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Realization of any finite jet in a scalar semilinear parabolic equation on the ball in $\mathbb{R}^3$

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1. - Introduction

Consider the scalar semilinear parabolic equation

$$u_t = \Delta u + g(x, u, \nabla u) \tag{1.1}$$

on the unit ball $D = \{x \in \mathbb{R}^N : |x| < 1\}$ in $\mathbb{R}^N$. Assuming some regularity of g, equation (1.1), subject to either of the boundary conditions

$$u = 0 \quad \text{on } \partial D \text{ (Dirichlet)}, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial D \text{ (Neumann)}, \tag{1.3}$$

where n is the unit normal vector field on ∂D, defines a local dynamical system on an appropriate Banach space [He].

The present paper is a continuation of our previous results [Po 3, 4] that aim at showing dynamic complexity of this boundary value problem. This is not a trivial task, because the considered dynamical system, nonetheless that it is infinite-dimensional, is defined by a special (scalar, local) equation. The structure of the equation may significantly restrict the class of dynamical systems that are admissible for the given problem. One of such restrictions concerning (1.1), (1.2) (or (1.3)) is that "most" trajectories converge to an equilibrium [Po 1]. In fact, any possible interesting dynamics must occur within a Lipschitz hypersurface in the phase space [Po 2, Ta]. For more special types of equations (1.1), there are yet stronger implications on the dynamics (see [Po 3] for a discussion). Our attempt in this and the previous papers [Po 3, 4] has been to show that, in the general case, there are few (if any) restrictions other than the

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mentioned one. This is done via solving an inverse problem, finding an equation for which a dynamics or some mode interaction bifurcation is prescribed. Note that inverse problems of this type have been addressed for different types of equations in [F-P, Ha 1, 2, Da 2].

In [Po 3], we have proved that any finite jet of an arbitrary N-dimensional vector field can be realized in (1.1), (1.2). More specifically, for any $k > 0$, and for any C^k vector field H on $\mathbb{R}^N$ with $H(0) = 0$, one can find a function g such that (1.1), (1.2) has an invariant N-dimensional manifold through the equilibrium $u \equiv 0$, and the Taylor expansion of the vector field given by the flow of (1.1), (1.2) on this invariant manifold coincides, in appropriate coordinates, with the Taylor expansion of H, up to k-th order terms. Having this result, one would expect that (1.1), (1.2) is dynamically no simpler than the vector fields on $\mathbb{R}^N$. For (1.1), (1.3) this conclusion has been explicitly proved [Po 4]: Any vector field on $\mathbb{R}^N$ can be realized as the vector field given by the flow of (1.1), (1.3) on some invariant manifold.

In both these results, the dimension of the spatial domain D and that of the phase space of the vector field, which (or whose jet) is being realized in the given problem, coincide. In the present paper, we prove that any jet (without restriction on the dimension of the source space) can be realized in the problem (1.1), (1.2) on the three-dimensional ball already. This result shows that for $N = 3$ (and therefore also for $N \geq 3$, as we explain in Section 4), the dimension of the spatial domain does not pose a limitation on complexity of the dynamics of (1.1), (1.2).

For $N = 2$, the method of this paper does not apply. We leave open the question whether two dimensions are enough for realization of any jet. As for $N = 1$, the dynamics of such equations is known to be very simple (see [Ze, Ma] and other references in [Po 3]).

The paper is organized as follows. In Section 2, we introduce a basic notation and state the main theorems. Section 3 is one part of the proof of the jet-realization result. Using a procedure from [Po 3], we reduce the proof to the problem of finding a function $a(x)$ such that the operator $u \mapsto \Delta u + a(x)u$ has some specific properties. Section 4 is mostly devoted to the proof of existence of such a function $a(x)$.

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2. - Main results

In the sequel, we focus our attention to the following equation of type (1.1)

$$u_t = \Delta u + a(x)u + g(x, \nabla u), \quad x \in D, \tag{2.1}$$

where D is the unit ball in $\mathbb{R}^3$, $a(x) \in C(\overline{D})$ and $g(x, y) \in C^{0,\infty}$, i.e., g is continuous on $\overline{D} \times \mathbb{R}^3$ together with all its partial derivatives with respect to y (usually $g(x, y)$ will be a polynomial in y). For definiteness, we chose Dirichlet boundary conditions (1.2), but all the subsequent results can be proved for Neumann boundary conditions with slight modifications in the proofs.

In this section we formulate our main results. First we put (2.1), (1.2) into the context of abstract analytic semigroups.

Pick a number $p > 3$ and write the boundary value problem (2.1), (1.2) as the abstract equation on $X := L_p(D)$:

$$u_t + Au = f(u). \tag{2.2}$$

Here A is the sectorial operator defined by the differential operator $\Delta + a(x)$ and Dirichlet boundary condition, and f is the Nemitskii operator defined by g. More specifically, one defines A with the domain

$$D(A) := W^{2,p}(D) \cap W_0^{1,p}(D) \tag{2.3}$$

by

$$Au = -\Delta u - a(x)u. \tag{2.4}$$

This is a sectorial operator on X [He]. For $1/2 < \alpha < 1$, the fractional power space X^α, corresponding to A, is the Sobolev-Slobodeckii space $W^{2\alpha,p}(D) \cap W_0^{1,p}(D)$ (see [Am, He]). Since $p > 3$, we can find $\alpha < 1$ such that the following embedding takes place

$$X^\alpha \hookrightarrow C^1(\overline{D}) \tag{2.5}$$

(see [Tr]). Then, defining $f : X^\alpha \to X$ by

$$f(u(\cdot))(x) = g(x, \nabla u(x)), \tag{2.6}$$

we clearly have $f \in C^\infty(X^\alpha, X)$. Thus, by [He], (2.2) defines a local semiflow on X^α.

As in [Po 3], we will use the phrase "realize a jet in (2.1), (1.2)", which we now explain.

Assume that A admits a decomposition $X = X_1 \oplus X_2$ into close invariant subspaces, with $X_1 \subset D(A)$ having finite dimension. Let $P : X \to X_1$ be the continuous projection along X_2. Note that $APu = PAu$ for any $u \in D(A)$.

Now fix an integer $k > 0$, and consider the finite-dimensional linear space $\mathcal{I}_0^k(X_1)$ of the k-jets on X_1 for which 0 is the source and target. Equivalently, any element $\mathcal{I}_0^k(X_1)$ can be understood as the Taylor expansion at 0 of a C^k-mapping $h : X_1 \to X_1$ such that $h(0) = 0$. This Taylor expansion (taken up to the order k) is called the k-jet of h.

We say that a jet $j^k \in \mathcal{I}_0^k(X_1)$ can be realized in (2.1), (1.2), if there exists a $g \in C^{0,\infty}$ such that $g(x, 0) \equiv 0$ and the corresponding abstract equation (2.2) has the following two properties:

(i) There exists a locally invariant manifold (i.e., a manifold consisting of portions of trajectories of (2.2)) of the form

$$W = \{u_1 + x(u_1) : u_1 \in U\}, \tag{2.7}$$

where U is a neighbourhood of 0 in X_1 and $x : X_1 \to X_2 \cap X^\alpha$ is a C^k mapping with $x(0) = 0$.

(ii) The k-jet of the C^k function $u_1 \longmapsto -Au_1 + Pf(u_1 + x(u_1)) : X_1 \to X_1$ is equal to the given jet j^k.

Note that the function in (ii) is the right-hand side of the projected equation

$$\dot{u}_1 = -Au_1 + Pf(u_1 + x(u_1)) \tag{2.8}$$

which represents the flow of (2.2) on the invariant manifold W: any solution of (2.2) on W is given by $u_1(t) + x(u_1(t))$, where $u_1(t)$ is a solution of (2.8). Thus to be able to realize any jet from $\mathcal{I}_0^k(X_1)$ means that we can achieve that (2.1), (1.2) has a locally invariant manifold of dimension $n = \dim X_1$ and, up to an error in terms of order $> k$, we can prescribe the vector field on this invariant manifold.

In order to state the main theorem, one more remark is needed.

If X_1 is the kernel of the operator A, then X_1 is finite dimensional and there exists a closed A-invariant subspace X_2 complementary to X_1 (i.e., $X = X_1 \oplus X_2$). This follows from the well-known fact that all eigenvalues of A have the same (and finite) algebraic and geometric multiplicities and are isolated points in the spectrum of A (see e.g. [Ka]).

THEOREM 1. *Let k, n be arbitrary positive integers. Then there exists a function $a(\cdot) \in C(\overline{D})$ such that the following two properties hold:*

(i) *The operator A defined by* (2.3), (2.4) *has an n-dimensional kernel X_1.*

(ii) *There exists a neighbourhood $\mathcal{B}$ of* 0 *in $\mathcal{I}_0^k(X_1)$, such that any jet in $\mathcal{B}$ can be realized in* (2.1), (1.2).

Let us remark that the fact that only jets sufficiently close to 0 can be realized is not a big restriction. With the help of time rescaling one can subsequently obtain any jet.

We now state another result, which is a linear version of Theorem 1.

THEOREM 2. *Let n ne an arbitrary positive integer. Then there exists a continuous function $a(x)$ and a $\delta > 0$ with the following property: For any complex numbers $\mu_1, \dots, \mu_n$ satisfying the relations*

$$|\mu_j| < \delta, \quad j = 1, \dots, n,$$

$$\mu_2 = \overline{\mu_1}, \dots, \mu_{2\ell} = \overline{\mu_{2\ell-1}} \quad \textit{(the bar denotes the complex conjugate)}$$

and

$$\mu_{2\ell+1}, \dots, \mu_n \in \mathbb{R},$$

there exist functions b_1, b_2, $b_3 \in C(\overline{D})$ such that the differential operator

$$Lu := \Delta u + a(x)u + \sum_{j=1}^{3} b_j(x)u_{x_j}$$

subject to Dirichlet boundary condition (cf. (2.3), (2.4)*) has $\mu_1, \dots, \mu_n$ as eigenvalues.*

From Theorem 1 it immediately follows that, for any integer m, one can find $a(x)$ and $g(x, y)$ linear in y such that (2.1), (2.2) has a trajectory dense in an m-torus. Just take $n = 2m$ and in Theorem 2 choose $\mu_{2j} = \overline{\mu_{2j-1}} = i\omega_j$, where the ω_j are rationally independent real numbers. The corresponding coefficients a, b_1, b_2, b_3 then define a linear equation (2.1), which has the required property.

In the nonlinear case, Theorem 1 enables one to apply bifurcation analysis involving higher order terms in the center manifold reduction, in order to find an m-torus which is in addition normally hyperbolic (see [Bi] and [Ch-H, Section 12.12.5]).

The proofs of Theorems 1, 2 are given in the following sections.

3. - A sufficient condition on the eigenfunctions

Assume that the operator A defined by (2.3), (2.4) has an n-dimensional kernel

$$X_1 = \operatorname{span}\{\phi_1, \dots, \phi_n\}.$$

The aim of this section is to derive a condition on the eigenfunctions $\phi_1, \dots, \phi_n$, which will imply that any small k-jet on X_1 can be realized in (2.1), (1.2).

Fix a positive integer k. We shall apply a result of [Po 3]. For this, we first introduce a Hilbert space of functions $g(x, y)$. Let Λ be the set of all $g(x, y)$ such that

$$g(x, y) = \sum_{m=1}^{k} a_m(x)(x_1 y_2 - x_2 y_1)^m, \tag{3.1}$$

$$x = (x_1, x_2, x_3) \in \overline{D}, \; y = (y_1, y_2, y_3) \in \mathbb{R}^3,$$

where $a_m(\cdot) \in H^2(D) = W^{2,2}(D)$. Since $H^2(D) \subset C(\overline{D})$, by the Sobolev embedding theorem, any $g \in \Lambda$ is a polynomial in y with continuous coefficients, hence $g \in C^{0,\infty}$. There is a one-to-one correspondence between $g \in \Lambda$ and the vector of coefficients

$$(a_1(\cdot), \dots, a_k(\cdot)) \in (H^2(D))^k.$$

We can thus make Λ a Hilbert space by equipping it with the topology of $(H^2(D))^k$ through the above identification.

Note that for any $g \in \Lambda$ and $u \in C^1(\overline{D})$ one has

$$g(x, \nabla u(x)) = \sum_{m=1}^{k} a_m(x)(u_\vartheta(x))^m, \tag{3.2}$$

where u_ϑ is the differential expression $x_1 u_{x_2} - x_2 u_{x_1}$ (ϑ comes from the spherical coordinates $x_1 = r\ \cos\vartheta\ \sin\gamma$, $x_2 = r\ \sin\vartheta\ \sin\gamma$, $x_3 = r\ \cos\gamma$).

We now apply the following result of [Po 3]. Any jet sufficiently close to 0 in $J_0^k(X_1)$ can be realized in (2.1), (1.2), provided the following condition is satisfied.

(SCP) For any polynomial $H : X_1 \to X_1$ of degree k satisfying $H(0) = 0$, there exists a $g \in \Lambda$ such that

$$\sum_{j=1}^{n} \phi_j(x) \int_D \phi_j(x) g(x, \nabla u_1(x)) \mathrm{d}x = H(u_1(\cdot)), \quad \text{for all } u_1 \in X_1. \tag{3.3}$$

For the reader's convenience, we briefly recall how this condition has been obtained.

If g is sufficiently close to 0 in Λ, then the corresponding abstract equation (2.2) has a locally invariant C^k manifold of the form (2.7) near the space X_1 (which is an invariant manifold for $g \equiv 0$). The vector field, given by (2.2) on this invariant manifold, is represented by an equation of the form (2.8) (with $Au_1 = 0$, because X_1 is the kernel of A). Denoting the k-jet of the right-hand side of this equation by $\Psi(g)$, we constitute a C^1 mapping $g \mapsto \Psi(g)$ from a neighbourhood of 0 in Λ into $J_0^k(X_1)$, satisfying $\Psi(0) = 0$. The image $R(\Psi)$ of this mapping consists of k-jets which can be realized in (2.1), (1.2). The condition (SCP) ensures that the derivative $\Psi'(0)$ is a surjective linear operator and, as a consequence, that $R(\Psi)$ contains a neighbourhood of 0 in $J_0^k(X_1)$. See [Po 3, Section 2] for details.

In what follows, we find an explicit condition on the eigenfunctions $\phi_1, \dots, \phi_n$ which will imply (SCP). First, using (3.2), we rewrite (3.3) as follows

$$\sum_{j=1}^{n} \phi_j(x) \sum_{m=1}^{k} \int_D \phi_j(x) a_m(x) (u_{1\vartheta})^m \mathrm{d}x = H(u_1(\cdot)). \tag{3.4}$$

In the real coordinates

$$u_1 = r_1\phi_1 + \ldots + r_n\phi_n \tag{3.5}$$

on X_1, (3.4) is equivalent to a system of equalities. To write this system down, we use the following notation. Put $\phi = (\phi_1, \ldots, \phi_n)$, $\phi_\vartheta = (\phi_{1\vartheta}, \ldots, \phi_{n\vartheta})$. For a nonnegative integer multiindex $\beta = (\beta_1, \ldots, \beta_n)$, let $|\beta| = \beta_1 + \ldots + \beta_n$ and

$$\phi_\vartheta^\beta = (\phi_{1\vartheta})^{\beta_1}(\phi_{2\vartheta})^{\beta_2}\ldots(\phi_{n\vartheta})^{\beta_n}.$$

Now, plugging (3.5) into (3.4) and comparing the coefficients of the resulting polynomials, we obtain for each $m = 1, \ldots, k$ a system of equalities

$$(a_m, \phi_j\phi_\vartheta^\beta) = \rho_{j\beta}, \ j = 1, \ldots, n, \ |\beta| = m, \tag{3.6}$$

where $\rho_{j\beta}$ are real numbers (coefficients of the components of the polynomial $H(u_1)$), and $(\cdot,\cdot)$ denotes the standard scalar product on $L_2(D)$. Thus (SCP) is equivalent to the property that, for any $m \in \{1, \ldots, k\}$ and any numbers $\rho_{j\beta}$, there exists a function $a_m \in H^2(D)$ such that (3.6) holds true. This last property is satisfied, provided the functions

$$\phi_j\phi_\vartheta^\beta, \ j = 1, \ldots, n, \ |\beta| = m, \tag{3.7}$$

are linearly independent. To show this, we adapt an argument used in a similar context in [F-P]. The functions (3.7) are elements of $L_2(D)(\phi_j \in C^2(\overline{D})$, by elliptic regularity). If they are linearly independent, then, using the Riesz representation theorem, for any $\rho_{j\beta}$ one finds an $a_m \in L_2(D)$ such that (3.6) holds true (i.e., the functional $u \mapsto (a_m, u)$ takes the prescribed values at the linearly independent functions (3.7)). The fact that sûch an a_m can be chosen in $H^2(D)$ follows from the observation that the assignment $a_m \mapsto (a_m, \phi_j\phi_\vartheta^\beta)$, $j = 1, \ldots, n$, $|\beta| = m$, defines a continuous linear mapping from $L_2(D)$ into a finite dimensional space. As we have just seen, this mapping is surjective, thus so is its restriction to the dense subspace $H^2(D)$.

The linear independence of the functions (3.7) is the sought explicit condition on the eigenfunctions. As we have shown, this condition implies (SCP) and hence enables us to realize all small k-jets in (2.1), (1.2). We state this result as

LEMMA 3.1. *Let $a(\cdot) \in C(\overline{D})$ be such that the operator A, defined by* (2.3), (2.4), *has an n-dimensional kernel $X_1 = \operatorname{span}\{\phi_1, \ldots, \phi_n\}$ and, for each $m = 1, \ldots, k$, the functions* (3.7) *are linearly independent.*

Then there exists a neighbourhood $\mathcal{B}$ in $I_0^k(X_1)$ such that any jet in $\mathcal{B}$ can be realized in (2.1), (1.2).

Let us remark that independence conditions similar to the one in Lemma 3.1 occur in the jet realization results of [F-P] and [Po 3]. The condition in

[F-P] involves eigenfunctions of the Sturm-Liouville operator $u_{xx} + a(x)u$. It was achieved, using a transversality theorem, by taking a generic potential $a(x)$. In the present situation, the genericity approach seems to be rather delicate, since we want to keep eigenvalues of higher multiplicity. The method we use here relies, similarly as in [Po 3], on the explicit knowledge of the eigenfunctions on the ball. However, unlike in [Po 3], the required property is not satisfied by the operator $\Delta + \lambda$, where λ is a constant, and an additional perturbation is needed. This problem will be considered in the next section.

We finish Section 3 with a result concerning linear equations.

LEMMA 3.2. *Let $a(\cdot) \in C(\overline{D})$ be such that the operator A defined by* (2.3), (2.4) *has a n-dimensional kernel $X_1 = \operatorname{span}\{\phi_1, \dots, \phi_n\}$ and the functions*

$$\phi_\iota \phi_{j\vartheta}, \quad \iota, j = 1, \dots, n$$

are linearly independent. Then there exists a $\delta > 0$ such that $a(\cdot)$ and δ have the property from the conclusion of Theorem 2.

Lemma 3.2 is a slightly stronger assertion than a special case ($k = 1$) of Lemma 3.1. In addition to this special case, on only has to verify that, if $g(x, y)$ is linear in y, then the function x in (2.7) can be chosen linear. This implies that the right-hand side of (2.8) is linear and hence prescribing its 1-jet means prescribing this function itself. The arguments are quite analogous to those in the proof of Proposition 2.2 in [Po 3]. We omit the details.

4. - Fulfillment of the sufficient condition

This section is devoted to the proof of

LEMMA 4.1. *For any positive integers k and n, there exists a function $a(\cdot) \in C(\overline{D})$ satisfying the hypotheses of Lemma* 3.1 *(hence also those of Lemma* 3.2*).*

Lemma 4.1, in conjunction with Lemmas 3.1, 3.2, implies Theorems 1, 2.

From now on we assume that k and n are fixed, arbitrarily chosen, positive integers. Lemma 4.1 will be proved using the following result.

LEMMA 4.2. *Consider the eigenvalue problem*

$$\Delta v + \lambda v = 0, \text{ on } D, \tag{4.1}$$

$$v|_{\partial D} = 0. \tag{4.2}$$

There exist an eigenvalue λ of multiplicity $2\ell + 1 \geq n$ and corresponding eigenfunctions $\psi_1, \dots, \psi_{2\ell+1}$, such that the following properties are satisfied:

(P1) *The functions*

$$\psi_i(x)\psi_j(x),\ \ 1 \le i \le j \le 2\ell + 1,$$

are linearly independent ($x \in D$).

(P2) *Set* $\psi = (\psi_1, \dots, \psi_n)$. *For any* $m \in \{1, \dots, k\}$ *the functions*

$$\psi_j \psi_\vartheta^\beta, \quad j = 1, \dots, n,\ \ |\beta| = m,$$

are linearly independent.

The proof of this lemma, which forms a major part of this section will be given later.

PROOF OF LEMMA 4.1. The outline is as follows. Take λ and $\psi_1, \dots, \psi_{2\ell+1}$ as in Lemma 4.2. We introduce a small perturbation to the eigenvalue problem (4.1), (4.2), by replacing λ by a continuous function $a(x) = \lambda + b(x)$, with $b(\cdot) \approx 0$. By continuous dependence of the spectrum, the operator A, defined by $-\Delta a(x)$ and Dirichlet boundary conditions, has exactly $2\ell + 1$ eigenvalues (counting multiplicity) near 0. The corresponding eigenfunctions satisfy $\phi_j \approx \psi_j$, thus the independence condition (P2) remains valid for ϕ's. Using (P1), we then prove that if $b(\cdot)$ is properly chosen, then the kernel of A is spanned by $\phi_1, \dots, \phi_n$, which gives us the desired conclusion.

Now we give the details. Denote $Y := \operatorname{span}\{\psi_1, \dots, \psi_{2\ell+1}\}$ and let $Q : L_2(D) \to Y$ be the orthogonal projection (assuming the usual scalar product on $L_2(D)$). For $b(\cdot) \in C(\overline{D})$, let A_b denote the operator on $L_2(D)$ defined by $-(\Delta + \lambda + b)$ and Dirichlet boundary conditions (cf. (2.3), (2.4)). It is well-known that A_b is a self-adjoint operator. For $b = 0$, the spaces $Y = \ker(A_0)$ and $Y^\perp = R(Q)$ are invariant under A_0. By standard perturbation results [Ka], there exists a neighbourhood $\mathcal{V}$ of 0 in $C(\overline{D})$ and a C^1 mapping $b \mapsto S_b : \mathcal{V} \to \mathcal{L}(Y, Y^\perp)$ ($\mathcal{L}$ denotes the space of bounded linear mappings) such that $S_0 = 0$, and, for each $b \in \mathcal{V}$, the space $Y_b := R(I + S_b)$ (I is the identity on Y) is invariant under A_b. This invariant space is spanned by the eigenfunctions $\phi_i := \psi_i + S_b\psi_i$, $i = 1, \dots, 2\ell + 1$, which correspond to $2\ell + 1$ eigenvalues of A_b near 0, and all the remaining eigenvalues of A_b stay away from 0 as b varies in $\mathcal{V}$. Since $\psi_i + S_b\psi_i$ depends continuously on b even in the C^1-norm (by [Ka] and regularity of the eigenfunctions), we can shrink $\mathcal{V}$, if necessary, such that for any $b \in \mathcal{V}$, the property, (P2) remains valid if ψ's are replaced by ψ's. We now prove that, for an appropriately chosen b, the kernel of A_b is spanned by the functions $\psi_i + S_b\psi_i$, $i = 1, \dots n$. Actually, we prove the stronger claim that the symmetric matrix of the restriction

$$A_b|_{Y_b} \in \mathcal{L}(Y_b, Y_b),$$

with respect to the basis $\psi_i + S_b\psi_i$, $i = 1, \dots, 2\ell + 1$, can be prescribed arbitrarily, up to a scalar multiple. Indeed, this matrix is equal to the matrix of the operator

$$Z_b := QA_b(I + S_b) \in \mathcal{L}(Y, Y),$$

with respect to the basis ψ_i, $i = 1, \dots, 2\ell + 1$ (note that $Q(I + S_b) = I$, because Q is the orthogonal projection). Clearly, $Z_0 = 0$ and $b \mapsto Z_b$ is a C^1 mapping. We compute the derivative of this mapping at $b = 0$. For any $e \in C(\overline{D})$, we have

$$Te := \left(\frac{\mathrm{d}}{\mathrm{d}b} Z_b|_{b=0} \right) e = Q \left(\frac{\mathrm{d}}{\mathrm{d}b} A_b|_{b=0} \right) e$$

because $QA_0 = 0$. Now, since the derivative of A_b is simply the multiplication operator $v(\cdot) \mapsto e(\cdot)v(\cdot)$, we find that the matrix of $Te \in \mathcal{L}(Y, Y)$, with respect to the basis ψ_i, is equal to

$$\left((\phi_i, e\phi_j)\right)_{i,j=1}^{2\ell+1} = \left((e, \phi_i\phi_j)\right)_{i,j=1}^{2\ell+1}.$$

From this and (P1) we obtain, using the Riesz representation theorem similarly as in Section 3, that $e \mapsto Te$ is a surjective linear operator. Consequently, by the local surjectivity theorem (see e.g., [Be]), the image of the mapping $b \mapsto Z_b$ contains a neighbourhood of $Z_0 = 0$, which proves the claim.

To complete the proof, we choose a $b \in \mathcal{V}$ such that the kernel of $A_b|_{Y_b}$, hence also that of A_b, is spanned by $\phi_i = \psi_i + S_b\psi_i$, $i = 1, \dots, n$. Then $a = \lambda + b$ is the sought function. □

We now prepare the proof of Lemma 4.2 by recalling some properties of the eigenfunctions of the Laplacian. It is a standard knowledge (see [Co-H]) that the eigenvalues of the problem (4.1), (4.2) form a countable set of real numbers

$$\lambda_{\ell j}, \quad \ell = 0, 1, 2, \ j = 1, 2, \dots,$$

with the unique accumulation point $+\infty$. Each $\lambda_{\ell j}$ has the odd multiplicity $2\ell+1$, the corresponding eigenspace being spanned by the functions (in the spherical coordinates $x_1 = r \cos\vartheta \sin\gamma$, $x_2 = r \sin\vartheta \sin\gamma$, $x_3 = r \cos\gamma$, $\vartheta \in (0, 2\pi)$, $\gamma \in (0, \pi)$)

$$J_j(r) w_q(\vartheta, \gamma), \quad q = 0, 1, \dots, 2\ell, \tag{4.3}$$

where $J_j(r)$ is a smooth function of $r > 0$ with isolated zeros ($J_j(r)$ is a solution of a second order linear equation), and the w_q form a basis for the $(2\ell + 1)$-dimensional space of the spherical harmonics of the ℓ-th order. We shall use the following explicit expression of the functions w_q (see [Sm])

$$w_q(\vartheta, \gamma) = P_{\ell q}(\cos\gamma) \ \cos(q\vartheta), \quad q = 0, 1, \dots, \ell, \tag{4.4}$$

$$w_q(\vartheta, \gamma) = P_{\ell(q-\ell)}(\cos\gamma) \ \sin((q - \ell)\vartheta), \quad q = \ell + 1, \dots, 2\ell, \tag{4.5}$$

where

$$P_{\ell q}(\xi) = (1-\xi)^{q/2} \frac{\mathrm{d}^{\ell+q}}{\mathrm{d}\xi^{\ell+q}} \left((\xi^2 - 1)^\ell \right), \quad \text{for } |\xi| \le 1. \tag{4.6}$$

Note that $(\ell! 2^\ell)^{-1} P_{\ell 0}$ is the Legendre polynomial of degree ℓ.

PROOF OF LEMMA 4.2. The proof is split into two parts.

Part (A). We prove that the property (P1) is satisfied for any eigenvalue $\lambda = \lambda_{\ell j}$.

Part (B). We prove that if ℓ is sufficiently large, then one can write the eigenfunctions corresponding to $\lambda_{\ell j}$ is such an order that (P2) holds.

Part (A): By (4.3), it is sufficient to prove linear independence of the functions

$$w_q(\vartheta,\gamma) w_m(\vartheta,\gamma), \quad 0 \le q \le m \le 2\ell,$$

i.e., the functions (we omit the subscript ℓ in $P_{\ell q}$)

$$\begin{aligned} &P_q(\xi)P_m(\xi)\cos(q\vartheta)\cos(m\vartheta), && 0 \le q \le m \le \ell, \\ &P_q(\xi)P_m(\xi)\sin(q\vartheta)\sin(m\vartheta), && 1 \le q \le m \le \ell, \\ &P_q(\xi)P_m(\xi)\cos(q\vartheta)\sin(m\vartheta), && 0 \le q \le \ell,\ 1 \le m \le \ell. \end{aligned} \tag{4.7}$$

Here $\xi \in (-1,1)$, $\vartheta \in (0, 2\pi)$.

Since elementary operations on a system of functions do not affect their linear independence, instead of (4.7) we can consider the following functions

$$\begin{aligned} &P_q(\xi)P_m(\xi)\cos(q+m)\vartheta, && 0 \le q \le m \le \ell, \\ &P_q(\xi)P_m(\xi)\cos(m-q)\vartheta, && 1 \le q \le m \le \ell, \\ &P_q(\xi)P_m(\xi)\sin(q+m)\vartheta, && 0 \le q \le m \le \ell,\ (m,q) \ne (0,0), \\ &P_q(\xi)P_m(\xi)\sin(m-q)\vartheta, && 0 \le q < m \le \ell. \end{aligned} \tag{4.8}$$

Suppose that these functions are linearly dependent, i.e., some their nontrivial linear combination is identically equal to zero. In order to derive a contradiction, we first look at this linear combination as a linear combination of the functions

$$\cos(j\vartheta),\ \sin(s\vartheta), \quad j = 0, 1, \dots, \ell,\ s = 1, \dots, \ell,$$

with coefficients depending on ξ. Since these trigonometric functions are linearly independent, we must have all the ξ-dependent coefficients identically equal, to zero. Now, each of these ξ-dependent coefficients is a linear combination (over $\mathbb{R}$) of certain products of the functions $P_j(\xi)$. If we prove that these products are linearly independent, then the latter linear combinations are trivial, therefore the original linear combination of the functions (4.8) must have been trivial, a contradiction. So (P1) will be established, provided we prove that the products which occur in arbitrary of the ξ-dependent coefficients are linearly independent.

Fix any $\nu \in \{0, 1, \dots, 2\ell\}$. As one easily finds, the ξ-dependent coefficient of the function $\cos(\nu\vartheta)$, as well as of $\sin(\nu\vartheta)$, is a linear combination of the following products

$$P_q(\xi)P_m(\xi), \quad 0 \le q \le m \le \ell, \ m + q = \nu, \tag{4.9}$$

$$P_q(\xi)P_m(\xi), \quad 1 \le q \le m \le \ell, \ m - q = \nu. \tag{4.10}$$

We prove that these products are linearly independent. (Note that the set of functions (4.10) is void if $\nu \ge \ell$). First we use (4.6) to express these functions as

$$(1 - \xi^2)^{\nu/2} P_0^{(q)}(\xi) P_0^{(\nu-q)}(\xi), \quad \max\{0, \nu - \ell\} \le q \le [\nu/2], \tag{4.11}$$

$$(1 - \xi^2)^{\nu/2+m} P_0^{(m)}(\xi) P_0^{(\nu+m)}(\xi), \quad 1 \le m \le \ell - \nu. \tag{4.12}$$

Here $^{(q)}$ denotes the q-th derivative and $[\cdot]$ is the integer part. Of course, linear independence of this functions is unaffected if we divide them by $(1 + \xi^2)^{\nu/2}$. In the forthcoming calculations, it is convenient to formulate the property of linear independence of (4.11), (4.12) in the following equivalent form. Denote

$$\tau_\nu := \max\{0, \nu - \ell\},$$
$$T_\nu := \min\{\ell, \nu\}.$$

It is easy to see that the functions (4.11), (4.12) are linearly independent if and only if the following property holds.

$(\mathrm{LI})_\nu$ For any real numbers

$$c_q, \quad q = \tau_\nu, \tau_\nu + 1, \dots, T_\nu,$$
$$d_m, \quad m = 1, \dots, \ell - \nu, \ \text{(this set is void if } \nu \ge \ell)$$

such that

$$\sum_{q=\tau_\nu}^{T_\nu} c_q P_0^{(q)}(\xi) P_0^{(\nu-q)}(\xi) + \sum_{m=1}^{\ell-\nu} d_m (1 - \xi^2)^m P_0^{(m)}(\xi) P_0^{(\nu+m)}(\xi) \equiv 0 \tag{4.13$_\nu$}$$

and

$$(c_{\tau_\nu}, c_{\tau_\nu+1}, \dots, c_{T_\nu}) = (c_{T_\nu}, c_{T_\nu-1}, \dots, c_{\tau_\nu}), \tag{4.14$_\nu$}$$

one has

$$\begin{aligned} c_q &= 0, & q &= \tau_\nu, \tau_\nu + 1, \dots, T_\nu, \\ d_m &= 0, & m &= 1, \dots, \ell - \nu. \end{aligned} \tag{4.15}$$

(Note that $(4.14)_\nu$ requires that the coefficients of the terms that occur in $(4.13)_\nu$ twice are equal).

We prove $(\mathrm{LI})_\nu$ by induction with respect to ν. It is obvious that $(\mathrm{LI})_{2\ell}$ holds ($P_0^{(\ell)}(\xi)$ is a nonzero constant). Suppose that $(\mathrm{LI})_{\nu+1}$ holds for some $0 \le \nu < 2\ell$. We prove that $(\mathrm{LI})_\nu$ holds. Let numbers c_q, d_m satisfy $(4.13)_\nu$, $(4.14)_\nu$. We have to show that this implies (4.15). We shall proceed as follows. Differentiating $(4.13)_\nu$, we obtain an identity $(4.13)_{\nu+1}$ with coefficients satisfying $(4.14)_{\nu+1}$. Then we use the induction hypothesis to prove that all the c_q, d_m vanish.

First recall the following equality for the Legendre polynomial $P_0(\xi)$ (see e.g., [Sm, Section IV.131]).

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\{(1-\xi^2)^m P_0^{(m)}(\xi)\} = -(\ell+m)(\ell-m+1)(1-\xi^2)^{m-1}P_0^{(m-1)}(\xi).$$

Using this, differentiation of $(4.13)_\nu$ gives (omitting the argument ξ)

$$\begin{aligned} 0 \equiv &\sum_{q=\tau_\nu}^{T_\nu} c_q (P_0^{(q+1)}P_0^{(\nu+1-q-1)} + P_0^{(q)}P_0^{(\nu+1-q)}) \\ &+\sum_{m=1}^{\ell-\nu} d_m\Bigg\{-(\ell+m)(\ell-m+1)(1-\xi^2)^{m-1}P_0^{(m-1)}P_0^{(\nu+1+m-1)} \\ &+(1-\xi^2)^m P_0^{(m)}P_0^{(\nu+1+m)}\Bigg\}, \end{aligned}$$

i.e.,

$$0 \equiv \sum_{q=\tau_\nu}^{T_\nu+1} \hat{c}_q P_0^{(q)}P_0^{(\nu+1-q)} + \sum_{m=1}^{\ell-\nu-1} \hat{d}_m (1-\xi^2)^m P_0^{(m)}P_0^{(\nu+1+m)}, \tag{4.16}$$

where

$$\hat{c}_q = c_{q-1} + c_q, \quad q = \tau_\nu + 1, \dots, T_\nu, \tag{4.17}$$

$$\hat{d}_m = d_m - (\ell+m+1)(\ell-m)\mathrm{d}_{m+1}, \quad m = 1, \dots, \ell-\nu-1, \tag{4.18}$$

$$\hat{c}_{\tau_\nu} = \hat{c}_{T_\nu+1} = \begin{cases} c_{\tau_\nu} = c_{T_\nu}, & \text{if } \nu \ge \ell \\ c_{\tau_\nu} - \dfrac{d_1\ell(\ell+1)}{2}, & \text{if } \nu < \ell. \end{cases} \tag{4.19}$$

(In (4.16), we have used the fact that $P_0^{(\ell+1)} = 0$).

We distinguish two cases

a) $\nu < \ell$,

b) $\nu \ge \ell$.

In the case a), we have $\tau_{\nu+1} = \tau_\nu = 0$ and $T_{\nu+1} = T_\nu + 1 = \nu + 1$. Thus, (4.16) is an identity $(4.13)_{\nu+1}$ with the coefficients $\hat{c}_q$, $\hat{d}_m$. By (4.17), (4.19) and $(4.14)_\nu$, these coefficients satisfy $(4.14)_{\nu+1}$. Hence, by the induction hypothesis, all the $\hat{c}_q$ and $\hat{d}_m$ vanish, i.e.,

$$c_0 = \frac{d_1 \ell(\ell+1)}{2}, \tag{4.20}$$

$$c_q = -c_{q-1}, \quad q = 1, \dots, \nu, \tag{4.21}$$

$$(\ell + m + 1)(\ell - m) d_{m+1} = d_m, \quad m = 1, \dots, \ell - \nu 1. \tag{4.22}$$

If ν is odd, then (4.21) in conjunction with $(4.14)_\nu$ implies

$$c_q = 0, \quad q = 0, \dots, \nu.$$

By this and (4.20), (4.22), we then also have

$$d_m = 0, \quad m = 1, \dots, \ell - \nu.$$

So all the desired equalities hold.

Now let ν be even. Suppose that (4.15) fails. By (4.20)-(4.22), this means that $d_1 \neq 0$. In order to derive a contradiction, we employ two other properties of the polynomial $P_0(\xi)$.

(A1). If $\ell = \deg P_0$ is odd (respectively even) then $P_0(\xi)$ is an odd respectively even) function.

(A2). If ℓ is odd (respectively even) then the sequence $P_0'(0)$, $P_0'''(0), \dots, P_0^{(\ell)}(0)$ (respectively $P_0(0)$, $P_0''(0), \dots, P_0^{(\ell)}(0)$) is oscillating (i.e., each two adjacent terms have different nonzero signs).

Both these properties are easily seen from the formula (4.6) for $P_0 = P_{0\ell}$. By (A1) and (4.21), plugging $\xi = 0$ in $(4.13)_\nu$ gives

$$c_0 \left(P_0^{(0)} P_0^{(\nu)} + P_0^{(2)} P_0^{(\nu-2)} + \dots + P_0^{(\nu)} P_0^{(0)} \right)$$
$$+ d_2 P_0^{(2)} P_0^{(\nu+2)} + d_4 P_0^{(4)} P_0^{(\nu+4)} + \dots + d_{\ell-\nu} P_0^{(\ell-\nu)} P_0^{(\ell)} = 0, \text{ if } \ell \text{ is even,}$$

and

$$-c_0 \left(P_0^{(1)} P_0^{(\nu-1)} + P_0^{(3)} P_0^{(\nu-3)} + \dots + P_0^{(\nu-1)} P_0^{(1)} \right)$$
$$+ d_1 P_0^{(1)} P_0^{(\nu+1)} + d_3 P_0^{(3)} P_0^{(\nu+3)} + \dots + d_{\ell-\nu} P_0^{(\ell-\nu)} P_0^{(\ell)} = 0, \text{ if } \ell \text{ is odd.}$$

In either case, this leads to a contradiction. Indeed, by (4.20), (4.22), $c_0, d_1, d_2, \dots d_{\ell-\nu}$ have the same signs and, by (A2), the following holds:

(a) If ℓ is even, then the values at $\xi = 0$ of $P_0^{(0)} P_0^{(\nu)}$, $P_0^{(2)} P_0^{(\nu-2)}, \dots, P_0^{\nu} P_0^{(0)}$, $P_0^{(2)} P_0^{(\nu+2)}$, $P_0^{(4)} P_0^{(\nu+4)}, \dots, P_0^{(\ell-\nu)} P_0^{(\ell)}$ have the same sign;

(b) If ℓ is odd, then the values at $\xi = 0$ of $P_0^{(1)}P_0^{(\nu-1)}$, $P_0^{(3)}P_0^{(\nu-3)}, \dots, P_0^{(\nu-1)}$ $P_0^{(1)}$ have the same sign, so do the values of $P_0^{(1)}P_0^{(\nu+1)}$, $P_0^{(3)}P_0^{(\nu+3)}, \dots,$ $P_0^{(\ell-\nu)}P_0^{(\ell)}$, but the signs of these two groups of values are different.

This clearly contradicts the latter equalities. We have thus completed the proof of (4.15) for $\nu < \ell$.

Now consider the case $\nu > \ell$. We have $T_{\nu+1} = T_\nu = \ell$ and $\tau_{\nu+1} = \tau_\nu + 1 = = \nu + 1 - \ell$. In (4.16), the second sum is missing (as it is in $(4.13)_\nu$). Moreover, the first and last term of the first sum are identical to 0 (because they involve the $(\ell+1)$-th derivative of P_0). So again, (4.16) is the identity $(4.13)_\nu$ with the coefficients

$$\hat{c}_q, \quad q = \nu + 1 - \ell, \dots, \ell,$$

which satisfy $(4.14)_{\nu+1}$ (see (4.17), $(4.14)_\nu$). By the induction hypothesis, we must have

$$\hat{c}_q = c_{q-1} + c_q = 0,$$

i.e.,

$$c_{q-1} = -c_q, \quad q = \nu + 1 - \ell, \dots, \ell.$$

If $2\ell - \nu = \ell - (\nu - \ell)$ is odd, then this, in conjunction with $(4.14)_\nu$, immediately implies that all the c_q vanish. If $2\ell - \nu$ is even, then arguments involving (A1), (A2), similar to those in the case $\nu < \ell$, give the same conclusion. This completes the induction argument and thereby the part (A) of the proof of Lemma 4.2.

For the part (B) we need the following lemma. Recall that n and k are fixed positive integers.

LEMMA 4.3. *There exist positive integers $s_1, \dots, s_n$ such that, for any integers $\iota_1, \dots, \iota_n$ satisfying*

$$0 \neq |\iota_1| + \dots + |\iota_n| \leq 2(k+1),$$

one has

$$s_1\iota_1 + \dots + s_n\iota_n \neq 0.$$

The proof is easy and is left to the reader (take s_j far from one another).
Let $s_1, \dots, s_n$ be as in Lemma 4.3. Choose ℓ with

$$\ell \geq \max\{s_1, \dots, s_n\}.$$

Then the eigenspace corresponding to the eigenvalue $\lambda = \lambda_{\ell 1}$ of (4.1), (4.2) contains the functions

$$\psi_j = J_1(r)P_{\ell s_j}(\xi)\sin(s_j\vartheta), \quad j = 1, \dots, n,$$

where $\xi = \cos\ \gamma \in (-1, 1)$, $\vartheta \in (0, 2\pi)$ (see (4.3), (4.5)). We prove that these functions satisfy the property (P2). To this end, fix an arbitrary $\kappa \in \{1, \ldots, k\}$. We have to prove that, if

$$\sum_{\substack{j=1,\ldots,n \\ |\beta|=\kappa}} \hat{c}_{j\beta}\psi_j\psi_\vartheta^\beta \equiv 0, \tag{4.23}$$

for some $\hat{c}_{j\beta} \in \mathbb{R}$, then

$$\hat{c}_{j\beta} = 0, \quad j = 1, \ldots, n, \ |\beta| = \kappa.$$

To simplify the notation, let

$$\sigma_j(\vartheta) := \sin(s_j\vartheta),$$

$$\varsigma_j(\vartheta) := \cos(s_j\vartheta),$$

and, as usual, $\varsigma := (\varsigma_1, \ldots, \varsigma_n)$. Further denote

$$\mathbf{P}(\xi) := (P_{\ell s_1}(\xi), \ldots, P_{\ell s_n}(\xi)). \tag{4.24}$$

Pick an $r_0 \in (0, 1)$ such that $J_l(r_0) \neq 0$. Using the above notation and fixing $r = r_0$ in (4.23), we obtain

$$\sum_{\substack{j=1,\ldots,n \\ |\beta|=\kappa}} c_{j\beta}\sigma_j(\vartheta)P_{\ell s_j}(\xi)\varsigma^\beta(\vartheta)\mathbf{P}^\beta(\xi) \equiv 0, \tag{4.25}$$

where the $c_{j\beta}$ are nonzero multiples of the $\hat{c}_{j\beta}$.

The first step in proving that such a linear dependence relation is possible only if all the coefficients vanish is somewhat similar to an argument in part (A). We view (4.25) as a linear combination of certain functions of ξ with ϑ-dependent coefficients. We prove that these functions of ξ are linearly independent, thus all the ϑ-dependent coefficients must vanish identically. Then we show this to imply that all the $c_{j\beta}$ vanish.

In order to bring together the terms in (4.25) that involve the same functions of ξ, we introduce the following equivalence relation $\simeq$ on the set of multiindices $(j, \beta) \in \{1, \ldots, n\} \times \{\beta : |\beta| = \kappa\}$:

$$(j, \beta) \simeq (m, \hat{\beta}) \quad \text{iff} \quad \beta_1 s_1 + \ldots + \beta_n s_n + s_j = \hat{\beta}_1 s_1 + \ldots + \hat{\beta}_n s_n + s_m.$$

Note that, by the choice of $s_1, \ldots, s_n$ (see Lemma 4.3), $(j, \beta) \simeq (m, \hat{\beta})$ if and only if $\beta + \epsilon_j = \hat{\beta} + \epsilon_m$, where $\epsilon_m = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on the m-th position. This implies that no equivalence class $\mathcal{V}$ of the relation $\simeq$ contains two distinct elements of the form (j, β), $(j, \hat{\beta})$. Let r denote the number of the equivalence classes. Index the classes by $p = 1, \dots, r$. To each equivalence class $\mathcal{V}_p$, we associate the nonnegative integer multiindex

$$c^p = (c_1^p, \dots, c_n^p) := \beta + \epsilon_j, \tag{4.26}$$

where (j, β) is an element of $\mathcal{V}_p$. By the definition of $\simeq$, this multiindex is well defined and the assignment $p \mapsto c^p$ is one to one. Note that $|c^p| = \kappa + 1$.

Using the equivalence classes, we can rewrite (4.25) as follows

$$\begin{aligned} 0 &\equiv \sum_{p=1}^{r} \sum_{(j,\beta)\in\mathcal{V}_p} c_{j\beta}\sigma_j(\vartheta)\varsigma^\beta(\vartheta)P_{\ell s_j}(\xi)\mathbf{P}^\beta(\xi) \\ &\equiv \sum_{p=1}^{r} \mathbf{P}^{c^p}(\xi) \sum_{(j,\beta)\in\mathcal{V}_p} c_{j\beta}\sigma_j(\vartheta)\varsigma^\beta(\vartheta). \end{aligned} \tag{4.27}$$

We now claim that the functions

$$\mathbf{P}^{c^p}(\xi), \quad p = 1, \dots, r \tag{4.28}$$

are linearly independent. Indeed, by (4.24), (4.6), we have

$$\mathbf{P}^{c^p}(\xi) = (1 - \xi^2)^{\frac{sc^p}{2}}\mathbf{Q}_p(\xi), \tag{4.29}$$

where $sc^p = s_1c_1^p + \dots + s_nc_n^p$ and $\mathbf{Q}_p(\xi)$ is a finite product of functions from the set $\{P'_{\ell 0}(\xi), P''_{\ell 0}(\xi), \dots, P^{(\ell)}_{\ell 0}(\xi)\}$ (each of these functions may occur in $\mathbf{Q}_p(\xi)$ repeatedly). Since the Legendre polynomial $P_{\ell 0}(\xi)$ has all its zeros in $(-1, 1)$ [Sm], so does any of the derivatives $P'_{\ell 0}(\xi), P''_{\ell 0}(\xi), \dots, P^{(\ell)}_{\ell 0}(\xi)$. Therefore

$$\mathbf{Q}_p(1) \neq 0, \quad \text{for } p = 1, \dots, r. \tag{4.30}$$

Next, by injectivity of $p \to c^p$ and by the choice of $s_1, \dots, s_n$, we have

$$sc^p \neq sc^q \quad \text{if } p \neq q.$$

This, in conjuction with (4.30), (4.29), clearly implies linear independence of functions (4.28).

This linear independence and (4.27) now imply that, for each $p = 1, \dots, r$, we have

$$\sum_{(j,\beta)\in\mathcal{V}_p} c_{j\beta}\sigma_j(\vartheta)\varsigma^\beta(\vartheta) \equiv 0. \tag{4.31}$$

Dividing (4.31) by $\varsigma^{c^p}(\xi)$, by definition of the equivalence classes and the multiindex c^p, we obtain

$$\sum_{(j,\beta)\in\mathcal{V}_p} c_{j\beta}\frac{\sigma_j(\vartheta)}{\varsigma_j(\vartheta)} = 0. \tag{4.30}$$

But the functions

$$\frac{\sigma_j(\vartheta)}{\varsigma_j(\vartheta)} = \operatorname{tg}(s_j\vartheta)$$

are linearly independent (the s_j are mutually distinct). Since $\mathcal{V}_p$ does not contain any two distinct elements (j,β), $(j,\hat{\beta})$, (4.32) implies

$$c_{j\beta} = 0 \quad \text{for all } (j,\beta) \in \mathcal{V}_p.$$

Since the last equalities hold for all the equivalence classes, it follows that all the $c_{j\beta}$ vanish. This completes the part (B) of the proof of Lemma 4.4. □

We finish this section with remarks concerning higher dimensional domains. Without going into details, we explain how one can treat the problem (2.1), (1.2) with D replaced by a domain $\Omega \subset \mathbb{R}^N$, $N \geq 4$. Similarly as in the three-dimensional case above, in order to prove statements like Theorems 1, 2, it suffices to find a domain Ω and a potential $a(\cdot)$ with the properties as in Lemma 3.1 (where ϑ is from the spherical coordinates in the first three components of the space variable $x \in \Omega$, cf. Section 3). Such a function $a(\cdot)$ can be found in the same way as we did it in the three-dimensional case, provided we find a domain Ω, such that for the problem (4.1), (4.2), with D replaced by Ω, the conclusion of Lemma 4.2 holds. The latter can be easily achieved on a cylindrical domain $\Omega = D \times \mathcal{B}$, where $\mathcal{B}$ is a convex domain in $\mathbb{R}^{N-3}$ and D is the 3-ball. In this case, we can separate variables to find that the eigenvalues of the Laplacian on Ω, with Dirichlet boundary condition, are of the form $\lambda + \hat{\lambda}$, where λ is an eigenvalue of (4.1), (4.2) and $\hat{\lambda}$ is an eigenvalue of

$$\begin{aligned} &\Delta v + \lambda v = 0 && \text{on } \mathcal{B}, \\ &v = 0 && \text{on } \partial\mathcal{B}. \end{aligned} \tag{4.31}$$

By appropriately rescaling $\mathcal{B}$ (replacing $\mathcal{B}$ by $\alpha\mathcal{B}$), we can achieve that if $\hat{\lambda}_1$ is the first (hence simple) eigenvalue of (4.31) and λ is a chosen eigenvalue of (4.1), (4.2) with multiplicity $2\ell + 1$, then $\lambda + \hat{\lambda}_1$ has the same multiplicity $2\ell + 1$. Choosing λ as in Lemma 4.2, the eigenfunctions corresponding to $\lambda + \hat{\lambda}_1$ have the form $\psi_j(\overline{x})z(\overline{\overline{x}})$ ($\overline{x} \in D$, $\overline{\overline{x}} \in \mathcal{B}$), where z is the eigenfunction of (4.31) corresponding to $\hat{\lambda}_1$, hence the independence conditions analogous to (P1), (P2) are satisfied.

Once we have found a domain with the required properties, we can go on to find such a domain with smooth boundary. For this a (nonsmooth) domain perturbation theory is needed. As is well-known, the eigenvalues and the eigenfunctions depend continuously on the domain [Co-H] (see [Da] and references therein for more recent results on domain perturbations). Thus replacing Ω by a "nearby" domain $\tilde{\Omega}$ we do not destroy the independence properties of the eigenfunctions. One has to ensure, however, that the eigenvalue $\lambda + \hat{\lambda}_1$ perturbs to an eigenvalue with the same multiplicity. For this, the $O(3)$-symmetry can be employed. Defining the action of $O(3)$ over function spaces by $(\Gamma u)(\overline{x},\overline{\overline{x}}) = u(\Gamma\overline{x},\overline{\overline{x}})$, $\overline{x} \in \mathbb{R}^3$, $\overline{\overline{x}} \in \mathbb{R}^{N-3}$, $\Gamma \in O(3)$, and considering the domains satisfying

$$\begin{pmatrix} \Gamma & 0 \\ 0 & I \end{pmatrix} \Omega = \Omega$$

for each $\Gamma \in O(3)$, we make this an equivariant problem. Since the eigenspace corresponding to the eigenvalue $\lambda+\hat{\lambda}_1$ of the unperturbed problem is actually the space of spherical harmonics of certain order multiplied by a single symmetric function, the above action is irreducible on this eigenspace (see e.g., [Va, G-S]). This should ensure that the perturbed eigenvalue retains the multiplicity of the unperturbed eigenvalue. (Unfortunately, we do not have a reference for such an equivariant domain perturbation problem).

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