}|_r] \|g - g_u\|_\infty + \beta (L h_r(\beta) + h_\vartheta(\beta))^2 |\Omega|.$$

We can now state the limit theorem to be proved.

**THEOREM 14.** *Under our stated assumptions, as  $\beta \rightarrow \infty$ ,  $\{S_f(\beta)\}$  converges to  $S_{g_u}$  with respect to the Hausdorff metric, and*

$$\mathcal{H}^1(S_f(\beta)) \rightarrow \mathcal{H}^1(S_{g_u}).$$

*We mean by this that, for any  $\varepsilon > 0$ , there exists  $\beta' < \infty$  such that, if  $\beta > \beta'$  and  $f$  is a minimizer of  $E$  for some  $g \in Y(\beta)$ , then  $d_H(S_f, S_{g_u}) < \varepsilon$  and  $|\mathcal{H}^1(S_f) - \mathcal{H}^1(S_{g_u})| < \varepsilon$ . Furthermore,  $\lim_{\beta \rightarrow \infty} \sup_{g \in Y(\beta)} \beta \int (f - g_u)^2 = 0$ .*

Since we will need to vary  $g$  and  $\beta$ , we will use  $E(f, \beta, g)$  to denote a particular evaluation of  $E$  and  $E(\beta, g)$  to fix  $\beta$  and  $g$  when  $f$  is considered a free variable. Before presenting the technical arguments, we provide a sketch of the main ideas.

The first few results establish convergence of minimizers of  $E$  to  $g_u$  in various senses somewhat weaker than that stated by the theorem. In particular, Lemmas 15, 16 and 17 establish that, if  $f_n, \beta_n, g_n$  are sequences such that  $\beta_n \rightarrow \infty$ ,  $g_n \in Y(\beta_n)$ , and  $f_n$  minimizes  $E(\beta_n, g_n)$ , then  $f_n$  converges to  $g_u$  in  $L^1(\Omega)$ ,  $Jf_n \rightarrow Jg_u$  weakly as radon measures, and  $\nabla f_n \rightarrow \nabla g_u$  weakly in  $L^1(\Omega; \mathbb{R}^2)$  and strongly in  $L^2(\Omega; \mathbb{R}^2)$ . It is also shown that  $\mathcal{H}^1(S_{f_n}) \rightarrow \mathcal{H}^1(S_{g_u})$  and that for any set  $A \subset \Omega$  which is positively separated from  $S_{g_u}$ ,  $\lim_{n \rightarrow +\infty} \mathcal{H}^1(S_{f_n} \cap A) = 0$ . These results alone are enough to ensure that, for  $\beta$  large enough,  $S_{g_u} \subset [S_{f_n}]_\varepsilon$ , but we defer the statement of this fact until the end of the proof. The opposite containment, namely  $[S_{f_n}]_\varepsilon \subset S_{g_u}$ , does not follow directly, and it is the proving of this statement which constitutes most of the difficulty of the proof.

Theorem 3 provides conditions under which one can assert, for a given  $x \in \Omega$  and  $f \in SBV(\Omega)$ , that  $x \notin \mathcal{S}_f$ . Our goal is to show that for  $\beta$  sufficiently large these conditions can be satisfied for each  $x \in \Omega \setminus [\mathcal{S}_{g_u}]_\varepsilon$ . To accomplish this we must, for each such  $x$ , find a  $\rho$  such that the following are all satisfied

$$(11) \quad F(u, \overline{B}_\rho(x)) \leq \xi \rho,$$

$$(12) \quad \lim_{t \rightarrow 0^+} t^{-1} \Psi(u, \overline{B}_t(x)) = 0,$$

$$(13) \quad \Psi(u, \overline{B}_t(x)) \leq \eta t, \quad \forall t \leq \rho,$$

where  $F$  and  $\Psi$  are defined as in Section 2.3. Let  $v^t \in SBV(\Omega)$  be such that it is equal to  $f$  outside of  $B_t(x)$  and minimizes  $\int_{B_t(x)} |\nabla v^t|^2 + \mathcal{H}^1(\mathcal{S}_{v^t} \cap \overline{B}_t(x))$  where  $0 \leq t \leq \rho(x)$ . Since  $f$  minimizes  $E$ , we know that

$$\beta \int_{B_t(x)} (v^t - g)^2 + F(v^t, \overline{B}_t(x)) \geq \beta \int_{B_t(x)} (f - g)^2 + F(f, \overline{B}_t(x));$$

thus, if a bound of the form  $\|v^t - f\|_\infty \leq h$  is established, then one can obtain a bound on  $\Psi(u, \overline{B}_t(x))$  of the form  $\beta \pi h t^2$ . Since  $t$  is bounded above by  $\rho$ , if we can choose  $\rho$  as a function of  $\beta$  and show that  $h \rho$  decays faster than  $\frac{1}{\beta}$ , then the condition (13) can be met. The conditions (11) and (12) then follow easily.

The remainder of this sketch is devoted to describing how we can achieve the desired estimates on  $\|v^t - f\|_\infty$ . The function  $v^t$  can be bounded by its boundary conditions, i.e., by a bound on  $f$  restricted to  $\partial B_t(x)$ . Thus, our goal is now to achieve estimates on  $f$  restricted to  $B_\rho(x)$ . To get strong bounds on the range of  $f$  on  $\partial B_\rho(x)$  it is extremely helpful to have  $\overline{\mathcal{S}}_f \cap \partial B_\rho = \emptyset$ . We accomplish this essentially through Lemma 23.

In Lemma 23 we consider small balls of radius  $\rho$  around a point  $x \in \Omega \setminus \overline{\mathcal{S}}_{g_u}$ . We let  $\rho$  be a function of  $\beta$  of the form  $\rho = \beta^{-\gamma}$  for some positive constant  $\gamma$ . We use the notation  $J(u, \beta, \rho, x) = \beta \int_{B_\rho(x)} u^2 + F(u, \overline{B}_\rho(x))$  and establish the

result

$$\sup_{x \in K} J(f - g_s, \beta, \beta^{-\gamma}, x) \leq c \beta^{-2\gamma}$$

where  $g_s$  is a smoothed version of  $g_u$ . There are two key observations we wish to make concerning the proof of this lemma. The first is that the proof involves redefining  $f$  in a ball  $B_\rho(x)$ , where  $\rho \simeq \beta^{-\gamma}$ , and then using the fact that  $f$  itself minimizes  $E$  to obtain estimates. In order to redefine  $f$  in a useful way, we use two ideas. At points disjoint from  $\overline{\mathcal{S}}_f$ ,  $f$  satisfies  $\Delta f = \beta(f - g)$ . Very roughly speaking, solutions to equations of this form look like smoothed versions of  $g$

where the smoothing is done over a ball of radius  $\beta^{-\frac{1}{2}}$ . The assumptions ensure that  $g$  tracks  $g_u$  reasonably closely, so a reasonable candidate for  $f$  might be a smoothed version of  $g_u$  found by convolving  $g_u$  with a mollifier with support in a ball of radius  $\beta^{-\frac{1}{2}}$ . This is precisely how we define the function which will be denoted by  $g_s$ . The redefined  $f$  is formed by continuously transforming  $f$  into  $g_s$  inside the ball  $B_\rho$ . To obtain estimates on the energy associated with the new  $f$ , we need to estimate  $f$  on  $\partial B_\rho(x)$ . Comparing the cost of the original  $f$  with the redefined  $f$  inside the ball  $B_\rho$  requires a bound on the ratio of the contribution to  $E$  occurring from the interior of the ball to the density of that occurring from the boundary of the ball. Proposition 20 provides us with a means of choosing the radius of the ball to guarantee that this ratio is somewhat controlled.

The second key idea in the proof of Lemma 23 involves estimating  $\nabla w$ , where  $w = f - g_s$ . This is achieved partly through noting  $\nabla w \, dx = Dw - Jw$ . Bounds on  $Dw$  are relatively easy to obtain using integration by parts. Thus, having bounds on  $Jw$  can yield bounds on  $\nabla w$ . Lemma 19 states that, for any compact  $K \subset \Omega$  disjoint from  $\overline{S}_{g_u} \cup \partial\Omega$ , if  $g_n, f_n, \beta_n$  are sequences such that  $\beta_n \uparrow +\infty$ ,  $g_n \in Y(\beta_n)$ , and  $f_n$  minimizes  $E(\beta_n, g_n)$ , then

$$\lim_{n \rightarrow +\infty} \sup_{x \in K} |f_n^+(x) - f_n^-(x)| = 0.$$

This result yields the required estimates.

The final stage of the proof is essentially carried out in Lemma 25, i.e. in this lemma desired bounds on  $v^t$  are established. The remaining results tie up some loose ends, such as extending the results by a reflection argument to include points  $x$  which may lie arbitrarily close to  $\partial\Omega$ .

### 3.2. Preliminary results

The first few results in this section are largely consequences of the compactness and lower-semicontinuity theorems for  $SBV(\Omega)$  functions, due to Ambrosio [2, 3], which were quoted in Section 2.5.

Let  $E^*(\beta, g)$  denote the minimal value of  $E(\beta, g)$ . By simply substituting  $g_u$  for  $f$ , we get the following bound

$$(14) \quad E^*(\beta, g) \leq \beta \int_{\Omega} (g - g_u)^2 + \int_{\Omega} |\nabla g_u|^2 + \mathcal{H}^1(S_{g_u}).$$

LEMMA 15. *If  $f_n, \beta_n, g_n$  are sequences such that  $\beta_n \uparrow +\infty$ ,  $g_n \in Y(\beta_n)$ , and  $f_n$  minimizes  $E(\beta_n, g_n)$ , then*

$$\begin{aligned} f_n &\rightarrow g_u && \text{in } L^1(\Omega) \\ Jf_n &\rightarrow Jg_u && \text{weakly as Radon measures} \\ \nabla f_n &\rightarrow \nabla g_u && \text{weakly in } L^1(\Omega; \mathbb{R}^2). \end{aligned}$$

PROOF. From the triangle inequality and equation (14), we obtain

$$\begin{aligned} \beta_n \int_{\Omega} (f_n - g_u)^2 &\leq \beta_n \int_{\Omega} (g_n - g_u)^2 + E^*(\beta_n, g_n) \\ &\leq 2\beta_n \int_{\Omega} (g_n - g_u)^2 + \int_{\Omega} |\nabla g_u|^2 + \mathcal{H}^1(\mathcal{S}_{g_u}). \end{aligned}$$

Since  $\beta_n \rightarrow \infty$ , it follows from equation (8) that  $f_n$  converges to  $g_u$  in  $L^2(\Omega)$ . Noting  $|\Omega| < \infty$ , we conclude  $f_n$  converges to  $g_u$  in  $L^1(\Omega)$ . The other statements follow from Theorem 8.  $\square$

LEMMA 16. *If  $f_n, \beta_n, g_n$  are sequences such that  $\beta_n \uparrow +\infty, g_n \in Y(\beta_n)$ , and  $f_n$  minimizes  $E(\beta_n, g_n)$ , then*

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{\Omega} |\nabla f_n|^2 &= \int_{\Omega} |\nabla g_u|^2 \\ \lim_{n \rightarrow +\infty} \mathcal{H}^1(\mathcal{S}_{f_n}) &= \mathcal{H}^1(\mathcal{S}_{g_u}). \end{aligned}$$

PROOF. By (14) and equation (8), we have

$$(15) \quad \limsup_{n \rightarrow \infty} \int_{\Omega} |\nabla f_n|^2 + \mathcal{H}^1(\mathcal{S}_{f_n}) \leq \int_{\Omega} |\nabla g_u|^2 + \mathcal{H}^1(\mathcal{S}_{g_u}).$$

Theorem 9 yields the inequality,

$$(16) \quad \int_{\Omega} |\nabla g_u|^2 + \mathcal{H}^1(\mathcal{S}_{g_u}) \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla f_n|^2 + \mathcal{H}^1(\mathcal{S}_{f_n}).$$

An examination of the proof reveals that each term is lower-semicontinuous separately, i.e.,  $\int_{\Omega} |\nabla g_u|^2 \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} |\nabla f_n|^2$  and  $\mathcal{H}^1(\mathcal{S}_{g_u}) \leq \liminf_{n \rightarrow +\infty} \mathcal{H}^1(\mathcal{S}_{f_n})$ .

(This can also easily be seen by rescaling  $f_n$  and  $g_u$  and noting that (15) and (16) still hold). From this the lemma follows.  $\square$

COROLLARY. *If  $A \subset \Omega$  is any borel set such that  $\text{dist}(A, \overline{\mathcal{S}_{g_u}} \cup \partial\Omega) > 0$ , then*

$$\lim_{n \rightarrow +\infty} \mathcal{H}^1(\mathcal{S}_{f_n} \cap A) = 0.$$

PROOF. For some  $\varepsilon > 0$ ,  $A \cap [\mathcal{S}_{g_u}]_{\varepsilon} = \emptyset$ . From Lemma 15 we conclude  $f_n \rightarrow g_u$  in  $L^1([\mathcal{S}_{g_u}]_{\varepsilon})$ . We can now apply essentially the same argument as in Lemma 16 to conclude  $\mathcal{H}^1(\mathcal{S}_{g_u}) \leq \liminf_{n \rightarrow \infty} \mathcal{H}^1(\mathcal{S}_{f_n} \cap [\mathcal{S}_{g_u}]_{\varepsilon})$ , but the result of Lemma 16 states  $\mathcal{H}^1(\mathcal{S}_{g_u}) = \lim_{n \rightarrow \infty} \mathcal{H}^1(\mathcal{S}_{f_n})$  so it follow that  $\lim_{n \rightarrow \infty} \mathcal{H}^1(\mathcal{S}_{f_n} \cap A) = 0$ .  $\square$

We append the additional notation  $e = f - g_u$ .

LEMMA 17. *If  $f_n$ ,  $\beta_n$ ,  $g_n$  are sequences such that  $\beta_n \uparrow +\infty$ ,  $g_n \in Y(\beta_n)$ , and  $f_n$  minimizes  $E(\beta_n, g_n)$ , then*

$$\lim_{n \rightarrow +\infty} \beta_n \int_{\Omega} e_n^2 + \int_{\Omega} |\nabla e_n|^2 = 0.$$

PROOF. From Lemma 16 and inequality (14), we conclude

$$\lim_{n \rightarrow \infty} E(f_n, \beta_n, g_n) = \int_{\Omega} |\nabla g_u|^2 + \mathcal{H}^1(S_{g_u})$$

and, hence,  $\lim_{n \rightarrow \infty} \beta_n \int_{\Omega} (f_n - g_n)^2 = 0$ . Since

$$\beta_n \int_{\Omega} (e_n)^2 \leq 2 \left( \beta_n \int_{\Omega} (f_n - g_n)^2 + \beta_n \int_{\Omega} (g_n - g_u)^2 \right)$$

and  $\lim_{n \rightarrow \infty} \beta_n \int_{\Omega} (g_n - g_u)^2 = 0$ , by equation (8), we obtain

$$\lim_{n \rightarrow \infty} \beta_n \int_{\Omega} (e_n)^2 = 0.$$

Now,

$$\int_{\Omega} |\nabla e_n|^2 = \int_{\Omega} |\nabla f_n|^2 - 2 \int_{\Omega} \nabla f_n \cdot \nabla g_u + \int_{\Omega} |\nabla g_u|^2$$

and since, by Lemma 16, we have  $\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla f_n|^2 = \int_{\Omega} |\nabla g_u|^2$ , the desired result follows if  $\lim_{n \rightarrow \infty} \int_{\Omega} \nabla f_n \cdot \nabla g_u = \int_{\Omega} |\nabla g_u|^2$ . It was assumed that  $\int_{\Omega} |\nabla g_u|^2 < \infty$ , so if we define  $\chi_N = \{|\nabla g_u|^2 \leq N\}$  then, by the monotone convergence theorem, we have  $\lim_{N \rightarrow \infty} \int_{\Omega} |\nabla g_u|^2 (1 - \chi_N) = 0$ . Thus, for any  $\varepsilon > 0$ , we can choose  $N$

sufficiently large so that  $2 \left( \int_{\Omega} |\nabla g_u|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla g_u|^2 (1 - \chi_N) \right)^{\frac{1}{2}} < \varepsilon$ . Now

$$\begin{aligned} \left| \int_{\Omega} \nabla f_n \cdot \nabla g_u - \int_{\Omega} |\nabla g_u|^2 \right| &\leq \left| \int_{\Omega} \nabla f_n \cdot \nabla g_u \chi_N - \int_{\Omega} |\nabla g_u|^2 \chi_N \right| \\ &\quad + \left| \int_{\Omega} \nabla f_n \cdot \nabla g_u (1 - \chi_N) \right| + \int_{\Omega} |\nabla g_u|^2 (1 - \chi_N). \end{aligned}$$

Since  $\nabla f_n \rightarrow \nabla g_u$  weakly in  $L^1(\Omega, \mathbb{R}^2)$ , we have  $\lim_{n \rightarrow +\infty} \int_{\Omega} \nabla f_n \cdot \nabla g_u \chi_N = \int_{\Omega} |\nabla g_u|^2 \chi_N$ . Using the Schwartz inequality, we obtain

$$\limsup_{n \rightarrow \infty} \left| \int_{\Omega} \nabla f_n \cdot \nabla g_u (1 - \chi_N) \right| \leq \left( \int_{\Omega} |\nabla f_n|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla g_u|^2 (1 - \chi_N) \right)^{\frac{1}{2}}.$$

Again we use Lemma 16, which states that  $\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla f_n|^2 = \int_{\Omega} |\nabla g_u|^2$ , to obtain

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \left| \int_{\Omega} \nabla f_n \cdot \nabla g_u - \int_{\Omega} |\nabla g_u|^2 \right| &\leq \left( \int_{\Omega} |\nabla g_u|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla g_u|^2 (1 - \chi_N) \right)^{\frac{1}{2}} \\ &\quad + \int_{\Omega} |\nabla g_u|^2 (1 - \chi_N) \\ &\leq 2 \left( \int_{\Omega} |\nabla g_u|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla g_u|^2 (1 - \chi_N) \right)^{\frac{1}{2}} \\ &\leq \varepsilon. \end{aligned}$$

Since  $\varepsilon$  is arbitrary, the proof is complete.  $\square$

At several points in the sequel, it will be necessary to obtain a uniform bound on the trace of an *SBV* function on some circle  $C \subset \Omega$ . In all cases the circle will be disjoint from the closure of the jump set of the function, and the function under consideration will be at least Lipschitz continuous on  $C$ . The following proposition is essentially a Sobolev inequality and will provide us with such a bound. Its proof is elementary, and we include it for sake of completeness.

PROPOSITION 18. *If  $u \in C^{0,1}(\partial B_\rho)$ ,  $\pi\rho \geq 2\beta^{-\frac{1}{2}}$ , and  $\beta \int_{\partial B_\rho} u^2 d\mathcal{H}^1 + \int_{\partial B_\rho} \dot{u}^2 d\mathcal{H}^1 \leq p$ , then*

$$(17) \quad \max_{y \in \partial B_\rho} |u(y)| \leq (2p^2\beta^{-1})^{-\frac{1}{4}}.$$

PROOF. Let  $\bar{u} = \max_{y \in \partial B_\rho(x)} |u(y)|$ . Since  $\beta \int_{\partial B_\rho} u^2 d\mathcal{H}^1 \leq p$ , it follows that

$$\beta \left(\frac{\bar{u}}{2}\right)^2 \cdot \mathcal{H}^1 \left\{ y \in B_\rho(x) : |u| > \frac{\bar{u}}{2} \right\} \leq p.$$

If  $|u(x)| \geq \frac{\bar{u}}{2}$  for all  $x \in \partial B_\rho$ , then we get  $\beta \left(\frac{\bar{u}}{2}\right)^2 2\pi\rho \leq p$ . Assuming  $\pi\rho \geq 2\beta^{-\frac{1}{2}}$ , we obtain (17). Now, if there is an  $x \in \partial B_\rho$  such that  $|u(x)| \leq \frac{\bar{u}}{2}$ , then it follows that

$$p \geq \int_{\partial B_\rho(x)} \dot{u}^2 d\mathcal{H}^1 \geq 8 \left(\frac{\bar{u}}{2}\right)^2 \left( \mathcal{H}^1 \left\{ y \in B_\rho(x) : |u| > \frac{\bar{u}}{2} \right\} \right)^{-1} \geq 8 \left(\frac{\bar{u}}{2}\right)^4 \beta p^{-1}$$

and again (17) is fulfilled.  $\square$

The next lemma will give us some control over the measure  $Jf$ . We show that the jump height of  $f$  at points of  $\mathcal{S}_f$  positively separated from  $\mathcal{S}_{g_u}$  (assuming they exist) must tend to zero as  $\beta$  tends to infinity.

LEMMA 19. *Let  $K \subset \Omega$  be any compact set disjoint from  $\bar{\mathcal{S}}_{g_u} \cup \partial\Omega$  and let  $f_n, \beta_n, g_n$  be sequences such that  $\beta_n \uparrow +\infty$ ,  $g_n \in Y(\beta_n)$ , and  $f_n$  minimizes  $E(\beta_n, g_n)$ ; it then follows that*

$$\lim_{n \rightarrow +\infty} \sup_{x \in K} |f_n^+(x) - f_n^-(x)| = 0.$$

PROOF. Let  $\delta = \frac{1}{2} \text{dist}(K, (\mathcal{S}_{g_u} \cup \partial\Omega))$  and let  $L$  be the Lipschitz constant associated with  $g_u$  on  $[K]_\delta$ . Given  $\varepsilon > 0$ , let  $\rho = \min\left(\frac{1}{4}L^{-1}\varepsilon, \delta\right)$ . Define  $\eta_n = \sup_{x \in K} \left( \beta_n \int_{B_\rho(x)} e_n^2 + \int_{B_\rho(x)} |\nabla e_n|^2 \right)$  where as before  $e = f - g_u$ . Lemma 17 asserts that  $\lim_{n \rightarrow +\infty} \eta_n = 0$ ; thus, by the corollary to Lemma 16 we conclude  $\exists N$  such



that, if  $n \geq N$ , the following are all satisfied,

$$(18) \quad \sup_{x \in K} \mathcal{H}^1(B_\rho(x) \cap \mathcal{S}_{f_n}) \leq \frac{\rho}{4}$$

$$(19) \quad \beta_n^{-\frac{1}{4}} \left( \frac{8\eta_n}{\rho} \right)^{\frac{1}{2}} \leq \frac{\varepsilon}{4}$$

$$(20) \quad \sup_{x \in K} \|(g_n - g_u)\chi_{B_\rho(x)}\|_\infty \leq \frac{\varepsilon}{4}.$$

Since  $f_n$  satisfies  $\mathcal{H}^1(\overline{\mathcal{S}}_{f_n} \setminus \mathcal{S}_{f_n}) = 0$  (according to Theorem 11), we conclude that for each  $x \in K$  and  $n \geq N$  there exists  $\rho_n(x) \in \left(\frac{\rho}{2}, \rho\right)$  such that  $\overline{\mathcal{S}}_{f_n} \cap B_{\rho_n(x)}(x) = \emptyset$  and  $\beta_n \int_{\partial B_{\rho_n(x)}(x)} e_n^2 + \int_{\partial B_{\rho_n(x)}(x)} |\nabla e_n|^2 \leq \frac{4}{\rho} \eta_n$ . Now, since  $f_n \in C^1(\Omega \setminus \overline{\mathcal{S}}_{f_n})$  and  $g_u$  is a Lipschitz function on  $[K]_\delta$ , we conclude  $e_n$  is a Lipschitz function on  $\partial B_{\rho_n(x)}(x)$ . We can now apply Proposition 18 (assuming  $n$  is large enough so that  $\pi\rho \geq 2\beta_n^{-\frac{1}{2}}$ ) and inequality (19) to obtain

$$(21) \quad \sup_{x \in K} \max_{y \in \partial B_{\rho_n(x)}(x)} |e_n(y)| \leq \frac{\varepsilon}{4}.$$

Let

$$t_n(x) = \max \left( \sup_{y \in B_{\rho_n(x)}(x)} g_n(y), \max_{y \in \partial B_{\rho_n(x)}(x)} f_n(y) \right).$$

From inequalities (20) and (21), we obtain

$$t_n(x) \leq \sup_{y \in B_{\rho_n(x)}(x)} g_u(y) + \frac{\varepsilon}{4}.$$

Suppose now that, for some  $y \in B_{\rho_n(x)}(x) \setminus \overline{\mathcal{S}}_{f_n}$ , we have  $f_n(y) > t_n(x)$ . Define

$$\tilde{f}_n(y) = \begin{cases} f_n(y) & y \in \Omega \setminus B_{\rho_n(x)}(x) \\ f_n(y) \vee t_n(x) & y \in \overline{B}_{\rho_n(x)}(x). \end{cases}$$

It follows that  $|\nabla \tilde{f}_n| \leq |\nabla f_n|$  almost everywhere and  $\mathcal{H}^1(\mathcal{S}_{\tilde{f}_n}) \leq \mathcal{H}^1(\mathcal{S}_{f_n})$ .

However, since  $f_n \in C^1(\Omega \setminus \overline{\mathcal{S}}_{f_n})$ , it is also true that  $\int_{\Omega} (\tilde{f}_n - g_n)^2 < \int_{\Omega} (f_n - g_n)^2$  which contradicts  $f_n$  being a minimizer of  $E$ . We conclude that  $f_n(y) \leq t_n$  for all  $y \in B_{\rho_n(x)}(x)$ ; thus,

$$\sup_{y \in B_{\rho_n(x)}(x)} f_n(y) \leq \sup_{y \in B_{\rho_n(x)}(x)} g_u(y) + \frac{\varepsilon}{4}.$$

Using a similar argument, it can be shown that

$$\inf_{y \in B_{\rho_n(x)}(x)} f_n(y) \geq \inf_{y \in B_{\rho_n(x)}(x)} g_u(y) - \frac{\varepsilon}{4},$$

and from this we conclude

$$\begin{aligned} \sup_{y \in B_{\rho_n(x)}(x)} (f_n^+(y) - f_n^-(y)) &\leq \sup_{y \in B_{\rho_n(x)}(x)} g_u(y) - \inf_{y \in B_{\rho_n(x)}(x)} g_u(y) + \frac{\varepsilon}{2} \\ &\leq 2Lr + \frac{\varepsilon}{2} \\ &\leq \varepsilon. \end{aligned}$$

The lemma now follows from the arbitrariness of  $\varepsilon$ .  $\square$

We now introduce some further notation. For  $\bar{B}_\rho(x) \in \Omega$  and  $u \in SBV(\Omega)$ , we define

$$J(u, \beta, \rho, x) = \beta \int_{B_\rho(x)} u^2 + \int_{B_\rho(x)} |\nabla u|^2 + \mathcal{H}^1(\bar{S}_u \cap B_\rho(x))$$

and, wherever it exists,

$$J'(u, \beta, \rho, x) = \beta \int_{\partial B_\rho(x)} u^2 d\mathcal{H}^1 + \int_{\partial B_\rho(x)} |\nabla u|^2 d\mathcal{H}^1 + \mathcal{H}^0(\bar{S}_u \cap \partial B_\rho(x)).$$

The proposition to follow provides us with a means of determining a  $\rho$  for a given  $x \in \Omega$  at which the ratio  $\frac{J'(u, \beta, \rho, x)}{J(u, \beta, \rho_1, x)}$  can be bounded. This will be important when we do surgery on minimizers of  $E$ .

**PROPOSITION 20.** *Suppose we are given  $u \in SBV(\bar{B}_{\rho_2}(x))$ . Let  $0 < \rho_1 < \rho_2$  and assume  $J(u, \beta, \rho_1, x) > 0$ . Then there exists  $\rho \in (\rho_1, \rho_2)$  such that*

$$J'(u, \beta, \rho, x) \leq 2(\rho_2 - \rho_1)^{-1} \log \left( \frac{J(u, \beta, \rho_2, x)}{J(u, \beta, \rho_1, x)} \right) J(u, \beta, \rho, x).$$

**PROOF.** Define

$$t = \inf_{\rho \in (\rho_1, \rho_2)} \frac{J'(u, \beta, \rho, x)}{J(u, \beta, \rho_1, x)}$$

and

$$\hat{J}(\rho) = J(u, \beta, \rho, x) + \int_{\rho_1}^{\rho} J'(u, \beta, r, x) d\mathcal{H}^1(r).$$

$\hat{J}(\rho)$  is a nondecreasing absolutely continuous function of  $\rho$  and  $\frac{\partial}{\partial \rho} \hat{J}(\rho) =$

$J'(u, \beta, \rho, x)$  for almost all  $\rho \in (\rho_1, \rho_2)$ . Thus for almost all  $\rho \in (\rho_1, \rho_2)$  we have  $\frac{\partial}{\partial \rho} \hat{J}(\rho) \geq tJ(u, \beta, \rho, x)$ . Later we will establish the relation

$$(22) \quad J(u, \beta, \rho, x) \geq \hat{J}(\rho),$$

but first we show how this implies the desired result. For almost all  $\rho \in (\rho_1, \rho_2)$ , we have

$$\frac{\partial}{\partial \rho} (e^{-(\rho-\rho_1)t} \hat{J}(\rho)) = e^{-(\rho-\rho_1)t} \frac{\partial}{\partial \rho} \hat{J}(\rho) - t (e^{-(\rho-\rho_1)t} \hat{J}(\rho)) \leq 0$$

by the definition of  $t$ . Thus we obtain  $\hat{J}(\rho_2) \geq \hat{J}(\rho_1) \exp(\rho_2 - \rho_1)t$ . Note  $\hat{J}(\rho_1) = J(u, \beta, \rho_1, x)$ ; then, by using equation (22), one obtains  $\hat{J}(\rho_2) \leq J(u, \beta, \rho_2, x)$ .

The lemma now follows by choosing  $\rho$  such that  $\frac{J'(u, \beta, \rho, x)}{J(u, \beta, \rho_1, x)} < 2t$ .

To prove inequality (22), we first note that

$$\begin{aligned} \beta \int_{B_\rho(x)} u^2 + \int_{B_\rho(x)} |\nabla u|^2 &= \int_{\rho_1}^{\rho} \left[ \beta \int_{\partial B_\rho(x)} u^2 d\mathcal{H}^1 + \int_{\partial B_\rho(x)} |\nabla u|^2 d\mathcal{H}^1 \right] d\mathcal{H}^1 \\ &\quad + \beta \int_{B_{\rho_1}(x)} u^2 + \int_{B_{\rho_1}(x)} |\nabla u|^2 \end{aligned}$$

by the Tonelli-Fubini theorem. Thus, all we need establish is

$$(23) \quad \mathcal{H}^1(\bar{\mathcal{S}}_u \cap (B_\rho(x) \setminus B_{\rho_1}(x))) \leq \int_{\rho_1}^{\rho} \mathcal{H}^0(\bar{\mathcal{S}}_u \cap \partial B_r(x)) dr.$$

To simplify the notation, we will assume  $x$  is the origin, and denote  $\bar{\mathcal{S}}_u \cap (B_\rho \setminus B_{\rho_1})$  by  $\Gamma$ . For any  $\varepsilon, \delta > 0$ , we can find a collection of sets  $\{U_i\}$  such that  $\Gamma \subset \bigcup_i U_i$ ,  $\text{diam}(U_i) < \delta$ , and  $\mathcal{H}_\delta^1(\Gamma) + \varepsilon \geq \sum_i \text{diam}(U_i)$ . Let  $\chi_{i,r}$  be the indicator function of the condition  $U_i \cap \partial B_r \neq \emptyset$ . It follows by definition that  $\mathcal{H}_\delta^0(\Gamma \cap \partial B_r) \leq \sum_i \chi_{i,r}$ . Now  $\int_{\rho_1}^{\rho} \chi_{i,r} dr \leq \text{diam}(U_i)$ ; hence,

$$\int_{\rho_1}^{\rho} \mathcal{H}_\delta^0(\Gamma \cap \partial B_r) \leq \sum_i \text{diam}(U_i) \leq \mathcal{H}_\delta^1(\Gamma) + \varepsilon \leq \mathcal{H}^1(\Gamma) + \varepsilon.$$

Since  $\varepsilon$  is arbitrary, we have in fact

$$\int_{\rho_1}^{\rho} \mathcal{H}_\delta^0(\Gamma \cap \partial B_r) \leq \mathcal{H}^1(\Gamma).$$

Now, consider any sequence  $\delta_n \downarrow 0$ . For each  $r$ , the sequence  $\mathcal{H}_{\delta_n}^0(\Gamma \cap \partial B_r)$  is monotonically increasing to  $\mathcal{H}^0(\Gamma \cap \partial B_r)$ ; thus, by the monotone convergence theorem we have

$$\lim_{n \rightarrow \infty} \int_{\rho_1}^{\rho} \mathcal{H}_{\delta_n}^0(\Gamma \cap \partial B_r) \, dr = \int_{\rho_1}^{\rho} \mathcal{H}^0(\Gamma \cap \partial B_r) \, dr$$

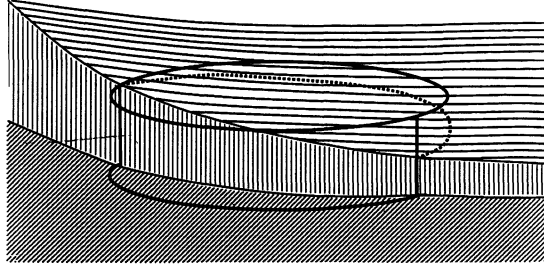
which completes the proof. □

In order to get some bounds on the contribution to  $E$  occurring in certain subsets of  $\Omega$ , we will redefine  $f$  in various balls in  $\Omega$ . To facilitate this, we will introduce some more notation.

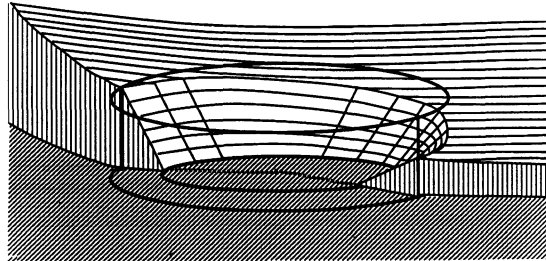
Suppose  $u \in SBV(\Omega)$  and  $\bar{B}_\rho(x) \subset \Omega$ . We will introduce polar coordinates  $r, \theta$  centered at  $x$ . For  $0 < \rho' < \rho$ , we define

$$(24) \quad \begin{aligned} \Phi(\rho, \rho', r, \theta) &= 1 \wedge \left( \frac{r - \rho'}{\rho - \rho'} \vee 0 \right) \\ \hat{u}(\rho, r, \theta) &= \begin{cases} u(r, \theta) & (r, \theta) \in \Omega \setminus \bar{B}_\rho \\ u(\rho, \theta) & \text{otherwise} \end{cases} \\ \tilde{u}(\rho, \rho', r, \theta) &= \Phi(\rho, \rho', r, \theta) \hat{u}(\rho, r, \theta). \end{aligned}$$

Figure 2 illustrates this definition.



$u$ , and the ball  $B_\rho$ ,



$\hat{u}$ , and the balls  $B_\rho$  and  $B_{\rho'}$

Figure 2 - Illustration of construction of  $\hat{u}$  from  $u$

LEMMA 21. Let  $B_\rho(x) \subset\subset \Omega$  and let  $u \in SBV(\Omega)$  satisfy  $u \in C^{0,1}(\Omega \setminus \overline{S}_u)$ . Then, with  $\tilde{u}$  defined as above, we have

$$J(\tilde{u}, \beta, \rho, x) \leq ([\beta(\rho - \rho')]^{-1} + (\rho - \rho'))J'(u, \beta, \rho, x).$$

PROOF. The following inequality is easily derived

$$\int_{B_\rho(x)} \Phi^2 \hat{u}^2 \leq \frac{1}{2}(\rho - \rho') \int_{\partial B_\rho(x)} u^2 d\mathcal{H}^1.$$

Note that  $\nabla \hat{u} \cdot \nabla \Phi = 0$ , so

$$\int_{B_\rho(x)} |\nabla(\Phi \hat{u})|^2 = \int_{B_\rho(x)} \Phi^2 |\nabla \hat{u}|^2 + \int_{B_\rho(x)} |\nabla \Phi|^2 \hat{u}^2.$$

Some straightforward algebra now verifies

$$\int_{B_\rho(x)} |\nabla(\Phi \hat{u})|^2 \leq \frac{1}{2}(\rho - \rho') \int_{\partial B_\rho(x)} |\nabla u|^2 d\mathcal{H}^1 + \frac{1}{\rho - \rho'} \int_{\partial B_\rho(x)} u^2 d\mathcal{H}^1.$$

Finally, because of the regularity assumption on  $u$ , it follows that, if  $u_1$  is the restriction of  $u$  to  $\partial B_\rho(x)$ , then, as a member of  $SBV(\partial B_\rho(x))$ , the function  $u_1$  satisfies  $S_{u_1} \subset \overline{S}_u$ . Thus, we obtain

$$\mathcal{H}^1(S_{\tilde{u}} \cap B_\rho(x)) \leq (\rho - \rho')\mathcal{H}^0(\partial B_\rho(x) \cap \overline{S}_u).$$

Together, these inequalities constitute the proof of the lemma.  $\square$

We will construct smoothed versions of  $g_u$ . Wherever  $x \notin \overline{S}_f$ , we have  $\Delta f = \beta(f - g)$ ; thus, roughly speaking,  $f$  is a smoothed version of  $g$  where the support of the smoothing occurs over a region of radius  $\beta^{-\frac{1}{2}}$ . We will compare the optimal  $f$  to such a smoothed version of  $g_u$ ; however, it is more convenient to use a mollifier to do the smoothing than to consider the solution to a p.d.e..

Let  $\eta \in C_0^\infty(B_1)$  be a positive, symmetric function satisfying  $\int_{B_1} \eta = 1$ . If  $x \in \Omega$

and  $0 < \beta^{-\frac{1}{2}} < \text{dist}(x, \partial\Omega)$ , then we can define

$$g_s(x) = \int_{B_{\beta^{-\frac{1}{2}}}(x)} \beta \eta\left(\beta^{\frac{1}{2}}(x - y)\right) g_u(y) dy.$$

By definition,  $\eta$  is uniformly continuous. We will denote the modulus of continuity of  $\eta$  by  $c_\eta$ ; i.e.,  $|\eta(x) - \eta(y)| \leq c_\eta|x - y|$  for all  $x, y \in \mathbb{R}^2$ .

LEMMA 22. Let  $g_u \in SBV(\Omega)$  satisfy our assumptions (Assumption 2 in particular) and  $K \subset \Omega$  be a compact set such that  $K \cap \overline{S}_{g_u} = \emptyset$ . Define  $\delta = \frac{1}{2} \text{dist}(K, S_{g_u} \cup \partial\Omega)$  and denote by  $L$  the Lipschitz constant associated with  $[K]_\delta$ . If  $\beta^{-\frac{1}{2}} < \delta$ , then the following estimates hold:

$$\begin{aligned} \sup_{y \in K} |g_s(y) - g_u(y)| &\leq L\beta^{-\frac{1}{2}} \\ \sup_{y \in K} |\nabla g_s(y)| &\leq L \\ \sup_{y \in K} |\Delta g_s(y)| &\leq \sqrt{2}\pi c_\eta L\beta^{\frac{1}{2}}. \end{aligned}$$

PROOF. Let  $x \in K$ , then

$$\begin{aligned} |g_s(x) - g_u(x)| &\leq \left| \int_{B_{\beta^{-\frac{1}{2}}}(x)} \beta\eta\left(\beta^{\frac{1}{2}}(x-y)\right) g_u(y) \, dy - g_u(x) \right| \\ &\leq \left| \int_{B_{\beta^{-\frac{1}{2}}}(x)} \beta\eta\left(\beta^{\frac{1}{2}}(x-y)\right) (g_u(y) - g_u(x)) \, dy \right| \\ &\leq \left| \int_{B_{\beta^{-\frac{1}{2}}}(x)} \beta^{\frac{1}{2}}\eta\left(\beta^{\frac{1}{2}}(x-y)\right) L \, dy \right| \leq L\beta^{-\frac{1}{2}} \end{aligned}$$

proving the first statement.

Since  $g_u$  is Lipschitz in  $[K]_\delta$ ,  $\nabla g_u$  exists almost everywhere (in the strong sense) on  $[K]_\delta$  and satisfies  $|\nabla g_u| \leq L$ . Now

$$\begin{aligned} \nabla g_s(x) &= \beta \int_{B_{\beta^{-\frac{1}{2}}}(x)} \nabla_x \eta\left(\beta^{\frac{1}{2}}(x-y)\right) g_u(y) \, dy \\ &= \beta \int_{B_{\beta^{-\frac{1}{2}}}(x)} \eta\left(\beta^{\frac{1}{2}}(x-y)\right) \nabla_y g_u(y) \, dy \end{aligned}$$

and the second statement follows.

Let  $e_1, e_2$  be the standard basis for  $\mathbb{R}^2$ . We have

$$\begin{aligned} &\left| \frac{\partial}{\partial x_1} g_s(x + \varepsilon e_1) - \frac{\partial}{\partial x_1} g_s(x) \right| \\ &= \beta \int_{B_{\beta^{-\frac{1}{2}}}(x)} \left( \eta\left(\beta^{\frac{1}{2}}(x + \varepsilon e_1 - y)\right) - \eta\left(\beta^{\frac{1}{2}}(x - y)\right) \right) \frac{\partial}{\partial y_1} g_u(y) \, dy \\ &\leq \beta^{\frac{3}{2}} c_\eta \varepsilon \int_{B_{\beta^{-\frac{1}{2}}}(x)} \frac{\partial}{\partial y_1} g_u(y) \, dy. \end{aligned}$$

Thus we get

$$|\Delta g_s(x)| \leq \beta^{\frac{3}{2}} c_\eta \int_{B_{\beta^{-\frac{1}{2}}}(x)} \left| \frac{\partial}{\partial y_1} g_u(y) \right| + \left| \frac{\partial}{\partial y_2} g_u(y) \right| dy \leq \sqrt{2\pi} \beta^{\frac{1}{2}} c_\eta L.$$

This completes the proof of the lemma.  $\square$

### 3.3. Main results

We are now ready to establish the most important estimate in the proof. One important consequence of the following lemma is that it shows that, if there remain small pieces of the boundary  $\mathcal{S}_f$  disjoint from  $[\mathcal{S}_{g_u}]_\varepsilon$ , then they are sparsely placed.

LEMMA 23. *Let  $g_u \in SBV(\Omega)$  satisfy our assumptions and  $K \subset \Omega$  be a compact set such that  $K \cap \bar{\mathcal{S}}_{g_u} = \emptyset$ . Define  $\delta = \frac{1}{2} \text{dist}(K, \mathcal{S}_{g_u} \cup \partial\Omega)$ , denote by  $L$  the Lipschitz constant associated with  $[K]_\delta$ , and set  $c = \pi(8(1 + L(1 + \sqrt{2\pi}c_\eta)))^2$ . Given  $0 < \gamma < \frac{1}{2}$ , there exists a constant  $\beta' < \infty$  such that, if  $\beta \geq \beta'$  and  $f$  minimizes  $E(\beta, g)$  for some  $g \in Y(\beta)$ , then*

$$\sup_{x \in K} J(f - g_s, \beta, \beta^{-\gamma}, x) \leq c\beta^{-2\gamma}.$$

PROOF. Assume the lemma is false. There exist a  $K$  and  $\gamma$  satisfying the conditions of the lemma and a sequence of quadruples  $\{(g_n, f_n, \beta_n, x_n)\}$  such that  $\beta_n \uparrow +\infty$ ,  $g_n \in Y(\beta_n)$ ,  $x_n \in K$ ,  $f_n$  minimizes  $E(\beta_n, g_n)$ , and

$$(25) \quad J(w_n, \beta_n, \beta_n^{-\gamma}, x_n) > c\beta_n^{-2\gamma},$$

for each  $n$ , where we have used the notation  $w_n = f_n - g_s$ . Note that since  $\beta$  depends on  $n$ , so does  $g_s$ .

Without loss of generality, we assume that  $\beta_n \geq 1$  and  $2\beta_n^{-\gamma} + \beta_n^{-\frac{1}{2}} < \delta$ , so, by Lemma 22, the following estimates hold

$$(26) \quad \sup_{y \in [K]_{2\beta_n^{-\gamma}}} |g_s(y) - g_u(y)| \leq L\beta_n^{-\frac{1}{2}}$$

$$(27) \quad \sup_{y \in [K]_{2\beta_n^{-\gamma}}} |\nabla g_s(y)| \leq L$$

$$(28) \quad \sup_{y \in [K]_{2\beta_n^{-\gamma}}} |\Delta g_s(y)| \leq \sqrt{2\pi} c_\eta L \beta_n^{\frac{1}{2}}.$$

Defining  $e = f - g_u$  as before, and applying the estimate given above along with the triangle inequality, we obtain

$$\begin{aligned}
J(w_n, \beta_n, 2\beta_n^{-\gamma}, x_n) &\leq 2 \left\{ \beta \int_{B_{2\beta_n^{-\gamma}}(x_n)} e_n^2 + \int_{B_{2\beta_n^{-\gamma}}(x_n)} |\nabla e_n|^2 \right. \\
&\quad \left. + \beta \int_{B_{2\beta_n^{-\gamma}}(x_n)} (g_u - g_s)^2 + \int_{B_{2\beta_n^{-\gamma}}(x_n)} |\nabla(g_u - g_s)|^2 \right\} \\
&\quad + \mathcal{H}^1(\mathcal{S}_{f_n} \cap B_{2\beta_n^{-\gamma}}(x_n)) \\
&\leq 2 \left( \beta \int_{B_{2\beta_n^{-\gamma}}(x_n)} e_n^2 + \int_{B_{2\beta_n^{-\gamma}}(x_n)} |\nabla e_n|^2 \right) + 24L^2\pi\beta_n^{-2\gamma} \\
&\quad + \mathcal{H}^1(\mathcal{S}_{f_n} \cap B_{2\beta_n^{-\gamma}}(x_n)).
\end{aligned}$$

By Lemma 17 and the corollary to Lemma 16, we can assert that for  $n$  sufficiently large

$$(29) \quad J(w_n, \beta_n, 2\beta_n^{-\gamma}, x_n) \leq 1.$$

From this we can conclude, by Proposition 20, that for all such  $n$  there exists  $\rho_n \in (\beta_n^{-\gamma}, 2\beta_n^{-\gamma})$  such that

$$J'(w_n, \beta_n, \rho_n, x_n) \leq 2\beta_n^\gamma \log \frac{\beta_n^{2\gamma}}{c} J(w_n, \beta_n, \rho_n, x_n).$$

Let  $N_1$  be such that, if  $n \geq N_1$ , then equation (29) holds and, also,  $\beta_n^\gamma \log \frac{\beta_n^{2\gamma}}{c} < \frac{\beta_n^{\frac{1}{2}}}{32}$ . We now have, for  $n \geq N_1$ ,

$$(30) \quad J'(w_n, \beta_n, \rho_n, x_n) \leq \beta_n^{\frac{1}{2}} \frac{1}{16} J(w_n, \beta_n, \rho_n, x_n).$$

Let us define  $\tilde{w}_n$  as in (24) with  $\rho'_n = \rho_n - \beta_n^{-\frac{1}{2}}$  with the balls centered at  $x_n$ ; i.e., we introduce polar coordinates  $r, \theta$  centered at  $x_n$  and set

$$\begin{aligned}
\Phi(\rho_n, \rho'_n, r, \theta) &= 1 \wedge \left( \frac{r - \rho'_n}{\rho_n - \rho'_n} \vee 0 \right) \\
\hat{w}_n(\rho_n, r, \theta) &= \begin{cases} w_n(r, \theta) & (r, \theta) \in \Omega \setminus \bar{B}_{\rho_n}(x_n) \\ w_n(\rho_n, \theta) & \text{otherwise} \end{cases} \\
\tilde{w}_n(\rho_n, \rho'_n, r, \theta) &= \Phi(\rho_n, \rho'_n, r, \theta) \hat{w}_n(\rho_n, r, \theta).
\end{aligned}$$



From Lemma 21, we obtain

$$J(\tilde{w}_n, \beta_n, \rho_n, x_n) \leq 2\beta_n^{-\frac{1}{2}} J^1(w_n, \beta_n, \rho_n, x_n);$$

applying (30), we derive

$$(31) \quad J(\tilde{w}_n, \beta_n, \rho_n, x_n) \leq \frac{1}{8} J(w_n, \beta_n, \rho_n, x_n).$$

Let  $\tilde{f}_n = g_s + \tilde{w}_n$ . Note that in  $\Omega \setminus B_{\rho_n}(x_n)$  we have  $\tilde{f}_n = f_n$ . Since  $f_n$  is a minimizer of  $E(f, \beta_n)$  we have  $E(f_n, \beta_n) \leq E(\tilde{f}_n, \beta_n)$ . We can express this in terms of  $w_n$  as

$$\begin{aligned} J(w_n, \beta_n, \rho_n, x_n) &\leq J(\tilde{w}_n, \beta_n, \rho_n, x_n) + 2\beta_n \int_{B_{\rho_n}(x_n)} (\tilde{w}_n - w_n)(g_s - g) \\ &\quad + 2 \int_{B_{\rho_n}(x_n)} (\nabla \tilde{w}_n - \nabla w_n) \cdot \nabla g_s. \end{aligned}$$

Substituting from equation (31), we get

$$(32) \quad \begin{aligned} \frac{7}{16} J(w_n, \beta_n, \rho_n, x_n) &\leq \beta_n \int_{B_{\rho_n}(x_n)} (\tilde{w}_n - w_n)(g_s - g) \\ &\quad + \int_{B_{\rho_n}(x_n)} (\nabla \tilde{w}_n - \nabla w_n) \cdot \nabla g_s. \end{aligned}$$

Note that equations (30) and (29) together imply that  $S_{w_n - \tilde{w}_n} \cap \partial B_{\rho_n}(x_n)$  has at most finitely many points; thus, we can apply the general result for *SBV* functions, given in equation (7), to get

$$(33) \quad \begin{aligned} &\int_{B_{\rho_n}(x_n)} (\nabla w_n - \nabla \tilde{w}_n) \cdot \nabla g_s \\ &= \int_{B_{\rho_n}(x_n)} D(w_n - \tilde{w}_n) \cdot \nabla g_s - \int_{B_{\rho_n}(x_n)} J(w_n - \tilde{w}_n) \cdot \nabla g_s \\ &= \int_{B_{\rho_n}(x_n)} (w_n - \tilde{w}_n) \Delta g_s - \int_{B_{\rho_n}(x_n)} J(w_n - \tilde{w}_n) \cdot \nabla g_s \end{aligned}$$

where we have used the notation

$$\int_{B_\rho} J u \cdot \phi = \int_{B_\rho \cap S_u} (u^+ - u^-) \phi \nu_u \, d\mathcal{H}^1.$$

It is clear that  $\sup_{x \in B_{\rho_n}(x_n)} (\tilde{w}_n^+(x) - \tilde{w}_n^-(x)) \leq \sup_{x \in B_{\rho_n}(x_n)} (f_n^+(x) - f_n^-(x))$ . From Lemma 19, we conclude that there exists  $N_2$  sufficiently large so that, if  $n \geq N_2$ , then  $\sup_{x \in B_{\rho_n}(x_n)} (f_n^+(x) - f_n^-(x)) \leq \frac{1}{18L}$ . Recall that  $|\nabla g_s| \leq L$  for all  $x \in [K]_{\rho_n}$ ; one now obtains

$$(34) \quad \left| \int_{B_{\rho_n}(x_n)} J w_n \cdot \nabla g_s \right| \leq \frac{1}{18} \mathcal{H}^1(\mathcal{S}_{f_n} \cap B_{\rho_n}(x_n)) \leq \frac{1}{18} J(w_n, \beta_n, \rho_n, x_n)$$

and

$$(35) \quad \left| \int_{B_{\rho_n}(x_n)} J \tilde{w}_n \cdot \nabla g_s \right| \leq \frac{1}{18} \mathcal{H}^1(\mathcal{S}_{\tilde{w}_n} \cap B_{\rho_n}(x_n)) \leq \frac{1}{18} J(\tilde{w}_n, \beta_n, \rho_n, x_n).$$

Let  $N = \max(N_1, N_2)$  and assume henceforth that  $n \geq N$ . Combining equations (35) and (34) and substituting from equation (31), we obtain

$$\left| \int_{B_{\rho_n}(x_n)} J(w_n - \tilde{w}_n) \cdot \nabla g_s \right| \leq \frac{1}{16} J(w_n, \beta_n, B_{\rho_n}, x_n).$$

Substituting this into equation (33) and in turn substituting the result into equation (32), we obtain

$$(36) \quad \frac{3}{8} J(w_n, \beta_n, \rho_n, x_n) \leq \int_{B_{\rho_n}(x_n)} (\tilde{w}_n - w_n)(\beta_n(g_s - g) + \Delta g_s).$$

Since

$$\lim_{n \rightarrow \infty} \beta_n^{\frac{1}{2}} \sup_{y \in [K]_{2\beta_n^{-\gamma}}} |g_u(y) - g(y)| = 0,$$

we can assume that  $N$  is sufficiently large so that, for  $n \geq N$ ,

$$\sup_{y \in [K]_{2\beta_n^{-\gamma}}} |g_u(y) - g(y)| \leq \beta_n^{\frac{1}{2}}.$$

From this and the estimates given in equations (26) and (28), we now have

$$(37) \quad \sup_{x \in B_{\rho_n}(x_n)} |\beta_n(g_s - g) + \Delta g_s| \leq \beta_n^{\frac{1}{2}} \frac{1}{8} \sqrt{\frac{c}{\pi}}.$$

Using the Schwartz inequality, we can derive

$$(38) \quad \begin{aligned} \int_{B_{\rho_n}(x_n)} |w_n| &\leq \beta_n^{\frac{1}{2}} \sqrt{\pi \rho_n^2} \left( \beta_n \int_{B_{\rho_n}(x_n)} w_n^2 \right)^{\frac{1}{2}} \\ &\leq \beta_n^{\frac{1}{2}} \sqrt{\pi \rho_n} (J(w_n, \beta_n, \rho_n, x_n))^{\frac{1}{2}}. \end{aligned}$$

Similarly,

$$\begin{aligned}
 \int_{B_{\rho_n}(x_n)} |\tilde{w}_n| &\leq \beta_n^{\frac{1}{2}} \int_{\partial B_{\rho_n}(x_n)} |\tilde{w}_n| \, d\mathcal{H}^1 \\
 (39) \qquad &\leq \beta_n^{-1} \sqrt{2\pi\rho_n} \left( \beta_n \int_{\partial B_{\rho_n}(x_n)} w_n^2 \, d\mathcal{H}^1 \right)^{\frac{1}{2}} \\
 &\leq \beta_n^{-1} \sqrt{2\pi\rho_n} (J^1(w_n, \beta_n, \rho_n, x_n))^{\frac{1}{2}} \\
 &\leq \beta_n^{\frac{3}{4}} \sqrt{2\pi\rho_n} \left( \frac{1}{16} J(w_n, \beta_n, \rho_n, x_n) \right)^{\frac{1}{2}}
 \end{aligned}$$

where in the last step we have used equation (30). Combining equations (37), (38), and (39) and substituting into equation (36), we obtain

$$\frac{3}{8} J(w_n, \beta_n, \rho_n, x_n) \leq \frac{1}{8} \sqrt{c\rho_n} \left( 1 + \frac{\sqrt{2}\beta_n^{-\frac{1}{4}}}{4\sqrt{\rho_n}} \right) (J(w_n, \beta_n, \rho_n, x_n))^{\frac{1}{2}}.$$

Now, since  $\rho_n \geq \beta_n^{-\gamma}$ ,  $\gamma < \frac{1}{2}$ , and  $\beta_n \geq 1$ , it follows that  $\left( 1 + \frac{\sqrt{2}\beta_n^{-\frac{1}{4}}}{4\sqrt{\rho_n}} \right) < \frac{3}{2}$ ,

and we now obtain

$$\frac{1}{4} J(w_n, \beta_n, \rho_n, x_n) < \frac{1}{8} \sqrt{c\rho_n} (J(w_n, \beta_n, \rho_n, x_n))^{\frac{1}{2}}.$$

Noting  $\rho_n \leq 2\beta_n^{-\gamma}$ , we conclude

$$J(w_n, \beta_n, \rho_n, x_n) < c\beta_n^{-2\gamma}$$

which contradicts equation (25).  $\square$

We are now ready to demonstrate that the conditions required to prove  $x \notin \mathcal{S}_f$  can be met simultaneously for each  $x \in K \subset \Omega \setminus \bar{\mathcal{S}}_{g_u}$  when  $\beta$  is sufficiently large. We first recall some notation and some important results on *SBV* functions quoted in Section 2. Let  $u \in SBV(\Omega)$ . For every compact set  $K \subset \Omega$ , we set

$$F(u, K) = \int_K |\nabla u|^2 + \mathcal{H}^1(\mathcal{S}_u \cap K)$$

and

$$\Phi(u, K) = \inf\{F(v, K) : v \in SBV(\Omega), v = u \in \Omega \setminus K\}.$$

The deviation from minimality is defined as

$$\Psi(u, K) = F(u, K) - \Phi(u, K).$$

LEMMA 24. *There exist universal constants  $\xi, \eta > 0$  such that, if  $u \in SBV(\Omega)$ ,  $B_\rho(x) \subset\subset \Omega$  for some  $\rho > 0$ , and each of the following three conditions hold:*

$$(40) \quad F(u, \overline{B}_\rho(x)) \leq \xi\rho,$$

$$(41) \quad \lim_{t \rightarrow 0^+} t^{-1} \Psi(u, \overline{B}_t(x)) = 0,$$

$$(42) \quad \Psi(u, \overline{B}_t(x)) \leq \eta t, \text{ for all } t \leq \rho;$$

then  $x \notin \mathcal{S}_u$ .

PROOF. This is just a combination of Theorems 2 and 3.  $\square$

LEMMA 25. *Let  $g_u \in SBV(\Omega)$  satisfy our assumptions and  $K \subset \Omega$  be a compact set such that  $K \cap \overline{\mathcal{S}}_g = \emptyset$ . There exists a constant  $\beta' < \infty$  such that, if  $\beta \geq \beta'$  and  $f$  is a minimizer of  $E(\cdot, \beta)$  with  $g \in Y(\beta)$ , then*

$$\mathcal{S}_f \cap K = \emptyset.$$

PROOF. Assume the lemma is false. Then there exists a  $K$  satisfying the conditions of the lemma and a sequence of quadruples  $\{(g_n, f_n, \beta_n, x_n)\}$  such that  $\beta_n \uparrow +\infty$ ,  $g_n \in Y(\beta)$ ,  $f_n$  minimizes  $E(\beta_n, g_n)$ , and  $x_n \in K \cap \mathcal{S}_{f_n}$ . Define  $\delta = \frac{1}{2} \text{dist}(K, \mathcal{S}_{g_n} \cup \partial\Omega)$ , denote by  $L$  the Lipschitz constant associated with  $[K]_\delta$ , and define  $c$  as in Lemma 23. Fix any real  $\gamma$  satisfying  $\frac{1}{4} < \gamma < \frac{1}{2}$ . By Lemma 23, we can assume

$$(43) \quad J(w_n, \beta_n, \beta_n^{-\gamma}, x_n) < c\beta_n^{-2\gamma}$$

for each  $n$ , where we have again used the notation  $w_n = f_n - g_n$ . Furthermore, for convenience we make the assumption  $c\beta_n^{-\gamma} < \frac{1}{4}$ . Define  $R = \left\{ \rho \in \left[ \frac{1}{2}\beta_n^{-\gamma}, \beta_n^{-\gamma} \right] : \overline{\mathcal{S}}_{f_n} \cap \partial B_\rho(x_n) = \emptyset \right\}$ . From equation (43), we have  $\mathcal{H}^1(\mathcal{S}_{f_n} \cap \beta_n^{-\gamma}) < c\beta_n^{-2\gamma}$  and thus,  $|R| > \frac{1}{2}\beta_n^{-\gamma} - c\beta_n^{-2\gamma} > \frac{1}{4}\beta_n^{-\gamma}$  by assumption. Since

$$\int_R \left[ \beta \int_{\partial B_\rho(x_n)} w_n^2 d\mathcal{H}^1 + \int_{\partial B_\rho(x_n)} |\nabla w_n|^2 d\mathcal{H}^1 \right] d\rho \leq J(w_n, \beta_n, \beta_n^{-\gamma}, x_n),$$

there exists a  $\rho_n \in \left[ \frac{1}{2}\beta_n^{-\gamma}, \beta_n^{-\gamma} \right]$  such that

$$(44) \quad \beta \int_{\partial B_\rho(x_n)} w_n^2 d\mathcal{H}^1 + \int_{\partial B_\rho(x_n)} |\nabla w_n|^2 d\mathcal{H}^1 < 4c\beta_n^{-\gamma}.$$

From the existence and regularity results for minimizers of  $E$ , we deduce  $w_n$  is  $C^1$  on  $\partial B_{\rho_n}(x_n)$ , so from Proposition 18 and equation (44) we conclude

$$\max_{y \in \partial B_{\rho_n}(x_n)} |w_n(y)| \leq \sqrt{8c} \beta_n^{-\frac{1}{4} - \frac{\gamma}{2}}.$$

Our goal in the remainder of the proof is to show that the three conditions of Lemma 24 are satisfied for  $n$  sufficiently large with  $u = f$ ,  $\rho = \rho_n$  and  $x = x_n$ , thus obtaining a contradiction with  $x_n \in \mathcal{S}_{f_n}$ . Now

$$\begin{aligned} F(f_n, \overline{B}_{\rho_n}(x_n)) &= \int_{B_{\rho_n}(x_n)} |\nabla f_n|^2 + \mathcal{H}^1(\mathcal{S}_{f_n} \cap B_{\rho_n}(x_n)) \\ &\leq 2 \left( \int_{B_{\rho_n}(x_n)} |\nabla w_n|^2 + |\nabla g_s|^2 \right) + \mathcal{H}^1(\mathcal{S}_{f_n} \cap B_{\rho_n}(x_n)) \\ &\leq 2(c + 4\pi L^2) \beta_n^{-2\gamma} \end{aligned}$$

where we have used the facts  $|\nabla g_s| \leq L$  and  $\rho_n \leq \beta^{-\gamma}$ . Condition (40) is thus satisfied as long as  $2(c + 4\pi L^2) \beta_n^{-\gamma} \leq \xi$ , which is clearly true for  $n$  sufficiently large. Consider a fixed  $n$ , let  $0 < t \leq \rho_n$ , and let  $v^t \in SBV(\Omega)$  realize  $\Phi(f_n, \overline{B}_t(x_n))$ , i.e.,  $v^t(x) = f_n(x)$  for all  $x \in \Omega \setminus \overline{B}_t(x_n)$  and  $F(v^t, \overline{B}_t(x_n)) = \Phi(f_n, \overline{B}_t(x_n))$ . Since  $f_n$  is a minimizer of  $E(\beta_n, g_n)$ , we have

$$\beta_n \int_{\overline{B}_t(x_n)} (f_n - g_n)^2 + F(f_n, \overline{B}_t(x_n)) \leq \beta_n \int_{\overline{B}_t(x_n)} (v^t - g_n)^2 + F(v^t, \overline{B}_t(x_n)).$$

Let  $\underline{g}_n$  be the infimum and  $\overline{g}_n$  the supremum of  $g_n$  in  $\overline{B}_{\rho_n}(x_n)$ . Set,

$$d_n = \sup_{y \in \partial B_{\rho_n}(x_n)} |w_n(y)| + \sup_{y \in \partial B_{\rho_n}(x_n)} |g_n(y) - g_s(y)|.$$

Using the same truncation argument as in the proof of Lemma 19, it is easy to establish

$$\forall x \in \overline{B}_{\rho_n}(x_n), f_n(x) \in \left[ \underline{g}_n - d_n, \overline{g}_n + d_n \right]$$

and essentially the same argument shows

$$\forall x \in \overline{B}_{\rho_n}(x_n), v^t(x) \in \left[ \inf_{x \in \overline{B}_{\rho_n}(x_n)} f_n(x), \sup_{x \in \overline{B}_{\rho_n}(x_n)} f_n(x) \right].$$

Thus we obtain

$$\begin{aligned} & \sup_{x \in \overline{B}_t(x_n)} |v^t - g_n| \leq \overline{g}_n - \underline{g}_n + 2d_n \\ & \leq 2 \left( \rho_n L + \sqrt{8c} \beta_n^{-\frac{1}{4} - \frac{\gamma}{2}} + L \beta_n^{-\frac{1}{2}} \right) + 4 \| (g_n - g_u) \chi_{\overline{B}_{\rho_n}(x_n)} \|_{\infty}. \end{aligned}$$

Because of the assumed equation (9) and the relations  $\gamma < \frac{1}{2}$  and  $\rho_n \leq \beta_n^{-\gamma}$ , there exists an  $N$  such that, if  $n \geq N$ , then

$$\sup_{x \in \overline{B}_{\rho_n}(x_n)} |v^t - g_n| \leq (1 + 3L) \beta_n^{-\gamma}$$

and hence

$$\Psi(f_n, \overline{B}_t(x)) \leq (1 + 3L)^2 \beta_n^{1-2\gamma} \pi t^2.$$

Condition (41) of Lemma 24 is clearly satisfied. Also,  $\Psi(f_n, \overline{B}_t(x)) \leq \gamma t$  as long as  $(1 + 3L)^2 \beta_n^{1-2\gamma} \pi t^2 < \gamma t$ , i.e., for all  $t < \frac{\gamma}{(1 + 3L)^2 \pi \beta_n^{1-2\gamma}}$ . Now, since  $\Psi \leq F$  and  $F(f_n, B_t(x_n)) \leq F(f_n, B_{\rho_n}(x_n))$  we have

$$\Psi(f_n, B_t(x_n)) \leq 2(c + \pi L^2) \beta_n^{-2\gamma},$$

and for  $t > \frac{2(c + \pi L^2) \beta_n^{-2\gamma}}{\gamma}$  we have  $\Psi(f_n, \overline{B}_t(x)) \leq \gamma t$ . Thus, equation (42) of Lemma 24 is satisfied if  $\frac{2(c + \pi L^2) \beta_n^{-2\gamma}}{\gamma} < \frac{\gamma}{(1 + 3L)^2 \pi \beta_n^{1-2\gamma}}$ . Since  $\gamma > \frac{1}{4}$ , this inequality is satisfied for  $n$  sufficiently large, and the proof is now complete. Note that had we set  $\gamma > \frac{1}{3}$ , then the first bound would have been sufficient since, for  $n$  large enough, we would have  $\rho_n \leq 2\beta_n^{-\gamma} \leq \frac{\gamma}{(1 + 3L)^2 \pi \beta_n^{1-2\gamma}}$ .  $\square$

Finally, we are ready to state the theorem to which the previous effort has been directed. Lemma 25 almost gives the theorem directly; the only problem is that, as stated, the lemma requires  $K$  be disjoint from the boundary of  $\Omega$ . Fortunately, the result can be extended to the boundary by a reflection argument.

**THEOREM 26.** *Let  $g_u \in SBV(\Omega)$  satisfy our assumptions and assume  $\Omega$  is a rectangle. Given  $\varepsilon > 0$ , there exists a constant  $\beta' < \infty$  such that, if  $\beta \geq \beta'$  and  $f$  is a minimizer of  $E(\beta, g)$  for some  $g \in Y(\beta)$ , then*

$$S_f \subset [S_{g_u}]_{\varepsilon}.$$

**PROOF.** Let  $\Omega_3$  be a rectangle with the same center and proportions as  $\Omega$  but 3 times the length. Similarly define  $\Omega_2$  with twice the length of  $\Omega$ . Define

$g_3$  and  $f_3$  on  $\Omega_3$  by reflection of  $g$  and  $f$  respectively. Note that  $g_3$  satisfies our assumptions on  $\Omega_3$ . Define

$$E_3(u, \beta) = \beta \int_{\Omega_3} (g_3 - u)^2 + \int_{\Omega_3} |\nabla u|^2 + \mathcal{X}^1(\mathcal{S}_u).$$

Since  $f$  minimizes  $E$ , we have from Lemma 13 that  $\mathcal{X}^1(\partial\Omega \cap \bar{\mathcal{S}}_f) = 0$ , and it follows that  $E_3(f_3, \beta) = 9E(f, \beta)$ . If  $u \in SBV(\Omega_3)$  such that  $E_3(u, \beta) < 9E(f, \beta)$  then, by restriction, reflection, and/or rotation, we could find  $u \in SBV(\Omega)$  such that  $E(u, \beta) < E(f, \beta)$ . Thus,  $E_3(u, \beta) \geq 9E(f, \beta)$  and  $f_3$  therefore minimizes  $E_3(\cdot, \beta)$ .

Let  $A = \bar{\Omega} \cap [\mathcal{S}_g]_\varepsilon$  and let  $A_3$  be the reflection of  $A$  onto  $\Omega_3$ .  $\bar{\Omega}_2 \setminus A_3$  is a compact subset of  $\Omega_3$ , disjoint from  $\bar{\mathcal{S}}_{g_3}$ . Thus, by Lemma 25 there exists  $\beta' < \infty$  such that if  $\beta > \beta'$ , then  $\mathcal{S}_{f_3} \cap \bar{\Omega}_2 \setminus A_3 = \emptyset$  and hence,  $\mathcal{S}_f \subset [\mathcal{S}_{g_u}]_\varepsilon$ .  $\square$

The next lemma establishes that the opposite containment also holds.

LEMMA 27. *Let  $g_u \in SBV(\Omega)$  satisfy our assumptions and assume  $\Omega$  is a rectangle. Given  $\varepsilon > 0$  there exists a constant  $\beta' < \infty$  such that, if  $\beta \geq \beta'$  and  $f$  is a minimizer of  $E(\beta, g)$  for some  $g \in Y(\beta)$ , then*

$$\mathcal{S}_{g_u} \subset [\mathcal{S}_f]_\varepsilon.$$

PROOF. Assume that the lemma is false. Then there exists an  $\varepsilon > 0$ , a sequence of minimizers  $\{f_n\}$  with  $\beta_n \rightarrow \infty$ , and  $g_n \in Y(\beta_n)$  such that there is a corresponding sequence of points  $x_n \in \mathcal{S}_{g_n}$  such that  $\text{dist}(x_n, \mathcal{S}_{f_n}) \geq \varepsilon$ . Let  $y$  be a cluster point of the sequence  $\{x_n\}$ . We can find  $x \in \mathcal{S}_{g_u}$  satisfying  $|x - y| < \frac{\varepsilon}{2}$ . Thus there exists a subsequence  $\{f_{n_k}\}$  such that  $\text{dist}(x, \mathcal{S}_{f_{n_k}}) \geq \frac{\varepsilon}{2}$ . This then contradicts weak convergence of  $Jf_{n_k}$  to  $Jg_u$  which was proved in Lemma 15.  $\square$

We are now ready to conclude the proof of Theorem 14.

PROOF OF THEOREM 14. The central result,  $\limsup_{\beta \rightarrow \infty} \sup_{g \in Y(\beta)} d_H(\mathcal{S}_f(\beta), \mathcal{S}_{g_u}) = 0$ , was established in Theorem 26 and Lemma 27. The result,

$$\limsup_{\beta \rightarrow \infty} \sup_{g \in Y(\beta)} |\mathcal{X}^1(\mathcal{S}_f) - \mathcal{X}^1(\mathcal{S}_{g_u})| = 0,$$

was proved in Lemma 16, as was  $\limsup_{\beta \rightarrow \infty} \sup_{g \in Y(\beta)} \left| \int_{\Omega} |\nabla f|^2 - \int_{\Omega} |\nabla g_u|^2 \right| = 0$ . Combining

these results, equation (14) yields  $\limsup_{\beta \rightarrow \infty} \sup_{g \in Y(\beta)} \beta \int_{\Omega} (f - g)^2 = 0$ . Together with

equation (8), this implies  $\limsup_{\beta \rightarrow \infty} \sup_{g \in Y(\beta)} \beta \int_{\Omega} (f - g_u)^2 = 0$ . This concludes the proof.

#### 4. - Minimizers of $E_0$

A theorem similar to that proved above can be established for the functional  $E_0$  when the underlying image  $g_u$  is a piecewise constant function. In [17] a constructive proof of such a limit theorem was given. Here we will show how the proof for this case can be given in a framework similar to that used for the piecewise smooth case. Our assumptions are essentially the same as before except for the necessary adjustment to the piecewise constant case.

##### 4.1. Problem formulation

In general, by a minimizer of  $E_0$  we mean a minimizer such as described in Theorem 7. In particular, we recall that  $\Gamma$  is a union of a finite number of  $C^{1,1}$  curves. Actually, the only regularity we require of  $\Gamma$  is that it be closed. Again, it will be convenient also to refer back to the formulation in the *SBV* setting. Thus, we will also consider that  $f \in SBV(\Omega)$ ,  $\Gamma = \overline{S}_f \cap \Omega$ , and hence that  $\mathcal{H}^1(\Omega \cap \overline{S}_f \setminus S_f) = 0$ . We will denote the connected components of  $\Omega \setminus \Gamma$  by  $\{\Omega_k\}$ . The function  $f$  will be constant on each  $\Omega_k$ ; we denote this constant by  $f(\Omega_k)$ . The set  $\Gamma$  thus determines a segmentation. Without loss of generality, we will set the parameter  $\alpha = 1$  and subsequently drop it from our notation.

##### 4.1.1. Assumptions on the domain

The assumptions we require on the domain do not go beyond those needed for Theorem 7, the existence theorem. For convenience we will therefore assume that our domain is an open rectangle. For the results of this section we will need the following isoperimetric inequality: there is a constant  $\zeta > 0$  such that, if  $A \in \Omega$  is a Caccioppoli set, then

$$\mathcal{H}^1(\partial^* A \cap \Omega) \geq \zeta \min \left\{ |A|^{\frac{1}{2}}, |\Omega \setminus A|^{\frac{1}{2}} \right\}.$$

We remark that it is enough that this inequality be satisfied when  $A$  is a polygon for it to hold for all Caccioppoli sets.

Suppose  $\gamma$  is a connected component of some minimizer  $\Gamma$  of  $E_0$  which is some positive distance from  $\partial\Omega$ . Let  $O$  be the connected component of  $\mathbb{R}^2 \setminus \gamma$  containing  $\mathbb{R}^2 \setminus \Omega$ . By the set bounded by  $\gamma$  we mean  $F = \mathbb{R}^2 \setminus O = \Omega \setminus O$ . If  $\gamma$  is not separated from the boundary of  $\Omega$  by a positive distance, then  $\overline{\gamma} \cap \partial\Omega$  is some finite set of points  $p_1, \dots, p_m$ . Since the boundary of  $\Omega$  is a Jordan curve, we can assume the points are ordered along the boundary. Thus  $\partial\Omega \setminus \{p_1, \dots, p_m\}$  consists of  $m$  segments of the boundary which we denote  $s_1, \dots, s_m$ . For any  $i \in \{1, \dots, m\}$ , we define the set  $O_i$  as the connected component of  $\mathbb{R}^2 \setminus \{\gamma \cup (\partial\Omega \setminus s_i)\}$  containing  $\mathbb{R}^2 \setminus \overline{\Omega}$ . Set  $F_i = \Omega \setminus O_i$ . We can now define the set bounded by  $\gamma$ ,  $F$ , to be an  $F_i$  of minimal area. (We choose it arbitrarily if it is not unique). These definitions are illustrated in Figure 3.



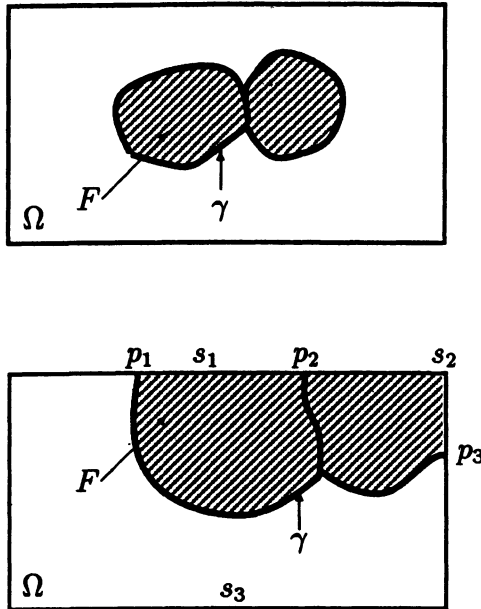


Figure 3 - Sets bounded by a curve

If  $\mathcal{H}^1(\gamma) < \zeta \sqrt{\frac{|\Omega|}{2}}$ , then we can conclude from the isoperimetric inequality that  $F$  is uniquely defined. In either case, the isoperimetric inequality implies that

$$(45) \quad \mathcal{H}^1(\gamma) \geq \zeta |F|^{\frac{1}{2}}.$$

4.1.2. Assumptions on the image

The following is our assumption on the underlying image.

ASSUMPTION 1.  $g_u \in L^\infty(\Omega) \cap SBV(\Omega)$ ,  $\mathcal{H}^1(\mathcal{S}_{g_u}) < \infty$ , and  $\nabla g_u = 0$ .

From Lemma 6 we conclude that this implies that there exists a Caccioppoli partition of  $\Omega$ ,  $\{R_i\}$ , and a bounded sequence of real numbers  $\{a_i\}$  satisfying  $a_i \neq a_j$  for  $i \neq j$  such that  $g_u = \sum_i a_i \chi_{R_i}$  almost everywhere in  $\Omega$ . Without loss of generality, we assume that equality holds everywhere.

### 4.1.3. The noise model

Our assumptions on the noise are the same as in the piecewise smooth case. The following are the properties we require of  $Y(\beta)$ , the class of admissible observations,

$$(46) \quad \lim_{\beta \rightarrow \infty} \sup_{g \in Y(\beta)} \beta \int_{\Omega} (g - g_u)^2 = 0,$$

and

$$(47) \quad \forall \varepsilon > 0, \quad \lim_{\beta \rightarrow \infty} \sup_{g \in Y(\beta)} \|(g - g_u)(1 - \chi_{[S_{g_u}]_\varepsilon})\|_\infty = 0.$$

To replace this with a model based on smearing and additive noise, we need only assume that the Minkowski content of  $S_{g_u}$  is finite. The argument is essentially the same as in the piecewise smooth case.

The statement of the limit theorem is almost identical to that of Theorem 14.

**THEOREM 28.** *Under our stated assumptions, as  $\beta \rightarrow \infty$ ,  $\{S_f(\beta)\}$  converges to  $S_{g_u}$  with respect to the Hausdorff metric, and*

$$\mathcal{H}^1(S_f(\beta)) \rightarrow \mathcal{H}^1(S_{g_u}).$$

*I.e., for any  $\varepsilon > 0$  there exists  $\beta' < \infty$  such that, if  $\beta > \beta'$  and  $f$  is a minimizer of  $E_0$  for some  $g \in Y(\beta)$ , then  $d_H(S_f, S_{g_u}) < \varepsilon$  and  $|\mathcal{H}^1(S_f) - \mathcal{H}^1(S_{g_u})| < \varepsilon$ . Furthermore  $\lim_{\beta \rightarrow \infty} \sup_{g \in Y(\beta)} \beta \int_{\Omega} (f - g_u)^2 = 0$ .*

### 4.2. Preliminary results

**PROPOSITION 29.** *Given a countable set  $\{a_i : i = 0, 1, \dots\} \subset \mathbb{R}$ , a nonnegative  $l_1$  sequence  $\{r_i : i = 1, 2, \dots\}$ , and constants  $c_1, c_2 > 0$ , there exists a nondecreasing function  $h : (0, \infty) \rightarrow [0, \infty)$  satisfying  $\lim_{t \rightarrow 0^+} \frac{h(t)}{\sqrt{t}} = 0$  such that, for any sequence  $\{p_i : i = 0, 1, \dots\}$  satisfying*

$$p_0 \geq c_1 \text{ and } r_i \geq p_i \geq 0 \text{ for } i > 0,$$

$$\sum_{i=0}^{\infty} p_i = 1, \text{ and}$$

$$\sum_{i=0}^{\infty} p_i (a_i - \hat{a})^2 < c_2 t \quad \text{where } \hat{a} = \sum_{i=0}^{\infty} p_i a_i,$$

we have

$$|\hat{a} - a_0| < h(t).$$

PROOF. We define the constant  $b = \sum_{i=1}^{\infty} r_i$ . We assume  $b > 0$  (the result is trivial otherwise). Define  $h_1 : (0, \infty] \rightarrow [0, b]$  by

$$h_1(t) = \sum_{\{i:0 < |a_i - a_0| < t\}} r_i.$$

Clearly,  $h_1(t)$  is nondecreasing in  $t$ . We claim  $h_1$  is continuous from the left and  $\lim_{t \rightarrow 0^+} h_1(t) = 0$ . For any  $\varepsilon > 0$ , there exists  $N < \infty$  such that  $\sum_{i=N}^{\infty} r_i < \varepsilon$ . For  $t \leq \min\{|a_i - a_0| : 0 < i < N\}$  we have  $h_1(t) < \varepsilon$  proving the second part of the claim. Given  $t > 0$ , let  $\delta = \min\{t - |a_i - a_0| : 0 < i < N \text{ and } t - |a_i - a_0| > 0\}$ ; for  $t' \in (t - \delta, t)$  we have  $h_1(t) - h_1(t') < \varepsilon$ , proving the first part of the claim.

Define  $h_2 : (0, \infty) \rightarrow [b^{-\frac{1}{2}}, \infty]$  by

$$h_2(t) = \sup \left\{ c : h_1(x\sqrt{t}) \leq \frac{1}{x^2}, \forall x < c \right\}.$$

Since  $h_1$  is nondecreasing and bounded above by  $b$ ,  $h_2$  is nonincreasing and bounded below by  $b^{-\frac{1}{2}}$ . The function  $h_2$  is finite for finite  $t$  since  $\lim_{x \rightarrow \infty} h_1(x) = b$ .

For any  $N < \infty$ , there exists  $\eta > 0$  such that  $\varepsilon \leq \eta \Rightarrow h_1(\varepsilon) \leq \frac{1}{N^2}$ . Thus,  $N\sqrt{t} \leq \eta \Rightarrow h_1(N\sqrt{t}) \leq \frac{1}{N^2}$  and, since  $h_1$  is nondecreasing while  $\frac{1}{x^2}$  is decreasing, we have  $t \leq \left(\frac{\eta}{N}\right)^2 \Rightarrow h_2(t) \geq N$ . We conclude  $\lim_{t \rightarrow 0^+} h_2(t) = \infty$ .

Also, since  $h_1$  is continuous from the left we have  $0 < x \leq h_2(t) \Rightarrow h_1(x\sqrt{t}) \leq \frac{1}{h_2^2(t)}$ .

Define  $h_3 : (0, \infty) \rightarrow [b^{-\frac{1}{2}}, \infty)$  by

$$h_3(t) = \max \left( b^{-\frac{1}{2}}, h_2(t) - \sqrt{\frac{c_2}{c_1}} \right).$$

Consider the case  $\hat{a} \geq a_0$ . We have

$$\begin{aligned} \sum_{\{i:a_i \leq \hat{a}\}} p_i(\hat{a} - a_i) &= \sum_{\{i:0 < a_i - \hat{a} < h_3(t)\sqrt{t}\}} p_i(a_i - \hat{a}) + \sum_{\{i:h_3(t)\sqrt{t} \leq a_i - \hat{a}\}} p_i(a_i - \hat{a}) \\ &\leq \sum_{\{i:0 < |a_i - a_0| < h_2(t)\sqrt{t}\}} p_i |a_i - \hat{a}| + \frac{1}{h_3(t)\sqrt{t}} \sum_{i=0}^{\infty} p_i (a_i - \hat{a})^2 \\ (48) \quad &\leq h_1(h_2(t)\sqrt{t}) h_2(t)\sqrt{t} + \frac{c_2 t}{h_3(t)\sqrt{t}} \\ &\leq \left( \frac{1}{h_2(t)} + \frac{c_2}{h_3(t)} \right) \sqrt{t}. \end{aligned}$$

For the first inequality we used the obvious bound  $|\hat{a} - a_0| < \sqrt{\frac{c_2}{c_1}t}$ . Define  $h(t) = \frac{1}{c_1} \left( \frac{1}{h_2(t)} + \frac{c_2}{h_3(t)} \right) \sqrt{t}$ . That  $h$  is nondecreasing follows from the fact that  $h_2$  (and hence  $h_3$ ) is nonincreasing. Also, since  $\lim_{t \rightarrow 0^+} h_2(t) = \infty$ , and hence  $\lim_{t \rightarrow 0^+} h_3(t) = \infty$ , it follows that  $\lim_{t \rightarrow 0^+} \frac{h(t)}{\sqrt{t}} = 0$ . Now,  $p_0(\hat{a} - a_0) \leq \sum_{\{i: a_i \leq \hat{a}\}} p_i(\hat{a} - a_i)$  and from (48) we get  $|a_0 - \hat{a}| \leq h(t)$ . The case  $\hat{a} \leq a_0$  can be treated similarly, yielding the same result. This completes the proof of the proposition.  $\square$

The set  $\{a_i\}$  will represent the set of values of the image  $g_c$ . If  $\{a_i\}$  has finite cardinality, then slightly stronger results can be obtained using the same line of proof.

For each  $\Omega_k$  in a segmentation, we define the following constants

$$p_i^k \triangleq \frac{|\Omega_k \cap R_i|}{|\Omega_k|}.$$

Note that  $\sum_{i=1}^{\infty} p_i^k = 1$ .

Let  $E_0^*(\beta, g)$  denote the minimal value of  $E_0(\beta, g)$ . Substituting  $g_u$  for  $f$ , we get the following bound

$$(49) \quad E_0^*(\beta, g) \leq \beta \int_{\Omega} (g - g_u)^2 + \mathcal{M}^1(\mathcal{S}_{g_u}).$$

LEMMA 30. *Given  $\xi > 0$  and  $i \geq 0$  there exists a function*

$$H : (0, \infty) \rightarrow [0, \infty)$$

*satisfying  $\lim_{\beta \rightarrow \infty} \sqrt{\beta}H(\beta) = 0$  such that, if  $\Gamma_\beta$  is a minimizer of  $E_0$  for some  $g \in Y(\beta)$  and  $\Omega_k$  is a connected component of  $\Omega \setminus \Gamma_\beta$  satisfying  $|\Omega_k \cap R_i| \geq \xi$ , then*

$$|f(\Omega_k) - a_i| \leq H(\beta).$$

PROOF. For convenience of notation we set  $i = 0$  and re-enumerate the other  $a_i$  starting from 1. Let  $\hat{a} = \sum_{i=0}^{\infty} p_i^k a_i = \frac{1}{|\Omega_k|} \int_{\Omega_k} g_u$ . We have

$$|f(\Omega_k) - a_0| \leq |f(\Omega_k) - \hat{a}| + |\hat{a} - a_0|.$$

The first term can be bounded as follows,

$$|f(\Omega_k) - \hat{a}| = \left| \frac{1}{|\Omega_k|} \int_{\Omega_k} (g - g_u) \right| \leq \frac{1}{|\Omega_k|^{\frac{1}{2}}} \left( \int_{\Omega_k} (g - g_u)^2 \right)^{\frac{1}{2}}.$$

To bound the second term we first note,

$$\sum_{i=0}^{\infty} p_i^k (\hat{a} - a_i)^2 = \int_{\Omega_k} (\hat{a} - g_u)^2$$

and, since  $f$  is constant in  $\Omega_k$  and  $\hat{a} = \frac{1}{|\Omega_k|} \int_{\Omega_k} g_u$ , we conclude,

$$\frac{1}{2} \int_{\Omega_k} (\hat{a} - g_u)^2 \leq \frac{1}{2} \int_{\Omega_k} (f - g_u)^2 \leq \int_{\Omega_k} (f - g)^2 + \int_{\Omega_k} (g - g_u)^2.$$

Applying our assumptions and equation (49), we see that for all  $\beta$  sufficiently large there is a constant  $c > 0$  such that,

$$\sum_{i=0}^{\infty} p_i^k (\hat{a} - a_i)^2 \leq c\beta^{-1}.$$

Define  $r_i = \min \left\{ 1, \frac{|R_i|}{\xi} \right\}$ . Clearly  $p_i^k \leq r_i$  and  $\sum_{i=1}^{\infty} r_i < \infty$ ; note also that  $p_0^k \geq \frac{\xi}{|\Omega|}$ . We can now apply Proposition 29 to conclude there exists a function  $h : (0, \infty) \rightarrow [0, \infty)$  satisfying  $\lim_{\beta \rightarrow \infty} \sqrt{\beta} h(\beta) = 0$  such that  $|\hat{a} - a_0| \leq h(\beta)$ . Set  $H(\beta) = h(\beta) + \xi^{-\frac{1}{2}} \sup_{g \in Y(\beta)} \left( \int_{\Omega_k} (g - g_u)^2 \right)^{\frac{1}{2}}$ , and the result follows.  $\square$

LEMMA 31. *If  $f_n, \beta_n, g_n$  are sequences such that  $\beta_n \uparrow +\infty, g_n \in Y(\beta_n)$ , and  $f_n$  minimizes  $E_0(\beta_n, g_n)$ , then*

$$\begin{aligned} f_n &\rightarrow g_u && \text{in } L^1(\Omega) \\ Jf_n &\rightarrow Jg_u && \text{weakly as Radon measures.} \end{aligned}$$

PROOF. From the triangle inequality and equation (49), we obtain

$$\beta_n \int_{\Omega} (f_n - g_u)^2 \leq \beta_n \int_{\Omega} (g_n - g_u)^2 + E^*(\beta_n) \leq 2\beta_n \int_{\Omega} (g_n - g_u)^2 + \mathcal{H}^1(\mathcal{S}_{g_u}).$$

Since  $\beta_n \rightarrow \infty$ , it follows from the assumed equation (46) that  $f_n$  converges to  $g_u$  in  $L^2(\Omega)$ . Noting  $|\Omega| < \infty$ , we conclude  $f_n$  converges to  $g_u$  in  $L^1(\Omega)$ . The other statement follows from Theorem 8.  $\square$

LEMMA 32. *If  $f_n$ ,  $\beta_n$ ,  $g_n$  are sequences such that  $\beta_n \uparrow +\infty$ ,  $g_n \in Y(\beta_n)$ , and  $f_n$  minimizes  $E_0(\beta_n, g_n)$ , then*

$$\lim_{n \rightarrow \infty} \mathcal{H}^1(\mathcal{S}_{f_n}) = \mathcal{H}^1(\mathcal{S}_{g_u}).$$

PROOF. By equations (49) and (46), we have

$$\limsup_{n \rightarrow \infty} \mathcal{H}^1(\mathcal{S}_{f_n}) \leq \mathcal{H}^1(\mathcal{S}_{g_u}).$$

Theorem 9 yields

$$\mathcal{H}^1(\mathcal{S}_{g_u}) \leq \liminf_{n \rightarrow \infty} \mathcal{H}^1(\mathcal{S}_{f_n}),$$

completing the proof.  $\square$

COROLLARY. *If  $A \subset \Omega$  is any Borel set such that  $\text{dist}(A, \overline{\mathcal{S}_{g_u}} \cup \partial\Omega) > 0$ , then*

$$\lim_{n \rightarrow +\infty} \mathcal{H}^1(\mathcal{S}_{f_n} \cap A) = 0.$$

PROOF. For some  $\varepsilon > 0$ ,  $A \cap [\mathcal{S}_{g_u}]_\varepsilon = \emptyset$ . From Lemma 15 we conclude  $f_n \rightarrow g_u$  in  $L^1([\mathcal{S}_{g_u}]_\varepsilon)$ . We can now apply essentially the same argument as in Lemma 32 to conclude  $\mathcal{H}^1(\mathcal{S}_{g_u}) \leq \liminf_{n \rightarrow \infty} \mathcal{H}^1(\mathcal{S}_{f_n} \cap [\mathcal{S}_{g_u}]_\varepsilon)$ . But the result of Lemma 32 states  $\mathcal{H}^1(\mathcal{S}_{g_u}) = \lim_{n \rightarrow \infty} \mathcal{H}^1(\mathcal{S}_{f_n})$ , so it follows that  $\lim_{n \rightarrow \infty} \mathcal{H}^1(\mathcal{S}_{f_n} \cap A) = 0$ .  $\square$

### 4.3. Main results

THEOREM 33. *For any  $\varepsilon > 0$  there exists a constant  $\beta' < \infty$  such that, if  $\beta > \beta'$ , then  $\Gamma_\beta \subset [\mathcal{S}_{g_u}]_\varepsilon$  for any  $\Gamma_\beta$  which minimizes  $E_0$  with  $g \in Y(\beta)$ .*

PROOF. Assume the theorem is false. There exists some  $\varepsilon > 0$  and a sequence of optimal  $\Gamma_{\beta_n}$  (for some  $g_n \in Y(\beta_n)$ ) with  $\beta_n \uparrow \infty$  such that  $\Gamma_{\beta_n} \not\subset [\mathcal{S}_{g_u}]_\varepsilon \neq \emptyset$  for each  $n$ . Since only finitely many  $R_i$  can satisfy  $R_i \not\subset [\mathcal{S}_{g_u}]_\varepsilon$ , there exists some single  $R_i$  and a subsequence (which we denote the same way) such that  $\Gamma_{\beta_n} \cap R_i \setminus [\mathcal{S}_{g_u}]_\varepsilon \neq \emptyset$  for each  $n$ . Let  $G$  represent an arbitrary connected component of  $R_i \setminus [\mathcal{S}_{g_u}]_\varepsilon$ . The set  $G$  is a subset of some connected component of  $R_i \setminus [\mathcal{S}_{g_u}]_{\varepsilon/2}$  which we denote  $\hat{G}$ . It follows that  $|\hat{G}| \geq \frac{\pi\varepsilon^2}{16}$  and, hence, there are only finitely many distinct  $\hat{G}$ . (Note that for  $G$  containing points at distance greater than  $\varepsilon$  from  $\partial\Omega$  we have  $|\hat{G}| \geq \frac{\pi\varepsilon^2}{4}$ ; we potentially loose a factor of 4 when  $G$  is near a corner of the rectangle  $\Omega$ ). We can assume therefore that there is some single  $\hat{G}$  such that  $\Gamma_{\beta_n} \cap \hat{G} \neq \emptyset$  for all  $n$ . Let  $\{C_j^n\}$  be the set of connected components of  $\Gamma_{\beta_n}$  satisfying  $C_j^n \cap \hat{G} \neq \emptyset$  and let  $\hat{\Gamma}_{\beta_n} = \Gamma_{\beta_n} \setminus \bigcup_j C_j^n$ . We will denote by  $\hat{\Omega}_k^n$  the connected component of  $\Omega \setminus \hat{\Gamma}_{\beta_n}$  which is a superset of

$\hat{G}$ . Some subset of the  $\Omega_k^n$  lying in  $\hat{\Omega}_k^n$ , whose union we denote by  $O_n$ , are the sets bounded by the  $C_j^n$ . It follows from the isoperimetric inequality that

$$(50) \quad |O_n| \leq \frac{1}{\xi^2} \left( \max_j \mathcal{H}^1(C_j^n) \right) \sum_j \mathcal{H}^1(C_j^n).$$

From the corollary to Lemma 32 we conclude  $\lim_{n \rightarrow \infty} \max_j \mathcal{H}^1(C_j^n) = 0$ , and now, since  $\sum_i \mathcal{H}^1(C_j^n)$  is bounded, we can conclude that for  $n$  large enough  $|O_n| \leq \frac{\pi \varepsilon^2}{32}$ . Hence, there is some  $\Omega_k^n \subset \hat{\Omega}_k^n$  satisfying  $|\Omega_k^n \cap R_i| \geq \frac{\pi \varepsilon^2}{8}$ . Let  $H$  be the function from Lemma 30 with  $\xi = \frac{\pi \varepsilon^2}{32}$  and  $i$  defined by the subscript of  $R_i$ . We now have

$$\begin{aligned} E_0(\hat{\Gamma}_{\beta_n}) - E_0(\Gamma) &\leq \beta_n \int_{O_n} (f_n(\Omega_k) - g_n)^2 - \sum_j \mathcal{H}^1(C_j^n) \\ &\leq 2\beta_n \left( |O_n| (f_n(\Omega_k) - a_i)^2 + \int_{O_n} (g_n - g_u)^2 \right) - \sum_i \mathcal{H}^1(C_j^n). \end{aligned}$$

Since  $\mathcal{H}^1(C_j) \leq \text{diam}(C_j)$ , it follows that, for  $n$  sufficiently large,  $O_n \subset R_i \setminus [\mathcal{S}_{g_u}]_{\frac{\varepsilon}{4}}$ . From the assumed inequality (47), we conclude that, for all  $n$  sufficiently large,  $2\beta_n \int_{O_n} (g_n - g_u)^2 \leq \frac{1}{2}|O_n|$ . Thus, for  $n$  sufficiently large we obtain, from equation (50),

$$E_0(\hat{\Gamma}_{\beta_n}) - E_0(\Gamma) \leq \left[ \frac{1}{\xi^2} \max_j \mathcal{H}^1(C_j^n) 2\beta_n H^2(\beta_n) - \frac{1}{2} \right] \sum_j \mathcal{H}^1(C_j^n).$$

Since the term in square brackets is negative for  $n$  sufficiently large, while  $\sum_j \mathcal{H}^1(C_j^n)$  is positive, we get a contradiction of the optimality of  $\Gamma$ . This completes the proof of the theorem.  $\square$

LEMMA 34. *Let  $g_u \in SBV(\Omega)$  satisfy our assumptions and assume  $\Omega$  is a rectangle. Given  $\varepsilon > 0$  there exists a constant  $\beta' < \infty$  such that, if  $\beta \geq \beta'$  and  $f$  is a minimizer of  $E(\cdot, \beta)$  with  $g \in Y(\beta)$ , then*

$$\mathcal{S}_{g_u} \subset [\mathcal{S}_f]_\varepsilon.$$

PROOF. The proof is identical to that of Lemma 27.  $\square$

PROOF OF THEOREM 28. Theorem 33 and Lemma 34 establish  $d_H(\Gamma_\beta, \bar{\mathcal{S}}_{g_u}) < \varepsilon$  while Lemma 32 asserts  $|\mathcal{H}^1(\Gamma_\beta) - \mathcal{H}^1(\mathcal{S}_{g_u})| < \varepsilon$  for all

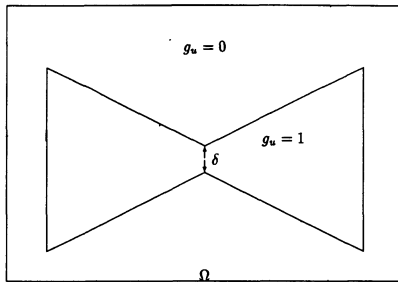
$\beta > \beta'$  for some  $\beta' < \infty$ . In the course of the proof we have shown that  $\limsup_{\beta \rightarrow \infty} \sup_{g \in Y(\beta)} |E_0^*(\beta, g) - \mathcal{X}^1(\mathcal{S}_{g_u})| = 0$  as well as  $\limsup_{\beta \rightarrow \infty} \sup_{g \in Y(\beta)} |\mathcal{X}^1(\Gamma_\beta) - \mathcal{X}^1(\mathcal{S}_{g_u})| = 0$ ; we conclude from this that  $\limsup_{\beta \rightarrow \infty} \sup_{g \in Y(\beta)} \beta \int_{\Omega} (f - g)^2 = 0$ . Since  $\limsup_{\beta \rightarrow \infty} \sup_{g \in Y(\beta)} \int_{\Omega} (g - g_u)^2 = 0$  by assumption, we have  $\limsup_{\beta \rightarrow \infty} \sup_{g \in Y(\beta)} \beta \int_{\Omega} (f - g_u)^2 = 0$ .

4.4. Weakening the noise constraints

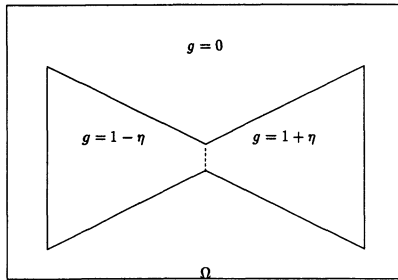
An obviously relevant question concerning the limit theorem is: how tight are the estimates on the noise and smearing decay rates?

4.4.1. A counter-example

In this section we show that our decay requirements on the additive noise are tight in the sense that, for any  $c > 0$  and arbitrary rectangular domain, we can find a piece-wise constant function  $g_u$  such that, if we allow additive  $L^\infty$  noise with a norm bounded above by  $c\beta^{-1}$ , then the optimal boundaries  $\Gamma_\beta$  need not converge to the discontinuity set of  $g_u$  in Hausdorff metric as  $\beta$  tends to infinity. Consider the function  $g_u$  as illustrated in Figure 4. There is some constant  $a > 0$  such that the area of the region over which  $g_u = 1$  is greater than  $a$  for all  $\delta > 0$ . We choose  $\delta > 0$  small enough to satisfy  $2\delta < c^2a$ .



The function  $g_u$



The function  $g$

Figure 4 - The counter-example



Now we let the observed image be as in Figure 4 where  $\eta = c\beta^{-\frac{1}{2}}$ . Suppose we have a sequence of solutions,  $\{\Gamma_{\beta_n}\}$ , to the piecewise constant variational problem converging to the discontinuity set of  $g_u$  in Hausdorff metric as  $\beta_n \rightarrow \infty$ . Let  $f_n$  be the associated minimizing function for the given  $\Gamma_{\beta_n}$ . It follows that

$$\lim_{n \rightarrow \infty} \beta_n \int_{\Omega} (f_n - g)^2 \geq c^2 a$$

and

$$\liminf_{n \rightarrow \infty} \mathcal{H}^1(\Gamma_n) \geq \mathcal{H}^1(\mathcal{S}_{g_u}).$$

Let  $\gamma$  be the dashed line of length  $\delta$  in Figure 4. Now consider  $\Gamma' = \mathcal{S}_{g_u} \cup \gamma$ ; it satisfies  $E_0(\Gamma') = \mathcal{H}^1(\mathcal{S}_{g_u}) + \delta$  for all  $n$ . Since we have  $\delta < c^2 a$  by construction, we get a contradiction. Thus, for this example, the sequence  $\{\Gamma_{\beta_n}\}$  cannot be a sequence of minimizers of  $E_0$ .

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AT&T Bell Labs  
600 Mountain Ave.  
2C-370 Murray Hill  
New Jersey