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On the Evolution Equations of Viscous Gaseous Stars

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1. - Introduction

In this paper we study the non-stationary motion of a star regarded as a compressible viscous fluid with self gravitation, bounded by a free surface. The star is supposed to occupy a given bounded domain Ω_0 of $\mathbb{R}^3$ at the initial time $t = 0$, while for each subsequent instant t it occupies the domain Ω_t, not known a priori. The equations governing the motion, obtained by the three laws of conservation (momentum, mass and energy), are the following (see for instance Serrin [7] and for detailed discussions in astrophysical context Ledoux-Walraven [1]):

$$(1.1)\quad \overline{\rho}[\dot{u} + (u \cdot \nabla)u + \nabla\overline{\phi} - \overline{b}] = -\nabla p + \sum_j \mu \mathrm{D}_j(\mathrm{D}_j u + \nabla u_j) + \left(\varsigma - \frac{2}{3}\mu\right) \nabla \operatorname{div} u$$

in $D_T \equiv \{(t, y) \in]0, T[\times\mathbb{R}^3 \mid y \in \Omega_t\}$,

$$(1.2)\qquad \dot{\overline{\rho}} + \operatorname{div}(\overline{\rho}u) = 0 \qquad \text{in } D_T,$$

$$(1.3)\qquad \begin{aligned} c_v\overline{\rho}(\dot{\overline{\theta}} + u \cdot \nabla\overline{\theta}) = {} & -\overline{\theta}\frac{\partial p}{\partial\overline{\theta}} \operatorname{div} u + \chi\Delta\overline{\theta} + \overline{\rho}\,\overline{r} + \frac{\mu}{2}\sum_{i,j}(\mathrm{D}_i u_j + \mathrm{D}_j u_i)^2 \\ & + \left(\varsigma - \frac{2}{3}\mu\right)(\operatorname{div} u)^2 \end{aligned}$$

in D_T.

The unknowns are the density $\overline{\rho} = \overline{\rho}(t, y)$, the fluid velocity $u = u(t, y) = {}^t(u_1, u_2, u_3)$, the temperature $\overline{\theta} = \overline{\theta}(t, y)$ and the domain Ω_t. Here $\dot{u}$ represents the time derivative. The external force field per unit mass $\overline{b}$ and the heat supply

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per unit mass per unit time $\overline{r}$ are known functions defined in $]0, T_0[\times\mathbb{R}^3$. The pressure $p = p(\overline{\rho}, \overline{\theta})$ and the specific heat at constant volume $c_v = c_v(\overline{\rho}, \overline{\theta})$ are given functions depending on the density and the temperature; the viscosity coefficients μ and ς and the coefficient of heat conductivity χ are assumed to be constant and to satisfy $\mu > 0$, $\varsigma > 0$, $\chi > 0$. Moreover $\overline{\phi}$ represents the Newtonian gravitational potential given by

$$\overline{\phi}(t, y) = -\kappa \int_{\mathbb{R}^3} \frac{\overline{\rho}(t, z)}{|y - z|} \mathrm{d}z, \tag{1.4}$$

κ standing for the constant of gravitation. We consider the following boundary conditions. The velocity satisfies a dynamical condition expressing the continuity of stress across the free boundary:

$$-pn_i^t + \mu \sum_j (\mathrm{D}_i u_j + \mathrm{D}_j u_i) n_j^t + \left(\varsigma - \frac{2}{3}\mu\right) \operatorname{div} u \, n_i^t = -\overline{p} n_i^t \tag{1.5}$$

on $S_T \equiv \{(t, y) \in]0, T[\times\mathbb{R}^3 \,|\, y \in \partial\Omega_t\}$.

Here $\overline{p}$ means the external pressure, a known function defined in $]0, T_0[\times\mathbb{R}^3$; $n^t = n^t(y)$ is the unit outward normal vector to $\partial\Omega_t$ at the point $y \in \partial\Omega_t$. The free boundary $\partial\Omega_t$ must be subjected to another kinematic condition, namely

(1.6) at each instant t of time it consists of the very same particles.

For a discussion on the above two boundary conditions see, for instance, Wehausen-Laitone [9]. We consider also the following boundary condition for the temperature:

$$\chi \frac{\partial \overline{\theta}}{\partial n} = h(\hat{\theta} - \overline{\theta}) \qquad \text{on } S_T, \tag{1.7}$$

where the external temperature $\hat{\theta}$ is a known function defined in $]0, T_0[\times\mathbb{R}^3$ and h is a given positive constant. Finally we consider the following initial conditions:

$$u(0, y) = u_0(y) \qquad \text{in } \Omega_0, \tag{1.8}$$

$$\overline{\rho}(0, y) = \rho_0(y) \qquad \text{in } \Omega_0, \tag{1.9}$$

$$\overline{\theta}(0, y) = \theta_0(y) \qquad \text{in } \Omega_0. \tag{1.10}$$

The free boundary problem for compressible Navier-Stokes equations (without considering self-gravitation) has been studied by P. Secchi and A. Valli [6] and by A. Tani [8]. T. Makino [4] investigated the Cauchy problem for the equations describing the evolution of a star regarded as an isentropic ideal gas with self gravitation (i.e., equations (1.2), (1.1) without the viscous terms and without considering (1.3); moreover, it is assumed that $p = K\rho^\gamma$, γ being the adiabatic exponent). In the present paper we find a solution of the above problem in a space of Sobolev type for short time. The proof of existence is obtained by linearization and by a fixed point argument as in [6]; to simplify the proof, the solution is found to exist in a Sobolev space with less regularity. In particular, if we neglect in (1.1) the term with the gravitational potential and do not consider (1.4) (i.e., we set $\kappa = 0$), we obtain a simpler proof of the result in [6].

As usual in free boundary problems, it is convenient to write the problem in the Lagrangian formulation, so that the domain of the unknowns becomes fixed in time.

Let $\eta(t,\cdot):\overline{\Omega}_0 \to \mathbb{R}^3$ be the solution of

$$(1.6)' \qquad \begin{aligned} &\dot\eta(t,x) = u(t,\eta(t,x)) && \text{in }]0,T[\times\overline{\Omega}_0,\\ &\eta(0,x) = x && \text{in } \overline{\Omega}_0, \end{aligned}$$

so that $(t,y) = (t,\eta(t,x))$ for a suitable $x \in \Omega_0$. Then $\Omega_t = \eta(t,\Omega_0)$ and, if η is an homeomorphism, $\eta(t,\partial\Omega_0) = \partial[\eta(t,\Omega_0)]$. Hence condition (1.6) can be substituted by (1.6)$'$. If we set $v(t,x) = u(t,\eta(t,x))$, $\rho(t,x) = \overline{\rho}(t,\eta(t,x))$, $\theta(t,x) = \overline{\theta}(t,\eta(t,x))$, $\phi(t,x) = \overline{\phi}(t,\eta(t,x))$, $b(t,x) = \overline{b}(t,\eta(t,x))$, $r(t,x) = \overline{r}(t,\eta(t,x))$, $p'(t,x) = \overline{p}(t,\eta(t,x))$, $\theta'(t,x) = \hat\theta(t,\eta(t,x))$, problem (1.1)-(1.10) becomes

$$(1.11) \qquad \begin{aligned} \rho(\dot v_i + a_{ki}\mathrm{D}_k\phi - b_i) = &- a_{ki}\mathrm{D}_k p + \mu a_{kj}\mathrm{D}_k(a_{sj}\mathrm{D}_s v_i + a_{si}\mathrm{D}_s v_j)\\ &+ (\varsigma - \tfrac{2}{3}\mu)a_{ki}\mathrm{D}_k(a_{sj}\mathrm{D}_s v_j) \quad \text{in } Q_T \equiv]0,T[\times\Omega_0, \end{aligned}$$

$$(1.12) \qquad \dot\rho + \rho a_{ki}\mathrm{D}_k v_i = 0 \qquad \text{in } Q_T,$$

$$(1.13) \qquad \begin{aligned} c_v(\rho,\theta)\rho\dot\theta = &- \theta\frac{\partial p}{\partial\theta}(\rho,\theta)a_{ki}\mathrm{D}_k v_i + \chi a_{ki}\mathrm{D}_k(a_{si}\mathrm{D}_s\theta) + \rho r\\ &+ \frac{\mu}{2}\sum_{i,j}(a_{kj}\mathrm{D}_k v_i + a_{ki}\mathrm{D}_k v_j)^2 + \left(\varsigma - \frac{2}{3}\mu\right)(a_{ki}\mathrm{D}_k v_i)^2 \text{ in } Q_T, \end{aligned}$$

$$(1.14) \qquad \dot\eta = v \qquad \text{in } Q_T,$$

$$(1.15) \qquad \phi(t,x) = -\kappa\int_{\Omega_0}\frac{\rho(t,\xi)}{|\eta(t,x)-\eta(t,\xi)|}|\det[\mathrm{D}\eta]|d\xi \qquad \text{in } Q_T,$$

$$(1.16)\qquad -pN_i + \mu(a_{ki}\mathrm{D}_k v_j + a_{kj}\mathrm{D}_k v_i)N_j + \left(\varsigma - \frac{2}{3}\mu\right) a_{kj}\mathrm{D}_k v_j N_i = -p' N_i$$

on $\Sigma_T \equiv]0,T[\times\partial\Omega_0$,

$$(1.17)\qquad \chi a_{ki}\mathrm{D}_k\theta N_i = h(\theta' - \theta) \qquad \text{on } \Sigma_T,$$

$$(1.18)\qquad v(0) = u_0 \qquad \text{in } \Omega_0,$$

$$(1.19)\qquad \rho(0) = \rho_0 \qquad \text{in } \Omega_0,$$

$$(1.20)\qquad \theta(0) = \theta_0 \qquad \text{in } \Omega_0,$$

$$(1.21)\qquad \eta(0) = \mathrm{Id} \qquad \text{in } \Omega_0 \text{ (Id identity function in } \Omega_0).$$

All indices run through 1, 2, 3; here and in the sequel, we adopt the Einstein convention about summation over repeated indices. The coefficients $a_{ki}(t,x)$ are the entries (k,i) of the Jacobian matrix $[\mathrm{D}\eta]^{-1}$ (where $\mathrm{D}\eta$ has the term $\mathrm{D}_k\eta_i$ in the i-th row, k-th column) and $N(t,x)$ is the normal to $\partial[\eta(t,\Omega_0)]$ calculated in $\eta(t,x)$, i.e., $N(t,x) = n^t(\eta(t,x))$. When problem (1.11)-(1.21) is solved, we can find a solution to the original problem (1.1)-(1.10) if η is a regular enough homeomorphism. We shall see in Theorems A and B the precise results.

Set $B_R \equiv \{x \in \mathbb{R}^3 \mid |x| < R\}$. Let us denote with $C^0(\overline{\Omega}_0)$ the space of continuous (and bounded) functions on $\overline{\Omega}_0$ and with $C^k(\overline{\Omega}_0)$ (k positive integer) the space of functions with derivatives up to order k in $C^0(\overline{\Omega}_0)$. Moreover, if m is a positive integer, $H^m(\Omega_0)$ is the Sobolev space of functions with m derivatives in $L^2(\Omega_0)$; we shall denote its norm by $\|\cdot\|_m$. For the definitions of $H^s(\Omega_0)$ and $H^s(\partial\Omega_0)$ (s not integer) see [2]. If X is a Banach space, $L^2(0,T;X)$, $L^\infty(0,T;X)$, $H^m(0,T;X)$, $H^s(0,T;X)$ are the spaces of X-valued functions in L^2, L^∞, and H^m, H^s respectively. $C^\alpha([0,T];X)$ is the space of X-valued Hölder continuous functions with exponent α. We shall denote by $|\cdot|_{p,m,T}$ the norm of $L^p(0,T;H^m(\Omega_0))$, $1 \le p \le +\infty$, by $\|\cdot\|_{m,m/2,Q_T}$ the norm in the space

$$H^{m,m/2}(Q_T) \equiv L^2(0,T;H^m(\Omega_0)) \cap H^{m/2}(0,T;L^2(\Omega_0)),$$

by $\|\cdot\|_{s,s/2,\Sigma_T}$ the norm in the space

$$H^{s,s/2}(\Sigma_T) \equiv L^2(0,T;H^s(\partial\Omega_0)) \cap H^{s/2}(0,T;L^2(\partial\Omega_0)).$$

The norm in $H^s(0,T;X)$ (s not integer) is defined in this way:

$$\|w\|^2_{H^s(0,T;X)} \equiv \|w\|^2_{H^{[s]}(0,T;X)} + \int_0^T\int_0^T \frac{\|w(t)-w(\tau)\|^2_X}{|t-\tau|^{1+2(s-[s])}}\mathrm{d}t\,\mathrm{d}\tau.$$

We shall prove the following results.

THEOREM A. *Let Ω_0 be a bounded connected open subset of $\mathbb{R}^3$, locally situated on one side of its boundary $\partial\Omega_0$; we assume $\partial\Omega_0 \in C^3$. Suppose that*

$$\bar{b} \in L^2(0,T_0;H^1(B_R)) \cap L^2(0,T_0;C^0(\overline{B}_R)),$$

$$\bar{r} \in L^2(0,T_0;H^1(B_R)) \cap L^2(0,T_0;C^0(\overline{B}_R))$$

for each $R>0$, $p \in C^2$, $c_v \in C^2$, $c_v > 0$,

$$\overline{p} \in H^{3/4}(0,T_0;H^1(B_R)) \cap L^\infty(0,T_0;H^2(B_R)),$$

$$\hat{\theta} \in H^{3/4}(0,T_0;H^1(B_R)) \cap L^\infty(0,T_0;H^2(B_R))$$

for each $R>0$, $u_0 \in H^2(\Omega_0)$, $\rho_0 \in H^2(\Omega_0)$ with $\min_{x\in\overline{\Omega}_0} \rho_0(x) \equiv m > 0$, $\theta_0 \in H^2(\Omega_0)$. Assume that the (necessary) compatibility conditions

$$\text{(1.22)} \qquad \begin{aligned} &\mu[\mathrm{D}_i(u_0)_j + \mathrm{D}_j(u_0)_i]n_j^0 + \left(\varsigma - \frac{2}{3}\mu\right) \operatorname{div} u_0\, n_i^0 \\ &\qquad = [p(\rho_0,\theta_0) - \overline{p}(0)]n_i^0 \qquad \text{on } \partial\Omega_0, \end{aligned}$$

for $i = 1,2,3$,

$$\text{(1.23)} \qquad \chi\frac{\partial\theta_0}{\partial n} = h(\hat{\theta}(0) - \theta_0) \qquad \text{on } \partial\Omega_0,$$

are satisfied.

Then there exist $T' \in]0,T_0]$,

$$v \in L^2(0,T';H^3(\Omega_0)) \cap H^1(0,T';H^1(\Omega_0)),$$

$\rho \in H^1(0,T';H^2(\Omega_0))$ with $\dot{\rho} \in L^\infty(0,T';H^1(\Omega_0))$ such that $\rho > 0$ in $\overline{Q}_{T'}$,

$$\theta \in L^2(0,T';H^3(\Omega_0)) \cap H^1(0,T';H^1(\Omega_0))$$

and a diffeomorphism

$$\eta \in H^1(0,T';H^3(\Omega_0)) \cap H^2(0,T';H^1(\Omega_0))$$

such that (v,ρ,θ,η) is a solution of (1.11)-(1.21).

A direct consequence of Theorem A is the following result (see [6] for a sketch of proof).

THEOREM B. *If the hypotheses of Theorem A hold, then there exists $T' \in]0,T_0]$, and for each $t \in [0,T']$ there exist a diffeomorphism $x \to \eta(t,x)$, a* *domain* $\Omega_t = \eta(t, \Omega_0)$, *a velocity field* $u(t,\cdot)$, *a temperature* $\overline{\theta}(t,\cdot)$ *and a density* $\overline{\rho}(t,\cdot)$ *which are solution of* (1.1)-(1.10). *Moreover* $\partial\Omega_t$ *is of class* C^1, *and we have*

$$\eta \in H^1(0,T';H^3(\Omega_0)) \cap H^2(0,T';H^1(\Omega_0)),$$

$$\begin{aligned}
&\mathrm{D}^k u \in L^2(D_{T'}),\ \mathrm{D}^k\overline{\theta} \in L^2(D_{T'}) && \textit{for } k = 0, 1, 2, 3,\\
&\mathrm{D}^k \dot{u} \in L^2(D_{T'}),\ \mathrm{D}^k\dot{\overline{\theta}} \in L^2(D_{T'}) && \textit{for } k = 0, 1,\\
&\mathrm{D}^k \overline{\rho} \in L^2(D_{T'}),\ \mathrm{D}^k\dot{\overline{\rho}} \in L^2(D_{T'}) && \textit{for } k = 0, 1, 2,\\
&\|\mathrm{D}^k\dot{\overline{\rho}}\|_{L^2(\Omega_t)} \in L^\infty(0,T') && \textit{for } k = 0, 1,
\end{aligned}$$

with $\overline{\rho} > 0$ *in* $\overline{D}_{T'}$.

2. - Proof of Theorem A

We prove the existence of a solution to (1.11)-(1.21) by a fixed point argument, following the approach of [6]. From now on each constant $c,\ c_i,\ c_i',\ C_i,\ C_i',\ T_i,\ T'$ will depend at most on the data of the problem $\Omega_0,\ T_0,\ \mu,\ \varsigma,\ \chi,\ u_0,\ \rho_0,\ \theta_0,\ p,\ c_v,\ \overline{b},\ \overline{p},\ \overline{r}$. Moreover we shall assume the outward unit normal n^0 to $\partial\Omega_0$ extended in a regular way, i.e., $n^0 \in C^2(\overline{\Omega}_0)$. Define the operators

$$A_i(x,\mathrm{D})w = -\frac{1}{\rho_0}\left\{\mu \mathrm{D}_j(\mathrm{D}_j w_i + \mathrm{D}_i w_j) + \left(\varsigma - \frac{2}{3}\mu\right)\mathrm{D}_i \operatorname{div} w\right\}, \tag{2.1}$$

$$L(x,\mathrm{D})\Theta = -\frac{\chi}{\rho_0 c_0}\Delta\Theta, \tag{2.2}$$

and the boundary operator

$$B_i(x,\mathrm{D})w = \mu(\mathrm{D}_i w_j + \mathrm{D}_j w_i)n_j^0 + \left(\varsigma - \frac{2}{3}\mu\right)\operatorname{div} w\, n_i^0, \tag{2.3}$$

where $c_0(x) = c_v(\rho_0(x), \theta_0(x))$. First we consider the linear problem

$$\begin{cases} \dot{w} + Aw = F & \text{in } Q_T\\ Bw = G & \text{on } \Sigma_T\\ w(0) = u_0 & \text{in } \Omega_0. \end{cases} \tag{2.4}$$

Define the operator A in $H^1(\Omega_0)$ setting

$$D(A) = \{w \in H^3(\Omega_0) \mid Bw = 0 \text{ on } \partial\Omega_0\}.$$

To solve problem (2.4), we want to apply Theorem 3.2, chap. 4 of Lions-Magenes [3] for $H = H^1(\Omega_0)$. Hence, some estimates are needed for the solution $w \in D(A)$ of the problem

$$Aw + \lambda w = f \tag{2.5}$$

where $f \in H^1(\Omega_0)$, $\lambda \in \mathbb{C}$. We introduce the bilinear form

$$a_\lambda(w,u) = \frac{\mu}{2}\int (\mathrm{D}_i w_j + \mathrm{D}_j w_i)(\mathrm{D}_i \overline{u}_j + \mathrm{D}_j \overline{u}_i) + \left(\varsigma - \frac{2}{3}\mu\right)\int \operatorname{div} w \ \operatorname{div} \overline{u} + \lambda \int \rho_0 w \overline{u}.$$

Here and in the sequel $\int$ denotes integration over Ω_0. Then the weak formulation of (2.5) is given by

$$a_\lambda(w,u) = \int \rho_0 f \overline{u} \qquad \text{for any } u \in H^1(\Omega_0).$$

As in [6], we have

LEMMA 2.1. *If* Re $\lambda \geq \lambda_0 \equiv \frac{1}{m}\min\left(\frac{\mu}{2}, \frac{3}{4}\varsigma\right)$, *$a_\lambda$ is a bounded and coercive form in $H^1(\Omega_0)$. The coerciveness constant is independent of λ.*

LEMMA 2.2. *For any $\lambda \in \mathbb{C}$ with* Re $\lambda \geq \lambda_0$, *$\lambda + A$ is an isomorphism from $D(A)$ (endowed with the graph norm) into $H^1(\Omega_0)$. Moreover, for any solution $w \in D(A)$ and for any $\lambda \in \mathbb{C}$ with* Re $\lambda > \lambda_0 + 1$,

$$\|w\|_0 \leq \frac{c}{|\lambda|}\|f\|_0, \tag{2.6}$$

where c does not depend on λ.

LEMMA 2.3. *If* Re $\lambda > \lambda_0 + 1$, *for any solution $w \in D(A)$ we have*

$$\|w\|_1 \leq \frac{c}{|\lambda|+1}\|f\|_1, \tag{2.7}$$

where c does not depend on λ.

PROOF. From Lemma 2.1 one has

$$\|w\|_1^2 \leq c|a_\lambda(w,w)| \leq c\|\rho_0\|_2\|f\|_0\|w\|_0$$

which gives

$$\|w\|_1 \leq c\|f\|_1, \tag{2.8}$$

with c independent of λ. Taking the scalar product in $L^2(\Omega_0)$, of (2.5) by $\rho_0 A\overline{w}$, gives

$$\|\sqrt{\rho_0}Aw\|_0^2 + \lambda \int \rho_0 w_i A_i \overline{w} = \int \rho_0 f_i A_i \overline{w}. \tag{2.9}$$

Integrating by parts yields

$$\begin{aligned}\int \rho_0 w_i A_i \overline{w} &= \frac{\mu}{2}\sum_{i,j}\int |\mathrm{D}_i w_j + \mathrm{D}_j w_i|^2 + \left(\varsigma - \frac{2}{3}\mu\right)\int |\operatorname{div} w|^2 \\ &\geq m\lambda_0 \sum_{i,j}\int |\mathrm{D}_i w_j + \mathrm{D}_j w_i|^2\end{aligned}$$

(see (2.9), (2.10) and the proof of Lemma 3.1 in [6]). From Korn's inequality

$$\|w\|_1^2 \leq K\left[\sum_{i,j}\int |\mathrm{D}_i w_j + \mathrm{D}_j w_i|^2 + \int |w|^2\right] \tag{2.10}$$

for any $w \in H^1(\Omega_0)$, we obtain

$$\int \rho_0 w_i A_i \overline{w} \geq m\lambda_0 \left\{\frac{1}{K}\|\nabla w\|_0^2 - \left(1 - \frac{1}{K}\right)\|w\|_0^2\right\}. \tag{2.11}$$

Moreover

$$\begin{aligned}\left|\int \rho_0 f_i A_i \overline{w}\right| &= \left|\frac{\mu}{2}\sum_{i,j}\int [\mathrm{D}_i f_j + \mathrm{D}_j f_i][\mathrm{D}_i \overline{w}_j + \mathrm{D}_j \overline{w}_i]\right. \\ &\quad \left. + \left(\varsigma - \frac{2}{3}\mu\right)\int \operatorname{div} f \ \operatorname{div} \overline{w}\right| \leq c\|w\|_1\|f\|_1.\end{aligned} \tag{2.12}$$

Observing that $\int \rho_0 w_i A_i \overline{w} \geq 0$, Re $\lambda > 0$, from (2.9)-(2.12) we obtain

$$\begin{aligned}m\lambda_0|\lambda|\left\{\frac{1}{K}\|\nabla w\|_0^2 - \left(1 - \frac{1}{K}\right)\|w\|_0^2\right\} &\leq \left|\|\sqrt{\rho_0}Aw\|_0^2 + \lambda\int \rho_0 w_i A_i \overline{w}\right| \\ &\leq c\|w\|_1\|f\|_1,\end{aligned}$$

from which it follows

$$|\lambda|\|w\|_1 \leq c[\|f\|_1 + |\lambda|\|w\|_0]. \tag{2.13}$$

From (2.6), (2.8), (2.13) we obtain (2.7). □

LEMMA 2.4. *Let* $F \in L^2(0,T;H^1(\Omega_0))$, $G \in H^{3/2,3/4}(\Sigma_T)$, $u_0 \in H^2(\Omega_0)$ *and the compatibility conditions*

$$\mu[\mathrm{D}_i(u_0)_j + \mathrm{D}_j(u_0)_i]n_j^0 + \left(\varsigma - \frac{2}{3}\mu\right)\operatorname{div} u_0\, n_i^0 = G_i(0) \quad \text{on } \partial\Omega_0 \tag{2.14}$$

be satisfied. Then there exists a unique solution

$$w \in L^2(0,T;H^3(\Omega_0)) \cap H^1(0,T;H^1(\Omega_0))$$

of (2.4). *Moreover*

$$|w|_{2,3,T} + |\dot{w}|_{2,1,T} \leq C_0[|F|_{2,1,T} + \|G\|_{3/2,3/4,\Sigma_T} + \|G(0)\|_1 + \|u_0\|_2] \tag{2.15}$$

where the constant C_0 *does not depend on* T.

PROOF. The trace $G(0)$ on $\partial\Omega_0$ belongs to $H^{1/2}(\partial\Omega_0)$ so that it is possible to find a function $\Phi \in H^{3/2,3/4}(]0,+\infty[\times\partial\Omega_0)$ such that $\Phi(0) = G(0)$ and

$$\|\Phi\|_{3/2,3/4,\Sigma_{+\infty}} \leq c\|G(0)\|_{1/2,\partial\Omega_0},$$

where the constant c does not depend on T and $\|\cdot\|_{s,\partial\Omega_0}$ is the norm in $H^s(\partial\Omega_0)$.

Now, we can extend $G - \Phi$ from $[0,T]\times\partial\Omega_0$ to $\mathbb{R}\times\partial\Omega_0$ in such a way that the extension $P(G-\Phi) \in H^{3/2,3/4}(\mathbb{R}\times\partial\Omega_0)$, $P(G-\Phi) = 0$ for $t < 0$ and

$$\|P(G-\Phi)\|_{3/2,3/4,\Sigma_{+\infty}} \leq c\|G-\Phi\|_{3/2,3/4,\Sigma_T},$$

where the constant c does not depend on T (extension by reflection around $t = T$: see Lions-Magenes [2], Theorem 2.2, chap. 1 and Theorem 11.3, chap. 1). Hence we have extended G to $\overline{G} = P(G-\Phi) + \Phi$ and $\overline{G} \in H^{3/2,3/4}(]0,+\infty[\times\partial\Omega_0)$ with

$$\|\overline{G}\|_{3/2,3/4,\Sigma_{+\infty}} \leq c[\|G\|_{3/2,3/4,\Sigma_T} + \|G(0)\|_{1/2,\partial\Omega_0}]. \tag{2.16}$$

Now, compatibility conditions (2.14) are necessary and sufficient to find a function $W \in H^{3,3/2}(]0,+\infty[\times\Omega_0)$ such that

$$\begin{cases} BW = \overline{G} & \text{on }]0+\infty[\times\partial\Omega_0 \\ W(0) = u_0 & \text{in } \Omega_0 \end{cases}$$

and satisfying the estimate

$$\|W\|_{3,3/2,+\infty} \leq c[\|\overline{G}\|_{3/2,3/4,\Sigma_{+\infty}} + \|u_0\|_2], \tag{2.17}$$

where the constant c does not depend on T (see [6]). Let us consider the problem

$$\begin{cases} \dot{V} + AV = F - \dot{W} - AW & \text{in } Q_T \\ BV = 0 & \text{on } \Sigma_T \\ V(0) = 0 & \text{in } \Omega_0. \end{cases}$$

Since Lemma 2.2 and Lemma 2.3 hold, we can apply Theorem 3.2, chap. 4 of [3], and find a solution

$$V \in L^2(0,T;H^3(\Omega_0)) \cap H^1(0,T;H^1(\Omega_0))$$

such that

$$|V|_{2,3,T} + |\dot{V}|_{2,1,T} \leq c[|F|_{2,1,T} + ||W||_{3,3/2,+\infty}], \tag{2.18}$$

where the constant c does not depend on T. The function $w = V + W$ is the solution of (2.4); from (2.16)-(2.18) we have

$$|w|_{2,3,T} + |\dot{w}|_{2,1,T} \leq c[|F|_{2,1,T} + ||G||_{3/2,3/4,\Sigma_T} + ||G(0)||_{1/2,\partial\Omega_0} + ||u_0||_2].$$

Finally,

$$||G(0)||_{1/2,\partial\Omega_0} \leq c||G(0)||_1$$

gives (2.15). □

In a similar way we solve the problem

$$\begin{cases} \dot{\Theta} + L\Theta = H & \text{in } Q_T; \\ \chi\dfrac{\partial\Theta}{\partial n} = K & \text{on } \Sigma_T; \\ \Theta(0) = \theta_0 & \text{in } \Omega_0. \end{cases} \tag{2.19}$$

We obtain

LEMMA 2.5. *Let $H \in L^2(0,T;H^1(\Omega_0))$, $K \in H^{3/2,3/4}(\Sigma_T)$, $\theta_0 \in H^2(\Omega_0)$ and the compatibility condition*

$$\chi\frac{\partial\theta_0}{\partial n} = K(0) \qquad \text{on } \partial\Omega_0$$

be satisfied. Then there exists a unique solution

$$\Theta \in L^2(0,T;H^3(\Omega_0)) \cap H^1(0,T;H^1(\Omega_0))$$

of (2.19). *Moreover*

$$|\Theta|_{2,3,T} + |\dot{\Theta}|_{2,1,T} \leq C_0'[|H|_{2,1,T} + ||K||_{3/2,3/4,\Sigma_T} + ||K(0)||_1 + ||\theta_0||_2] \tag{2.20}$$

where the constant C_0' does not depend on T.

We state a proposition which will be useful in the sequel (see [6]).

PROPOSITION 2.6. *Let $w \in L^2(0,T;H^k(\Omega_0))$ with*

$$\frac{\partial^s w}{\partial t^s} \equiv w^{(s)} \in L^2(0,T;H^r(\Omega_0)).$$

Then, for each $0 \le j \le s$, $j \in \mathbb{R}$,

$$\begin{aligned} &\sup_{t\in[0,T]} \|w(t)\|_\beta + \sum_{0\le j\le s} |w^{(j)}|_{2,\beta_j,T} + [w]_{j,\beta_j,T} \\ &\le c\left[|w|_{2,k,T} + |w^{(s)}|_{2,r,T} + \sum_{0\le j<s-1/2} \|w^{(j)}(0)\|_{\gamma_j}\right] \end{aligned} \tag{2.21}$$

where $\beta = \left(1 - \frac{1}{2s}\right)k + \frac{r}{2s}$, $\beta_j = \left(1 - \frac{j}{s}\right)k + \frac{j}{s}r$, $\gamma_j = \beta + \beta_j - k$, and

$$[w]_{j,\beta_j,T} \equiv \left(\int_0^T\int_0^T \frac{\|w^{[j]}(t) - w^{[j]}(\tau)\|_{\beta_j}^2}{|t-\tau|^{1+2(j-[j])}}\mathrm{d}t\,\mathrm{d}\tau\right)^{1/2}.$$

The constant c does not depend on T.

Set now

$$\begin{aligned} \frac{E}{2} \equiv{}& \|\overline{p}(0)n^0\|_1 + \|p(\rho_0,\theta_0)n^0\|_1 + \|h\hat{\theta}(0)\|_1 + \|h\theta_0\|_1 + \|u_0\|_2 + \|\rho_0\|_2 \\ &+ \|\theta_0\|_2 + \|\mathrm{Id}\|_1. \end{aligned} \tag{2.22}$$

Let η be the solution of

$$\begin{aligned} \dot\eta &= v \qquad \text{in } Q_T, \\ \eta(0) &= \mathrm{Id} \qquad \text{in } \Omega_0; \end{aligned}$$

it is easily verified that there exist constants C_1 and C_2 such that if, for an arbitrary $T \le T_0$, v, ρ and η satisfy $|v|_{2,3,T} + |\dot v|_{2,1,T} \le C_0 E$, $\sup_{t\in[0,T]} \|\rho(t)\|_2 \le E$, $\det[\mathrm{D}\eta(t,x)] \ge \frac{1}{2}$ in $\overline{Q}_T$, $v(0) = u_0$, then

$$\left(\int_0^T \|\rho(t)a_{ki}(t)\mathrm{D}_k v_i(t)\|_2^2 \mathrm{d}t\right)^{1/2} \le C_1, \tag{2.23}$$

$$\sup_{t\in[0,T]} \|\rho(t)a_{ki}(t)\mathrm{D}_k v_i(t)\|_1 \le C_2, \tag{2.24}$$

where $a_{ki} = ([D\eta]^{-1})_{ki}$. The constants C_1 and C_2 do not depend, as usual, on T. Set now

$$R_T \equiv \Big\{(v,\rho,\theta) \mid |v|_{2,3,T} + |\dot v|_{2,1,T} \le C_0 E,\ v(0) = u_0,$$
$$\sup_{t\in[0,T]} \|\rho(t)\|_2 \le E,\ |\dot\rho|_{2,2,T} \le C_1,\ \sup_{t\in[0,T]} \|\dot\rho(t)\|_1 \le C_2,$$
$$\rho(0) = \rho_0,\ \rho(t,x) \ge \frac{m}{2} \text{ in } \overline{Q}_T,\ |\theta|_{2,3,T} + |\dot\theta|_{2,1,T} \le C_0' E,\ \theta(0) = \theta_0\Big\}.$$

First, we show that $R_T \neq \emptyset$ for each T. Proceeding as in the proof of Lemma 2.4, we see that there exist two functions v', θ', in

$$L^2(0,+\infty; H^3(\Omega_0)) \cap H^1(0,+\infty; H^1(\Omega_0))$$

such that $v'(0) = u_0$, $\theta'(0) = \theta_0$. Moreover

$$|v'|_{2,3,+\infty} + |\dot v'|_{2,1,+\infty} \le c_1[\|u_0\|_2 + \|Bu_0\|_1],$$
$$|\theta'|_{2,3,+\infty} + |\dot\theta'|_{2,1,+\infty} \le c_1' \left[\|\theta_0\|_2 + \left\|\chi\frac{\partial\theta_0}{\partial n}\right\|_1\right].$$

The constants c_1, c_1' are easily seen to be less than C_0 and C_0', respectively. Using the compatibility conditions (1.22), (1.23) we have

$$|v'|_{2,3,+\infty} + |\dot v'|_{2,1,+\infty} \le C_0[\|u_0\|_2 + \|\overline{p}(0)n^0\|_1 + \|p(\rho_0,\theta_0)n^0\|_1] \le C_0\frac{E}{2},$$
$$|\theta'|_{2,3,+\infty} + |\dot\theta'|_{2,1,+\infty} \le C_0'[\|\theta_0\|_2 + \|h\hat\theta(0)\|_1 + \|h\theta_0\|_1] \le C_0'\frac{E}{2}.$$

Hence $(v',\rho_0,\theta') \in R_T$ for any T. We can now construct a map Λ defined in R_T. We shall show that it has a fixed point, namely a solution of our problem. Take $(v^*,\rho^*,\theta^*) \in R_T$. Let η^* be the solution of

$$\begin{aligned} \dot\eta^* &= v^* && \text{in } Q_T,\\ \eta^*(0) &= \mathrm{Id} && \text{in } \Omega_0, \end{aligned}$$

that is $\eta^*(t,x) = x + \int_0^t v^*(s,x)\mathrm{d}s$. Moreover, if $T \le T_0$,

$$\|\eta^*(t)\|_3 \le \|\mathrm{Id}\|_1 + C_0 E t^{1/2} \le \frac{E}{2} + C_0 E T_0^{1/2} \qquad \text{for } t \in [0,T].$$

Hence, for each $T \le T_0$, $(v^*,\rho^*,\theta^*) \in R_T$ yields

$$\sup_{t\in[0,T]} \|\eta^*(t)\|_3 \le \frac{E}{2} + C_0 E T_0^{1/2}.$$

Furthermore, there exists a constant $C_3 \geq 1$ such that, for an arbitrary $T \leq T_0$, $(v^*, \rho^*, \theta^*) \in R_T$ implies

$$\|\mathrm{D}\eta^*\|_{C^{1/2}([0,T];C^0(\overline{\Omega}_0))} \leq C_3,$$
$$\|\det \mathrm{D}\eta^*\|_{C^{1/2}([0,T];C^0(\overline{\Omega}_0))} \leq 6C_3^3.$$

Since

$$\begin{aligned}|\det[\mathrm{D}\eta^*(t,x)] - 1| &= |\det[\mathrm{D}\eta^*(t,x)] - \det[\mathrm{D}\eta^*(0,x)]| \\ &\leq \|\det \mathrm{D}\eta^*\|_{C^{1/2}([0,T];C^0(\overline{\Omega}_0))} t^{1/2},\end{aligned}$$

and for any pair of orthonormal vectors $\tau_1(x)$, $\tau_2(x)$ we have

$$\begin{aligned}&\big|\,|\mathrm{D}\eta^*(t,x)\tau_1(x) \wedge \mathrm{D}\eta^*(t,x)\tau_2(x)| - 1\big| \\ &= \Big|\,|\mathrm{D}\eta^*(t,x)\tau_1(x) \wedge \mathrm{D}\eta^*(t,x)\tau_2(x)| \\ &\quad - |\mathrm{D}\eta^*(0,x)\tau_1(x) \wedge \mathrm{D}\eta^*(0,x)\tau_2(x)|\,\Big| \\ &\leq 2\|D\eta^*\|^2_{C^{1/2}([0,T];C^0(\overline{\Omega}_0))} t^{1/2},\end{aligned} \tag{2.25}$$

we can find $T_1 \in]0, T_0]$ such that

$$2C_3^2 T_1^{1/2} \leq 6C_3^3 T_1^{1/2} \leq \frac{1}{2};$$

hence, for $T \leq T_1$, $(v^*, \rho^*, \theta^*) \in R_T$ yields

$$\det[\mathrm{D}\eta^*(t,x)] \geq \frac{1}{2} \qquad \text{in } \overline{Q}_T, \tag{2.26}$$

$$|\mathrm{D}\eta^*(t,x)\tau_1(x) \wedge \mathrm{D}\eta^*(t,x)\tau_2(x)| \geq \frac{1}{2} \qquad \text{in } \overline{Q}_T, \tag{2.27}$$

for any pair of orthonormal vectors $\tau_1(x)$, $\tau_2(x)$. Finally we have

$$\eta^*(t,x) - \eta^*(t,y) = x - y + \int_0^t [v^*(s,x) - v^*(s,y)]\mathrm{d}s$$

and consequently

$$|\eta^*(t,x) - \eta^*(t,y)| \geq |x-y| - \int_0^t |v^*(s,x) - v^*(s,y)|\mathrm{d}s.$$

Since

$$\int_0^t |v^*(s,x) - v^*(s,y)| \mathrm{d}s \le C_4 \int_0^t \|v^*(s)\|_3 |x-y| \mathrm{d}s$$
$$\le C_4 \left(\int_0^t \|v^*(s)\|_3^2 \mathrm{d}s \right)^{1/2} |x-y| t^{1/2},$$

we can find an instant $T_2 \in]0, T_1]$ such that, for $T \le T_2$, $(v^*, \rho^*, \theta^*) \in R_T$ implies

$$|\eta^*(t,x) - \eta^*(t,y)| \ge \frac{1}{2}|x-y| \qquad \forall t \in [0,T],\ \forall x,y \in \overline{\Omega}_0, \tag{2.28}$$

i.e., $\eta^*(t,\cdot)$ is injective for any $t \in [0,T]$. Define now the operators

$$A_i^*(t,x,\mathrm{D})w \equiv -\frac{1}{\rho^*}\Big\{ \mu a_{kj}^* \mathrm{D}_k (a_{sj}^* \mathrm{D}_s w_i + a_{si}^* \mathrm{D}_s w_j)$$
$$+ \left(\varsigma - \frac{2}{3}\mu\right) a_{ki}^* \mathrm{D}_k (a_{sj}^* \mathrm{D}_s w_j) \Big\},$$
$$L^*(t,x,\mathrm{D})\Theta \equiv -\frac{\chi}{\rho^* c^*} a_{kj}^* \mathrm{D}_k (a_{sj}^* \mathrm{D}_s \Theta),$$

and the boundary operator

$$B_i^*(t,x,\mathrm{D})w \equiv \mu(a_{kj}^* \mathrm{D}_k w_i + a_{ki}^* \mathrm{D}_k w_j) N_j^* + \left(\varsigma - \frac{2}{3}\mu\right) a_{kj}^* \mathrm{D}_k w_j N_i^*,$$

where $a_{kj}^* = a_{kj}^*(t,x)$ is the entry (k,j) of the matrix $[\mathrm{D}\eta^*]^{-1}$, $N^* = N^*(t,x)$ is the unit outward normal to $\eta^*(t, \partial\Omega_0)$ in $\eta^*(t,x)$, $c^*(t,x) \equiv c_v(\rho^*(t,x), \theta^*(t,x))$. Since (v^*, ρ^*, θ^*) satisfy the initial conditions (1.18)-(1.21), we have

$$A^*(0,x,\mathrm{D}) = A(x,\mathrm{D}), \qquad L^*(0,x,\mathrm{D}) = L(x,\mathrm{D}), \qquad B^*(0,x,\mathrm{D}) = B(x,\mathrm{D}),$$
$$a_{ki}^*(0,x) = \delta_{ki}, \qquad N^*(0,x) = n^0(x).$$

Consider the following problems

$$\begin{cases} \dot v_i + A_i v = A_i v^* - A_i^* v^* - a_{ki}^* \mathrm{D}_k \phi^* - \frac{1}{\rho^*} a_{ki}^* \mathrm{D}_k p^* + \overline{b}_i^* \equiv F_i^* & \text{in } Q_T; \\ B_i v = B_i v^* - B_i^* v^* + p^* N_i^* - \overline{p}^* N_i^* \equiv G_i^* & \text{on } \Sigma_T; \\ v(0) = u_0 & \text{in } \Omega_0; \end{cases} \tag{2.29}$$

$$(2.30)\quad \begin{cases} \dot\theta + L\theta = L\theta^* - L^*\theta^* - \dfrac{\theta^*}{\rho^* c^*}\overline{\mathrm{D}}_2 p^* a^*_{kj}\mathrm{D}_k v^*_j + \dfrac{\overline{r}^*}{c^*} \\ \qquad + \dfrac{\mu}{2\rho^* c^*}\sum_{ij}(a^*_{kj}\mathrm{D}_k v^*_i + a^*_{ki}\mathrm{D}_k v^*_j)^2 \\ \qquad + \dfrac{1}{\rho^* c^*}\left(\varsigma - \frac{2}{3}\mu\right)(a^*_{ki}\mathrm{D}_k v^*_i)^2 \equiv H^* & \text{in } Q_T; \\ \chi\dfrac{\partial\theta}{\partial n} = \chi\dfrac{\partial\theta^*}{\partial n} - \chi a^*_{ki}\mathrm{D}_k\theta^* N^*_i + h(\hat\theta^* - \theta^*) \equiv K^* & \text{on } \Sigma_T; \\ \theta(0) = \theta_0 & \text{in } \Omega_0; \end{cases}$$

$$(2.31)\quad \begin{cases} \dot\rho = -\rho^* a^*_{ki}\mathrm{D}_k v^*_i & \text{in } Q_T; \\ \rho(0) = \rho_0 & \text{in } \Omega_0; \end{cases}$$

where

$$(2.32)\quad \phi^*(t,x) \equiv -\kappa\int_{\Omega_0}\frac{\rho^*(t,\xi)}{|\eta^*(t,x)-\eta^*(t,\xi)|}|\det[\mathrm{D}\eta^*(t,\xi)]|\mathrm{d}\xi, \qquad \forall(t,x)\in Q_T,$$

$$p^*(t,x) \equiv p(\rho^*(t,x),\theta^*(t,x)),\ \ \overline{b}^*(t,x) \equiv \overline{b}(t,\eta^*(t,x)),$$

$$\overline{r}^*(t,x) \equiv \overline{r}^*(t,\eta^*(t,x)),\ \ \overline{\mathrm{D}}_2 p^*(t,x) \equiv \frac{\partial p}{\partial\overline{\theta}}(\rho^*(t,x),\theta^*(t,x)),$$

$$\hat\theta^*(t,x) \equiv \hat\theta(t,\eta^*(t,x)),\ \ \overline{p}^*(t,x) \equiv \overline{p}(t,\eta^*(t,x)).$$

First, we note that, if $(v^*,\rho^*,\theta^*)\in R_T$, then

$$\eta^* \in H^1(0,T;H^3(\Omega_0)) \cap H^2(0,T;H^1(\Omega_0))$$

and $|\dot\eta^*|_{2,3,T} + |\ddot\eta^*|_{2,1,T} \le C_0 E$. Hence $\det \mathrm{D}\eta^* \in H^1(0,T;H^2(\Omega_0))$ and, since for $T \le T_2$ (2.26) holds, we also have $a^*_{ki} \in H^1(0,T;H^2(\Omega_0))$ with $\dot a^*_{ki} \in H^1(0,T;L^2(\Omega_0))$; by interpolation $\dot a^*_{ki} \in C^0([0,T];H^1(\Omega_0))$. The norms of all these functions are bounded by some constants depending on the data of the problem but, from Proposition 2.6, independent of T. Assume $T \le T_2$; we want now to solve (2.29). Since by (2.26), (2.28) η^* is a diffeomorphism, instead of (2.32) we can consider

$$(2.33)\qquad \overline{\phi}^*(t,z) \equiv \phi^*(t,(\eta^*)^{-1}(t,z)) = -\kappa\int_{\Omega^*_t}\frac{\rho^*(t,(\eta^*)^{-1}(t,y))}{|z-y|}\mathrm{d}y,$$

for each $t\in[0,T]$, $z\in\Omega^*_t \equiv \eta^*(t,\Omega_0)$. For any $t\in[0,T]$, extend $\rho^*(t,\cdot)$ to $\mathbb{R}^3$ by 0 out of Ω_0. The changement of variables $z \to x = (\eta^*)^{-1}(t,z)$ shows that the function $(t,z)\to\rho^*(t,(\eta^*)^{-1}(t,z))$ belongs to $L^\infty(0,T;L^p(\mathbb{R}^3))$ for any $p>1$. Each norm is bounded by a constant depending on the data but independent of T. By potential theoretic estimates $\nabla\overline{\phi}^* \in L^\infty(0,T;W^{1,p}(\mathbb{R}^3))$ for any $p>3/2$.

Hence we obtain $\nabla\phi^* \in L^\infty(0,T;W^{1,p}(\Omega_0))$ for any $p > 3/2$, with each norm bounded by a constant independent of T. In particular,

$$\nabla\phi^* \in L^\infty(0,T;H^1(\Omega_0)) \cap L^\infty(Q_T).$$

Now we can estimate F^* in $L^2(0,T;H^1(\Omega_0))$. We have

$$|F^*|_{2,1,T} \leq c\left\{\left|\frac{\rho^*-\rho_0}{\rho^*\rho_0}\right|_{\infty,2,T}|\rho^*A^*v^*|_{2,1,T} + \left\|\frac{1}{\rho_0}\right\|_2 |(\rho^*A^* - \rho_0 A)v^*|_{2,1,T}\right.$$
$$\left. + |a^*_{ki}\mathrm{D}_k\phi^*|_{2,1,T} + \left|\frac{1}{\rho^*}a^*_{ki}\mathrm{D}_k p^*\right|_{2,1,T} + |\bar{b}|_{2,1,T,R}\right\}$$

where $|\bar{b}|_{2,1,T,R}$ is the norm in $L^2(0,T;H^1(B_R))$ and R is such that

$$\|\eta^*\|_{L^\infty(Q_T)} \leq c\left(\frac{E}{2} + C_0 E T_0^{1/2}\right) \leq R.$$

Recalling that, if $f \in H^1(0,T;X)$, where X is a Banach space, we have

$$\|f(t) - f(0)\|_X \leq |\dot{f}|_{2,X,T}t^{1/2}$$

and that, if $f \in L^\infty(0,T;X)$, we have

$$|f|_{2,X,T} \leq |f|_{\infty,X,T}T^{1/2},$$

we obtain

$$|F^*|_{2,1,T} \leq c\left[T^{1/2} + |\bar{b}|_{2,1,T,R}\right]. \tag{2.34}$$

Next we estimate the boundary term G^* in $H^{3/2,3/4}(\Sigma_T)$. We have

$$\|G^*\|_{3/2,3/4,\Sigma_T} \leq |G^*|_{2,3/2,\Sigma_T} + [G^*]_{3/4,0,\Sigma_T}$$

where

$$[G^*]_{3/4,0,\Sigma_T} \equiv \left(\int_0^T\int_0^T \frac{\|G^*(t) - G^*(\tau)\|^2_{0,\partial\Omega_0}}{|t-\tau|^{5/2}}\mathrm{d}t\,\mathrm{d}\tau\right)^{1/2}.$$

Since by conditions (2.26), (2.28) $\eta^*(t,\cdot)$ is a diffeomorphism, for each $t \in [0,T]$, in each local chart $\psi = \psi(\xi_1,\xi_2)$ of $\partial\Omega_0$ the unit vector N^* can be written as

$$N^*(t,x) = \frac{\mathrm{D}\eta^*(t,x)\tau_1(x) \wedge \mathrm{D}\eta^*(t,x)\tau_2(x)}{|\mathrm{D}\eta^*(t,x)\tau_1(x) \wedge \mathrm{D}\eta^*(t,x)\tau_2(x)|},$$

where

$$\tau_1(x) \equiv \frac{\frac{\partial \psi}{\partial \xi_1}(\psi^{-1}(x))}{\left|\frac{\partial \psi}{\partial \xi_1}(\psi^{-1}(x))\right|},$$

$$\tau_2(x) \equiv \frac{\frac{\partial \psi}{\partial \xi_2}(\psi^{-1}(x)) - \left[\frac{\partial \psi}{\partial \xi_2}(\psi^{-1}(x))\tau_1(x)\right]\tau_1(x)}{\left|\frac{\partial \psi}{\partial \xi_2}(\psi^{-1}(x)) - \left[\frac{\partial \psi}{\partial \xi_2}(\psi^{-1}(x))\tau_1(x)\right]\tau_1(x)\right|}.$$

From the estimates on v^*, η^*, we obtain

$$|N^*|_{\infty,3/2,\Sigma_T} \leq c, \tag{2.35}$$

$$\|N^*(t) - N^*(s)\|_{3/2,\partial\Omega_0} \leq c|t-s|^{1/2} \qquad \forall t, s \in [0,T]. \tag{2.36}$$

Hence we can proceed as for F^* obtaining

$$|G^*|_{2,3/2,\Sigma_T} \leq cT^{1/2}. \tag{2.37}$$

To estimate the seminorm $[G^*]_{3/4,0,\Sigma_T}$ is more complicated; we observe that

$$[N^*]_{3/4,1/2,\Sigma_T} \leq cT^{3/4},$$

that $\dot{a}^*_{ki}$ bounded in $L^\infty(0,T;H^1(\Omega_0))$ gives

$$[a^*_{ki}(\cdot) - a^*_{ki}(0)]_{3/4,1,\Sigma_T} \leq cT^{3/4},$$

and that $\mathrm{D}v^*$ is bounded in $H^{3/4}([0,T];L^2(\partial\Omega_0))$. Using also (2.36) we thus obtain

$$[Bv^* - B^*v^*]_{3/4,0,\Sigma_T} \leq cT^{1/2}. \tag{2.38}$$

Next we observe that $\dot{\rho}^*$ bounded in $L^\infty([0,T];H^1(\Omega_0))$ yields

$$[\rho^*]_{3/4,0,\Sigma_T} \leq cT^{3/4};$$

moreover, by using Proposition 2.6, $\theta^* \in H^1(0,T;H^1(\Omega_0))$ gives

$$[\theta^*]_{3/4,0,\Sigma_T} \leq \sqrt{2}T^{1/4-\varepsilon}[\theta^*]_{1-\varepsilon,0,\Sigma_T} \leq cT^{1/4-\varepsilon},$$

where $0 < \varepsilon < 1/4$. Hence we obtain

$$[p^*N^*]_{3/4,0,\Sigma_T} \leq cT^{1/4-\varepsilon}. \tag{2.39}$$

For the last term in G^* we have

$$[\overline{p}^*N^*]_{3/4,0,\Sigma_T} \leq c[\overline{p}]_{3/4,1,T,R} + c|\overline{p}|_{\infty,2,T_0,R}T^{3/4}. \tag{2.40}$$

Hence from (2.38)-(2.40) we obtain

$$[G^*]_{3/4,0,\Sigma_T} \leq c\left[T^{1/4-\varepsilon} + [\overline{p}]_{3/4,1,T,R}\right].$$

Lemma 2.4 yields that there exists the solution v of (2.29) such that

$$|v|_{2,3,T} + |\dot{v}|_{2,1,T} \leq C_0\Big[c(T^{1/2} + |\bar{b}|_{2,1,T,R}) + cT^{1/2} + c(T^{1/4-\varepsilon} + [\overline{p}]_{3/4,1,T,R}) + \frac{E}{2}\Big].$$

Hence we can find $T_3 \in]0, T_2]$ such that for an arbitrary $T \leq T_3$, $(v^*, \rho^*, \theta^*) \in R_T$ implies

$$|v|_{2,3,T} + |\dot{v}|_{2,1,T} \leq C_0 E.$$

Now we proceed with the estimates for θ. First we consider the estimate of H^* in $L^2(0,T;H^1(\Omega_0))$. Observe that $H^{3/2+\varepsilon}(\Omega_0) \subset L^\infty(\Omega_0)$, $0 < \varepsilon < 1/2$, so that if $f \in H^{3/2+\varepsilon}(\Omega_0)$, $g \in H^1(\Omega_0)$ then $fg \in H^1(\Omega_0)$. We have

$$\begin{aligned}|H^*|_{2,1,T} \leq c\Bigg\{ & \left|\frac{\rho^* - \rho_0}{\rho^*\rho_0}\right|_{\infty,2,T} |\rho_0 L\theta^*|_{2,1,T} \\ & + \left|\frac{1}{\rho^*}\right|_{\infty,2,T} \left|\frac{c^* - c_0}{c^* c_0}\right|_{\infty,3/2+\varepsilon,T} |\rho_0 c_0 L\theta^*|_{2,1,T} \\ & + \left|\frac{1}{\rho^* c^*}\right|_{\infty,2,T} |(\rho_0 c_0 L - \rho^* c^* L^*)\theta^*|_{2,1,T} \\ & + \left|\frac{\theta^*}{\rho^* c^*}\overline{\mathrm{D}}_2 p^* a^*_{kj} \mathrm{D}_k v^*_j\right|_{2,1,T} + \left|\frac{\overline{r}^*}{c^*}\right|_{2,1,T} \\ & + \left|\frac{\mu}{2\rho^* c^*}\sum_{ij}(a^*_{kj}\mathrm{D}_k v^*_i + a^*_{ki}\mathrm{D}_k v^*_j)^2\right|_{2,1,T} \\ & + \left|\frac{1}{\rho^* c^*}\left(\varsigma - \frac{2}{3}\mu\right)(a^*_{ki}\mathrm{D}_k v^*_i)^2\right|_{2,1,T}\Bigg\}.\end{aligned}$$

Observe that, by interpolation,

$$\theta^* \in H^{3/4-\varepsilon/2}(0,T;H^{3/2+\varepsilon}(\Omega_0)) \subset C^{1/4-\varepsilon/2}([0,T];H^{3/2+\varepsilon}(\Omega_0)),$$

$0 < \varepsilon < 1/2$, so that

$$\left|\frac{c^* - c_0}{c^* c_0}\right|_{\infty,3/2+\varepsilon,T} \leq c\left[|\rho^* - \rho_0|_{\infty,2,T} + |\theta^* - \theta_0|_{\infty,3/2+\varepsilon,T}\right] \leq cT^{1/4-\varepsilon/2}.$$

Concerning the quadratic terms in $\mathrm{D}v^*$, we observe that by interpolation $\mathrm{D}v^*$ is bounded in $H^{1/4-\varepsilon/2}(0,T;H^{3/2+\varepsilon}(\Omega_0)) \subset L^{4/(1+2\varepsilon)}(0,T;H^{3/2+\varepsilon}(\Omega_0))$, where $0<\varepsilon<1/2$; hence

$$|\mathrm{D}_i v_j^* \mathrm{D}_k v_h^*|_{2,1,T} \le c|\mathrm{D}v^*|_{\infty,1,T}|\mathrm{D}v^*|_{4/(1+2\varepsilon),3/2+\varepsilon,T} T^{1/4-\varepsilon/2} \le cT^{1/4-\varepsilon/2};$$

as usual the constants do not depend on T. In a similar way we estimate the term containing $\overline{\mathrm{D}}_2 p^*$. The other terms can be treated in a straightforward manner. Thus we obtain

$$|H^*|_{2,1,T} \le c[T^{1/4-\varepsilon/2} + |\overline{r}|_{2,1,T,R}].$$

The estimate of K^* in $H^{3/2,3/4}(\Sigma_T)$ is similar to the one of G^*. We obtain

$$||K^*||_{3/2,3/4,\Sigma_T} \le c[T^{1/4-\varepsilon} + [\hat{\theta}]_{3/4,1,T,R}],$$

where $0<\varepsilon<1/4$. Hence, from Lemma 2.5 there exists the solution θ of (2.30) such that

$$|\theta|_{2,3,T} + |\dot{\theta}|_{2,1,T} \le C_0' \left[c(T^{1/4-\varepsilon/2} + |\overline{r}|_{2,1,T,R}) + c(T^{1/4-\varepsilon} + [\hat{\theta}]_{3/4,1,T,R}) + \frac{E}{2} \right],$$

where we can take one same ε, $0<\varepsilon<1/4$. Then there exists an instant $T_4 \in]0,T_3]$ such that for an arbitrary $T \le T_4$,

$$|\theta|_{2,3,T} + |\dot{\theta}|_{2,1,T} \le C_0' E.$$

Now we consider problem (2.31). Let $T \le T_4$. From (2.23) we have

$$\left(\int_0^T ||\rho^*(t) a_{ki}^*(t) \mathrm{D}_k v_i^*(t)||_2^2 \mathrm{d}t \right)^{1/2} \le C_1$$

so that

$$\left(\int_0^T ||\rho^*(t) a_{ki}^*(t) \mathrm{D}_k v_i^*(t)||_{L^\infty(\Omega_0)}^2 \mathrm{d}t \right)^{1/2} \le C_1'.$$

Let $T' \in]0,T_4]$ such that

$$(C_1 + C_1')(T')^{1/2} \le \min\left(\frac{m}{2}, \frac{E}{2}\right).$$

The solution of (2.31) is given by

$$\rho(t,x) = \rho_0(x) - \int_0^t \rho^*(s,x) a_{ki}^*(s,x) \mathrm{D}_k v_i^*(s,x) \mathrm{d}s;$$

hence $(v^*, \rho^*, \theta^*) \in R_{T'}$ implies

$$\rho(t,x) \geq \rho_0(x) - \int_0^{T'} \|\rho^*(s) a^*_{ki}(s) \mathrm{D}_k v^*_i(s)\|_{L^\infty(\Omega_0)} \mathrm{d}s$$

$$\geq m - C_1'(T')^{1/2} \geq \frac{m}{2} \text{for each } (t,x) \in \overline{Q}_T.$$

Moreover

$$\sup_{t\in[0,T']} \|\rho(t)\|_2 \leq \|\rho_0\|_2 + \int_0^{T'} \|\rho^*(s) a^*_{ki}(s) \mathrm{D}_k v^*_i(s)\|_2 \mathrm{d}s \leq \frac{E}{2} + C_1(T')^{1/2} \leq E.$$

From (2.23), (2.24), we have

$$|\dot{\rho}|_{2,2,T'} \leq C_1, \qquad \sup_{t\in[0,T']} \|\dot{\rho}(t)\|_1 \leq C_2.$$

Hence we have proved that the map $\Lambda : (v^*, \rho^*, \theta^*) \to (v, \rho, \theta)$ satisfies $\Lambda(R_{T'}) \subseteq R_{T'}$. Let us introduce the space

$$X \equiv \{(v, \rho, \theta) | v, \rho, \theta \in C^0([0, T']; H^{2-\varepsilon}(\Omega_0))\},$$

where ε is a fixed small positive parameter, say $0 < \varepsilon < 1/2$.

LEMMA 2.7. *$R_{T'}$ is a convex and compact subset of X.*

PROOF. $R_{T'}$ is obviously convex and bounded in $Y \times Y \times H^1(0, T'; H^2(\Omega_0))$, where

$$Y \equiv L^2(0, T'; H^3(\Omega_0)) \cap H^1(0, T'; H^1(\Omega_0)).$$

The space Y is continuously embedded in

$$C^0([0, T']; H^2(\Omega_0)) \cap C^{\varepsilon/2}([0, T']; H^{2-\varepsilon}(\Omega_0)),$$

which is, from Ascoli-Arzelà's and Rellich's theorems, compactly embedded in $C^0([0, T']; H^{2-\varepsilon}(\Omega_0))$. Analogously, $H^1(0, T'; H^2(\Omega_0))$ is compactly embedded in $C^0([0, T']; H^{2-\varepsilon}(\Omega_0))$. Hence $R_{T'}$ is relatively compact in X. Finally, it is easily verified that $R_{T'}$ is closed in X. □

LEMMA 2.8 *The map Λ is continuous from the topology of X into the topology of $C^0([0, T']; L^2(\Omega_0))$.*

PROOF. Suppose that $(v^*_n, \rho^*_n, \theta^*_n) \in R_{T'}$ converge in X to (v^*, ρ^*, θ^*) and let $(v_n, \rho_n, \theta_n) \equiv \Lambda(v^*_n, \rho^*_n, \theta^*_n)$, $(v, \rho, \theta) \equiv \Lambda(v^*, \rho^*, \theta^*)$.We observe that $\eta^*_n \to \eta^*$

in $C^1([0,T'];H^{2-\varepsilon}(\Omega_0))$ (with obvious notation). Assume that E is an extension operator from Ω_0 to $\mathbb{R}^3$, bounded in $H^{1-\varepsilon}$ and H^2, i.e.,

$$E \in \mathcal{L}(H^{1-\varepsilon}(\Omega_0), H^{1-\varepsilon}(\mathbb{R}^3) \cap \mathcal{L}(H^2(\Omega_0), H^2(\mathbb{R}^3)).$$

If $f \in H^{1-\varepsilon}(\Omega_0)$, $g \in H^2(\Omega_0)$, we have (see [5])

$$\|fg\|_{1-\varepsilon} \le \|Ef\, Eg\|_{H^{1-\varepsilon}(\mathbb{R}^3)} \le c\|Ef\|_{H^{1-\varepsilon}(\mathbb{R}^3)}\|Eg\|_{H^2(\mathbb{R}^3)} \le c\|f\|_{1-\varepsilon}\|g\|_2,$$

so that $fg \in H^{1-\varepsilon}(\Omega_0)$. Since $\mathrm{D}\eta_n^* \to \mathrm{D}\eta^*$ in $C^1([0,T'];H^{1-\varepsilon}(\Omega_0))$ and $\mathrm{D}\eta_n^*$ is bounded in $H^1(0,T';H^2(\Omega_0))$, then $a_{ki}^{*(n)} \to a_{ki}^*$ in $H^1(0,T';H^{1-\varepsilon}(\Omega_0))$. Observe also that

$$\|N_n^* - N^*\|_{L^2(\partial\Omega_0)} \le c\|\eta_n^* - \eta^*\|_{2-\varepsilon}.$$

Take the difference between the equations for (v_n,ρ_n,θ_n) and (v,ρ,θ), multiply by $\rho_0(v_n - v)$, $(\rho_n - \rho)$, $\rho_0 c_0(\theta_n - \theta)$ respectively, and integrate over Ω_0. Integrating by parts and using Korn's inequality (2.10), from the equations for the velocity we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho_0}(v_n - v)\|_0^2 + m\lambda_0\left[\frac{1}{K}\|\nabla(v_n - v)\|_0^2 - \left(1 - \frac{1}{K}\right)\|v_n - v\|_0^2\right]$$
$$\le \int_{\partial\Omega_0} [G_n^* - G^* - B(v_n^* - v^*)](v_n - v) + \int \rho_0(F_n^* - F^*)(v_n - v),$$

with obvious notations. Estimating the right-hand side, after long but straightforward calculations, gives

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho_0}(v_n - v)\|_0^2 + \frac{m\lambda_0}{K}\|v_n - v\|_1^2 \\
&\le \varepsilon_0\|v_n - v\|_1^2 + C(\varepsilon_0)\|v_n - v\|_0^2 \\
&\quad + C(\varepsilon_0)\Big\{ |v_n^* - v^*|_{\infty,2-\varepsilon}^2 + |\rho_n^* - \rho^*|_{\infty,2-\varepsilon}^2 + |\theta_n^* - \theta^*|_{\infty,2-\varepsilon}^2 \\
&\quad + |\eta_n^* - \eta^*|_{\infty,2-\varepsilon}^2 + (1 + \|v_n^*\|_3^2)|a_n^* - a^*|_{\infty,1-\varepsilon}^2 \\
&\quad + \|\nabla(\phi_n^* - \phi^*)\|_0^2 + \|\overline{b}_n^* - \overline{b}^*\|_0^2 + \|\overline{p}_n^* - \overline{p}^*\|_{L^2(\partial\Omega_0)}^2 \Big\},
\end{aligned}$$

where ε_0 is a small positive parameter. Choosing $\varepsilon_0 = \frac{m\lambda_0}{2K}$, we obtain by Gronwall's lemma

$$\begin{aligned}
\|(v_n - v)(t)\|_0^2 \le c\{&|v_n^* - v^*|_{\infty,2-\varepsilon}^2 + |\rho_n^* - \rho^*|_{\infty,2-\varepsilon}^2 \\
&+ |\theta_n^* - \theta^*|_{\infty,2-\varepsilon}^2 + |\eta_n^* - \eta^*|_{\infty,2-\varepsilon}^2 + |a_n^* - a^*|_{\infty,1-\varepsilon}^2 \\
&+ |\nabla(\phi_n^* - \phi^*)|_{2,0,T'}^2 + |\overline{b}_n^* - \overline{b}^*|_{2,0,T'}^2 + |\overline{p}_n^* - \overline{p}^*|_{2,0,\Sigma_{T'}}^2\},
\end{aligned} \tag{2.41}$$

for any $t \in]0, T']$, where

$$\phi_n^*(t,x) \equiv -\kappa \int_{\Omega_0} \frac{\rho_n^*(t,\xi)}{|\eta_n^*(t,x) - \eta_n^*(t,\xi)|} |\det D\eta_n^*(t,\xi)| d\xi,$$

$\overline{b}_n^*(t,x) \equiv \overline{b}(t, \eta_n^*(t,x))$, $\overline{p}_n^*(t,x) \equiv \overline{p}(t,\eta_n^*(t,x))$. Fix an arbitrary parameter $\varepsilon_1 > 0$. We observe that $\eta_n^*(t,x) \to \eta^*(t,x)$ for each $(t,x) \in Q_{T'}$. Since $\overline{b} \in L^2(0,T_0; C^0(\overline{B}_R))$, $\overline{p} \in L^\infty(0,T_0;C^0(\overline{B}_R))$, by Lebesgue's theorem we can find $n_1 > 0$ such that

$$[|\overline{b}_n^* - \overline{b}^*|_{2,0,T'}^2 + |\overline{p}_n^* - \overline{p}^*|_{2,0,\Sigma_{T'}}^2] < \varepsilon_1 \tag{2.42}$$

for each $n > n_1$. Consider now the term with the gravitational potential. Recalling the boundedness of η_n^*, η^*, we obtain

$$\begin{aligned} |\nabla(\phi_n^* - \phi^*)|_{2,0,T'} \leq c\big[&\|\nabla\overline{\phi}_n^*\|_{L^\infty(0,T';W^{1,p}(\mathbb{R}^3))} |\eta_n^* - \eta^*|_{\infty,2-\varepsilon} \\ &+ \|\nabla(\overline{\phi}_n^* - \overline{\phi}^*)\|_{L^2(]0,T'[\times\mathbb{R}^3)} \\ &+ |\nabla(\overline{\phi}^*(\,\cdot\,,\eta_n^*) - \overline{\phi}^*(\,\cdot\,,\eta^*))|_{2,0,T'}\big], \end{aligned} \tag{2.43}$$

where $p > 3$ so that $W^{1,p}(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$, $\overline{\phi}_n^*(t,z) \equiv \phi_n^*(t,(\eta_n^*)^{-1}(t,z))$, $\overline{\phi}^*(t,z) \equiv \phi^*(t,(\eta^*)^{-1}(t,z))$. Recall that $\nabla\overline{\phi}_n^*$, $\nabla\overline{\phi}^* \in L^\infty(0,T';W^{1,p}(\mathbb{R}^3))$ for any $p > 3/2$; here and in the right hand side of (2.43) the gradients of $\overline{\phi}_n^*$, $\overline{\phi}^*$ are with respect to the variable z. Consider the second term in the right-hand side of (2.43). $(\overline{\phi}_n^* - \overline{\phi}^*)(t,z)$ satisfies the equation

$$\Delta(\overline{\phi}_n^* - \overline{\phi}^*)(t,z) = 4\pi\kappa(\overline{\rho}_n^* - \overline{\rho}^*)(t,z) \tag{2.44}$$

for each $t \in [0,T']$, $z \in \mathbb{R}^3$, where $\overline{\rho}_n^*(t,z) \equiv \rho_n^*(t,(\eta_n^*)^{-1}(t,z))$ and similarly for $\overline{\rho}^*$. We multiply (2.44) by $\overline{\phi}_n^* - \overline{\phi}^*$, integrate over $\mathbb{R}^3$, integrate by parts. Using the estimate $\|f\|_{L^6(\mathbb{R}^3)} \leq c\|\nabla f\|_{L^2(\mathbb{R}^3)}$ we obtain

$$\begin{aligned} &\|\nabla(\overline{\phi}_n^* - \overline{\phi}^*)(t)\|_{L^2(\mathbb{R}^3)} \\ &\leq c\|(\overline{\rho}_n^* - \overline{\rho}^*)(t)\|_{L^{6/5}(\mathbb{R}^3)} \\ &\leq c\{|\rho_n^* - \rho^*|_{\infty,2-\varepsilon} \\ &\quad + \|\rho^*(t,(\eta_n^*)^{-1}(t,\,\cdot\,)) - \rho^*(t,(\eta^*)^{-1}(t,\,\cdot\,))\|_{L^{6/5}(\mathbb{R}^3)}\}. \end{aligned} \tag{2.45}$$

Here ρ^* is considered 0 out of Ω_0, so that we define

$$\rho^*(t,(\eta_n^*)^{-1}(t,z)) \equiv 0 \qquad \text{if } z \notin \eta_n^*(t,\Omega_0),$$
$$\rho^*(t,(\eta^*)^{-1}(t,z)) \equiv 0 \qquad \text{if } z \notin \eta^*(t,\Omega_0).$$

Let $z \in \eta_n^*(t,\Omega_0) \cap \eta^*(t,\Omega_0)$, define $x_n \equiv (\eta_n^*)^{-1}(t,z)$, $x \equiv (\eta^*)^{-1}(t,z)$, i.e., $z = \eta_n^*(t,x_n) = \eta^*(t,x)$. Then

$$\begin{aligned} |x_n - x| &\le \int_0^t |v_n^*(s,x_n) - v^*(s,x)|\mathrm{d}s \\ &\le c|v_n^* - v^*|_{\infty,2-\varepsilon} + C_4 \left(\int_0^t \|v^*(s)\|_3^2 \mathrm{d}s \right)^{1/2} |x_n - x| t^{1/2} \\ &\le c|v_n^* - v^*|_{\infty,2-\varepsilon} + \frac{1}{2}|x_n - x|, \end{aligned}$$

from which we obtain

$$|x_n - x| \le c|v_n^* - v^*|_{\infty,2-\varepsilon}$$

(see the calculations which lead to (2.28)). On the other hand, the Lebesgue measure of $\eta_n^*(t,\Omega_0)\Delta\eta^*(t,\Omega_0)$ (here Δ stands for the symmetric difference of the two sets) goes to zero as $n \to +\infty$, uniformly for $t \in [0,T']$. Then, from the continuity of ρ^*, we can find $n_2 \ge n_1$ such that

$$\|\nabla(\overline{\phi}_n^* - \overline{\phi}^*)(t)\|^2_{L^2(]0,T'[\times \mathbf{R}^3)} < \varepsilon_1 \tag{2.46}$$

for each $n > n_2$. Finally, since $\nabla\overline{\phi}^* \in L^\infty(0,T';C^0(\overline{B}_R))$, by Lebesgue's theorem we can find $n_3 > n_2$ such that

$$|\nabla(\overline{\phi}^*(\,\cdot\,,\eta_n^*) - \overline{\phi}^*(\,\cdot\,,\eta^*))|_{2,0,T'} < \varepsilon_1 \tag{2.47}$$

for each $n > n_3$. Hence, from (2.41)-(2.43), (2.45)-(2.47) we obtain the convergence of v_n to v in $C^0([0,T'];L^2(\Omega_0))$. In an analogous way we obtain the convergence of θ_n to θ. The convergence of ρ_n to ρ is obtained by a direct computation. □

By a compactness argument, Λ is continuous from the topology of X into the topology of X. We can finally apply the Schauder's fixed point theorem, and find a fixed point $(v,\rho,\theta) = \Lambda(v,\rho,\theta)$, that is a solution of our problem. The proof of theorem A is complete.

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