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**$S^k$-valued maps minimizing the $L^p$ norm of the gradient with free discontinuities**

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# $S^k$-Valued Maps Minimizing the $L^p$ Norm of the Gradient with Free Discontinuities

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## 1. - Introduction

In this paper, we study two new variational problems involving $S^k$-valued maps, where $S^k = \{z \in \mathbb{R}^{k+1};\ |z| = 1\}$. We deal with *free discontinuity problems* since a solution is a pair $(K, u)$, where $K$ is a (*a-priori* unknown) closed set and $u$ is a map suitably smooth outside of $K$ (see [11]). These problems can be regarded as a possible schematization of problems in mathematical physics in which both volume forces and surface tensions are present.

In the scalar case, recently two free discontinuity problems have been studied by E. De Giorgi and the authors in [13], [9]. The main theorem in [13] can be seen as an existence and partial regularity result for a Neumann-type problem, whereas in [9] we consider a Dirichlet-type problem.

The main results of this paper are presented in Theorems 4.1, 4.10 and 5.4. For brevity's sake here we illustrate only a simplified version of these results.

NEUMANN-TYPE PROBLEM. *Let $n \in \mathbb{N}$, $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $k \in \mathbb{N}$. Assume that $g \in L^\infty(\Omega; \mathbb{R}^{k+1})$. Then, for every $p > 1$ and $q \geq 1$, there exists at least one pair $(K_0, u_0)$ minimizing the functional*

$$\int_{\Omega\setminus K} |\nabla u|^p \mathrm{d}y + \int_{\Omega\setminus K} |u - g|^q \mathrm{d}y + \mathcal{H}^{n-1}(K \cap \Omega)$$

*in the class of the admissible pairs $(K, u)$ with $K \subset \mathbb{R}^n$ closed and $u \in C^1(\Omega\setminus K; S^k)$. Moreover, for every minimizing pair $(K_0, u_0)$, the singular set $K_0 \cap \Omega$ is $(\mathcal{H}^{n-1}, n-1)$ rectifiable and there exists $\varepsilon_1 > 0$ such that for every essential minimizing pair $(K', u')$ (see Remark 4.6) we have*

$$(1.1) \qquad \liminf_{\rho\to 0} \rho^{1-n} \mathcal{H}^{n-1}(K' \cap \overline{B}_\rho(x)) \geq \varepsilon_1 \qquad \textit{for every } x \in K' \cap \Omega$$

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*and*

$$\lim_{\rho\to 0} \frac{|\{x \in \Omega; \operatorname{dist}(x, Q) < \rho\}|}{2\rho} = \mathcal{H}^{n-1}(Q) \tag{1.2}$$

*for every compact set* $Q \subset K' \cap \Omega$.

DIRICHLET-TYPE PROBLEM. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set with* $C^1$ *boundary and let* $w$ *be a* $C^1(\partial\Omega; S^k)$ *map. Then for every* $p > 1$ *there exists at least one pair* $(K_0, u_0)$ *minimizing the functional*

$$\int_{\Omega\setminus K} |\nabla u|^p \mathrm{d}y + \mathcal{H}^{n-1}(K \cap \overline{\Omega})$$

*in the class of the admissible pairs* $(K, u)$ *with* $K \subset \mathbb{R}^n$ *closed,*

$$u \in C^1(\Omega\setminus K; S^k) \cap C^0(\overline{\Omega}\setminus K; S^k)$$

*such that* $u = w$ *in* $\partial\Omega\setminus K$. *Moreover the properties* (1.1) *and* (1.2), *even at the boundary points, hold for every essential minimizing pair.*

While the passage from the case of a scalar function $u$ considered in [13] and [9] to the case of a vector function $u$ without constraints does not present any problem, the introduction of the constraint $|u| = 1$ forces meaningful changes in the proofs. The interest in considering such a constraint is connected to recent studies on minima of non-convex functionals and on possible applications of these studies to the theory of liquid crystals.

In order to prove the above cited theorem for the Neumann-type problem, we take into account the idea of the so-called direct methods in the Calculus of Variations. Therefore we join to the functional to be minimized a new functional defined on a class of special functions of bounded variation (the class $SBV(\Omega; S^k)$), where a suitable topology can be found such that the new functional is both lower semicontinuous and coercive. Then Theorem 4.1 and Theorem 4.10 are established by proving first the existence of a solution for a minimum problem for the new functional in the class $SBV(\Omega; S^k)$ (see Lemma 4.3) and after by proving that the minimum of the new functional is also the minimum of our original functional. The same method and further estimates near the boundary points allow us to solve a Dirichlet-type problem (Theorem 5.4).

The plan of the exposition is the following.

In the second section we recall the definition and some properties of the class $SBV(\Omega)$ and we adapt some results proved in [13], [9] to the case of vector valued functions considered here. In particular we state the Poincaré-Wirtinger type inequality (Theorem 2.5) which is an essential tool in the rest of this paper.

The third section is devoted to the study of properties of quasi-minima in $SBV_{\mathrm{loc}}(\Omega; S^k)$ of some new integral functionals. In Theorem 3.6 we study the behaviour of blow-up sequences. The lack of convexity of the target space forces us to modify the arguments used in [13], [9] (see Lemma 3.5). In Theorem 3.11 we prove some properties of the singular sets of quasi-minima.

In the fourth section we prove the existence theorem for a Neumann-type problem by using the direct methods in the Calculus of Variations and the partial regularity properties estabilished in Theorem 3.11. In Lemma 4.9 and Theorem 4.10 we prove some results which, among other things, can be useful to approximate the functionals that we consider by elliptic ones, more suitable in numerical computations. In the scalar case this approximation has been already obtained in [6].

In the fifth section we sketch the proof of the existence theorem for a Dirichlet-type problem (Theorem 5.4).

For a comparison with other variational problems that deal with seeking a minimizer, under suitable conditions, of the energy

$$\int_{\Omega} |\nabla u|^2 \mathrm{d}y$$

in the Sobolev space $W^{1,2}(\Omega; S^k)$, we mention a few results from the large literature on this topic.

We recall that such problems appear both in the study of harmonic maps (see [27], [28], [14]) as in the theory of liquid crystals (see [10], [15], [22], [30]). The minimizers are, in general, singular and, according to the works of R. Schoen and K. Uhlenbeck [27], [28] and also of M. Giaquinta and E. Giusti [17], the singular set has Hausdorff dimension at most $(n-3)$ and it is discrete for $n = 3$. Moreover, in the case $n = 3$ and $k = 2$, H. Brézis, J.M. Coron and E.H. Lieb [8] have shown that the minimizers map small spheres around any singular point into $S^2$ with topological degree plus or minus one. The singularities appear not only for topological reasons, but because they enable to reduce energy. Actually, R. Hardt and F.- H. Lin [24] have shown that, in general, a Lavrentiev phenomenon does occur. As a further step towards an understanding of the geometry of energy minimizers F.J. Almgren and E.H. Lieb have estimated in [2] the number of points of discontinuity which a minimizer can have.

In the static theory of liquid crystals a nematic liquid crystal is a fluid in a container $\Omega$, which is formed by rodlike molecules whose directions are specified by a unit vector field (the map $u : \Omega \subset \mathbb{R}^3 \to S^2$), these directions are fixed along the boundary and the configuration assumes a position which minimizes the Oseen-Frank functional (see J.L. Ericksen [15]). The harmonic maps appear as models for the director of a nematic liquid crystal with equal Oseen-Frank constants. The results of [17], [27], [28] have been generalized to the case of the Oseen-Frank functional by R. Hardt, D. Kinderlehrer and F.-H. Lin [22] and in the case of maps that minimize the $L^p$ norm of the gradient for $p > 1$ by R. Hardt and F.-H. Lin [23].

Another approach to the study of the energy of maps from $\Omega \subset \mathbb{R}^3$ to $S^2$

has been proposed in [7] with the introduction of a relaxed energy and in [19], [20] in which smooth functions are regarded as graphs, or, more precisely, as cartesian currents. In these contexts the minimizers have, in general, singularities of Hausdorff dimension at most one.

In the case of the free discontinuity problems which we deal with here, we prove that the essential minimizing pairs have a singular set which, if not empty, has Hausdorff dimension equal to $(n-1)$ (see Lemma 4.9 or the property (1.1)). It is well-known that for $n \geq 3$ the map $u_0(y) = \frac{y}{|y|}$ is a minimizer in $B_1 = \{y \in \mathbb{R}^n; |y| < 1\}$ for $\int_{B_1} |\nabla u|^2 dy$ over the class of all $W^{1,2}$ maps with $|u| = 1$ and boundary data $w(y) = y$ (the identity map on $\partial B_1$) and obviously it shows an isolated singularity at the origin. We remark that the pair $(\{0\}, u_0)$ is not a solution of the Dirichlet-type problem with boundary data $w(y) = y$ for the functional, here considered,

$$\int_{B_1 \setminus K} |\nabla u|^2 dy + \mathcal{H}^{n-1}(K \cap \overline{B}_1).$$

Indeed, if we define for $0 < r \leq 1$

$$u_r(y) = \begin{cases} \dfrac{y}{|y|} & \text{if } |y| \geq r \\ (1,0,\dots,0) & \text{if } |y| < r, \end{cases}$$

then

$$\int_{B_1 \setminus \{|y|=r\}} |\nabla u_r|^2 dy + \mathcal{H}^{n-1}(\{|y| = r\} \cap \overline{B}_1) < \int_{B_1} |\nabla u_0|^2 dy.$$

It is an open problem to describe the geometric structure of the singular set of an essential minimizing pair. In the scalar case we refer to [11] for some related conjectures which, at our knowledge, are still open.

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## 2. - Preliminary results for functions in $SBV(\Omega; \mathbb{R}^m)$

Given an open set $\Omega \subseteq \mathbb{R}^n$, we define, following [12], the class of special functions of bounded variation $SBV(\Omega; \mathbb{R}^m)$ and we point out a few of its properties.

For a given set $E \subset \mathbb{R}^n$ we denote by $\chi_E$ its characteristic function, by $\overline{E}$ its topological closure and by $\partial E$ its topological boundary; moreover we denote by $\mathcal{H}^{n-1}(E)$ its $(n-1)$-dimensional Hausdorff measure and by $|E|$ its Lebesgue outer measure. If $\Omega$, $\Omega'$ are open subsets in $\mathbb{R}^n$, with $\Omega \subset\subset \Omega'$ we mean that $\overline{\Omega}$ is compact and $\overline{\Omega} \subset \Omega'$.

We indicate by $B_\rho(x)$ the ball $\{y \in \mathbb{R}^n; |y-x| < \rho\}$, and we set $B_\rho = B_\rho(0)$, $\omega_n = |B_1|$. By $(e^1, \ldots, e^n)$ we denote the canonical base of $\mathbb{R}^n$ and by $(\hat{e}^1, \ldots, \hat{e}^m)$ the canonical base of $\mathbb{R}^m$. Let $u : \Omega \to \mathbb{R}^m$ be a Borel function; for $x \in \Omega$ and $z \in \tilde{\mathbb{R}}^m = \mathbb{R}^m \cup \{\infty\}$ (the one point compactification of $\mathbb{R}^m$) we say (following [12]) that $z$ is the approximate limit of $u$ at $x$, and we write

$$z = \operatorname{ap}\lim_{y \to x} u(y),$$

if

$$g(z) = \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} g(u(y))\mathrm{d}y}{|B_\rho|}$$

for every $g \in C^0(\tilde{\mathbb{R}}^m)$; this definition is equivalent to other ones existing in the literature (e.g. 2.9.12 in [16]).

The set

$$S_u = \{x \in \Omega;\ \operatorname{ap}\lim_{y \to x} u(y) \quad \text{does not exist}\}$$

is a Borel set, of negligible Lebesgue measure; for brevity's sake we denote by $\tilde{u} : \Omega \backslash S_u \to \tilde{\mathbb{R}}^m$ the function

$$\tilde{u}(x) = \operatorname{ap}\lim_{y \to x} u(y).$$

Let $x \in \Omega \backslash S_u$ be such that $\tilde{u}(x) \in \mathbb{R}^m$; we say that $u$ is approximately differentiable at $x$ if there exists a linear map $\nabla u(x) : \mathbb{R}^n \to \mathbb{R}^m$ (the approximate differential of $u$ at $x$) such that

$$\operatorname{ap}\lim_{y \to x} \frac{|u(y) - \tilde{u}(x) - \nabla u(x)(y-x)|}{|y-x|} = 0.$$

If $u$ is a smooth function then $\nabla u$ is the differential. In the following with the notation $|\nabla u|$ we mean the euclidean norm of $\nabla u$

$$|\nabla u(x)| = \left( \sum_{j=1}^{m} \sum_{i=1}^{n} (\nabla u(x)(e^i) \cdot \hat{e}^j)^2 \right)^{1/2}$$

for almost all $x \in \Omega$.

Now we recall some properties related to scalar valued functions.

For every $u \in L^1_{\mathrm{loc}}(\Omega)$ we define (see [25])

$$\int_\Omega |\mathrm{D}u| = \sup \left\{ \int_\Omega u \operatorname{div} \phi \,\mathrm{d}y;\ \phi \in C^1_0(\Omega; \mathbb{R}^n),\ |\phi| \le 1 \right\}.$$

By $BV(\Omega)$ we denote the Banach space of all functions $u$ of $L^1(\Omega)$ with $\int_\Omega |Du| < +\infty$.

It is well-known that $u \in BV(\Omega)$ iff $u \in L^1(\Omega)$ and its distributional derivative $Du$ is a bounded vector measure. For the main properties of the functions of bounded variation we refer e.g. to [16], [21], [25].

Here we recall only that for every $u \in BV(\Omega)$ the following properties hold:

$S_u$ is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable (see [16], 4.5.9(16));

$\mathcal{H}^{n-1}(\{x \in \Omega; \tilde{u}(x) = \infty\}) = 0$ (see [16], 4.5.9(3));

$\nabla u$ exists a.e. on $\Omega$ and coincides with the Radon-Nikodym derivative of $Du$ with respect to the Lebesgue measure (see [16], 4.5.9(26));

for $\mathcal{H}^{n-1}$ almost all $x \in S_u$ there exist $\nu = \nu_u(x) \in \partial B_1$, $u^+(x) \in \mathbb{R}$ and $u^-(x) \in \mathbb{R}$ (outer and inner trace, respectively, of $u$ at $x$ in the direction $\nu$) such that

$$\lim_{\rho \to 0} \rho^{-n} \int_{\{y \in B_\rho(x); y\cdot\nu > 0\}} |u(y) - u^+(x)| dy = 0,$$

$$\lim_{\rho \to 0} \rho^{-n} \int_{\{y \in B_\rho(x); y\cdot\nu < 0\}} |u(y) - u^-(x)| dy = 0,$$

and

$$\int_\Omega |Du| \geq \int_\Omega |\nabla u| dy + \int_{S_u \cap \Omega} |u^+ - u^-| d\mathcal{H}^{n-1} \tag{2.1}$$

(see [16], 4.5.9(17),(22),(15)).

Following [12], we define a class of special functions of bounded variation which are characterized by a property stronger than (2.1).

DEFINITION 2.1. We define $SBV(\Omega)$ as the class of all functions

$$u \in BV(\Omega)$$

such that

$$\int_\Omega |Du| = \int_\Omega |\nabla u| dy + \int_{S_u \cap \Omega} |u^+ - u^-| d\mathcal{H}^{n-1}. \tag{2.2}$$

We remark that the well-known Cantor-Vitali function has bounded variation, but it does not satisfy (2.2).

REMARK 2.2. Let $u \in BV(\Omega)$ and set $u_a = (u \wedge a) \vee (-a)$ for $a \in (0, +\infty)$. The following properties hold:

$$|\nabla u_a| \leq |\nabla u| \text{ a.e. on } \Omega, \qquad \mathcal{H}^{n-1}\left((S_{u_a} \backslash S_u) \cap \Omega\right) = 0,$$

$$\int_\Omega |Du_a| \le \int_\Omega |Du|,$$

$$\int_\Omega |\nabla u| dy = \lim_{a\to+\infty} \int_\Omega |\nabla u_a| dy, \qquad \mathcal{H}^{n-1}(S_u \cap \Omega) = \lim_{a\to+\infty} \mathcal{H}^{n-1}(S_{u_a} \cap \Omega),$$

$$\int_\Omega |Du| = \lim_{a\to+\infty} \int_\Omega |Du_a|.$$

Moreover, for $u \in BV(\Omega)$, it holds:

$u \in SBV(\Omega)$ iff $u_a \in SBV(\Omega)$ for every $a \in (0, +\infty)$;

and more generally:

$u \in SBV(\Omega)$ iff $\phi(u) \in SBV(\Omega)$ for every $\phi : \mathbb{R} \to \mathbb{R}$ uniformly Lipschitz continuous with $\phi(0) = 0$.

Denote by $W^{1,p}(\Omega)$ ($p \ge 1$) the Sobolev space of functions $u \in L^p(\Omega)$ such that $Du \in L^p(\Omega; \mathbb{R}^n)$; then we remark that, for $u \in SBV(\Omega)$,

$$u \in W^{1,p}(\Omega) \quad \text{iff} \quad \mathcal{H}^{n-1}(S_u \cap \Omega) = 0 \quad \text{and} \quad \int_\Omega (|\nabla u|^p + |u|^p) dy < +\infty$$

(see e.g. [16], 4.5.9(30)).

For further results on the functions in $SBV(\Omega)$ we refer to [12], [3], [4], [5].

In this paper we deal with functions in the space $SBV(\Omega; \mathbb{R}^m)$, i.e. functions $u : \Omega \to \mathbb{R}^m$ whose components $u^i$ ($i = 1, \dots, m$) are functions in $SBV(\Omega)$. By $SBV_{\text{loc}}(\Omega; \mathbb{R}^m)$ we denote the space of all functions which belong to $SBV(\Omega'; \mathbb{R}^m)$ for every open set $\Omega' \subset\subset \Omega$. For every $u \in SBV(\Omega; \mathbb{R}^m)$ the following properties hold (see [3], [4]):

a) $S_u = \bigcup_{i=1}^m S_{u^i}$;

b) three Borel functions exist, $\nu : S_u \to S^{n-1}$, $u^+ : S_u \to \mathbb{R}^m$ and $u^- : S_u \to \mathbb{R}^m$, such that for $\mathcal{H}^{n-1}$-almost all $x \in S_u$

$$\lim_{\rho\to 0} \rho^{-n} \int_{\{y\in B_\rho(x); y\cdot\nu>0\}} |u(y) - u^+(x)| dy = 0, \tag{2.3}$$

$$\lim_{\rho\to 0} \rho^{-n} \int_{\{y\in B_\rho(x); y\cdot\nu<0\}} |u(y) - u^-(x)| dy = 0; \tag{2.4}$$

c) for $\mathcal{H}^{n-1}$-almost all $x \in S_u$ the triple $(u^+(x), u^-(x), \nu(x))$ is uniquely determined up to a permutation of $u^+(x), u^-(x)$ and a change of sign of $\nu(x)$. In particular, for every $u \in SBV(\Omega;\mathbb{R}^m)$ and $v \in SBV(\Omega;\mathbb{R}^k)$ we have $\nu_u = \pm\nu_v$ $\mathcal{H}^{n-1}$-a.e. in $S_u \cap S_v$.

LEMMA 2.3. *Let $\Omega \subset \mathbb{R}^n$ be open, $m \geq 1$ and $u \in L^\infty(\Omega;\mathbb{R}^m) \cap L^1(\Omega;\mathbb{R}^m)$. Let $K \subset \mathbb{R}^n$ be closed and assume $u \in C^1(\Omega\backslash K;\mathbb{R}^m)$ and*

$$\int_{\Omega\backslash K} |\nabla u| \mathrm{d}y + \mathcal{H}^{n-1}(K \cap \Omega) < +\infty.$$

*Then*

(i) $u \in SBV(\Omega;\mathbb{R}^m)$,

(ii) $S_u \cap \Omega \subset K$,

(iii) *three vector-valued Borel functions, $u^+, u^-$ and $\nu$, exist which are defined $\mathcal{H}^{n-1}$-a.e. on $K \cap \Omega$ and satisfy* (2.3)-(2.4).

We note only that (i) and (ii) are proved in Lemma 2.3 of [13] for $m = 1$ and that (iii) follows by the preceding property b) setting $u^+(x) = u^-(x) = \tilde{u}(x)$ for $x \in K\backslash S_u$ and $\nu$ an arbitrary constant direction on $K\backslash S_u$.

In this paper we use the following compactness theorem in $SBV(\Omega;\mathbb{R}^m)$, that is an obvious consequence of a result by L. Ambrosio (see Theorem 2.1 of [4]).

THEOREM 2.4. *Let $p > 1$, $n \in \mathbb{N}$, $n \geq 2$, $\Omega \subset \mathbb{R}^n$ open, $m \in \mathbb{N}$. Let $Q \subset \mathbb{R}^m$ be a compact set and let $(u_h) \subset SBV_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ be a sequence such that*

$$\sup_h \left\{ \int_\Omega |\nabla u_h|^p \mathrm{d}y + \mathcal{H}^{n-1}(S_{u_h} \cap \Omega) \right\} < +\infty,$$

*and $u_h(x) \in Q$ for almost all $x \in \Omega$. Then there exists a subsequence $(u_{h_i})$ converging in measure to a function $u \in SBV_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ with $u(x) \in Q$ for almost all $x \in \Omega$.*

We remark that the conclusions just asserted do not follow if $p = 1$, because in this case it is possible to approximate every $u \in BV(\Omega;\mathbb{R}^m)$ by a sequence of smooth functions (which are also functions of class $SBV(\Omega;\mathbb{R}^m)$).

In [13], section 3, a Poincaré-Wirtinger type inequality for scalar valued functions of the class $SBV$ in a ball and two consequences have been proved. Here we give the statements of analogous results for the vector valued case. The proofs are omitted because they are simple generalizations of those given in [13].

Let $B$ be a ball in $\mathbb{R}^n$, $n \geq 2$; for every measurable function $u : B \to \mathbb{R}^m$ with $u = (u^1, \ldots, u^m)$ and for $0 \leq s \leq |B|$, we set

$$u_*(s, B) = (u_*^1(s, B), \ldots, u_*^m(s, B))$$

where

$$u^i_*(s,B) = \inf\{t \in \mathbb{R};\ |\{u^i < t\} \cap B| \geq s\} \qquad (i = 1, \dots, m).$$

We define

$$\mathrm{med}(u,B) = u_*\left(\frac{1}{2}|B|, B\right);$$

moreover, for every $u \in SBV(B;\mathbb{R}^m)$ such that $(2\gamma_n \mathcal{H}^{n-1}(S_u \cap B))^{\frac{n}{n-1}} < \frac{1}{2}|B|$ we set

$$\tau'(u,B) = u_*\left((2\gamma_n \mathcal{H}^{n-1}(S_u \cap B))^{\frac{n}{n-1}}, B\right),$$
$$\tau''(u,B) = u_*\left(|B| - (2\gamma_n \mathcal{H}^{n-1}(S_u \cap B))^{\frac{n}{n-1}}, B\right),$$

where $\gamma_n$ is the isoperimetric constant relative to the balls of $\mathbb{R}^n$.

In the following, given $a, b \in \mathbb{R}^m$ we pose

$$a \wedge b = (\min(a^1, b^1), \dots, \min(a^m, b^m))$$

and

$$a \vee b = (\max(a^1, b^1), \dots, \max(a^m, b^m)).$$

THEOREM 2.5. *Let $B \subset \mathbb{R}^n$ be a ball, $n \geq 2$, $1 \leq p < n$ and $p^* = np/(n-p)$. Let $u \in SBV(B;\mathbb{R}^m)$, $2\gamma_n \mathcal{H}^{n-1}(S_u \cap B) < \left(\frac{1}{2}|B|\right)^{\frac{n-1}{n}}$, and*

$$\overline{u} = (u \wedge \tau''(u,B)) \vee \tau'(u,B).$$

*Then*

$$\left(\int_B |\overline{u} - \mathrm{med}(u,B)|^{p^*} \mathrm{d}y\right)^{1/p^*} \leq \frac{2\gamma_n p(n-1)}{n-p}\left(\int_B |\nabla u|^p \mathrm{d}y\right)^{1/p}.$$

We remark that, by the definition, $|\{u \neq \overline{u}\}| \leq 2(2\gamma_n \mathcal{H}^{n-1}(S_u \cap B))^{n/(n-1)}$.

THEOREM 2.6. *Let $B \subset \mathbb{R}^n$ be a ball, $u_h \in SBV(B;\mathbb{R}^m)$ for every $h \in \mathbb{N}$, $p > 1$, and let*

$$\sup_h \int_B |\nabla u_h|^p \mathrm{d}y < +\infty,$$

$$\lim_h \mathcal{H}^{n-1}(S_{u_h} \cap B) = 0.$$

*Then a subsequence $(u_{h_i})$ and a function $u_\infty \in W^{1,p}(B;\mathbb{R}^m)$ exist such that*

$$\lim_i \left[\overline{u}_{h_i} - \mathrm{med}(u_{h_i}, B)\right] = u_\infty \qquad \text{in } L^r(B;\mathbb{R}^m)$$

*for every* $1 \le r < np/(n-p)$ *if* $1 < p < n$, *and for every* $r \ge 1$ *if* $p \ge n$; *moreover*

$$\lim_i \left[u_{h_i} - \mathrm{med}(u_{h_i}, B)\right] = u_\infty \qquad a.e.\ on\ B$$

*and*

$$\int_B |\nabla u_\infty|^p \mathrm{d}y \le \liminf_i \int_B |\nabla u_{h_i}|^p \mathrm{d}y.$$

THEOREM 2.7. *Let* $n \in \mathbb{N}$, $n \ge 2$, $p > 1$, *let* $\Omega \subset \mathbb{R}^n$ *be an open set; let* $u \in SBV(\Omega;\mathbb{R}^m)$ *and* $x \in \Omega$. *If*

$$\lim_{\rho\to 0} \rho^{1-n} \left[ \int_{B_\rho(x)} |\nabla u|^p \mathrm{d}y + \mathcal{H}^{n-1}(S_u \cap B_\rho(x)) \right] = 0,$$

*then*

$$x \notin S_u \quad and \quad \tilde{u}(x) \in \mathbb{R}^m.$$

## 3. - A limit theorem and some estimates for quasi-minima in $SBV_{\mathrm{loc}}(\Omega; S^k)$.

*Assumptions*. In this section we will assume $n \in \mathbb{N}$, $n \ge 2$, $m \in \mathbb{N}$, $m \ge 2$, $p > 1$, $\Omega \subset \mathbb{R}^n$ open subset.

Let $\gamma : \Omega \times \mathbb{R}^m \times \mathbb{R}^m \times S^{n-1} \to (0, +\infty)$ be a Borel function with the property that there exist two constants $\alpha_0, \alpha_1$ such that

$$0 < \alpha_0 \le \gamma(y, u, v, \nu) \le \alpha_1$$

for every $y \in \Omega$, $u$, $v \in \mathbb{R}^m$, $\nu \in S^{n-1}$, and $\gamma(y, \cdot, \cdot, \nu)$ be positively 0-homogeneous.

DEFINITION 3.1. Let $u \in SBV_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ and $c > 0$. Let $Q \subset \Omega$ be closed. We set

$$\mathcal{F}(u, c, Q) = \int_Q |\nabla u|^p \mathrm{d}y + c \int_{S_u \cap Q} \gamma(y, u^+, u^-, \nu) \mathrm{d}\mathcal{H}^{n-1}.$$

For every $t > 0$ and for every $u$ such that $|u| = t$ a.e. in $\Omega$, we set

$$\Phi(u, c, Q, t) = \inf \big\{ \mathcal{F}(v, c, Q);\ v \in SBV_{\mathrm{loc}}(\Omega;\mathbb{R}^m),\ v = u \text{ in } \Omega \backslash Q,\ |v| = t \text{ a.e. in } \Omega \big\};$$

moreover, if $\Phi(u, c, Q, t) < +\infty$, we set

$$\Psi(u, c, Q, t) = \mathcal{F}(u, c, Q) - \Phi(u, c, Q, t).$$

We first state three technical lemmas, whose proofs are straightforward.

LEMMA 3.2. *Let $u \in SBV(B_r;\mathbb{R}^m)$. For every $c > 0$ and $t > 0$ the functions*

$$\rho \to \mathcal{F}(u, c, \overline{B}_\rho) \qquad \text{and} \qquad \rho \to \Psi(u, c, \overline{B}_\rho, t)$$

*are non-decreasing in $(0, r)$ and*

$$\Phi(u, c, \overline{B}_\rho, t) \leq c\alpha_1 n\omega_n \rho^{n-1}.$$

We remark only that the preceding inequality follows by choosing the admissible function

$$v = u(1 - \chi_{\overline{B}_\rho}) + t\hat{e}^m \chi_{\overline{B}_\rho}.$$

LEMMA 3.3. *Let $u \in SBV(B_r(x_0);\mathbb{R}^m)$, with $|u| = t$ a.e. in $B_r(x_0)$. For $\rho \in (0, r)$ and for every $x \in B_{r/\rho}$ set $u_\rho(x) = \rho^{(1-p)/p}\, u(x_0 + \rho x)$, then*

$$u_\rho \in SBV(B_{r/\rho};\mathbb{R}^m),$$

$$\mathcal{F}(u_\rho, c, \overline{B}_1) = \rho^{1-n}\mathcal{F}(u, c, \overline{B}_\rho(x_0))$$

*and*

$$\Phi(u_\rho, c, \overline{B}_1, \rho^{(1-p)/p}t) = \rho^{1-n}\Phi(u, c, \overline{B}_\rho(x_0), t).$$

LEMMA 3.4. *Let $u,\ v \in SBV(B_r;\mathbb{R}^m)$, $c > 0$ and $\rho \in (0, r)$. Suppose*

$$\mathcal{H}^{n-1}(S_u \cap \partial B_\rho) = \mathcal{H}^{n-1}(S_v \cap \partial B_\rho) = 0.$$

*Set*

$$w(x) = \begin{cases} u(x) & \text{if } x \in \overline{B}_\rho, \\ v(x) & \text{if } x \in B_r \backslash \overline{B}_\rho, \end{cases}$$

*then*

$$\mathcal{F}(w, c, \overline{B}_\rho) \leq \mathcal{F}(u, c, \overline{B}_\rho) + c\alpha_1 \mathcal{H}^{n-1}\left(\{\tilde{u} \neq \tilde{v}\} \cap \partial B_\rho\right).$$

Now we prove the following lemma.

LEMMA 3.5. *Let $t > 0$, $\epsilon \in (0,1)$; let $\Omega, \Omega', \Omega''$ be open sets of $\mathbb{R}^n$ such that $\Omega \subset\subset \Omega' \subset\subset \Omega''$. Let $u \in SBV_{\text{loc}}(\Omega'';\mathbb{R}^m)$ with $|u| = t$ a.e. in $\Omega''\backslash\overline{\Omega'}$, $u^m \geq (1-\epsilon)t$ a.e. in $\overline{\Omega'}$, and let $v \in W^{1,p}(\Omega'';\mathbb{R}^m)$ with $v^m \geq (1-\epsilon)t$ a.e. in $\Omega''$. Then a function $w \in SBV_{\text{loc}}(\Omega'';\mathbb{R}^m)$ exists with $|w| = t$ a.e. in $\Omega''$, $w = u$ a.e. in $\Omega''\backslash\overline{\Omega'}$ and such that, for every $c > 0$,*

$$\mathcal{F}(w, c, \overline{\Omega'}) \leq \frac{1}{(1-\epsilon)^p}\left[(1+\epsilon)\left(\mathcal{F}(v, c, \overline{\Omega'}) + \frac{\alpha_1}{\alpha_0}\,\mathcal{F}(u, c, \overline{\Omega'}\backslash\Omega)\right)\right.$$
$$\left. + \epsilon\left(\frac{2n_\epsilon}{d}\right)^p \int_{\overline{\Omega'}\backslash\Omega} |u - v|^p \mathrm{d}y\right],$$

*where* $n_\epsilon \in \mathbb{N}$ *and* $n_\epsilon > 2^{p-1}/\epsilon$, $d = \mathrm{dist}(\Omega, \mathbb{R}^n \setminus \Omega')$.

PROOF. Let $P(z) = tz/|z|$ for every $z \in \mathbb{R}^m \setminus \{0\}$. Fixed $\epsilon > 0$, let $n_\epsilon \in \mathbb{N}$ be such that $2^{p-1}/n_\epsilon < \epsilon$; let $\Omega_i$ $(i = 1, \dots, n_\epsilon)$ be open sets of $\mathbb{R}^n$ such that $\Omega \subset\subset \Omega_1 \subset\subset \dots \subset\subset \Omega_{n_\epsilon} = \Omega'$ and $\mathrm{dist}(\Omega_i, \mathbb{R}^n \setminus \Omega_{i+1}) = d/n_\epsilon$ $(i = 1, \dots, n_\epsilon - 1)$. Let $\varphi_i \in C_0^\infty(\Omega_{i+1})$ with $0 \leq \varphi_i \leq 1$, $\varphi_i = 1$ in a neighbourhood of $\overline{\Omega}_i$ and $|\nabla \varphi_i| \leq 2n_\epsilon/d$. Setting $w_i = P((1-\varphi_i)u + \varphi_i v)$, we have $w_i \in SBV_{\mathrm{loc}}(\Omega''; \mathbb{R}^m)$ (see [5]), $w_i = u$ a.e. in $\Omega'' \setminus \overline{\Omega}'$, $|w_i| = t$ a.e. in $\Omega''$,

$$\mathcal{H}^{n-1}(S_{w_i} \setminus (S_u \cap (\Omega'' \setminus \Omega))) = 0$$

and

$$\int_{\overline{\Omega}'} |\nabla w_i|^p \mathrm{d}y \leq t^p \int_{\overline{\Omega}'} \frac{|\nabla((1-\varphi_i)u + \varphi_i v)|^p}{|(1-\varphi_i)u + \varphi_i v|^p} \mathrm{d}y$$

$$\leq \frac{1}{(1-\epsilon)^p} \int_{\overline{\Omega}'} |\nabla((1-\varphi_i)u + \varphi_i v)|^p \mathrm{d}y$$

$$\leq \frac{1}{(1-\epsilon)^p} \left[ \int_{\overline{\Omega}' \setminus \Omega} |\nabla u|^p \mathrm{d}y + \int_{\overline{\Omega}'} |\nabla v|^p \mathrm{d}y + \int_{\Omega_{i+1} \setminus \Omega_i} |\nabla((1-\varphi_i)u + \varphi_i v)|^p \mathrm{d}y \right]$$

$$\leq \frac{1}{(1-\epsilon)^p} \left[ \int_{\overline{\Omega}' \setminus \Omega} |\nabla u|^p \mathrm{d}y + \int_{\overline{\Omega}'} |\nabla v|^p \mathrm{d}y \right.$$

$$\left. + 2^{p-1} \left( \int_{\Omega_{i+1} \setminus \Omega_i} |\nabla u|^p \mathrm{d}y + \int_{\Omega_{i+1} \setminus \Omega_i} |\nabla v|^p \mathrm{d}y + \left(\frac{2n_\epsilon}{d}\right)^p \int_{\Omega_{i+1} \setminus \Omega_i} |u - v|^p \mathrm{d}y \right) \right];$$

then it follows that

$$\min_{1 \leq i \leq n_\epsilon} \int_{\overline{\Omega}'} |\nabla w_i|^p \mathrm{d}y \leq \frac{1}{(1-\epsilon)^p} \left[ \int_{\overline{\Omega}' \setminus \Omega} |\nabla u|^p \mathrm{d}y + \int_{\overline{\Omega}'} |\nabla v|^p \mathrm{d}y \right.$$

$$\left. + \frac{2^{p-1}}{n_\epsilon} \left( \int_{\overline{\Omega}' \setminus \Omega} |\nabla u|^p \mathrm{d}y + \int_{\overline{\Omega}' \setminus \Omega} |\nabla v|^p \mathrm{d}y + \left(\frac{2n_\epsilon}{d}\right)^p \int_{\overline{\Omega}' \setminus \Omega} |u - v|^p \mathrm{d}y \right) \right]$$

$$\leq \frac{1}{(1-\epsilon)^p} \left[ (1+\epsilon) \left( \int_{\overline{\Omega}'} |\nabla v|^p \mathrm{d}y + \int_{\overline{\Omega}' \setminus \Omega} |\nabla u|^p \mathrm{d}y \right) \right.$$

$$+\epsilon\left(\frac{2n_\epsilon}{d}\right)^p \int_{\Omega_{i+1}\setminus\Omega_i} |u-v|^p \mathrm{d}y\Bigg].$$

Therefore a suitable function $w_i$ exists such that, setting $w = w_i$, we have

$$\mathcal{F}(w, c, \overline{\Omega}') = \int_{\overline{\Omega}'} |\nabla w|^p \mathrm{d}y + c \int_{S_w\cap\overline{\Omega}'} \gamma(y, w^+, w^-, \nu)\mathrm{d}\mathcal{H}^{n-1}$$

$$\leq \frac{1}{(1-\epsilon)^p}\left[(1+\epsilon)\left(\int_{\overline{\Omega}'} |\nabla v|^p \mathrm{d}y + \frac{\alpha_1}{\alpha_0}\mathcal{F}(u, c, \overline{\Omega}'\setminus\Omega)\right)\right.$$

$$\left. + \epsilon\left(\frac{2n_\epsilon}{d}\right)^p \int_{\overline{\Omega}'\setminus\Omega} |u-v|^p \mathrm{d}y\right].$$

Then the thesis follows immediately. q.e.d.

We are now in a position to prove the following limit theorem.

THEOREM 3.6. *Let $\overline{B}_r(x) \subset \Omega$, $u_h \in SBV(\Omega; \mathbb{R}^m)$, $c_h > 0$ and $t_h > 0$ for every $h \in \mathbb{N}$. Let $u_\infty \in W^{1,p}(B_r(x); \mathbb{R}^m)$ be such that*

(1) $\lim_h c_h = +\infty$, $\lim_h c_h t_h^{-q} = 0$ *where* $q = p^*$ *if* $p < n$ *or* $q \geq 1$ *if* $p \geq n$,

(2) $|u_h| = t_h$ *a.e. in* $B_r(x)$,

(3) $\lim_h \mathcal{F}(u_h, c_h, \overline{B}_\rho(x)) = \alpha(\rho) < +\infty$ *for almost all* $\rho < r$,

(4) $\lim_h \Psi(u_h, c_h, \overline{B}_\rho(x), t_h) = 0$ *for every* $\rho < r$,

(5) $\lim_h [u_h - \mathrm{med}(u_h, B_r(x))] = u_\infty$ *a.e. on* $B_r(x)$.

*Then there exists a hyperplane $\Sigma$ such that the image of $u_\infty$ lies essentially in $\Sigma$; moreover the function $u_\infty$ is $p$-harmonic in $B_r(x)$ (i.e. it minimizes the functional $\int_{B_r(x)} |\nabla u|^p \mathrm{d}y$ in $u_\infty + W_0^{1,p}(B_r(x); \mathbb{R}^m)$) and*

$$\alpha(\rho) = \int_{B_\rho(x)} |\nabla u_\infty|^p \mathrm{d}y$$

*for almost all $\rho < r$.*

PROOF. We may assume that $x = 0$, $\sup_{\rho<r}\alpha(\rho) < +\infty$ and moreover, up to rotations, that $\mathrm{med}(u_h, B_r) = \lambda_h \hat{e}^m$ with $\lambda_h \geq 0$. From the assumptions (3) and

(1), it follows that

$$\lim_h \mathcal{H}^{n-1}(S_{u_h} \cap B_r) = 0 \quad \text{and} \quad \sup_h \int_{B_r} |\nabla u_h|^p dy < +\infty;$$

so that, by Theorem 2.6,

$$\int_{B_\rho} |\nabla u_\infty|^p dy \le \lim_h \mathcal{F}(u_h, c_h, \overline{B}_\rho) = \alpha(\rho).$$

The proof will be completed by proving that the image of $u_\infty$ lies essentially in the hyperplane $\{z \in \mathbb{R}^m; z_m = 0\}$, i.e.

$$u_\infty^m \equiv 0 \quad \text{a.e. in} \quad B_r, \tag{3.1}$$

and that, for almost all $\rho < r$ and for every $v \in W^{1,p}(B_r;\mathbb{R}^m)$, $v = u_\infty$ in $B_r \backslash \overline{B}_\rho$, it follows

$$\alpha(\rho) \le \int_{B_\rho} |\nabla v|^p dy. \tag{3.2}$$

We prove (3.1) with an argument analogous to that used in [22] for proving Lemma 2.1. In the following with $\Lambda$ we denote a generic positive constant, which does not depend on $h$. Because

$$|u_h| = t_h \quad \text{a.e. in} \quad B_r, \quad \int_{B_r} |\nabla u_h|^p dy \le \Lambda, \;\; c_h \mathcal{H}^{n-1}(S_{u_h} \cap B_r) \le \Lambda,$$

setting $v_h = t_h^{-1} u_h$, we have

$$|v_h| = 1 \quad \text{a.e. in} \quad B_r, \quad \int_{B_r} |\nabla v_h|^p dy \le \Lambda t_h^{-p}, \;\; \mathcal{H}^{n-1}(S_{v_h} \cap B_r) \le \Lambda c_h^{-1}.$$

By Theorem 2.5 and the subsequent remark, we have

$$t_h^{p^*} \left(1 - \frac{\lambda_h}{t_h}\right)^{p^*} = t_h^{p^*} \frac{1}{|B_r \cap \{v_h = \overline{v}_h\}|} \int_{B_r \cap \{v_h = \overline{v}_h\}} \left(|\overline{v}_h| - \frac{\lambda_h}{t_h}\right)^{p^*} dy$$

$$\le \Lambda t_h^{p^*} \int_{B_r} |\overline{v}_h - \operatorname{med}(v_h, B_r)|^{p^*} dy \le \Lambda t_h^{p^*} \left( \int_{B_r} |\nabla v_h|^p dy \right)^{p^*/p} \le \Lambda,$$

where, without loss of generality, we have assumed $p < n$. Therefore we have

$$\lim_h \frac{\lambda_h}{t_h} = 1 \tag{3.3}$$

and, possibly by restriction to a subsequence, we may suppose

$$\lim_h t_h \left(1 - \frac{\lambda_h}{t_h}\right) = d \geq 0.$$

By using the trivial identity $|u_h - \mathrm{med}(u_h, B_r) + \mathrm{med}(u_h, B_r)|^2 = t_h^2$, we have

$$|u_h - \mathrm{med}(u_h, B_r)|^2 + 2\,(u_h - \mathrm{med}(u_h, B_r)) \cdot \mathrm{med}(u_h, B_r) = t_h^2 \left(1 - \frac{\lambda_h^2}{t_h^2}\right)$$

and also

$$\frac{|u_h - \mathrm{med}(u_h, B_r)|^2}{t_h} + 2\,(u_h - \mathrm{med}(u_h, B_r)) \cdot \frac{\mathrm{med}(u_h, B_r)}{t_h} = t_h \left(1 - \frac{\lambda_h}{t_h}\right)\left(1 + \frac{\lambda_h}{t_h}\right).$$

Then, letting $h \to +\infty$, by the hypothesis (5) and by (3.3) we obtain $0 + 2u_\infty \cdot \hat{e}^m = 2d$ a.e. in $B_r$, so that

$$d = \mathrm{med}(u_\infty^m, B_r) = \lim_h \mathrm{med}(u_h^m \lambda_h, B_r) = \lim_h \left[\mathrm{med}(u_h^m, B_r) - \lambda_h\right] = 0.$$

Finally we deduce $u_\infty^m \equiv 0$ a.e. in $B_r$.

Now we show (see next (3.5)) that $u_\infty$ is the limit in $L^p(B_r; \mathbb{R}^m)$ of a sequence of functions which are obtained by $u_h - \mathrm{med}(u_h, B_r)$ under suitable truncations. Fixing $\epsilon \in (0, 1)$, we define

$$\hat{u}_h = (\overline{u}_h^1, \ldots, \overline{u}_h^{m-1}, \hat{u}_h^m), \tag{3.4}$$

where $\overline{u}^i$ are defined as in Theorem 2.5 and

$$\hat{u}_h^m = u_h^m \vee \left(\tau'(u_h^m, B_r) \vee (1 - \epsilon) t_h\right) \wedge \tau''(u_h^m, B_r).$$

By (3.3) we have definitively

$$|\hat{u}_h - \mathrm{med}(u_h, B_r) - u_\infty| \leq |\overline{u}_h - \mathrm{med}(u_h, B_r) - u_\infty|,$$

and by Theorem 2.6 it follows

$$\lim_h \int_{B_r} |\hat{u}_h - \mathrm{med}(u_h, B_r) - u_\infty|^p \mathrm{d}y = 0. \tag{3.5}$$

Moreover, by Theorem 2.5, we have definitively, for $h$ such that

$$\lambda_h \geq \left(1 - \frac{\epsilon}{2}\right) t_h,$$

$$\begin{aligned}
&|\{u_h \neq \hat{u}_h\} \cap B_r| \leq |\{u_h \neq \overline{u}_h\} \cap B_r| + |\{\overline{u}_h \neq \hat{u}_h\} \cap B_r| \\
&\leq 2\left(2\gamma_n \mathcal{H}^{n-1}(S_{u_h} \cap B_r)\right)^{n/(n-1)} + \left|\{\tau'(u_h^m, B_r) \leq u_h^m \leq (1-\epsilon)t_h\} \cap B_r\right| \\
&\leq \Lambda\, c_h^{n/(1-n)} + \left(\frac{2}{\epsilon t_h}\right)^q \int\limits_{\{u_h^m \leq (1-\epsilon)t_h\} \cap B_r} |\overline{u}_h^m - \lambda_h|^q \mathrm{d}y \\
&\leq \Lambda\, c_h^{n/(1-n)} + \Lambda \left(\frac{2}{\epsilon t_h}\right)^q \left(\int\limits_{B_r} |\nabla \overline{u}_h|^p \mathrm{d}y\right)^{q/p} \leq \Lambda\left(c_h^{n/(1-n)} + (\epsilon t_h)^{-q}\right).
\end{aligned}$$

Multiplying by $c_h$ the preceding inequality and by the hypothesis (1), we obtain

$$\lim_h c_h |\{\hat{u}_h \neq u_h\} \cap B_r| = 0.$$

Being $c_h|\{\hat{u}_h \neq u_h\} \cap B_r| = c_h \int\limits_0^r \mathcal{H}^{n-1}(\{\tilde{\hat{u}}_h \neq \tilde{u}_h\} \cap \partial B_\rho)\mathrm{d}\rho$, possibly passing to a subsequence, we obtain

$$\lim_h c_h \mathcal{H}^{n-1}(\{\tilde{\hat{u}}_h \neq \tilde{u}_h\} \cap \partial B_\rho) = 0 \tag{3.6}$$

for almost all $\rho < r$.

Finally we may prove (3.2) by contradiction. By (3.1) we may assume, without loss of generality, that $v^m \equiv 0$ a.e. in $B_r$.

Since the function $\rho \to \alpha(\rho)$ is non-decreasing, it is also a continuous function for almost all $\rho < r$. Let $\rho' < r$ such that $\alpha(\cdot)$ is continuous in $\rho'$ and the hypothesis (3) is fulfilled. We suppose that a function $v$ exists such that $v \in W^{1,p}(B_r; \mathbb{R}^m)$, $v = u_\infty$ in $B_r \backslash \overline{B}_{\rho'}$ and for a $\sigma > 0$

$$\int\limits_{B_{\rho'}} |\nabla v|^p \mathrm{d}y < \alpha(\rho') - \sigma.$$

Let now $\rho > 0$ be such that $\rho' < \rho < r$ and $\alpha(\cdot)$ be continuous in $\rho$, let the hypotheses (3), (4) and the condition (3.6) be fulfilled and moreover

$$\alpha(\rho) - \alpha(\rho') < \frac{\sigma\,\alpha_0}{4\alpha_1}, \qquad \int\limits_{B_\rho \backslash B_{\rho'}} |\nabla v|^p \mathrm{d}y < \frac{\sigma}{4}.$$

By using the functions defined by (3.4) we set

$$u_h'(x) = \begin{cases} u_h(x) & \text{if } x \in B_r \backslash \overline{B}_\rho, \\ \hat{u}_h(x) & \text{if } x \in \overline{B}_\rho. \end{cases}$$

By Lemma 3.4 and by (3.6) it follows

$$
\begin{aligned}
&\lim_h \mathcal{F}(u'_h, c_h, \overline{B}_\rho \backslash B_{\rho'}) \\
(3.7)\qquad &\leq \lim_h \Big[\mathcal{F}(u_h, c_h, \overline{B}_\rho \backslash B_{\rho'}) + 2\alpha_1 c_h \mathcal{H}^{n-1}(\{\tilde{u}_h \neq \tilde{\tilde{u}}_h\} \cap \partial B_\rho)\Big] \\
&= \alpha(\rho) - \alpha(\rho') < \frac{\sigma\alpha_0}{4\alpha_1}.
\end{aligned}
$$

By Lemma 3.5 applied to the functions $u'_h$ and $v + \lambda_h \hat{e}^m$ for $h$ large enough we obtain a sequence of functions $w_h \in SBV(B_r; \mathbb{R}^m)$ such that $|w_h| = t_h$ a.e. in $B_r$, $w_h = u_h$ a.e. in $B_r \backslash \overline{B}_\rho$ and moreover, for a $n_\epsilon \in \mathbb{N}$, $n_\epsilon > 2^{p-1}/\epsilon$,

$$
(3.8)\qquad
\begin{aligned}
\mathcal{F}(w_h, c_h, \overline{B}_\rho) \leq \frac{1}{(1-\epsilon)^p} \Bigg[ &(1+\epsilon)\left(\mathcal{F}(v, c_h, \overline{B}_\rho) + \frac{\alpha_1}{\alpha_0} \mathcal{F}(u'_h, c_h, \overline{B}_\rho \backslash B_{\rho'})\right) \\
&+ \epsilon \left(\frac{2n_\epsilon}{\rho - \rho'}\right)^p \int_{B_\rho \backslash B_{\rho'}} |u'_h - v - \lambda_h \hat{e}^m|^p dy \Bigg].
\end{aligned}
$$

Since

$$
\mathcal{F}(u_h, c_h, \overline{B}_\rho) \leq \mathcal{F}(w_h, c_h, \overline{B}_\rho) + \Psi(u_h, c_h, \overline{B}_\rho, t_h),
$$

letting $h \to +\infty$, by the hypotheses (3) and (4) and by the conditions (3.5), (3.7) and (3.8) we obtain

$$
\alpha(\rho) \leq \frac{1+\epsilon}{(1-\epsilon)^p} \left( \int_{B_\rho} |\nabla v|^p dy + \frac{\sigma}{4} \right);
$$

hence, because of the arbitrariness of $\epsilon$,

$$
\alpha(\rho) \leq \int_{B_\rho} |\nabla v|^p dy + \frac{\sigma}{4} < \alpha(\rho') - \frac{\sigma}{2},
$$

and this is a contradiction because the function $\rho \to \alpha(\rho)$ is non-decreasing.

q.e.d.

COROLLARY 3.7. *Let* $\overline{B_r}(x) \subset \Omega$, $u_h \in SBV(\Omega; \mathbb{R}^m)$, $0 < \lambda_h < +\infty$ *and* $t_h > 0$ *for every* $h \in \mathbb{N}$. *Let* $u_\infty \in W^{1,p}(B_r(x); \mathbb{R}^m)$ *be such that*

(1) $\lim_h \mathcal{H}^{n-1}(S_{u_h} \cap \overline{B}_r(x)) = 0$, $\lim_h t_h = +\infty$, $\lim_h \lambda_h t_h^{-q} = 0$ *where* $q = p^*$ *if* $p < n$ *or* $q \geq 1$ *if* $p \geq n$,

(2) $|u_h| = t_h$ *a.e. in* $B_r(x)$,

(3) $\lim_h \mathcal{F}(u_h, \lambda_h, \overline{B}_\rho(x)) = \alpha(\rho) < +\infty$ *for almost all* $\rho < r$,

(4) $\lim_h \Psi(u_h, \lambda_h, \overline{B}_\rho(x), t_h) = 0$ *for every* $\rho < r$,

(5) $\lim_h [u_h - \mathrm{med}(u_h, B_r(x))] = u_\infty$ *a.e. on* $B_r(x)$.

*Under these conditions, the conclusion of Theorem 3.6 remains valid.*

PROOF. We may assume that $x = 0$. If $\limsup_h \lambda_h = +\infty$ the assertion follows by Theorem 3.6. If $\limsup_h \lambda_h < +\infty$, setting

$$c_h = \lambda_h \vee (\mathcal{H}^{n-1}(S_{u_h} \cap \overline{B}_r) + t_h^{-q})^{-1/2},$$

we have $\lim_h c_h = +\infty$ and also $\lim_h c_h t_h^{-q} = 0$. Indeed, for $h$ large enough, $c_h \leq t_h^{q/2}$. Passing to the limit, by the assumption (1), the preceding assertion follows.

We note that

$$\mathcal{F}(u_h, c_h, \overline{B}_\rho) = \mathcal{F}(u_h, \lambda_h, \overline{B}_\rho) + (c_h - \lambda_h) \int_{S_{u_h} \cap \overline{B}_\rho} \gamma(y, u_h^+, u_h^-, \nu)\, \mathrm{d}\mathcal{H}^{n-1},$$

and that for $h$ large enough

$$(c_h - \lambda_h) \int_{S_{u_h} \cap \overline{B}_\rho} \gamma(y, u_h^+, u_h^-, \nu)\, \mathrm{d}\mathcal{H}^{n-1} \leq c_h \alpha_1 \mathcal{H}^{n-1}(S_{u_h} \cap \overline{B}_\rho)$$
$$\leq \alpha_1 [\mathcal{H}^{n-1}(S_{u_h} \cap \overline{B}_\rho)]^{1/2}.$$

Hence, by the assumptions (1) and (3), we obtain $\lim_h \mathcal{F}(u_h, c_h, \overline{B}_\rho) = \alpha(\rho)$ for almost all $\rho < r$. In the same way we obtain also $\lim_h \Psi(u_h, c_h, \overline{B}_\rho) = 0$. Then the assertion follows by Theorem 3.6. q.e.d.

In the next Lemma 3.9 we shall use a result which is proved in a more general situation in [29] (see the Theorem in page 244).

THEOREM 3.8. *Let* $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ *be p-harmonic in* $\Omega$. *Let* $B_r$ *be a ball with* $r \in (0, 1]$ *such that* $B_r \subset \Omega$. *Then there exists a positive constant* $c_0$ *which depends only on* $n$, $m$ *and* $p$ *such that*

$$\operatorname*{ess\,sup}_{B_{\frac{2}{3}r}} |\nabla u|^p \leq \frac{c_0}{\omega_n r^n} \int_{B_r} (1 + |\nabla u|)^p \mathrm{d}y.$$

Let $k \in \mathbb{N}$, we define

$$SBV_{\mathrm{loc}}(\Omega; S^k) = \left\{ v \in SBV_{\mathrm{loc}}(\Omega; \mathbb{R}^{k+1}); |v| = 1 \text{ a.e. in } \Omega \right\}.$$

LEMMA 3.9. *(Decay lemma). For every* $\alpha \in (0, \frac{2}{3}]$, $\beta \in (0,1)$, *such that* $\alpha^\beta < 1/c_0(\omega_n + 1)2^{p-1}$, *and for every* $c > 0$, *there exist three positive constants* $\epsilon$, $\vartheta$ *and* $r$, *depending on* $n$, $k$, $p$, $c$, $\alpha_0$, $\alpha_1$, $\alpha$ *and* $\beta$, *such that: if* $\Omega \subset \mathbb{R}^n$ *is open,* $\rho \in (0, r]$, $\overline{B}_\rho(x) \subset \Omega$ *and if* $u \in SBV_{\text{loc}}(\Omega; S^k)$ *with*

$$\mathcal{H}^{n-1}(S_u \cap \overline{B}_\rho(x)) \leq \epsilon^p \rho^{n-1}$$

*and*

$$\Psi(u, c, \overline{B}_\rho(x), 1) \leq \vartheta \mathcal{F}(u, c, \overline{B}_\rho(x)),$$

*then*

$$\mathcal{F}(u, c, \overline{B}_{\alpha\rho}(x)) \leq \alpha^{n-\beta} \mathcal{F}(u, c, \overline{B}_\rho(x)).$$

PROOF. Suppose the lemma is not true. Then there exist $n \geq 2$, $\alpha \in (0, 2/3]$, $\beta \in (0,1)$, such that $\alpha^\beta < 1/c_0(\omega_n + 1)2^{p-1}$, $c > 0$, three sequences $(\epsilon_h)$, $(\vartheta_h)$ and $(\rho_h)$ such that $\lim_h \epsilon_h = \lim_h \vartheta_h = \lim_h \rho_h = 0$, a sequence $(u_h)$ in $SBV_{\text{loc}}(\Omega; S^k)$ and a sequence of balls $\overline{B}_{\rho_h}(x_h) \subset \Omega$, such that

$$\mathcal{H}^{n-1}(S_{u_h} \cap \overline{B}_{\rho_h}(x_h)) = \epsilon_h^p \rho_h^{n-1},$$

$$\Psi(u_h, c, \overline{B}_{\rho_h}(x_h), 1) \leq \vartheta_h \mathcal{F}(u_h, c, \overline{B}_{\rho_h}(x_h))$$

and

$$\mathcal{F}(u_h, c, \overline{B}_{\alpha\rho_h}(x_h)) > \alpha^{n-\beta} \mathcal{F}(u_h, c, \overline{B}_{\rho_h}(x_h)).$$

For each $h$, translating $x_h$ into the origin and blowing up, i.e. setting

$$v_h(x) = t_h u_h(x_h + \rho_h x) \qquad x \in B_1,$$

where

$$t_h = \sigma_h^{1/p} \rho_h^{(1-p)/p}, \qquad \sigma_h = \rho_h^{n-1}[\mathcal{F}(u_h, c, \overline{B}_{\rho_h}(x_h))]^{-1},$$

we have, by using Lemma 3.3,

$$\mathcal{H}^{n-1}(S_{v_h} \cap \overline{B_1}) = \epsilon_h^p,$$

$$\mathcal{F}(v_h, c\sigma_h, \overline{B}_1) = \rho_h^{1-n} \mathcal{F}(\sigma_h^{1/p} u_h, c\sigma_h, \overline{B}_{\rho_h}(x_h))$$

$$= \rho_h^{1-n} \sigma_h \mathcal{F}(u_h, c, \overline{B}_{\rho_h}(x_h)) = 1,$$

$$\Psi(v_h, c\sigma_h, \overline{B}_1, t_h) \leq \vartheta_h,$$

and also

$$\mathcal{F}(v_h, c\sigma_h, \overline{B}_\alpha) > \alpha^{n-\beta}. \tag{3.9}$$

By Theorem 2.6, there exist a subsequence of $(v_h)$, still denoted by $(v_h)$, and a function $v_\infty \in W^{1,p}(B_1; \mathbb{R}^{k+1})$ such that $\lim_h [v_h - \text{med}(v_h, B_1)] = v_\infty$ a.e. in $B_1$. In order to use Corollary 3.7 we prove that $\lim_h \sigma_h t_h^{-q} = 0$. We may assume $p < n$, so that $q = p^*$. Then, recalling that $\mathcal{F} = \Psi + \Phi$, by using Lemma 3.2 we have $\sigma_h \geq$ const $> 0$ and

$$\sigma_h t_h^{-q} = [\mathcal{F}(u_h, c, \overline{B}_{\rho_h}(x_h))]^{p/(n-p)} \rho_h^{-1} \leq \text{ const } \rho_h^{n(p-1)/(n-p)}.$$

By Corollary 3.7 we argue that the function $v_\infty$ is $p$-harmonic in $B_1$ and that

$$\limsup_h \mathcal{F}(v_h, c\sigma_h, \overline{B}_\alpha) \leq \int_{B_\alpha} |\nabla v_\infty|^p \, dy.$$

moreover

$$\int_{B_1} |\nabla v_\infty|^p dy \leq 1.$$

By Theorem 3.8 we have

$$\int_{B_\alpha} |\nabla v_\infty|^p dy \leq c_0 \alpha^n \int_{B_1} (1 + |\nabla v_\infty|)^p dy \leq 2^{p-1} c_0 \alpha^n (\omega_n + 1) < \alpha^{n-\beta},$$

whereas by (3.9) we have

$$\int_{B_\alpha} |\nabla v_\infty|^p dy \geq \alpha^{n-\beta}. \qquad \text{q.e.d.}$$

Obviously if in Lemma 3.9 we suppose $\mathcal{F}(u, c, \overline{B}_\rho(x)) \leq \epsilon^p \rho^{n-1}$ in place of $\mathcal{H}^{n-1}(S_u \cap \overline{B}_\rho(x)) \leq \epsilon^p \rho^{n-1}$ then the same thesis holds. Hence by an iteration argument as in [13], Lemma 4.10 and 4.11, we obtain the following result.

LEMMA 3.10. *Let $\alpha$, $\beta$, $c$, $\epsilon$, $\vartheta$ and $r$ be as in Lemma 3.9. Let $\rho \in (0, r]$, $\overline{B}_\rho(x) \subset \Omega$ and $u \in SBV_{\rm loc}(\Omega; S^k)$. Assume*

$$\mathcal{F}(u, c, \overline{B}_\rho(x)) \leq \epsilon^p \rho^{n-1},$$

$$\Psi(u, c, \overline{B_t}(x), 1) \leq \vartheta \epsilon^p (\alpha t)^{n-1} \qquad \textit{for every } t \leq \rho;$$

*moreover, assume that*

$$\lim_{t \to 0} t^{1-n} \Psi(u, c, \overline{B}_t(x), 1) = 0.$$

*Then*

$$\lim_{t \to 0} t^{1-n} \mathcal{F}(u, c, \overline{B}_t(x)) = 0.$$

Finally we prove a partial regularity theorem for functions in the space $SBV_{\rm loc}(\Omega; S^k)$ which satisfy the quasi-minimum condition (3.10).

THEOREM 3.11. *Let $c > 0$ and let $u \in SBV_{\mathrm{loc}}(\Omega; S^k)$. Assume that for every compact set $Q \subset \Omega$, $\mathcal{F}(u, c, Q) < +\infty$ and moreover that*

$$\lim_{\rho\to 0} \rho^{1-n} \sup_{x\in Q} \Psi(u, c, \overline{B}_\rho(x), 1) = 0. \tag{3.10}$$

*Set* $\Omega_0 = \left\{x \in \Omega;\ \lim_{\rho\to 0} \rho^{1-n}\mathcal{F}(u, c, \overline{B}_\rho(x)) = 0\right\}$, *then*

(i) $\Omega_0$ *is open and* $\overline{S}_u \cap \Omega \subseteq \Omega\backslash\Omega_0$,

(ii) $\mathcal{H}^{n-1}((\overline{S}_u \cap \Omega)\backslash S_u) = 0$.

PROOF. Let $x \in \Omega_0$ and let $\alpha$, $\beta$, $\epsilon$, $\vartheta$ and $r$ be as in Lemma 3.9. By the definition of $\Omega_0$ and by (3.10), there exists a positive $\rho < \frac{1}{2}$ dist$(x, \partial\Omega) \wedge r$ such that

$$\mathcal{F}(u, c, \overline{B_t}(x)) \le \epsilon^p 2^{1-n} t^{n-1}$$

and

$$\Psi(u, c, \overline{B_t}(y), 1) \le \vartheta\epsilon^p(\alpha t)^{n-1}$$

for every $t \le \rho$ and for every $y \in B_\rho(x)$. Then we infer that

$$\mathcal{F}(u, c, \overline{B}_{\rho/2}(y)) \le \mathcal{F}(u, c, \overline{B}_\rho(x)) \le \epsilon^p(\rho/2)^{n-1}$$

for every $y \in B_{\rho/2}(x)$. By Lemma 3.10 we conclude that $B_{\rho/2}(x) \subset \Omega_0$. Thus $\Omega_0$ is an open set. By Theorem 2.7 we have $S_u \subset \Omega\backslash\Omega_0$, and so $\overline{S_u} \cap \Omega \subset \Omega\backslash\Omega_0$. Finally (ii) follows by a covering argument (see e.g. Lemma 2.6 in [13]).

q.e.d.

## 4. - A Neumann-type problem

Let $n \in \mathbb{N}$, $n \ge 2$, let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $k \in \mathbb{N}$ and $p > 1$. To begin, we consider the class of the admissible pairs

$$\mathcal{A} = \left\{(K, u); K \subset \mathbb{R}^n \text{ closed},\ u \in C^1(\Omega\backslash K; S^k),\right.$$
$$\left.\int_{\Omega\backslash K} |\nabla u|^p dy + \mathcal{H}^{n-1}(K \cap \Omega) < +\infty\right\}.$$

We note that if $(K, u) \in \mathcal{A}$, by Lemma 2.3 it follows that

$$u \in SBV(\Omega; S^k) = \{v \in SBV(\Omega; \mathbb{R}^{k+1});\ |v| = 1 \text{ a.e. in } \Omega\}$$

and that three vector-valued Borel functions $u^+$, $u^-$, $\nu$ exist which are defined $\mathcal{H}^{n-1}$-a.e. on $K \cap \Omega$ and satisfy (2.3), (2.4).

Let $\psi : \mathbb{R}^{k+1} \times \mathbb{R}^n \to [0, +\infty)$ be a function such that $\psi$ is *lower semicontinuous, for every $u \in \mathbb{R}^{k+1}$ let $\psi(u, \cdot)$ be convex and positively 1-homogeneous, and moreover let two constants $\alpha_0$, $\alpha_1$ exist such that $0 < \alpha_0 \leq \psi(u, \nu) \leq \alpha_1$ for every $u \in \mathbb{R}^{k+1}$, $\nu \in S^{n-1}$ and $\psi(\cdot, \nu)$ be positively 0-homogeneous*. We define

$$\gamma(u, v, \nu) = \psi(u, \nu) + \psi(v, -\nu) \tag{4.1}$$

for every $u, v \in \mathbb{R}^{k+1}$, $\nu \in S^{n-1}$.

The main result of this section is the following existence theorem.

THEOREM 4.1. *Let $n \in \mathbb{N}$, $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $k \in \mathbb{N}$. Assume that $g \in L^\infty(\Omega; \mathbb{R}^{k+1})$; let $\gamma$ be given by* (4.1). *Then for every $p > 1$, $q \geq 1$, $\lambda > 0$ and $\mu > 0$ a pair $(K_0, u_0) \in \mathcal{A}$ exists such that*

$$\mathcal{G}(K_0, u_0) = \min_{(K,u) \in \mathcal{A}} \mathcal{G}(K, u),$$

*where*

$$\mathcal{G}(K, u) = \int_{\Omega \setminus K} |\nabla u|^p \mathrm{d}y + \mu \int_{\Omega \setminus K} |u - g|^q \mathrm{d}y + \lambda \int_{K \cap \Omega} \gamma(u^+, u^-, \nu) \mathrm{d}\mathcal{H}^{n-1}.$$

We prove this result by the direct methods in the Calculus of Variations. We shall use the following semicontinuity result proved in [4], (see Theorem 3.6 and Section 5.1).

THEOREM 4.2. *Let $\gamma$ be given by* (4.1). *Then for every $p > 1$, for every sequence $(u_h) \subset SBV(\Omega; \mathbb{R}^{k+1}) \cap L^\infty(\Omega; \mathbb{R}^{k+1})$ converging in $L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^{k+1})$ to $u \in SBV(\Omega; \mathbb{R}^{k+1})$ and satisfying the condition $\sup_h \|u_h\|_{L^\infty} < +\infty$, the following inequality holds*

$$\int_{\Omega} |\nabla u|^p \mathrm{d}y + \int_{S_u \cap \Omega} \gamma(u^+, u^-, \nu_u) \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_h \left[ \int_{\Omega} |\nabla u_h|^p \mathrm{d}y + \int_{S_{u_h} \cap \Omega} \gamma(u_h^+, u_h^-, \nu_{u_h}) \mathrm{d}\mathcal{H}^{n-1} \right].$$

Now we introduce a new functional defined for every $u \in SBV(\Omega; S^k)$ by

$$\overline{\mathcal{G}}(u) = \int_{\Omega} |\nabla u|^p \mathrm{d}y + \mu \int_{\Omega} |u - g|^q \mathrm{d}y + \lambda \int_{S_u \cap \Omega} \gamma(u^+, u^-, \nu) \mathrm{d}\mathcal{H}^{n-1}.$$

First we prove that at least one minimizer of the functional $\overline{\mathcal{G}}$ exists in $SBV(\Omega; S^k)$.

LEMMA 4.3. *Under the hypotheses of Theorem 4.1, there exists*

$$\min_{u \in SBV(\Omega;S^k)} \overline{\mathcal{G}}(u)$$

*and it is smaller than, or equal to,*

$$\inf_{(K,u)\in\mathcal{A}} \mathcal{G}(K,u).$$

PROOF. Let $(u_h) \subset SBV(\Omega; S^k)$ be a minimizing sequence for $\overline{\mathcal{G}}$. Since

$$\inf_{u \in SBV(\Omega;S^k)} \overline{\mathcal{G}}(u) < +\infty,$$

and

$$\int_\Omega |Du_h| = \int_\Omega |\nabla u_h| \mathrm{d}y + \int_{S_{u_h}\cap\Omega} |u_h^+ - u_h^-| \mathrm{d}\mathcal{H}^{n-1}$$

$$\leq |\Omega|^{(p-1)/p} \left( \int_\Omega |\nabla u_h|^p \mathrm{d}y \right)^{1/p} + \frac{1}{\alpha_0} \int_{S_{u_h}\cap\Omega} \gamma(u_h^+, u_h^-, \nu_{u_h}) \mathrm{d}\mathcal{H}^{n-1},$$

we conclude

$$\sup_h \int_\Omega |\mathrm{D}u_h| < +\infty.$$

By the compactness theorem in $BV(\Omega; \mathbb{R}^{k+1})$ (see e.g. [21], Theorem 1.19), there are a subsequence, still denoted by $(u_h)$, and a function $w \in BV(\Omega; \mathbb{R}^{k+1})$ such that $u_h \to w$ in $L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^{k+1})$. By Theorem 2.4 $w \in SBV(\Omega; S^k)$ and, by Theorem 4.2,

$$\overline{\mathcal{G}}(w) \leq \lim_h \overline{\mathcal{G}}(u_h),$$

thus $w$ is a minimizer for $\overline{\mathcal{G}}$ in $SBV(\Omega; S^k)$.
By Lemma 2.3, if $(K,u) \in \mathcal{A}$ then $u \in SBV(\Omega; S^k)$ and $S_u \cap \Omega \subset K$, hence we infer

$$\min_{u \in SBV(\Omega;S^k)} \overline{\mathcal{G}}(u) \leq \inf_{(K,u)\in\mathcal{A}} \mathcal{G}(K,u). \qquad \text{q.e.d.}$$

LEMMA 4.4. *Under the hypotheses of Theorem 4.1, if $w$ is a minimizer of the functional $\overline{\mathcal{G}}$ in $SBV(\Omega; S^k)$, then for every $\overline{B}_\rho(x) \subset \Omega$ the following estimates hold*

$$\Psi(w, \lambda, \overline{B}_\rho(x), 1) \leq \mu(\|g\|_{L^\infty} + 1)^q \omega_n \rho^n \tag{4.2}$$

*and*

$$\mathcal{F}(w,\lambda,\overline{B}_\rho(x)) \leq 2\lambda\alpha_1 n\omega_n\rho^{n-1} + \mu(\|g\|_{L^\infty}+1)^q\omega_n\rho^n. \tag{4.3}$$

PROOF. Let $v \in SBV(\Omega; S^k)$ with $v = w$ in $\Omega\backslash\overline{B}_\rho(x)$. By the minimality of $w$ we have

$$\mathcal{F}(w,\lambda,\overline{B}_\rho(x)) \leq \mathcal{F}(w,\lambda,\overline{B}_\rho(x)) + \mu \int_{B_\rho(x)} |w-g|^q \mathrm{d}y$$

$$\leq \mathcal{F}(v,\lambda,\overline{B}_\rho(x)) + \mu \int_{B_\rho(x)} |vg|^q \mathrm{d}y$$

$$\leq \mathcal{F}(v,\lambda,\overline{B}_\rho(x)) + \mu(\|g\|_{L^\infty}+1)^q\omega_n\rho^n,$$

thus, because of the arbitrariness of $v$, we infer (4.2). Choosing

$$v = w\chi_{\Omega\backslash\overline{B}_\rho(x)} + \chi_{\overline{B}_\rho(x)}\hat{e}^{k+1},$$

we deduce (4.3). q.e.d.

Now we state some regularity properties for a minimizer of the functional $\overline{\mathcal{G}}$.

LEMMA 4.5. *Under the hypotheses of Theorem* 4.1 *and if $w$ is a minimizer of the functional $\overline{\mathcal{G}}$ in $SBV(\Omega; S^k)$, then*

$$\tilde{w} \in C^1(\Omega\backslash\overline{S}_w; S^k) \quad \textit{and} \quad \mathcal{H}^{n-1}((\overline{S}_w \cap \Omega)\backslash S_w) = 0,$$

*where, for every $x \in \Omega\backslash S_w$, $\tilde{w}(x) = \operatorname{ap}\lim_{y\to x} w(y)$.*

PROOF. Let $B_\rho(x) \subset \Omega\backslash\overline{S}_w$; then $w \in W^{1,p}(B_\rho(x); S^k)$ and it is a minimizer of the functional

$$\int_{B_\rho(x)} |\nabla u|^p \mathrm{d}y + \mu \int_{B_\rho(x)} |u-g|^q \mathrm{d}y \tag{4.4}$$

among the functions $u$ in $w + W_0^{1,p}(B_\rho(x); S^k)$. By the inequality (4.3) and by Theorem 3.5.2 of [26] we have $\tilde{w} \in C^{0,(p-1)/p}(B_\rho(x); S^k)$. To infer the $C^1$ regularity of $\tilde{w}$ near the point $x$, we assume $\tilde{w}(x) = \hat{e}^{k+1}$ and choose $0 < \rho' < \rho$ so that $\tilde{w}(B_{\rho'}(x))$ is contained in $S^k \cap B_{1/2}(\hat{e}^{k+1})$. Then we may, in $B_{\rho'}(x)$, substitute

$$\tilde{w}^{k+1} = \left(1 - \sum_{i=1}^{k} (\tilde{w}^i)^2\right)^{1/2}$$

in the functional (4.4) to infer that $(\widetilde{w}^1, \ldots, \widetilde{w}^k)$ is a local minimizer in $B_{\rho'}(x)$ of the functional

$$\int_{B_{\rho'}(x)} \left( \sum_{i=1}^{k} \sum_{\alpha=1}^{n} (\nabla_\alpha \widetilde{w}^i)^2 + \frac{\sum_{\alpha=1}^{n} \left( \sum_{i=1}^{k} \widetilde{w}^i \nabla_\alpha \widetilde{w}^i \right)^2}{1 - \sum_{i=1}^{k} (\widetilde{w}^i)^2} \right)^{p/2} dy$$

$$+ \int_{B_{\rho'}(x)} \left( \sum_{i=1}^{k} (\widetilde{w}^i - g^i)^2 + \left( \sqrt{1 - \sum_{i=1}^{k} (\widetilde{w}^i)^2 g^{k+1}} \right)^2 \right)^{q/2} dy.$$

We now may adapt the proof of Theorem 4.3 in [18] in the case $p \geq 2$, or the proof of Theorem 1.2 in [1] in the case $1 < p < 2$, in order to obtain that $\nabla \widetilde{w}$ is locally Hölder continuous in $\Omega \backslash \overline{S}_w$. Moreover, taking into account (4.2), by Theorem 3.11 we infer also

$$\mathcal{H}^{n-1}((\overline{S}_w \cap \Omega) \backslash S_w) = 0. \qquad \text{q.e.d.}$$

PROOF OF THEOREM 4.1. Let $w \in SBV(\Omega; S^k)$ be a minimizer for $\overline{\mathcal{G}}$. By Lemma 4.5 the pair $(\overline{S}_w, \widetilde{w}) \in \mathcal{A}$ and moreover

$$\overline{\mathcal{G}}(w) = \mathcal{G}(\overline{S}_w, \widetilde{w}).$$

Since, by Lemma 4.3,

$$\overline{\mathcal{G}}(w) \leq \inf_{(K,u) \in \mathcal{A}} \mathcal{G}(K, u)$$

we conclude that the pair $(\overline{S}_w, \widetilde{w})$ gives the minimum of $\mathcal{G}$. q.e.d.

REMARK 4.6. We note that, if $(K, u) \in \mathcal{A}$ and minimizes $\mathcal{G}$, then there exists a unique pair $(K', u') \in \mathcal{A}$ with the following property: $K'$ is the smallest closed set contained in $K$ such that $u$ has an extension $u' \in C^1(\Omega \backslash K'; S^k)$; such a pair is called *essential minimizing pair.*

REMARK 4.7. If $(K, u) \in \mathcal{A}$ and minimizes $\mathcal{G}$, then, by Lemma 2.3, $u \in SBV(\Omega; S^k)$ and $\overline{S}_u \cap \Omega \subset K$. By the proof of Theorem 4.1 we conclude that, conversely, $u$ is a minimizer for $\overline{\mathcal{G}}$, $\overline{\mathcal{G}}(u) = \mathcal{G}(K, u)$, $\mathcal{H}^{n-1}((K \backslash \overline{S}_u) \cap \Omega) = 0$ and $u$ may be extended as a $C^1$ function to $\Omega \backslash \overline{S}_u$, hence $(\overline{S}_u, \widetilde{u})$ is an essential minimizing pair.

Finally we prove that for every essential minimizing pair of the functional $\mathcal{G}$, every compact contained in the singular set has $(n-1)$-dimensional Hausdorff measure which is equal to its Minkowski content (see Theorem 4.10). To this aim we give two estimates for the minimizers of the functional $\overline{\mathcal{G}}$.

LEMMA 4.8. *There exist two constants* $\epsilon, r_0 > 0$ *such that if* $w$ *is a minimizer of the functional* $\overline{\mathcal{G}}$ *in* $SBV(\Omega; S^k)$ *then for every* $x \in \overline{S}_w \cap \Omega$

$$\mathcal{F}(w, \lambda, \overline{B}_\rho(x)) \geq \epsilon^p \rho^{n-1}$$

*for every* $\rho < r_0 \wedge \operatorname{dist}(x, \partial\Omega)$.

PROOF. Let $\alpha$, $\beta$, $\epsilon$, $\vartheta$ and $r$ be as in Lemma 3.9. Let

$$r_0 = \frac{\vartheta \epsilon^p \alpha^{n-1}}{\mu(\|g\|_{L^\infty} + 1)^q \omega_n} \wedge r.$$

Were the lemma false, we would find $x \in \overline{S}_w \cap \Omega$ and a positive

$$\rho < r_0 \wedge \operatorname{dist}(x, \partial\Omega)$$

such that

$$\mathcal{F}(w, \lambda, \overline{B}_\rho(x)) < \epsilon^p \rho^{n-1}.$$

Since by (4.2)

$$\Psi(w, \lambda, \overline{B_t}(x), 1) \leq [\mu(\|g\|_{L^\infty} + 1)^q \omega_n t] t^{n-1} \leq \vartheta \epsilon^p (\alpha t)^{n-1}$$

for every $t \leq \rho$ and also

$$\lim_{t \to 0} t^{1-n} \Psi(w, \lambda, \overline{B}_t(x), 1) = 0,$$

then, by Lemma 3.10 and Theorem 3.11, we would have $x \notin \overline{S}_w \cap \Omega$. q.e.d.

LEMMA 4.9. *There exist two constants* $\epsilon_1, r_1 > 0$ *such that if* $w$ *is a minimizer of the functional* $\overline{\mathcal{G}}$ *in* $SBV(\Omega; S^k)$ *then for every* $x \in \overline{S}_w \cap \Omega$

$$\mathcal{H}^{n-1}(S_w \cap \overline{B}_\rho(x)) \geq \epsilon_1^p \rho^{n-1}$$

*for every* $\rho < r_1 \wedge \operatorname{dist}(x, \partial\Omega)$.

PROOF. Let $\alpha$, $\beta$, $\epsilon$, $\vartheta$ and $r$ be as in Lemma 3.9. Let $x \in \overline{S}_w \cap \Omega$. By Lemma 4.8 we have

$$\mathcal{F}(w, \lambda, \overline{B}_\rho(x)) \geq \epsilon^p \rho^{n-1} \quad \text{for every } \rho < r_0 \wedge \operatorname{dist}(x, \partial\Omega). \tag{4.5}$$

Let $c_1$ be such that the inequality (4.3) can be rewritten as

$$\mathcal{F}(w, \lambda, \overline{B}_\rho(x)) \leq c_1 \rho^{n-1}.$$

Let $\alpha' \in (0, 2/3]$ be such that $(\alpha')^\beta < 1/c_0(\omega_n + 1)2^{p-1}$ and $(\alpha')^{1-\beta} c_1 < \epsilon^p$. By Lemma 3.9 there exist three positive constants, which we denote by $\epsilon_1$,

$\vartheta_1$ and $r_1$, depending on $n$, $k$, $p$, $\lambda$, $\alpha_0$, $\alpha_1$, $\alpha'$ and $\beta$, such that $r_1 < r_0$, $\mu(\|g\|_{L^\infty}+1)^q\omega_n r_1/\epsilon^p < \vartheta_1$ and for every $\rho \in (0, r_1]$ with $\overline{B}_\rho(x) \subset \Omega$, if

$$\mathcal{H}^{n-1}(S_w \cap \overline{B}_\rho(x)) \le \epsilon_1^p \rho^{n-1}$$

and

$$\Psi(w, \lambda, \overline{B}_\rho(x), 1) \le \vartheta_1 \mathcal{F}(w, \lambda, \overline{B}_\rho(x))$$

then

$$\mathcal{F}(w, \lambda, \overline{B}_{\alpha'\rho}(x)) \le (\alpha')^{n-\beta} \mathcal{F}(w, \lambda, \overline{B}_\rho(x)). \tag{4.6}$$

Assume by contradiction that there exists a positive number $\rho < r_1$ so that

$$\mathcal{H}^{n-1}(S_w \cap \overline{B}_\rho(x)) < \epsilon_1^p \rho^{n-1}.$$

By (4.2) and (4.5)

$$\begin{aligned}\Psi(w, \lambda, \overline{B}_\rho(x), 1) &\le \mu(\|g\|_{L^\infty}+1)^q \omega_n \rho^n \\ &\le \frac{\mu(\|g\|_{L^\infty}+1)^q \omega_n \rho}{\epsilon^p} \mathcal{F}(w, \lambda, \overline{B}_\rho(x)) < \vartheta_1 \mathcal{F}(w, \lambda, \overline{B}_\rho(x)).\end{aligned}$$

Hence by (4.6) we have that

$$\mathcal{F}(w, \lambda, \overline{B}_{\alpha'\rho}(x)) \le (\alpha')^{n-\beta} \mathcal{F}(w, \lambda, \overline{B}_\rho(x)) \le (\alpha')^{1-\beta} c_1 (\alpha'\rho)^{n-1} < \epsilon^p (\alpha'\rho)^{n-1},$$

which contradicts (4.5). q.e.d.

THEOREM 4.10. *Let $(K, u) \in \mathcal{A}$ be a minimizing pair of the functional $\mathcal{G}$. Then*

(i) *$K \cap \Omega$ is $(\mathcal{H}^{n-1}, n-1)$ rectifiable;*

(ii) *there exists an essential minimizing pair $(K', u') \in \mathcal{A}$ such that for every compact set $Q \subset K' \cap \Omega$ the following equality holds*

$$\lim_{\rho \to 0} \frac{|\{x \in \Omega; \operatorname{dist}(x, Q) < \rho\}|}{2\rho} = \mathcal{H}^{n-1}(Q). \tag{4.7}$$

PROOF. By Remark 4.7 we have that $u$ is a minimizer of $\overline{\mathcal{G}}$ in $SBV(\Omega; S^k)$, $\mathcal{H}^{n-1}((K \setminus \overline{S}_u) \cap \Omega) = 0$ and $(\overline{S}_u, \tilde{u})$ is an essential minimizing pair. The assertion (i) immediately follows since $S_u$ is $(\mathcal{H}^{n-1}, n-1)$ rectifiable and $\mathcal{H}^{n-1}\left((\overline{S}_u \cap \Omega) \setminus S_u\right) = 0$. By choosing $K' = \overline{S}_u$ and $u' = \tilde{u}$, the equality (4.7) follows as in [6], Proposition 5.3, by using the uniform density estimate established in Lemma 4.9. q.e.d.

REMARK 4.11. We remark that the conditions on $\gamma$ (see (4.1)) are sufficient to obtain the semicontinuity result in Theorem 4.2. For more general integrands $\gamma$, for which also semicontinuity theorems hold, we refer to Theorems 3.6 and 4.1 in [4]. For these cases, if $\gamma$ is assumed even to be bounded above and below by two positive constants, one may repeat the discussions of this section in order to extend the preceding results.

## 5. - A Dirichlet-type problem

Let $\gamma$, $\mathcal{F}$ and $\Psi$ be as in Section 4; let assume $n$, $k \in \mathbb{N}$, $n \geq 2$, $p > 1$. For any $C^1$ function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\varphi(0) = 0 = |\nabla\varphi(0)|$, $Lip\,\varphi \leq 1$, let

$$\Omega_\varphi = \{x \in B_1;\ x_n > \varphi(x')\}$$

where $x' = (x_1, \dots, x_{n-1})$.

Arguing as in Theorem 3.7 and Corollary 3.8 of [9], we may modify the proof of Corollary 3.7 in order to obtain the following limit theorem.

THEOREM 5.1. *Let $\varphi_h : \mathbb{R}^{n-1} \to \mathbb{R}$ be a sequence of $C^1$ functions such that $\varphi_h(0) = 0 = |\nabla\varphi_h(0)|$, $Lip\,\varphi_h \leq 1$, $\lim_h \|\nabla\varphi_h\|_{L^\infty} = 0$. Let $\lambda_h > 0$ and let $t_h > 0$ such that $\lim_h t_h = +\infty$ and $\lim_h \lambda_h t_h^{-q} = 0$ where $q = p^*$ if $p < n$ or $q \geq 1$ if $p \geq n$. Let $w_h \in C^1(B_1;\mathbb{R}^{k+1})$ with $|w_h| = t_h$ a.e. in $B_1$ and $\lim_h(\|w_h - \mathrm{med}(w_h, B_1)\|_{L^\infty} + \|\nabla w_h\|_{L^\infty}) = 0$. Let $u_h \in SBV(B_1;\mathbb{R}^{k+1})$ with $|u_h| = t_h$ a.e. in $B_1$, such that $u_h = w_h$ in $B_1 \backslash \overline{\Omega}_{\varphi_h}$ for every $h \in \mathbb{N}$. Let $u_\infty \in W^{1,p}(B_1;\mathbb{R}^{k+1})$. Assume that*

(1) $\lim_h \mathcal{H}^{n-1}(S_{u_h} \cap B_1) = 0$,

(2) $\lim_h \mathcal{F}(u_h, \lambda_h, \overline{B}_\rho) = \alpha(\rho) < +\infty$ *for almost all* $\rho < 1$,

(3) $\lim_h \Psi(u_h, \lambda_h, \overline{\Omega}_{\varphi_h} \cap \overline{B}_\rho, t_h) = 0$ *for every* $\rho < 1$,

(4) $\lim_h [u_h - \mathrm{med}(u_h, B_1)] = u_\infty$ *a.e. on* $B_1$.

*Then $u_\infty \in C^0(B_1;\mathbb{R}^{k+1})$, there exists a hyperplane $\Sigma$ such that the image of $u_\infty$ lies essentially in $\Sigma$, $u_\infty = 0$ in $\{x_n \leq 0\} \cap B_1$, $u_\infty$ is p-harmonic in $\{x_n > 0\} \cap B_1$, and*

$$\alpha(\rho) = \int_{B_\rho} |\nabla u_\infty|^p \mathrm{d}y$$

*for almost all* $\rho < 1$.

We note that for the preceding function $u_\infty$ a reflection principle is true (for instance see [23], page 576). Then by Theorem 5.1 and by the reflection principle, we infer by contradiction, as in Lemma 3.9 of [9], the following decay estimate near a boundary point.

LEMMA 5.2. *For every $\alpha \in (0,2/3]$, $\beta \in (0,1)$, such that $\alpha^\beta < 1/c_0 \cdot(\omega_n+1)2^{p-1}$, and for every $\lambda > 0$, $L > 0$, there exist three positive constants $\epsilon$, $\vartheta$ and $r$, depending on $n$, $k$, $p$, $\lambda$, $\alpha_0$, $\alpha_1$, $\alpha$, $\beta$ and $L$, such that:*
*for every $\varphi \in C^1(\mathbb{R}^{n-1})$ with $\varphi(0) = 0 = |\nabla\varphi(0)|$, $Lip\,\varphi \le 1$, and for every $w \in C^1(B_1; S^k)$ with $Lip\, w \le L$, if $0 < \rho < r \wedge 1$, if $u \in SBV(B_1; S^k)$ such that $u = w$ in $B_\rho \backslash \overline{\Omega}_\varphi$, $\Psi(u, \lambda, \overline{\Omega}_\varphi \cap \overline{B}_\rho, 1) = 0$ and*

$$\mathcal{H}^{n-1}(S_u \cap B_\rho) \le \epsilon^p \rho^{n-1},$$

*then*

$$\mathcal{F}(u,\lambda,\overline{B}_{\alpha\rho}) \le \alpha^{n-\beta}\max\{\mathcal{F}(u,\lambda,\overline{B}_\rho),\ \vartheta\rho^{n-1}(Lip\,\varphi + Lip\,w)\}$$

*and*

$$\lim_{t\to 0} t^{1-n}\mathcal{F}(u,\lambda,\overline{B}_t) = 0.$$

By using Lemma 3.10 and Lemma 5.2 we obtain the following partial regularity theorem for a minimizer of a Dirichlet problem with free discontinuity set.

THEOREM 5.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\Omega \subset\subset B_R$ and let $\lambda > 0$. Assume that $\partial\Omega$ is a $C^1$ surface; let $w \in C^1(\partial\Omega; S^k)$ and let $w_e \in W^{1,1}(B_R; S^k)$ be an extension of $w$ with $\tilde{w}_e = w$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$. Assume that $u \in SBV(B_R; S^k)$ satisfies the conditions $u = w_e$ in $B_R \backslash \overline{\Omega}$ and $\Psi(u, \lambda, \overline{\Omega}, 1) = 0$.*
*Set $\Omega_0 = \left\{x \in \overline{\Omega};\ \lim_{\rho\to 0}\rho^{1-n}\mathcal{F}(u,\lambda,\overline{\Omega}\cap\overline{B}_\rho(x)) = 0\right\}$, then*

(i) *$\Omega_0$ is relatively open in $\overline{\Omega}$ and $\overline{S}_u \cap \overline{\Omega} \subseteq \overline{\Omega}\backslash\Omega_0$,*
(ii) $\mathcal{H}^{n-1}\left((\overline{S_u} \cap \overline{\Omega})\backslash S_u\right) = 0$.

We conclude this section by stating the existence of a solution for a Dirichlet problem with free discontinuity set.

Fixed $w \in C^1(\partial\Omega; S^k)$, we set

$$\mathcal{A}_w = \left\{(K,u) \in \mathcal{A};\ u \in C^0(\overline{\Omega}\backslash K; S^k),\ u = w \text{ in } \partial\Omega\backslash K\right\}$$

(for the definition of the class $\mathcal{A}$ see Section 4). Given $(K,u) \in \mathcal{A}_w$, by Lemma 2.3 three vector valued Borel functions $u^+$, $u^-$, $\nu$ exist which are defined $\mathcal{H}^{n-1}$-a.e. on $K \cap \Omega$ and satisfy (2.3) and (2.4). Moreover for $x \in K \cap \partial\Omega$ we denote by $\nu(x)$ the outer unit normal to $\partial\Omega$, and we set $u^+(x) = w(x)$ and

$$u^-(x) = \liminf_{\rho\to 0}\frac{1}{|\Omega\cap B_\rho(x)|}\int_{\Omega\cap B_\rho(x)} u(y)\mathrm{d}y.$$

Let $p > 1$, $\lambda > 0$ and let $\gamma$ be given by (4.1). For every $(K, u) \in \mathcal{A}_w$ we define the functional

$$\mathcal{G}_0(K,u) = \int_{\Omega\setminus K} |\nabla u|^p \mathrm{d}y + \lambda \int_{K\cap\overline{\Omega}} \gamma(u^+, u^-, \nu)\mathrm{d}\mathcal{H}^{n-1}.$$

THEOREM 5.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $C^1$ boundary and let $w \in C^1(\partial\Omega; S^k)$. Then a pair $(K_0, u_0) \in \mathcal{A}_w$ exists such that*

$$\mathcal{G}_0(K_0, u_0) = \min_{(K,u)\in\mathcal{A}_w} \mathcal{G}_0(K, u).$$

REMARK 5.5. For every $(K, u) \in \mathcal{A}_w$ minimizing $\mathcal{G}_0$, there exists a unique pair $(K', u') \in \mathcal{A}_w$ with the following property: $K'$ is the smallest closed set contained in $K$ such that $u$ has an extension $u' \in C^1(\Omega\setminus K'; S^k) \cap C^0(\overline{\Omega}\setminus K'; S^k)$ with $u' = w$ in $\partial\Omega\setminus K'$; such a pair is called *essential minimizing pair.*

As in Section 4 we may prove that for every essential minimizing pair of the functional $\mathcal{G}_0$ estimates similar to the ones of Lemma 4.8 and 4.9 hold even at the boundary points and the $(n-1)$-dimensional Hausdorff measure of the singular set is equal to its Minkowski content.

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## REFERENCES

[1] E. ACERBI - N. FUSCO, *Regularity for minimizers of non-quadratic functionals: the case* $1<p<2$, J. Math. Anal. Appl., **140** (1989), 115-135.

[2] F.J. ALMGREN JR. - E.H. LIEB, *Singularities of energy minimizing maps from the ball to the sphere: Examples, counterexamples, and bounds*, Ann. of Math., **128** (1988), 483-530.

[3] L. AMBROSIO, *A compactness theorem for a special class of functions of bounded variation*, Boll. Un. Mat. Ital., **3-B** (1989), 857-881.

[4] L. AMBROSIO, *Existence theory for a new class of variational problems*, Arch. Rational Mech. Anal., **111** (1990), 291-322.

[5] L. AMBROSIO - G. DAL MASO, *The chain rule for the distributional derivatives*, Proc. Amer. Math. Soc., to appear.

[6] L. AMBROSIO - V.M. TORTORELLI, *Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence*, Comm. Pure Appl. Math., **43** (1990), 999-1036.

[7] F. BETHUEL - H. BRÉZIS - J.M. CORON, *Relaxed energies for harmonic maps*, Preprint 1989.

[8] H. BRÉZIS - J.M. CORON - E.H. LIEB, *Harmonic Maps with Defects*, Comm. Math. Phys., **107** (1986), 679-705.

[9] M. CARRIERO - A. LEACI, *Existence theorem for a Dirichlet problem with free discontinuity set*, Nonlinear Anal. TMA, **15** (1990), 661-677.

[10] S. CHANDRASEKHAR, *Liquid Crystals*. Cambridge University Press, Cambridge, 1977.

[11] E. DE GIORGI, *Free Discontinuity Problems in Calculus of Variations*, Proc. Int. Meeting in J.L. Lions's honour, Paris, June 6-10, 1988, to appear.

[12] E. DE GIORGI - L. AMBROSIO, *Un nuovo tipo di funzionale del Calcolo delle Variazioni*, Atti Accad. Naz. Lincei, **82** (1988), 199-210.

[13] E. DE GIORGI - M. CARRIERO - A. LEACI, *Existence theorem for a minimum problem with free discontinuity set*, Arch. Rational Mech. Anal., **108** (1989), 195-218.

[14] J. EELLS - L. LEMAIRE, *Another report on harmonic maps*, Bull. London Math. Soc., **20** (1988), 385-524.

[15] J.L. ERICKSEN, *Equilibrium theory of liquid crystals*, Adv. Liq. Cryst., Vol. 2, G.H. Brown Ed., Academic Press, New York, 1976, 233-299.

[16] H. FEDERER, *Geometric Measure Theory*, Springer Verlag, Berlin, 1969.

[17] M. GIAQUINTA - E. GIUSTI, *The singular set of the minima of certain quadratic functionals*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **11** (1984), 45-55.

[18] M. GIAQUINTA - G. MODICA, *Remarks on the regularity of the minimizers of certain degenerate functionals*, Manuscripta Math., **57** (1986), 55-100.

[19] M. GIAQUINTA - G. MODICA - J. SOUČEK, *Cartesian currents and variational problems for mappings into spheres*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **16** (1989), 393-485.

[20] M. GIAQUINTA - G. MODICA - J. SOUČEK, *The Dirichlet energy of mappings with values into the sphere*, Manuscripta Math., **65** (1989), 489-507.

[21] E. GIUSTI, *Minimal Surfaces and Functions of Bounded Variation*. Birkhäuser Verlag, Boston, 1984.

[22] R. HARDT - D. KINDERLEHRER - F.-H. LIN, *Existence and partial regularity of static liquid crystal configurations*, Comm. Math. Phys., **105** (1986), 547-570.

[23] R. HARDT - F.-H. LIN, *Mappings minimizing the $L^p$ norm of the gradient*, Comm. Pure Appl. Math., **40** (1987), 555-588.

[24] R. HARDT - F.-H. LIN, *A remark on $H^1$ mappings*, Manuscripta Math., **56** (1986), 1-10.

[25] U. MASSARI - M. MIRANDA, *Minimal Surfaces of Codimension One*, North-Holland, Amsterdam, 1984.

[26] C.B. MORREY, *Multiple Integrals in the Calculus of Variations*, Springer Verlag, New York, 1966.

[27] R. SCHOEN - K. UHLENBECK, *A regularity theory for harmonic maps*, J. Differential Geom., **18** (1982), 307-335.

[28] R. Schoen - K. Uhlenbeck, *Boundary regularity and the Dirichlet problem of harmonic maps*, J. Differential Geom., **18** (1983), 253-268.

[29] P. Tolksdorff, *Everywhere regularity for some quasilinear systems with a lack of ellipticity*, Ann. Mat. Pura Appl., **134** (1983), 241-266.

[30] E. Virga, *Drops of nematic liquid crystals*, Arch. Rational Mech. Anal., **107** (1989), 371-390.

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