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Continuity of the Darcy's Law in the Low-Volume Fraction Limit

GRÉGOIRE ALLAIRE *

0. - Introduction

The derivation of Darcy's law through the homogenization of the Stokes equations in a porous medium is now well understood. Some ten years ago, J.B. Keller [10], J.L. Lions [14], and E. Sanchez-Palencia [19] showed that Darcy's law is the limit of the Stokes equations in a periodic porous medium, when the period goes to zero. Note that the assumption on the periodicity of the porous medium implies that in each period the fluid and the solid part have a size of the same order of magnitude. Their main tool was the celebrated two-scale method (see also [5]), which yields heuristic results. Almost at the same time, L. Tartar [22] proved the rigorous convergence of this limit process using his energy method introduced in [23] (see also [16]). Later on, T. Levy [12] and E. Sanchez-Palencia [18] showed that, for some porous media in which the solid part is smaller than the fluid part, the homogenization of the Stokes equations leads again to Darcy's law, but, with a permeability different from the first one. The rigorous convergence in this case was proved by the author [2], [3]. The purpose of this paper is to study the two permeability tensors associated to those two Darcy's laws, and to actually prove that there is a continuous transition between them.

Let us describe more precisely the setting and the main results of the present paper. The fluid part Ω_ε of a porous medium is obtained by removing, from a bounded domain Ω, a collection of periodically distributed and identical obstacles. We denote by ε the period, and by a_ε the obstacle size; each obstacle lies in a cubic cell $(-\varepsilon, +\varepsilon)^N$, and is similar to the same model obstacle T rescaled to size a_ε.

We consider a Stokes flow in Ω_ε under the action of an exterior force f,

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and with a no-slip condition on the boundaries of the obstacles

$$\begin{cases} \nabla p_\varepsilon - \Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \nabla \cdot u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where u_ε and p_ε are the velocity and the pressure of the fluid.

1) *Assume that* $a_\varepsilon = \eta\varepsilon$, *with* $\eta \in (0, 1)$.

The constant η is actually the obstacle size in the rescaled unit cell $P = (-1, +1)^N$. This case has been extensively studied with the two-scale method (see [10], [14], [19]): when ε goes to zero, the limit of the Stokes equations is the following Darcy's law

$$\begin{cases} u = A(f - \nabla p) & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

where u and p are the velocity and the pressure of the fluid. The matrix A is the so-called permeability tensor given by

$${}^t e_i A e_k = \frac{1}{2^N} \int_{P-\eta T} \nabla v_k : \nabla v_i,$$

where the functions $(v_k)_{1\le k\le N}$ are the solutions of the so-called cell problems

$$\begin{cases} \nabla p_k - \Delta v_k = e_k & \text{in } P - \eta T, \\ \nabla \cdot v_k = 0 & \text{in } P - \eta T, \\ v_k = 0 & \text{on } \partial(\eta T), \\ p_k, \ v_k \ P\text{-periodic}. \end{cases}$$

2) *Assume that* $a_\varepsilon \ll \varepsilon$.

We further assume that the obstacles are not too small

$$a_\varepsilon \gg \varepsilon^{N/(N-2)} \text{ if } N \ge 3, \text{ and } |\log a_\varepsilon|^{-1} \gg \varepsilon^2 \text{ if } N = 2.$$

(For smaller obstacles, different limit regimes occur; see [3], [12]). This second case has been studied by the author in [3], where he proved that when ε goes to zero, the limit of the Stokes equations is the following Darcy's law

$$\begin{cases} u = M^{-1}(f - \nabla p) & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

where, again, (u, p) are the velocity and pressure of the fluid, and M^{-1} is the permeability tensor. For each $k \in \{1, \cdots, N\}$, let us introduce the so-called local problem

$$\begin{cases} \nabla q_k - \Delta w_k = 0 & \text{in } \mathbb{R}^N - T, \\ \nabla \cdot w_k = 0 & \text{in } \mathbb{R}^N - T, \\ w_k = 0 & \text{on } \partial T, \\ w_k = e_k & \text{at infinity, if } N \geq 3, \\ w_k = (\log r)e_k & \text{at infinity, if } N = 2. \end{cases}$$

Denoting by F_k the drag force applied on T, i.e., $F_k = \int_{\partial T} \left(q_k n - \frac{\partial w_k}{\partial n} \right)$, where n is the interior normal vector of ∂T, the matrix M is given by

$$M e_k = \frac{1}{2^N} F_k,$$

or equivalently

$$M = \pi \mathrm{Id} \quad \text{if } N = 2, \qquad {}^t e_i M e_k = \frac{1}{2^N} \int_{\mathbb{R}^N - T} \nabla w_i : \nabla w_k \quad \text{if } N \geq 3.$$

Apparently there is no link between the formulae for A and M^{-1}. From a physical point of view, there should be one. Indeed, for an entire range of obstacle size smaller than ε, we always obtain a Darcy's law with the same permeability tensor M^{-1}; thus it seems natural that, even when the obstacle size is exactly of the order of magnitude of ε, one finds again the same permeability tensor. This is not quite true, but, anyway, there is a sort of continuity when passing from one case to the other. Actually we shall prove that, when η (the obstacle size in the rescaled unit cell) goes to zero, the permeability tensor $A(\eta)$ (obtained in the first case) converges, up to a suitable rescaling, to the other permeability tensor M^{-1} (obtained in the second case). Thus, together with [3], this result shows that there is a complete continuity of the limit regimes when the obstacle size varies from zero to the period ε (included). This process of letting η go to zero, after taking the homogenized limit, is called the low-volume fraction limit (see, e.g., [9], [21]). Our main result (Theorem 3.1 of the present paper) is summarized in the following

THEOREM. *Let (p_k, v_k) be the unique solution of the cell problem in the first case. Rescaling it, for $x \in (\eta^{-1}P - T)$ and for any space dimension, we define*

$$v_k^\eta(x) = \eta^{N-2} v_k(\eta x) \quad \textit{and} \quad p_k^\eta(x) = \eta^{N-1} p_k(\eta x).$$

Let (q_i, w_i) be the unique solution of the local problem in the second case. Then

(p_k^η, v_k^η) *converges weakly to* $\sum_{i=1}^{N} ({}^t e_i M^{-1} e_k)(q_i, w_i)$ *in*

$$[L^2_{\text{loc}}(\mathbb{R}^N - T)/\mathbb{R}] \times [H^1_{\text{loc}}(\mathbb{R}^N - T)]^N.$$

Furthermore we have

$$\begin{cases} \lim_{\eta \to 0} \eta^{N-2} A(\eta) = M^{-1} & \text{for } N \geq 3 \\ \lim_{\eta \to 0} \frac{1}{|\log \eta|} A(\eta) = M^{-1} & \text{for } N = 2. \end{cases}$$

Note that the solutions of the cell problem, in the first case, do not converge to the solutions of the local problem, in the second case, but rather to a linear combination of them (this fact has already been mentioned by E. Sanchez-Palencia in [18]). We give an explanation of it based on a new corrector result for the velocity (see Theorem 1.3). In [3], when $a_\varepsilon \ll \varepsilon$, we pointed out that the solutions of the local problem are the boundary layers of the Stokes flow around the obstacles. In Theorem 1.3 of the present paper, when $a_\varepsilon = \eta\varepsilon$, we see that the boundary layers actually are a linear combination of the solutions of the cell problem. Thus, from a physical point of view, it seems natural that Theorem 3.1 corresponds to a continuity between boundary layers, rather than between solutions of the cell, or local, problem.

Let us conclude this introduction by saying that the idea of the "low-volume fraction limit" goes back to R.S. Rayleigh [17]. Stokes flows through periodic arrays of obstacles have been studied by H. Hasimoto [9] and A.S. Sangani and A. Acrivos [21] (see also the references therein). Using different methods, restricted to the case of spherical obstacles, they obtained an asymptotic expansion of the permeability $A(\eta)$ (instead of just its limit, as we do here). Finally, we also mention that the framework of the present paper is similar to that introduced by T. Levy and E. Sanchez-Palencia in [13] and [20] (concerning the derivation of the Einstein formula for the viscosity of a suspension of particles).

NOTATIONS. Throughout this paper, C denotes various real positive constants which never depend on ε or η. Let $H^1_\#(P)$ (resp. $L^2_\#(P)$) denote the space of P-periodic functions of $H^1(P)$ (resp. $L^2(P)$). The duality products between $H^1_0(\Omega)$ and $H^{-1}(\Omega)$, and between $[H^1_0(\Omega)]^N$ and $[H^{-1}(\Omega)]^N$, are both denoted by $\langle \cdot, \cdot \rangle_{H^{-1},H^1_0(\Omega)}$. The duality product between $[H^1_\#(P)]^N$ and its dual, is denoted by $\langle \cdot, \cdot \rangle_{H^{-1}_\#,H^1_\#(P)}$. The canonical basis of $\mathbb{R}^N$ is denoted by $(e_k)_{1\leq k\leq N}$.

1. - Obstacles size of the order of the period

Throughout this paper, we consider a porous medium modelled by a periodic array of fixed and isolated obstacles. The present section is devoted to the case where the obstacles have a size of the same order of magnitude than the period. This situation has been extensively studied with the celebrated two-scale method (see [1], [10], [14], [19], [22]).

1.a) *Recall of previous results.*

Let us briefly describe the geometry under consideration; the porous medium Ω_ε is defined as follows. Let Ω be a bounded, connected, open set in $\mathbb{R}^N (N \geq 2)$, with Lipschitz boundary $\partial\Omega$, Ω being locally located on one side of its boundary. Let ε be a sequence of strictly positive reals which tends to zero. The set Ω is covered with a regular mesh of size 2ε, each cell being a cube P_i^ε, identical to $(-\varepsilon, +\varepsilon)^N$. At the center of each cube P_i^ε, entirely included in Ω, we put an obstacle T_i^ε of size of the same order of magnitude than the period ε. The fluid domain Ω_ε is obtained by removing from Ω all the holes T_i^ε: thus $\Omega_\varepsilon = \Omega - \bigcup_{i=1}^{N(\varepsilon)} T_i^\varepsilon$ (the total number of obstacles $N(\varepsilon)$ is of the order of ε^{-N}). By perforating only the cells which are entirely included in Ω, it follows that no obstacle meets the boundary $\partial\Omega$. Every obstacle T_i^ε is similar to the same model obstacle T rescaled to size $\eta\varepsilon$, where η is a positive constant. (We assume that T is a smooth closed set, which contains a small open ball, and which is, itself, included in the unit ball). When rescaled to size 1, the fluid cell $P_i^\varepsilon - T_i^\varepsilon$ is similar to the so-called unit cell $P - \eta T$, where P is equal to $(-1, +1)^N$. Thus, the constant η is actually the size of the obstacle *in the unit cell*, while $\eta\varepsilon$ is the size of the obstacle *in the ε-cell P_i^ε*. In the present section η is considered as a constant, although in Section 2 we shall study the limit when η goes to zero (the so-called low-volume fraction limit).

Let us consider a Stokes flow in Ω_ε under the action of an exterior force f, and with a Dirichlet boundary condition on $\partial\Omega_\varepsilon$. Let u_ε and p_ε be the velocity and the pressure of the fluid (its viscosity and density are set equal to 1). If $f \in [L^2(\Omega)]^N$, there is a unique solution $(u_\varepsilon, p_\varepsilon)$ in $[H_0^1(\Omega_\varepsilon)]^N \times [L^2(\Omega_\varepsilon)/\mathbb{R}]$ of the Stokes equations

$$(1.1) \qquad \begin{cases} \nabla p_\varepsilon - \Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \nabla \cdot u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Using the celebrated two-scale method, several authors ([10], [14], [19]) have heuristically shown that the limit, when ε goes to zero, of the Stokes problem (1.1) is the following Darcy's law

$$(1.2) \qquad \begin{cases} u = A(f - \nabla p) & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

where u ant p are the velocity and the pressure of the fluid, and the permeability tensor A is a symmetric, positive definite, matrix. (Problem (1.2) is a second order elliptic equation for the pressure p, and there exists a unique solution p in $H^1(\Omega)/\mathbb{R}$).

Furthermore, the permeability tensor A is given by an explicit expression involving the so-called local solutions $(v_k)_{1\leq k\leq N}$ of a Stokes flow in the unit cell $P-\eta T$ (see, e.g., Section 7.2 in [19])

$$ {}^t e_i A e_k = \frac{1}{2^N} \int_{P-\eta T} \nabla v_k : \nabla v_i. \tag{1.3}$$

For each $k \in [1, N]$, the velocity v_k is defined as the solution, in $[H^1_\#(P-\eta T)]^N$, of the local problem

$$\begin{cases} \nabla p_k - \Delta v_k = e_k & \text{in } P-\eta T \\ \nabla \cdot v_k = 0 & \text{in } P-\eta T \\ v_k = 0 & \text{on } \partial(\eta T) \\ p_k,\ v_k\ P\text{-periodic.} \end{cases} \tag{1.4}$$

Besides these work based on asymptotic expansions, Tartar [22] gave a rigorous proof of the convergence, using his energy method (see [23] or [16]). We recall his result

THEOREM 1.1. *Let $(u_\varepsilon, p_\varepsilon)$ be the solution of* (1.1), *and (u,p) be the solution of* (1.2). *Let $\tilde{u}_\varepsilon$ be the extension of the velocity u_ε defined by*

$$\tilde{u}_\varepsilon = u_\varepsilon \text{ in } \Omega_\varepsilon,\ \tilde{u}_\varepsilon = 0 \text{ in } \Omega - \Omega_\varepsilon.$$

There exists an extension P_ε of the pressure such that

$$\begin{cases} \dfrac{\tilde{u}_\varepsilon}{\varepsilon^2} \rightharpoonup u & \text{in } [L^2(\Omega)]^N \text{ weakly} \\ P_\varepsilon \to p & \text{in } L^2(\Omega)/\mathbb{R} \text{ strongly.} \end{cases} \tag{1.5}$$

REMARK 1.2. The main difficulty in Theorem 1.1 is the construction of a uniformly bounded extension of the pressure. L. Tartar built it using a theoretical "dual" argument (see [22]), and later on, R. Lipton and M. Avellaneda [15] made Tartar's extension explicit. Theorem 1.1 has been generalized in [1] to the case of a porous medium with a connected solid part.

1.b) *A corrector result for the velocity.*

In Theorem 1.1 the convergence of the pressure is strong, while that of the velocity is merely weak. A natural question arises: can we improve it by adding a so-called corrector to the velocity, and then obtain a strong convergence? The present subsection is devoted to this problem which, surprisingly, has not been addressed before (to the author's knowledge), although its resolution involves only elementary arguments. In the same setting as in Subsection 1.a, our main result is

THEOREM 1.3. *For $k \in [1, N]$, let v_k be the solution of the local problem* (1.4). *Let us define a function $v_k^\varepsilon \in [H^1(\Omega)]^N$ by*

$$v_k^\varepsilon(x) = v_k\left(\frac{x}{\varepsilon}\right) \text{ for } x \in \Omega. \tag{1.6}$$

Then, the corrector of the velocity is $\sum_{k=1}^{N}({}^t uA^{-1}e_k)v_k^\varepsilon$, *and we have*

$$\left(\frac{\tilde{u}_\varepsilon}{\varepsilon^2} - \sum_{k=1}^{N}({}^t uA^{-1}e_k)v_k^\varepsilon\right) \to 0 \text{ in } [L^2(\Omega)]^N \text{ strongly.} \tag{1.7}$$

REMARK 1.4. There is no assumption on the smoothness of the limit velocity u, and yet the corrector is well defined in $L^2(\Omega)$. As a matter of fact, we notice that, if the obstacle T is smooth, standard regularity results imply that the solution v_k of (1.4) belongs to $[L^\infty(P)]^N$. Thus, the sequence v_k^ε is bounded in $[L^\infty(\Omega)]^N$, and $({}^t uA^{-1}e_k)v_k^\varepsilon$ is bounded in $[L^2(\Omega)]^N$, without any further hypothesis.

Before proving Theorem 1.3, we need to establish some lemmas about the weak semi-continuity of the energy (which are related to the Γ-convergence introduced by E. De Giorgi [7]).

LEMMA 1.5. *Let z_ε be a sequence such that*

(i) $z_\varepsilon \rightharpoonup z$ *in* $[L^2(\Omega)]^N$ *weakly,*
(ii) $(\varepsilon\nabla z_\varepsilon)$ *is a bounded sequence in* $[L^2(\Omega)]^{N^2}$,
(iii) $(\varepsilon\nabla \cdot z_\varepsilon) \to 0$ *in* $L^2(\Omega)$ *strongly,*
(iv) $z_\varepsilon = 0$ *in* $\Omega - \Omega_\varepsilon$.

Then

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \int_\Omega |\nabla z_\varepsilon|^2 \geq \int_\Omega {}^t zA^{-1}z. \tag{1.8}$$

PROOF. We briefly sketch the proof, as it is a simple adaptation, to the case at hand, of a result due to D. Cioranescu and F. Murat [6] (see also Proposition 3.4.6 in [3]).

For any smooth function $\phi \in [D(\Omega)]^N$, let us define a real sequence

$$\Delta_\varepsilon = \varepsilon^2 \int_\Omega \left|\nabla\left(z_\varepsilon - \sum_{k=1}^{N}\phi_k v_k^\varepsilon\right)\right|^2. \tag{1.9}$$

We expend Δ_ε and obtain

$$\Delta_\varepsilon = \varepsilon^2 \int_\Omega |\nabla z_\varepsilon|^2 + \sum_{i,k=1}^N \varepsilon^2 \int_\Omega \phi_k \phi_i \nabla v_k^\varepsilon : \nabla v_i^\varepsilon$$

$$(1.10) \qquad - 2\varepsilon^2 \sum_{k=1}^N \int_\Omega \phi_k \nabla v_k^\varepsilon : \nabla z_\varepsilon + \sum_{i,k=1}^N \varepsilon^2 \int_\Omega \nabla \phi_k \cdot \nabla \phi_i v_k^\varepsilon \cdot v_i^\varepsilon$$

$$+ 2 \sum_{i,k=1}^N \varepsilon^2 \int_\Omega \nabla \phi_k v_k^\varepsilon \phi_i \nabla v_i^\varepsilon - 2\varepsilon^2 \int_\Omega \nabla z_\varepsilon : \left(\sum_{k=1}^N v_k^\varepsilon \nabla \phi_k \right).$$

Using standard estimates, it is easy to show that the last three terms of (1.10) go to zero as ε does. In order to pass to the limit in the second term of (1.10), we notice that the sequence $\varepsilon^2 \nabla v_k^\varepsilon : \nabla v_i^\varepsilon$ is periodic and bounded in $L^1(\Omega)$. Thus, as a measure, it converges to its mean value, and we obtain

$$\sum_{i,k=1}^N \varepsilon^2 \int_\Omega \phi_k \phi_i \nabla v_k^\varepsilon : \nabla v_i^\varepsilon \to \sum_{i,k=1}^N \int_\Omega \phi_k \phi_i \left(\frac{1}{|P|} \int_P \nabla v_k : \nabla v_i \right) = \int_\Omega {}^t\phi A \phi.$$

Integrating by parts the third term of (1.10) yields

$$\varepsilon^2 \int_\Omega \phi_k \nabla v_k^\varepsilon : \nabla z_\varepsilon = \varepsilon^2 \langle \Delta v_k^\varepsilon, \phi_k z_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} - \varepsilon^2 \int_\Omega z_\varepsilon \cdot \nabla v_k^\varepsilon \cdot \nabla \phi_k$$

$$(1.11) \qquad = \varepsilon^2 \langle \nabla p_k^\varepsilon - \Delta v_k^\varepsilon, \phi_k z_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} - \varepsilon^2 \int_\Omega z_\varepsilon \cdot \nabla v_k^\varepsilon \cdot \nabla \phi_k + \varepsilon^2 \int_\Omega p_k^\varepsilon z_\varepsilon \cdot \nabla \phi_k + \varepsilon^2 \int_\Omega p_k^\varepsilon \phi_k \nabla \cdot z_\varepsilon.$$

Recalling from (1.4) that $\varepsilon^2(\nabla p_k^\varepsilon - \Delta v_k^\varepsilon)$ is equal to e_k, we deduce from (1.11)

$$\varepsilon^2 \int_\Omega \phi_k \nabla v_k^\varepsilon : \nabla z_\varepsilon \to \int_\Omega \phi_k z \cdot e_k.$$

Finally, passing to the limit in (1.10) leads to

$$\liminf_{\varepsilon \to 0} \varepsilon^2 \int_\Omega |\nabla z_\varepsilon|^2 \geq 2 \int_\Omega z \cdot \phi - \int_\Omega {}^t\phi A \phi.$$

Replacing ϕ by $A^{-1}z$ gives the desired result.

Q.E.D.

LEMMA 1.6. *If we add to the hypotheses of Lemma 1.5 the assumption that*

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \int_\Omega |\nabla z_\varepsilon|^2 = \int_\Omega {}^t z A^{-1} z,$$

then, we obtain

$$\left(z_\varepsilon - \sum_{k=1}^{N} ({}^t z A^{-1} e_k) v_k^\varepsilon \right) \to 0 \textit{ in } [L^2(\Omega)]^N \textit{ strongly.} \tag{1.12}$$

PROOF. As Lemma 1.5 above, the present lemma is adapted from a result due to D. Cioranescu and F. Murat [6]. Using the new assumption while passing to the limit in (1.10) yields

$$\lim_{\varepsilon\to 0} \Delta_\varepsilon = \int_\Omega {}^t z A^{-1} z + \int_\Omega {}^t\phi A \phi - 2 \int_\Omega z \cdot \phi = \int_\Omega {}^t(z - A\phi) A^{-1} (z - A\phi). \tag{1.13}$$

Then, introducing a sequence ϕ_n of smooth functions which converges to $A^{-1}z$ in $[L^2(\Omega)]^N$, and passing to the limit in (1.13), we obtain

$$\lim_{\varepsilon\to 0} \varepsilon^2 \int_\Omega \left| \nabla \left(z_\varepsilon - \sum_{k=1}^{N} ({}^t z A^{-1} e_k) v_k^\varepsilon \right) \right|^2 = 0.$$

Using Poincaré inequality in Ω_ε (see, e.g., Lemma 1 in [22]) leads to the desired result.

Q.E.D.

PROOF OF THEOREM 1.3. Les ut check that the assumptions of Lemmas 1.5 and 1.6 are satisfied by the sequence of solutions $\tilde{u}_\varepsilon/\varepsilon^2$ and its limit u, solution of the Darcy's law (1.2). The only non-trivial point is the convergence of the energy. We have

$$\int_\Omega |\nabla \tilde{u}_\varepsilon|^2 = \int_\Omega f \cdot \tilde{u}_\varepsilon.$$

From Theorem 1.1 we know that

$$\int_\Omega f \cdot \frac{\tilde{u}_\varepsilon}{\varepsilon^2} \to \int_\Omega f \cdot u.$$

Then, using Darcy's law, and integrating by parts yields

$$\int_\Omega f \cdot u = \int_\Omega (A^{-1}u + \nabla p) \cdot u = \int_\Omega {}^t u A^{-1} u.$$

Finally we obtain

$$\varepsilon^2 \int_\Omega \left| \nabla \frac{\tilde{u}_\varepsilon}{\varepsilon^2} \right|^2 \to \int_\Omega {}^t u A^{-1} u.$$

Applying Lemma 1.6 gives the desired result.

Q.E.D.

2. - Obstacles size smaller than the period

In this second section we recall some results obtained in [3] about the homogenization of a Stokes flow in a porous medium made of periodically distributed obstacles smaller than the period. This is in contrast with the situation in Section 1 where both the obstacles size and the period were of the same order of magnitude.

In the present case, the porous medium Ω_ε is defined exactly as in Section 1 *except that* each hole T_i^ε is similar to the same model obstacle T rescaled to a size a_ε *smaller than the period*. In other words, we assume that the size a_ε satisfies

$$\lim_{\varepsilon \to 0} \frac{a_\varepsilon}{\varepsilon} = 0. \tag{2.1}$$

Now, as in [3], we define a ratio σ_ε

$$\begin{cases} \sigma_\varepsilon = \left(\dfrac{\varepsilon^N}{a_\varepsilon^{N-2}} \right)^{1/2} & \text{for } N \geq 3, \\ \sigma_\varepsilon = \varepsilon \left| \operatorname{Log} \left(\dfrac{a_\varepsilon}{\varepsilon} \right) \right|^{1/2} & \text{for } N = 2. \end{cases}$$

In addition to (2.1), we assume that

$$\lim_{\varepsilon \to 0} \sigma_\varepsilon = 0. \tag{2.2}$$

The assumption (2.2) implies that the obstacles are not too small, so that the homogenized limit of the Stokes flow is always the following Darcy's law

$$\begin{cases} u = M^{-1}(f - \nabla p) & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where (u, p) are the velocity and pressure of the fluid (problem (2.3) is a second order elliptic equation for the pressure p, and there exists a unique solution p in $H^1(\Omega)/\mathbb{R}$).

To compute the permeability tensor M^{-1}, we need to introduce the local problem for each $k \in \{1, \dots, N\}$

$$
\text{(2.4)} \qquad \begin{cases} \nabla q_k - \Delta w_k = 0 & \text{in } \mathbb{R}^N - T, \\ \nabla \cdot w_k = 0 & \text{in } \mathbb{R}^N - T, \\ w_k = 0 & \text{on } \partial T, \\ w_k = e_k & \text{at infinity, if } N \geq 3, \\ w_k = (\log r) e_k & \text{at infinity, if } N = 2. \end{cases}
$$

Denoting by F_k the drag force applied on T, i.e., $F_k = \int_{\partial T} \left(q_k n - \frac{\partial w_k}{\partial n} \right)$, where n is the interior normal vector of ∂T, the matrix M is given by

$$
\text{(2.5)} \qquad M e_k = \frac{1}{2^N} F_k.
$$

The formula (2.5) is valid in any dimension, but there is actually a big difference between the 2-D case and the others. For any obstacle T, it turns out that

$$
\text{(2.6)} \qquad \text{if } N = 2: \quad M = \pi \, \mathrm{Id}
$$

$$
\text{(2.7)} \qquad \text{if } N \geq 3: \quad {}^t e_i M e_k = \frac{1}{2^N} \int_{\mathbb{R}^N - T} \nabla w_i : \nabla w_k.
$$

The result (2.6) is a consequence of the well-known Stokes paradox and of the Finn-Smith paradox [8], while formula (2.7) is obtained through an integration by parts, valid only for a space dimension $N \geq 3$ (see [4] for details).

Now, we recall the precise convergence theorem [3].

THEOREM 2.1. *Assume the hole size satisfies* (2.1) *and* (2.2). *Let* $(u_\varepsilon, p_\varepsilon)$ *be the unique solution of the Stokes system* (1.1). *Let* $\tilde{u}_\varepsilon$ *be the extension of the velocity by* 0 *in* $\Omega - \Omega_\varepsilon$. *There exists an extension* P_ε *of the pressure such that*

$$
\begin{cases} \dfrac{\tilde{u}_\varepsilon}{\sigma_\varepsilon^2} \rightharpoonup u & \text{in } [L^2(\Omega)]^N \text{ weakly}, \\ P_\varepsilon \rightarrow p & \text{in } L^2(\Omega)/\mathbb{R} \text{ strongly}, \end{cases}
$$

where (u, p) *is the unique solution of the Darcy's law* (2.3).

REMARK 2.2. For *any* hole size, satisfying (2.1) and (2.2), we obtain a Darcy's law with the *same* permeability tensor, independent of the hole size. In other words, the limit problem is always the same Darcy's law (2.3) for all the range of sizes a_ε satisfying (2.1) and (2.2); the only difference between two sizes is the value σ_ε which rescales the velocity.

Let us mention also that assumption (2.2) is required to obtain a Darcy's law as the homogenized limit. If the limit of σ_ε is strictly positive and finite, then the limit problem is Brinkman's law, while, if σ_ε goes to infinity, then the Stokes equations are obtained at the limit (see [3] for more details).

We have not yet said anything about existence and uniqueness of solutions of the local problem (2.4). Let us begin by a definition: $D^{1,2}(\mathbb{R}^N - T)$ is defined as the completion, with respect to the L^2-norm of the gradient, of the space of all smooth functions with compact support in $\mathbb{R}^N - T$, i.e.,

$$D^{1,2}(\mathbb{R}^N - T) = \overline{\mathcal{D}(\mathbb{R}^N - T)}^{\|\nabla\phi\|_{L^2(\mathbb{R}^N)}}. \tag{2.8}$$

Though H^1 is the usual space of admissible velocities for a Stokes problem in a bounded domain, it is well-known (see Section 2, Chapter 2 of [11]) that $D^{1,2}$ is the "natural" space for a Stokes problem in an exterior domain. Now, we are in a position to state

LEMMA 2.3. *Let $\theta(x)$ be a smooth function equal to* 0 *on the obstacle T, and equal to* 1 *in a neighbourhood of infinity. If $N \geq 3$, the local problem* (2.4) *admits a unique solution $(q_k, w_k - \theta e_k)$ in $L^2(\mathbb{R}^N - T) \times [D^{1,2}(\mathbb{R}^N - T)]^N$. Furthermore, the boundary condition at infinity is satisfied through the following Sobolev embedding*

$$D^{1,2}(\mathbb{R}^N - T) \subset L^{\frac{2N}{N-2}}(\mathbb{R}^N - T). \tag{2.9}$$

If $N = 2$, there exists a unique solution (q_k, w_k) of (2.4) *which is the sum of two terms. The first one, (q_k^0, w_k^0), is the solution of* (2.4), *when the obstacle T is the unit ball, which is explicity given by*

$$\begin{cases} w_k^0 = x_k r f(r) e_r + g(r) e_k \\ q_k^0 = x_k r h(r) \end{cases} \tag{2.10}$$

$$\textit{with} \quad \begin{cases} f(r) = -\dfrac{1}{r^2} + \dfrac{1}{r^4} \\ g(r) = \operatorname{Log} r - \dfrac{1}{2r^2} + \dfrac{1}{2} \\ h(r) = -\dfrac{2}{r^2}. \end{cases}$$

The second one, (q_k', w_k'), is now the solution of a "difference" problem which admits a unique solution in $L^2(\mathbb{R}^2 - T) \times [D^{1,2}(\mathbb{R}^2 - T)]^2$

$$\begin{cases} \nabla q_k' - \Delta w_k' = \left(q_k^0 e_r - \dfrac{\partial w_k^0}{\partial r} \right) \delta_{B_1} = \dfrac{2}{\pi} e_k \delta_{B_1} & \textit{in } \mathbb{R}^2 - T \\ \nabla \cdot w_k' = 0 & \textit{in } \mathbb{R}^2 - T \\ w_k' = 0 & \textit{on } \partial T \\ w_k' = o(\log r) \textit{ at } \infty \left(\textit{i.e., } \dfrac{|w_k'|}{\log r} \to 0 \right), \end{cases} \tag{2.11}$$

where δ_{B_1} is the unit mass measure concentrated on the unit sphere ∂B_1 (recall that we have assumed that T is included in the ball B_1).

Furthermore, the boundary condition at infinity is satisfied through the following embedding

$$\phi \in D^{1,2}(\mathbb{R}^2 - T) \Rightarrow \frac{\phi}{(r+1)\log(r+2)} \in L^2(\mathbb{R}^2 - T), \tag{2.12}$$

which can be interpreted as $\phi = o(\log r)$, because $\log r$ *does not belong to the space on the right-hand-side of* (2.12).

For a proof of Lemma 2.3, we refer to Lemma 2.2 in [4] (see also [11] and [18]).

In the sequel we shall need a characterization of the space $D^{1,2}(\mathbb{R}^N - T)$ which is given by

LEMMA 2.4. *If $N \geq 3$, then*

$$D^{1,2}(\mathbb{R}^N - T) = \left\{\phi \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)/\phi = 0 \text{ on } T, \text{ and } \nabla\phi \in [L^2(\mathbb{R}^N)]^N\right\}.$$

If $N = 2$, then

$$D^{1,2}(\mathbb{R}^2 - T)$$

$$= \left\{\phi/\phi = 0 \text{ on } T, \ \frac{\phi}{(r+1)\log(r+2)} \in L^2(\mathbb{R}^2), \text{ and } \nabla\phi \in [L^2(\mathbb{R}^2)]^2\right\}.$$

PROOF. Thanks to (2.9) and (2.12), we already know that $D^{1,2}(\mathbb{R}^N - T)$ is included in the space on the right-hand-side (let us call it H). It remains to prove that any function of H is the limit of a sequence of smooth functions with compact support in $\mathbb{R}^N - T$, such that the sequence of their gradients is bounded in $L^2(\mathbb{R}^N - T)$.

Let ρ_n be a sequence of mollifiers with compact support in a ball of radius $1/n$. Let χ be a smooth cut-off function defined by

$$\begin{cases} \chi(r) = 1 & \text{for } r \leq 1, \\ \chi'(r) \leq 0 & \text{for } 1 \leq r \leq 2, \\ \chi(r) = 0 & \text{for } r \geq 2. \end{cases}$$

If $N \geq 3$, let $\chi_n(x) = \chi\left(\frac{|x|}{n}\right)$. For any $\phi \in H$, we define a sequence

$$\phi_n = (\phi * \rho_n)\chi_n.$$

The function $\phi * \rho_n$ is a regularization by convolution with a mollifier of the function ϕ, and it is well-known that $\phi * \rho_n$ converges almost everywhere to ϕ.

Thus

$$\phi_n \to \phi \quad \text{a.e. in } \mathbb{R}^N. \tag{2.13}$$

Using elementary properties of the convolution, we have

$$\nabla\phi_n = (\nabla\phi * \rho_n)\chi_n + (\phi * \rho_n)\nabla\chi_n \tag{2.14}$$

where $\nabla\phi * \rho_n$ is bounded in $L^2(\mathbb{R}^N)$, $\phi * \rho_n$ is bounded in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, and $\nabla\chi_n$ is bounded in $L^N(\mathbb{R}^N)$. Thus, (2.14) implies that $\nabla\phi_n$ is bounded in $L^2(\mathbb{R}^N)$, which, together with (2.13) yields the desired result.

If $N = 2$, let $\chi_n(x) = \chi\left(\frac{\log|x|}{\log n}\right)$. For any $\phi \in H$, we define a sequence

$$\phi_n = (\phi * \rho_n)\chi_n.$$

The convergence (2.13) still holds, but now, we bound (2.14) by noticing that $\nabla\phi * \rho_n$ is bounded in $L^2(\mathbb{R}^2)$, $\frac{\phi * \rho_n}{(r+1)\log(r+2)}$ is bounded in $L^2(\mathbb{R}^2)$, and $(r+1)\log(r+2)\nabla\chi_n$ is bounded in $L^\infty(\mathbb{R}^2)$. Thus, (2.14) implies that $\nabla\phi_n$ is bounded in $L^2(\mathbb{R}^2)$, which leads to the desired result.

We have cheated a bit because ϕ_n is equal to zero, *not in* T, but in a smaller subset which tends to T as n goes to infinity. Nevertheless, ϕ_n is equal to zero in T if the function ϕ is equal to zero in a neighbourhood of T. Thus we can remedy this, by approximating any function ϕ of H by a sequence of functions of H which are supported away of T, then apply the above result to each function of this sequence, and finally extract a diagonal sequence and conclude.

Q.E.D

3. - Main result: continuity of the permeability tensor

In Sections 1 and 2, under different hypotheses on the obstacle size, we have obtained, in both cases, a Darcy's law as the homogenized problem. The permeability tensors, associated to these Darcy's law, are given, either by (1.3) or (2.5). Obviously these formulae are completely different, and so are the permeability tensors M^{-1} and A. From a physical point of view, this statement seems paradoxical. Why should the permeability tensor be the same for a large range of obstacle sizes smaller than the period, and then have a different value when the obstacle size is equal to the period? We shall prove in this last section that, beyond this seemingly paradox, there is an actual continuity of the permeability tensor.

In Section 1 we took care of defining the size $\eta\varepsilon$ of the obstacles T_i^ε as the product of two contributions: first, ε is the size of the period (which goes to zero in the homogenization process), and second, η is the size of the obstacle *in the unit cell* (which is a constant during the homogenization process). Thus, after passing to the homogenized limit (i.e., ε goes to zero) in Section 1, we can take a second limit (the so-called low-volume fraction limit) as η goes to zero. We shall prove that, in this second limit process, the permeability tensor A (which depends on η, and, from now on, denoted by $A(\eta)$) converges (after a suitable rescaling) to the permeability tensor M^{-1} which was obtained in Section 2. More precisely, we obtain

THEOREM 3.1. *Let (p_k, v_k) be the unique solution of the cell problem* (1.4) *in $[L^2_\#(P)/\mathbb{R}] \times [H^1_\#(P)]^N$. Rescaling it, for $x \in (\eta^{-1}P - T)$ and for any space dimension, we define*

$$(3.1) \qquad v^\eta_k(x) = \eta^{N-2} v_k(\eta x) \text{ and } p^\eta_k(x) = \eta^{N-1} p_k(\eta x).$$

Let (q_i, w_i) be the unique solution of the local problem (2.4) *in Section 2. We define a linear combination of those solutions: $(\tilde{p}_k, \tilde{v}_k) = \sum_{i=1}^{N} ({}^t e_i M^{-1} e_k)(q_i, w_i)$. Then (p^η_k, v^η_k) converges weakly to $(\tilde{p}_k, \tilde{v}_k)$ in*

$$[L^2_{\text{loc}}(\mathbb{R}^N - T)/\mathbb{R}] \times [H^1_{\text{loc}}(\mathbb{R}^N - T)]^N.$$

Furthermore we have

$$(3.2) \qquad \begin{cases} \lim_{\eta \to 0} \eta^{N-2} A(\eta) = M^{-1} & \text{for } N \geq 3 \\ \lim_{\eta \to 0} \dfrac{1}{|\log \eta|} A(\eta) = M^{-1} & \text{for } N = 2. \end{cases}$$

REMARK 3.2. In order to compare the solutions of (1.4) and (2.4), we rescale the unit cell, involved in (1.4), to size $1/\eta$, so that in both case the obstacle has the same fixed size. In (3.2) we also need to rescale the matrix $A(\eta)$ in order to obtain M^{-1} as its limit. As we expected it, the permeability $A(\eta)$ increases (and even goes to infinity) as η tends to zero, i.e., as the volume of the obstacles decreases.

A striking aspect of (3.2) is that both A and M are the energies of some local problems (see formulae (1.3) and (2.7)), but A actually converges to the inverse of M. On the same token, let us emphasize that the rescaled solutions of the cell problem (1.4) do not converge to the solutions of the local problem (2.4), but rather to a linear combination of them. This can be explained in the light of the associated corrector results for the velocity. Such a result is usually of the form: if the sequence u_ε converges weakly to its limit u, then there exist some functions $(b^\varepsilon_k)_{1 \leq k \leq N}$ such that $u_\varepsilon - \sum_{k=1}^{N} b^\varepsilon_k u_k$ converges strongly to zero (where u_k is the k^{th} component of u). The functions $(b^\varepsilon_k)_{1 \leq k \leq N}$, which are equal to zero on the obstacles, are interpreted as the boundary layers around them. For example, in Theorem 1.2.3 and Remark 2.1.5 of [3] the solutions of (2.4) (rescaled to size a_ε) are used as boundary layers. On the contrary in Theorem 1.3, we do not introduce the solutions of the cell problem (1.4), but rather a linear combination of them. Actually, in the setting of Section 1, the exact boundary layers are $\left(\sum_{k=1}^{N}({}^t e_i A^{-1} e_k) v_k\right)_{1\le i\le N}$, and not the $(v_k)_{1\le i\le N}$ themselves. Thus, although both local problems are used to define the test functions in the proof of the convergence, their solutions do not have the same meaning. Only the solutions of (2.4) are boundary layers, while the solutions of (1.4) are the periodic flows under a unit constant force.

REMARK 3.3. Theorem 3.1 has already been stated, in a slightly different form, by E. Sanchez-Palencia [18], but proved only in the 3-D case. However, the convergence (3.2) of the permeability tensor is new. For a periodic distribution of spherical obstacles, the low-volume fraction limit has been investigated by H. Hasimoto [9], and A.S. Sangani and A. Acrivos [21]. Their method is different of ours: first, they construct an explicit fundamental solution of the Stokes problem, and second, they do an asymptotic expansion in η of it. Consequently, they also obtain an asymptotic expansion of $A(\eta)$ (which is a scalar matrix in that case), not restricted to the first term as in Theorem 3.1. Unfortunately, their method works only for spherical obstacles.

Before proving Theorem 3.1, we give a Poincaré inequality in $P - \eta T$.

LEMMA 3.4. *There exists a constant C, which depends only on T, such that, for any $v \in [H^1(P - \eta T)]^N$ satisfying $v = 0$ on the boundary $\partial(\eta T)$, we have*

$$\begin{cases} \|v\|_{L^2(P-\eta T)} \le \dfrac{C}{\eta^{(N-2)/2}} \|\nabla v\|_{L^2(P-\eta T)} & \text{for } N \ge 3, \\ \|v\|_{L^2(P-\eta T)} \le C(\log|\eta|)^{1/2} \|\nabla v\|_{L^2(P-\eta T)} & \text{for } N = 2. \end{cases}$$

The proof of this lemma is elementary, so we skip it (see Lemma 4.1 of E. Sanchez-Palencia [18], or adapt the ideas of Lemma 3.4.1 in [3]).

PROOF OF THEOREM 3.1. Let us first obtain some a priori estimates for the solutions of the cell problem (1.4). Taking two solutions v_k and v_i of (1.4), and integrating by parts leads to

$$\int_{P-\eta T} \nabla v_k : \nabla v_i = \int_{P-\eta T} v_i \cdot e_k. \tag{3.3}$$

Using Lemma 3.4, we deduce from (3.3)

$$\begin{cases} \|v_k\|_{L^2(P-\eta T)} \le \dfrac{C}{\eta^{(N-2)}} \text{ and } \|\nabla v_k\|_{L^2(P-\eta T)} \le \frac{C}{\eta^{(N-2)/2}} & \text{for } N \ge 3, \\ \|v_k\|_{L^2(P-\eta T)} \le C|\log \eta| \text{ and } \|\nabla v_k\|_{L^2(P-\eta T)} \le C|\log \eta|^{1/2} & \text{for } N = 2. \end{cases} \tag{3.4}$$

Besides, with the help of Lemma 2.2.4 in [3], it is not difficult to obtain the following estimate for the pressure

$$(3.5)\qquad \begin{cases} \|p_k\|_{L^2(P-\eta T)/\mathbb{R}} \leq \dfrac{C}{\eta^{(N-2)/2}} & \text{for } N \geq 3, \\ \|p_k\|_{L^2(P-\eta T)/\mathbb{R}} \leq C|\log \eta|^{1/2} & \text{for } N = 2. \end{cases}$$

Now, we rescale the unit cell $P - \eta T$ in $\eta^{-1}P - T$ in order to work in a domain where the obstacle T has a fixed size. Let us recall the definition (3.1) of the rescaled solutions of (1.4)

$$v_k^\eta(x) = \eta^{N-2} v_k(\eta x) \quad \text{and} \quad p_k^\eta(x) = \eta^{N-1} p_k(\eta x).$$

The functions (p_k^η, v_k^η) are solutions of the following Stokes system

$$(3.6)\qquad \begin{cases} \nabla p_k^\eta - \Delta v_k^\eta = \eta^N e_k & \text{in } \eta^{-1}P - T, \\ \nabla \cdot v_k^\eta = 0 & \text{in } \eta^{-1}P - T, \\ v_k^\eta = 0 & \text{on } \partial T, \\ p_k^\eta, v_k^\eta \quad \eta^{-1}P\text{-periodic}. \end{cases}$$

The problem is now to find the limit of (p_k^η, v_k^η). For that purpose, we separate in two cases, according to the space dimension.

1) $N \geq 3$.

Rescaling the estimates (3.4) and (3.5) yields

$$(3.7)\qquad \|p_k^\eta\|_{L^2(\eta^{-1}P-T)/\mathbb{R}} \leq C \text{ and } \|\nabla v_k^\eta\|_{L^2(\eta^{-1}P-T)} \leq C \text{ for } N \geq 3.$$

From (3.7) we easily deduce that there exists $(\tilde{p}_k, \tilde{v}_k)$ in $[L^2(\mathbb{R}^N - T)/\mathbb{R}] \times [H^1_{\text{loc}}(\mathbb{R}^N - T)]^N$ such that, up to a subsequence, we get

$$(p_k^\eta, v_k^\eta) \rightharpoonup (\tilde{p}_k, \tilde{v}_k) \text{ in } [L^2(\mathbb{R}^N - T)/\mathbb{R}] \times [H^1_{\text{loc}}(\mathbb{R}^N - T)]^N.$$

Moreover, multiplying equations (3.6) by a smooth function with compact support in $\mathbb{R}^N$, then passing to the limit as η goes to zero, we check that $(\tilde{p}_k, \tilde{v}_k)$ satisfies the following system

$$(3.8)\qquad \begin{cases} \nabla \tilde{p}_k - \Delta \tilde{v}_k = 0 & \text{in } \mathbb{R}^N - T, \\ \nabla \cdot \tilde{v}_k = 0 & \text{in } \mathbb{R}^N - T, \\ \tilde{v}_k = 0 & \text{on } \partial T. \end{cases}$$

In order to identify the limit $(\tilde{p}_k, \tilde{v}_k)$, we have to find what type of boundary condition is satisfied by $\tilde{v}_k$ at infinity.

Let us recall the following Sobolev inequality with critical exponent

$$\left\| v_k^\eta - \frac{1}{|\eta^{-1}P|} \int_{\eta^{-1}P} v_k^\eta \right\|_{L^{2N/(N-2)}(\eta^{-1}P)} \leq C \|\nabla v_k^\eta\|_{L^2(\eta^{-1}P)}, \tag{3.9}$$

where, for a matter of simplicity, v_k^η denotes both the original function defined in $\eta^{-1}P - T$, and its extension by zero in T. The key point in inequality (3.9) is that the constant C does not depend on η, because the Sobolev inequality with exactly the critical exponent $2N/(N-2)$ is invariant under dilatation. On the other hand, with the help of (3.4), we have

$$\left| \frac{1}{|\eta^{-1}P|} \int_{\eta^{-1}P} v_k^\eta \right| = \left| \frac{\eta^{N-2}}{|P|} \int_P v_k \right| \leq C.$$

Thus, by possibly extracting a subsequence, there exists a constant vector $c_k \in \mathbb{R}^N$ such that

$$\lim_{\eta \to 0} \frac{1}{|\eta^{-1}P|} \int_{\eta^{-1}P} v_k^\eta = c_k.$$

From inequality (3.9) and estimate (3.7), we deduce that for any bounded set ω in $\mathbb{R}^N$

$$\left\| v_k^\eta - \frac{1}{|\eta^{-1}P|} \int_{\eta^{-1}P} v_k^\eta \right\|_{L^{2N/(N-2)}(\omega)} \leq C. \tag{3.10}$$

Passing to the limit in (3.10), and using the weak lower semi-continuity of the $L^{2N/(N-2)}$ norm, yields

$$\|\tilde{v}_k - c_k\|_{L^{2N/(N-2)}(\omega)} \leq C$$

where C does not depend on ω. Hence, $\tilde{v}_k - c_k$ belongs to $[L^{2N/(N-2)}(\mathbb{R}^N)]^N$. From (3.7) we also know that $\nabla \tilde{v}_k$ belongs to $[L^2(\mathbb{R}^N)]^{N^2}$. Thus, using Lemma 2.4, we deduce that $\tilde{v}_k$ is a function of $[D^{1,2}(\mathbb{R}^N)]^N$. In other words, the boundary condition associated with (3.8) is

$$\tilde{v}_k = c_k \text{ at infinity.} \tag{3.11}$$

But the constant c_k is still unknown. In order to find its value, we compute the drag force corresponding to the Stokes problem (3.8) and (3.11). Let ϕ be a smooth function with compact support in the unit ball, and identically equal to 1 on the set T. Let us multiply equation (3.6) by ϕe_i

$$\int_{B_1} \nabla v_k^\eta : (\nabla \phi \otimes e_i) + \int_{\partial T} \left(p_k^\eta n - \frac{\partial v_k^\eta}{\partial n} \right) \cdot e_i = \int_{B_1} \eta^N \phi e_k \cdot e_i. \tag{3.12}$$

Passing to the limit in (3.12), and integrating by parts yields

$$\lim_{\eta\to 0}\int\limits_{\partial T}\left(p_k^\eta n-\frac{\partial v_k^\eta}{\partial n}\right)\cdot e_i=-\int\limits_{B_1}\nabla\tilde{v}_k:(\nabla\phi\otimes e_i)=\int\limits_{\partial T}\left(\tilde{p}_k n-\frac{\partial\tilde{v}_k}{\partial n}\right)\cdot e_i.$$

On the other hand, thanks to the periodic boundary condition, integrating the equation (3.6) gives

$$\int\limits_{\partial T}\left(p_k^\eta n-\frac{\partial v_k^\eta}{\partial n}\right)\cdot e_i=\int\limits_{\eta^{-1}P-T}\eta^N e_k=\frac{1}{|P-\eta T|}e_k.$$

Finally we obtain the value of the drag force

$$\int\limits_{\partial T}\left(\tilde{p}_k n-\frac{\partial\tilde{v}_k}{\partial n}\right)\cdot e_i=\frac{1}{2^N}e_k.$$

Using formula (2.7), it is easy to see that, for a given drag force, there exists a unique solution of the Stokes problem (3.8) in the space $L^2(\mathbb{R}^N-T)\times[D^{1,2}(\mathbb{R}^N-T)]^N$. Thus, we can identify the solution $(\tilde{p}_k,\tilde{v}_k)$ with the following sum of solutions (q_i,w_i) of the local problem (2.6)

$$(\tilde{p}_k,\tilde{v}_k)=\sum_{i=1}^{N}({}^te_iM^{-1}e_k)(q_i,w_i).$$

Hence the constant c_k is equal to $\sum_{i=1}^{N}({}^te_iM^{-1}e_k)e_i$. The limit $(\tilde{p}_k,\tilde{v}_k)$ is uniquely determined in $L^2(\mathbb{R}^N-T)\times[D^{1,2}(\mathbb{R}^N-T)]^N$, thus the entire sequence converges to that limit. Finally, the permeability tensor $A(\eta)$ is given by formula (1.3)

$${}^te_iA(\eta)e_k=\frac{1}{2^N}\int\limits_{P-\eta T}\nabla v_k:\nabla v_i=\frac{1}{2^N}\int\limits_{P-\eta T}e_i\cdot v_k.$$

Then, we obtain

$$\eta^{N-2}\ {}^te_iA(\eta)e_k=\frac{1}{|\eta^{-1}P|}\int\limits_{\eta^{-1}P-T}e_i\cdot v_k^\eta\to c_k\cdot e_i={}^te_iM^{-1}e_k,$$

which is the desired result.

2) $N=2$.

As we have already seen it in Lemma 2.3, the two-dimensional case is completely different from the other ones; for example, we cannot deduce uniform a priori estimates like (3.7) for (p_k^η, v_k^η). Note that, we could have done so, if we had further rescaled, and worked with $(p_k^\eta/|\log \eta|^{1/2}, v_k^\eta/|\log \eta|^{1/2})$ instead. Unfortunately, it turns out that the sequence $v_k^\eta/|\log \eta|^{1/2}$ converges to zero! Consequently, we follow the ideas of Lemma 2.3 (see also [4]), and we decompose the solution (p_k^η, v_k^η) of (3.6) in two terms. The first one is the solution $(p_k^{0\eta}, v_k^{0\eta})$ of the following Stokes problem in the ball $B_{\eta^{-1}}$ of radius $1/\eta$

$$(3.13)\qquad \begin{cases} \nabla p_k^{0\eta} - \Delta v_k^{0\eta} = 0 & \text{in } B_{\eta^{-1}} - B_1, \\ \nabla \cdot v_k^{0\eta} = 0 & \text{in } B_{\eta^{-1}} - B_1, \\ v_k^{0\eta} = 0 & \text{in } B_1, \\ v_k^{0\eta} = -(\log \eta) e_k & \text{in } \mathbb{R}^2 - B_{\eta^{-1}}. \end{cases}$$

We can explicitely compute $(p_k^{0\eta}, v_k^{0\eta})$ in $B_{\eta^{-1}} - B_1$

$$(3.14)\qquad \begin{cases} v_k^{0\eta} = x_k r f_\eta(r) e_r + g_\eta(r) e_k \\ p_k^{0\eta} = x_k r h_\eta(r) \end{cases}$$

$$(3.15)\qquad \text{with } \begin{cases} f_\eta(r) = -\dfrac{a(\eta)}{r^2} + \dfrac{b(\eta)}{r^4} + c(\eta) \\ g_\eta(r) = a(\eta) \log r - \dfrac{b(\eta)}{2r^2} - \dfrac{3c(\eta)}{2} r^2 + d(\eta) \\ h_\eta(r) = -\dfrac{2a(\eta)}{r^2} - 4c(\eta) \end{cases}$$

where

$$a(\eta) = \frac{\log \eta}{\log \eta + \dfrac{1-\eta^2}{1+\eta^2}}, \quad b(\eta) = \frac{a(\eta)}{1+\eta^2},$$

$$c(\eta) = \frac{\eta^2 a(\eta)}{1+\eta^2}, \quad \text{and} \quad d(\eta) = a(\eta) \frac{1+3\eta^2}{2(1+\eta^2)}.$$

It is easy to see that $a(\eta)$ converges to 1, as η goes to zero, and therefore, that the solutions $(p_k^{0\eta}, v_k^{0\eta})$ of (3.13) converge pointwise (and even uniformly in any compact subset of $\mathbb{R}^2$) to the functions (q_k^0, w_k^0) given in (2.10). Furthermore, an easy computation shows that

$$(3.16)\qquad \begin{cases} \left(p_k^{0\eta} e_r - \dfrac{\partial v_k^{0\eta}}{\partial r} \right)_{(r=1)} = -\dfrac{2a(\eta)}{(1+\eta^2)} \left[4\eta^2 (e_k \cdot e_r) e_r + (1-\eta^2) e_k \right] \\ \left(p_k^{0\eta} e_r - \dfrac{\partial v_k^{0\eta}}{\partial r} \right)_{(r=\eta^{-1})} = -\dfrac{2\eta a(\eta)}{(1+\eta^2)} \left[4 (e_k \cdot e_r) e_r - (1-\eta^2) e_k \right]. \end{cases}$$

Now, the second term is the difference defined by

$$(p_k'^{\eta}, v_k'^{\eta}) = \left(p_k^{\eta} - \frac{1}{\pi a(\eta)} p_k^{0\eta}, v_k^{\eta} - \frac{1}{\pi a(\eta)} v_k^{0\eta}\right).$$

The underlying idea of this decomposition in two terms is that we can obtain uniform a priori estimates for those "difference" functions. Let δ_{B_1} (resp. $\delta_{B_{\eta^{-1}}}$) be the unit mass measure concentrated on the boundary of the ball B_1 (resp. $B_{\eta^{-1}}$). The "difference" functions are solutions of the following Stokes system

$$\text{(3.17)}\quad \begin{cases} \nabla p_k'^{\eta} - \Delta v_k'^{\eta} = \eta^2 e_k - \dfrac{1}{\pi a(\eta)}\left(p_k^{0\eta} e_r - \dfrac{\partial v_k^{0\eta}}{\partial r}\right)\delta_{B_1} \\ \qquad\qquad + \dfrac{1}{\pi a(\eta)}\left(p_k^{0\eta} e_r - \dfrac{\partial v_k^{0\eta}}{\partial r}\right)\delta_{B_{1/\eta}} & \text{in } \eta^{-1}P - T, \\ \nabla \cdot v_k'^{\eta} = 0 & \text{in } \eta^{-1}P - T, \\ v_k'^{\eta} = 0 & \text{on } \partial T, \\ p_k'^{\eta}, v_k'^{\eta} \quad \eta^{-1}P\text{periodic}. \end{cases}$$

Let us prove that the solution $(p_k'^{\eta}, v_k'^{\eta})$ of (3.17) converges to $\left(\frac{q_k'}{\pi}, \frac{w_k'}{\pi}\right)$, where (q_k', w_k') is the solution of (2.11). We define two measures by

$$\begin{cases} \mu_\eta = -\dfrac{1}{\pi a(\eta)}\left(p_k^{0\eta} e_r - \dfrac{\partial v_k^{0\eta}}{\partial r}\right)\delta_{B_1}, \\ \nu_\eta = \eta^2 e_k + \dfrac{1}{\pi a(\eta)}\left(p_k^{0\eta} e_r - \dfrac{\partial v_k^{0\eta}}{\partial r}\right)\delta_{B_{\eta^{-1}}}. \end{cases}$$

Thanks to (3.16), it is obvious that the sequence μ_η is compactly supported on ∂B_1, and converges strongly to $\frac{2}{\pi} e_k \delta_{B_1}$ in $[H^{-1}(B_2)]^2$, and that the sequence ν_η has mean value zero in $\eta^{-1}P$. An integration by parts in (3.17) leads to

$$\text{(3.18)}\quad \begin{aligned} \int_{\eta^{-1}P-T} |\nabla v_k'^{\eta}|^2 &= \langle \mu_\eta, v_k'^{\eta}\rangle_{H_\#^{-1}, H_\#^1(\eta^{-1}P)} + \langle \nu_\eta, v_k'^{\eta}\rangle_{H_\#^{-1}, H_\#^1(\eta^{-1}P)} \\ &= \langle \mu_\eta, v_k'^{\eta}\rangle_{H_\#^{-1}, H_\#^1(\eta^{-1}P)} \\ &\quad + \langle \nu_\eta, v_k'^{\eta} - \frac{1}{|\eta^{-1}P|}\int_{\eta^{-1}P} v_k'^{\eta}\rangle_{H_\#^{-1}, H_\#^1(\eta^{-1}P)}. \end{aligned}$$

Using the fact that μ_η is a bounded sequence of measures supported on ∂B_1, and Poincaré inequality in $B_1 - T$, we bound the first term in the right-hand side of (3.18)

$$|\langle \mu_\eta, v_k'^\eta \rangle_{H_\#^{-1}, H_\#^1(\eta^{-1}P)}| \leq C \|v_k'^\eta\|_{L^2(\partial B_1)} \leq C \|\nabla v_k'^\eta\|_{L^2(B_1 - T)}.$$

The second term is more tricky. First we need to rescale it to size 1

$$\langle \nu_\eta, v_k'^\eta - \frac{1}{|\eta^{-1}P|} \int_{\eta^{-1}P} v_k'^\eta \rangle_{H_\#^{-1}, H_\#^1(\eta^{-1}P)} = \langle \nu, v_k' - \frac{1}{|P|} \int_P v_k' \rangle_{H_\#^{-1}, H_\#^1(P)}, \tag{3.19}$$

where the measure ν is defined by $\nu = e_k - \frac{2}{\pi(1+\eta^2)}[4(e_k \cdot e_r)e_r - (1-\eta^2)e_k]\delta_{B_1}$, and the function v_k' is given by $v_k'(x) = v_k'^\eta\left(\frac{x}{\eta}\right)$. Now we bound (3.19) using Poincaré-Wirtinger inequality in P and the fact that the measure ν is bounded independently of η

$$\left| \langle \nu, v_k' - \frac{1}{|P|} \int_P v_k' \rangle_{H_\#^{-1}, H_\#^1(P)} \right| \leq C \left\| v_k' - \frac{1}{|P|} \int_P v_k' \right\|_{H^1(P)} \leq C \|\nabla v_k'\|_{L^2(P)}.$$

In two dimensions, we have $\|\nabla v_k'\|_{L^2(P)} = \|\nabla v_k'^\eta\|_{L^2(\eta^{-1}P)}$, thus

$$\left| \langle \nu_\eta, v_k'^\eta - \frac{1}{|\eta^{-1}P|} \int_{\eta^{-1}P} v_k'^\eta \rangle_{H_\#^{-1}, H_\#^1(\eta^{-1}P)} \right| \leq C \|\nabla v_k'^\eta\|_{L^2(\eta^{-1}P)}.$$

Finally, we get from (3.18)

$$\|\nabla v_k'^\eta\|_{L^2(\eta^{-1}P)} \leq C. \tag{3.20}$$

Since $v_k'^\eta$ is a bounded sequence in $[H^1_{\text{loc}}(\mathbb{R}^2)]^2$, we extract a subsequence which converges to some limit w_k'/π. With the help of Lemma 2.2.4 in [3], it is easy to obtain a similar result for the pressure $p_k'^\eta$ which is bounded in $L^2(\mathbb{R}^2 - T)/\mathbb{R}$, and converges, up to a subsequence, to some limit q_k'/π. To see which equations are satisfied by those limit, we multiply (3.17) by a smooth function with compact support, and we pass to the limit as η goes to zero. Therefore we obtain

$$\begin{cases} \nabla q_k' - \Delta w_k' = 2e_k\delta_{B_1} & \text{in } \mathbb{R}^2 - T, \\ \nabla \cdot w_k' = 0 & \text{in } \mathbb{R}^2 - T, \\ w_k' = 0 & \text{on } \partial T. \end{cases} \tag{3.21}$$

Furthermore, we know that $q_k' \in L^2(\mathbb{R}^2 - T)/\mathbb{R}$ and $\nabla w_k' \in [L^2(\mathbb{R}^2 - T)]^4$, and if we prove that $\frac{w_k'}{(r+1)\log(r+2)}$ belongs to $[L^2(\mathbb{R}^2 - T)]^2$, then w_k' belongs to $[D^{1,2}(\mathbb{R}^2 - T)]^2$, according to Lemma 2.4, Fortunately, because $v_k'^\eta$ is equal to zero on ∂T, and satisfies periodic boundary condition on $\eta^{-1}\partial P$, it is easy to adapt a lemma of O.A. Ladyzhenskaya (see Section 1.4 in Chapter 1 of [11]), and to show that

$$\left\| \frac{v_k'^\eta}{(r+1)\log(r+2)} \right\|_{L^2(\eta^{-1}P-T)} \leq C\|\nabla v_k'^\eta\|_{L^2(\eta^{-1}P-T)}. \tag{3.22}$$

Both (3.20) and (3.22) implies that, passing to the limit, w_k' belongs to $[D^{1,2}(\mathbb{R}^2 - T)]^2$. Thus (q_k', w_k') is identified as the unique solution in $[L^2(\mathbb{R}^2 - T)/\mathbb{R}] \times [D^{1,2}(\mathbb{R}^2 - T)]^2$ of the Stokes system (2.11) (cf. Lemma 2.3).

Finally, we conclude that the sequence (p_k^η, v_k^η) of solutions of (3.6) is the sum of two terms which converge respectively to $(q_k^0/\pi, w_k^0/\pi)$ (particular solution given in (2.10)), and to $(q_k'/\pi, w_k'/\pi)$ (solution of (3.21)). In other words, the entire sequence (p_k^η, v_k^η) converges to $(q_k/\pi, w_k/\pi)$ unique solution of (2.4). It remains to check that $\frac{1}{|\log \eta|}A(\eta)$ converges to $M^{-1} = \frac{1}{\pi}$ Id. From the definition (1.3) of the permeability tensor $A(\eta)$, we obtain

$$\begin{aligned} {}^t e_i A(\eta) e_k &= \frac{1}{4}\int_{P-\eta T} \nabla v_k : \nabla v_i = \frac{1}{4}\int_{\eta^{-1}P-T} \nabla v_k^\eta : \nabla v_i^\eta \\ &= \frac{1}{4}\int_{\eta^{-1}P-T} \nabla v_k^{0\eta} : \nabla v_i^{0\eta} + \frac{1}{4}\int_{\eta^{-1}P-T} \nabla v_k^{0\eta} : \nabla v_i'^\eta \\ &\quad + \frac{1}{4}\int_{\eta^{-1}P-T} \nabla v_k'^\eta : \nabla v_i^{0\eta} + \frac{1}{4}\int_{\eta^{-1}P-T} \nabla v_k'^\eta : \nabla v_i'^\eta. \end{aligned} \tag{3.23}$$

Thanks to (3.20), the last term of (3.23) is bounded, while the second and the third ones are easily shown to grow at most in $|\log \eta|^{1/2}$ when η goes to zero. An integration by parts gives for the first one

$$\begin{aligned} \frac{1}{4}\int_{\eta^{-1}P-T} \nabla v_k^{0\eta} : \nabla v_i^{0\eta} &= -\frac{1}{4}\int_{\partial B_{\eta^{-1}}} \left(p_k^{0\eta} e_r - \frac{\partial v_k^{0\eta}}{\partial r} \right) \cdot (-\log \eta e_i) \\ &= -\pi a(\eta) \log \eta \, e_k \cdot e_i. \end{aligned}$$

This yields

$$\frac{1}{|\log \eta|}{}^t e_i A(\eta) e_k \to \pi e_k \cdot e_i,$$

which is the desired result.

Q.E.D.

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