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# On the $D^*$-extension and the Hartogs extension

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# On the $D^*$-Extension and the Hartogs Extension

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## Introduction

The extension of holomorphic maps is the fundamental problem of complex analytic geometry and complex analysis of several variables. Many mathematicians were interested in that problem and obtained big results. In the extension of holomorphic maps, the study of holomorphic maps that can be extended holomorphically to a hole or to an envelope of holomorphy was also frequently considered (see Kobayashi [12], Kwack [13],...).

In this paper, we establish some results concerning the extension of holomorphic maps to a hole and to an envelope of holomorphy. These results give relations between the hyperbolicity of complex spaces and the extension of holomorphic maps.

In the first section, we study characterizations of $D^*$-extension for compact complex spaces. We also prove a generalization of a Brody's theorem for compact complex spaces (see [2]).

In the second section, we prove a characterization of Hartogs extension for holomorphically convex Kahler spaces. That characterization is a generalization of an Ivaskowicz's theorem for holomorphically convex Kahler spaces (see Ivaskowicz [10]).

In the third section, we study a relation between the $D^*$-extension and the Hartogs extension.

In the fourth section, we prove a theorem on the inverse invariance of $D^*$-extension and Hartogs extension under some special holomorphic maps.

In the fifth section, we shall investigate the invariance of the $D^*$-extension and the Hartogs extension under some holomorphic maps, in particular for finite proper surjective maps.

In the sixth section, we investigate the $D^*$-extension and the Hartogs extension of 1-convex spaces.

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## 0. - Definitions

0.1. A complex space $X$ is called to have the $D^*$-extension property (shortly $D^*$-EP) if every holomorphic map $f : D^* = D\backslash\{0\} \to X$ can be extended to a holomorphic map $F : D \to X$.

0.2. A complex space $X$ is called to have the Hartogs extension property (shortly HEP) if every holomorphic map, from a Riemann domain over a Stein manifold into $X$, can be extended to an envelope of holomorphy of that map.

In this paper, we shall make use of properties of complex spaces as in Gunning-Rossi [7], and properties of the Kobayashi pseudodistance on complex spaces as in Kobayashi [12] or Kwack [13]. We also usually assume that complex spaces are connected and have a countable basic of open subsets.

For $0 < r \in R$, put $D_r = \{z \in \mathbb{C} : |z| < r\}$, $D_1 = D$.

## 1. - Characterizations of $D^*$-extension for compact complex spaces

First we have the following definitions.

DEFINITION 1.1.

(i) Let $M, N$ be complex spaces.
A map $f : M \to N$ is called a $C^\infty$-map if, for every local map $(U, \varphi)$ of $M$, $(V, \psi)$ of $N$, the map $\psi \circ f \circ \varphi^{-1}$ can be extended to a $C^\infty$-map from an open neighbourhood of $\varphi(U)$ in $\mathbb{C}^n$ into $\mathbb{C}^m$.

(ii) Let $M, N$ be complex spaces; $A \subset M$.
A map $f : A \to N$ is called a $C^\infty$-map if it can be extended to a $C^\infty$-map from an open neighbourhood of $A$ in $M$ into $N$.

(iii) Let $M$ be a complex space.
A map $f : M \to \mathbb{R}$ is called a $C^\infty$-map if, for every local map $(U, \varphi)$ of $M$, the map $f \circ \varphi^{-1}$ can be extended to a $\mathbb{C}^\infty$-map from an open neighbourhood of $\varphi(U)$ in $\mathbb{C}^n$ into $\mathbb{R}$.

(iv) Let $M$ be a complex space, $[a, b] \subset \mathbb{R}$.
A map $f : [a, b] \to M$ is called a holomorphic map (or holomorphic curve) if it can be extended to a holomorphic map from an open neighbourhood of $[a, b]$ in $\mathbb{C}$ into $M$.

(v) Let $M$ be a complex space, $[0, 1] \subset \mathbb{R}$.
A map $f : [0, 1] \to M$ is called a piecewise holomorphic map (or piecewise holomorphic curve) if there exist numbers $0 = a_0 < a_1 < \ldots < a_k = 1$ such that $f\big|_{[a_{j-1}, a_j]}$ is holomorphic for every $j$, $1 \leq j \leq k$.

PROPOSITION 1.2. *Let $\{U_\alpha\}_{\alpha \in A}$ be any open cover of a complex space $M$. Then there exists a $C^\infty$-partition of unity $\{\varphi_\alpha\}_{\alpha \in A}$ which is subordinate to the cover $\{U_\alpha\}_{\alpha \in A}$.*

PROOF. As in Narasimhan [14]. Q.E.D.

For every complex space $M$, we have the tangent space of $M$ denoted by $TM$ (see Fischer [4]). $TM$ is a complex space and the canonical projection $\pi : TM \to M$ is holomorphic.

DEFINITION 1.3.

(i) A Hermitian form on $M$ is a $C^\infty$-map $h : TM \oplus TM \to \mathbb{C}$ such that the restriction of $h$ on any fibre $T_pM \times T_pM$ is a Hermitian form, where $TM \oplus TM$ is the Whitney sum of two fibres.

(ii) A Hermitian form $h$ on $M$ is called a Hermitian structure on $M$ if $h(u,u) > 0$ for every $0 \neq u \in T_pM$, for any $p \in M$.

(iii) Let $M$ be a complex space with a Hermitian structure $h$.
For every $v \in TM$, we put $\|v\| = \sqrt{h(v,v)}$.
For every holomorphic map $f : X \to M$ ($X$ is open in $\mathbb{C}$), we put

$$\|f'(z_0)\| = \left\| f_* \left( \frac{\partial}{\partial z} \right)_{z=z_0} \right\|,$$

for every $z_0 \in X$.
Clearly $\|f'(z_0)\|$ is the norm $\|T_{z_0} f\|$ of the linear map

$$T_{z_0} f : T_{z_0} X \to T_{f(z_0)} M.$$

We have the following proposition.

PROPOSITION 1.4. *For any complex space there exists a Hermitian structure, and when the structure is given that space is called a Hermitian complex space.*

PROOF. Let $\mathcal{U}$ be a atlas of $M$: $\mathcal{U} = \{(U_i, \alpha_i)\}_{i \in I}$. Choose a $C^\infty$-partition of unity $\{\varphi_i\}_{i \in I}$ subordinate to the cover $\{U_i\}_{i \in I}$. We have a canonical injection

$$\Phi_i : TU_i \hookrightarrow U_i \times \mathbb{C}^{n_i}.$$

Put a map $\psi_i : TM \to \mathbb{C}^{n_i}$ as follows:

$$\psi_i|_{TU_i} = \pi_i \cdot \Phi_i,$$
$$\psi_i|_{TM \setminus TU_i} = 0,$$

where $\pi_i : U_i \times \mathbb{C}^{n_i} \to \mathbb{C}^{n_i}$ is the canonical projection. Let $q_i : \mathbb{C}^{n_i} \times \mathbb{C}^{n_i} \to \mathbb{C}$ be the canonical Hermitian form on $\mathbb{C}^{n_i}$. The function $h : TM \oplus TM \to \mathbb{C}$ is defined as follows:

$$h(u,v) = \sum_{i \in I} \varphi_i(p) \cdot q_i(\psi_i(u),\ \psi_i(v)), \qquad \text{for all } u, v \in T_pM.$$

Obviously $h$ is a Hermitian structure on $M$. Q.E.D.

DEFINITION 1.5. A Hermitian structure $h$, on a complex space $M$, that was constructed as in Proposition 1.4., is called a canonical Hermitian structure on $M$.

In this paper, we only consider canonical Hermitian structures on a complex space. For a complex space $M$, with a canonical Hermitian structure $h$, a distance between two points is defined as follows.

Let $\gamma : [a,b] \to M$ be a holomorphic curve. Let $\overline{\gamma} : U \to M$ be a holomorphic extension of $\gamma$, from an open neighbourhood $U$ of $[a,b]$ in $\mathbb{C}$, into $M$. Put

$$L_\gamma = \int_a^b \|\dot{\overline{\gamma}}(t)\| dt.$$

Let $\gamma : [a,b] \to M$ be a piecewise holomorphic curve, i.e. there exist numbers $a = a_0 < a_1 < \ldots < a_k = b$ such that $\gamma_i = \gamma\big|_{[a_{i-1},a_i]}$ is a holomorphic curve, for all $i$, $1 \leq i \leq k$. Put

$$L_\gamma = \sum_{i=1}^{k} L_{\gamma_i}.$$

Clearly $L_\gamma$ does not depend on the selection of numbers $a_1, \ldots, a_{k-1}$.

DEFINITION 1.6. Let $p, q \in M$. We say $\Omega_{p,q}$ is the set of all piecewise holomorphic curves $\gamma : [0,1] \to M$, with $\gamma(0) = p$, $\gamma(1) = q$.

Put $h_M(p,q) = \inf\{L_\gamma : \gamma \in \Omega_{p,q}\}$. Clearly if $M$ is a complex manifold, then $h_M$ is a Hermitian metric on $M$. Hence $h_M$ is a metric of $M$ and induces the given topology of $M$. But this is correct for any complex space with a metric constructed as above.

We have the following proposition.

PROPOSITION 1.7. *$h_M$ is a metric on $M$ and is called the Hermitian metric of $M$.*

PROOF.

(i) From the definition of $h_M$, we have

$$h_M(p,q) \geq 0$$
$$h_M(p,q) = h_M(q,p), \qquad \text{for every } p, q \in M.$$

(ii) If $\gamma_1 \in \Omega_{p,r}$, $\gamma_2 \in \Omega_{r,q}$, then a curve $\gamma : [0,1] \to M$, which is defined by

$$\gamma(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leq t \leq 1/2 \\ \gamma_2(2t-1) & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

is in $\Omega_{p,q}$. By $L_\gamma = L_{\gamma_1} + L_{\gamma_2}$, we have $h_M(p,q) \leq h_M(p,r) + h_M(r,q)$ for every $p, q, r \in M$.

(iii) Obviously $h_M(p,p) = 0$, for every $p \in M$. We must prove that $h_M(p,q) > 0$, if $p \neq q \in M$. Consider the atlas $\mathcal{U} = \{(U_i, \alpha_i)\}_{i \in I}$, that induces $h$. We can assume that $p \in V :=$ open $\subset \overline{V} :=$ compact $\subset \overset{\circ}{\widehat{\operatorname{supp} \varphi_i}} \subset U_i$ and $q \notin \overline{V}$. Assume that $\inf\limits_{p \in \overline{V}} \varphi_i(p) = \varepsilon > 0$. Consider any piecewise holomorphic curve in $\overline{V}$, which connects $p$ with a point of $\partial V$:

$$\gamma : [0,1] \to M, \;\; \gamma(0) = p, \;\; \gamma(1) \in \partial V, \;\; \gamma([0,1]) \subset \overline{V}.$$

Let $0 = a_0 < a_1 < \ldots < a_k = 1$ such that $\gamma_j = \gamma\big|_{[a_{j-1}, a_j]}$ is a holomorphic curve, for every $j$, $1 \leq j \leq k$. Let $\overline{\gamma}_j$ be a holomorphic extension of $\gamma_j$, from an open neighbourhood of $[a_{j-1}, a_j]$ in $\mathbb{C}$, into $M$. We have

$$\begin{aligned}
\|\dot{\overline{\gamma}}_j(t)\| &= \sqrt{h\left(\dot{\overline{\gamma}}_j(t), \dot{\overline{\gamma}}_j(t)\right)} \\
&= \sqrt{\sum_{s \in I} (\varphi_s \cdot \overline{\gamma}_j)(t) \cdot q_s \left[\psi_s\left(\dot{\overline{\gamma}}_j(t)\right), \psi_s\left(\dot{\overline{\gamma}}_j(t)\right)\right]} \\
&\geq \left\{\varepsilon \cdot q_i \left[\psi_i\left(\dot{\overline{\gamma}}_j(t)\right), \psi_i\left(\dot{\overline{\gamma}}_j(t)\right)\right]\right\}^{1/2} \\
&= \sqrt{\varepsilon} \cdot \|(\alpha_i \cdot \overline{\gamma}_j)'(t)\|,
\end{aligned}$$

for every $t \in [a_{j-1}, a_j]$. Hence

$$L_{\gamma_j} \geq \sqrt{\varepsilon} \cdot \int_{a_{j-1}}^{a_j} \|(\alpha_i \cdot \overline{\gamma}_j)'(t)\| \mathrm{d}t,$$

$$L_\gamma = \sum_{j=1}^{k} L_{\gamma_j} \geq \sqrt{\varepsilon} \cdot h_{\mathbb{C}^{n_i}} \left[\alpha_i(p), \;\; \alpha_i(\partial V)\right].$$

Since $\alpha_i(p) \notin \alpha_i(\partial V)$ and $h_{\mathbb{C}^{n_i}}$ is a metric on $\mathbb{C}^{n_i}$, we have

$$h_{\mathbb{C}^{n_i}} \left[\alpha_i(p), \;\; \alpha_i(\partial V)\right] > \delta > 0.$$

Hence $L_\gamma > \sqrt{\varepsilon} \cdot \delta$. Consider any piecewise holomorphic curve $\sigma : [0,1] \to M$ that connects $p$ and $q$. Then there exists a minimal number $r \in [0,1]$ such that $\sigma(r) \in \partial V$, thus $\sigma([0,r]) \subset \overline{V}$. Put $\sigma_1 = \sigma\big|_{[0,r]}$, then $L_\sigma \geq L_{\sigma_1}$. Consider a map

$$\begin{aligned}
v : [0,1] &\to [0,r] \\
t &\mapsto r \cdot t.
\end{aligned}$$

Put $\tilde{\sigma} = \sigma_1 \cdot v$; we have

$$L_\sigma \geq L_{\sigma_1} = L_{\tilde{\sigma}} > \sqrt{\varepsilon} \cdot \delta > 0,$$

then

$$L_\sigma > \sqrt{\varepsilon} \cdot \delta, \qquad \text{for every } \sigma \in \Omega_{p,q},$$

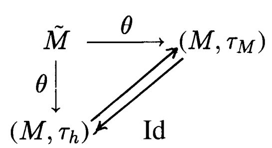
therefore

$$h_M(p,q) > 0, \qquad \text{for every } p \neq q \in M.$$

Q.E.D.

PROPOSITION 1.8. *For a Hermitian complex space $M$, the metric topology $\tau_h$, which was induced by $h_M$, coincides with the given topology $\tau_M$ of $M$.*

PROOF. Let $\theta : \tilde{M} \to M$ be the resolution of singularities of $M$. Hence $\tilde{M}$ is a complex manifold and $\theta$ is proper holomorphic surjective. Consider the diagram:




First we prove that the map $\theta : \tilde{M} \to (M, \tau_h)$ is continuous. Indeed, let $\tilde{x}_0 \in \tilde{M}$, $\{\tilde{x}_n\}_{n=1}^{\infty} \subset \tilde{M}$, $\{\tilde{x}_n\}_{n=1}^{\infty} \to \tilde{x}_0$. Take $K$, a compact neighbourhood of $\tilde{x}_0$ in $\tilde{M}$. Choose $\varepsilon > 0$ such that $B_\varepsilon = B_{\tilde{M}}(\tilde{x}_0, \varepsilon) \subset \operatorname{Int} K$. Since $\{\tilde{x}_n\}_{n=1}^{\infty} \to \tilde{x}_0$, we may assume that $\{\tilde{x}_n\}_{n=1}^{\infty} \subset B_\varepsilon$. Consider a map $T\theta : T\tilde{M} \to TM$; we have

$$\sup_{\tilde{x} \in K} \|T_{\tilde{x}}\theta\| = c < +\infty.$$

Consider any piecewise holomorphic curve $\gamma$ in $\tilde{M}$, which connects $\tilde{x}_n$ with $\tilde{x}_0$ and $\gamma([0,1]) \subset K$. Let $a_0 = 0 < a_1 < \ldots < a_k = 1$ and $\gamma_j = \gamma\big|_{[a_{j-1}, a_j]}$ be a holomorphic map, for every $j$, $1 \leq j \leq k$. Let $\overline{\gamma}_j$ be a holomorphic extension of $\gamma_j$ from an open neighbourhood of $[a_{j-1}, a_j]$ in $\mathbb{C}$ into $\tilde{M}$, for every $j$, $1 \leq j \leq k$. We have

$$\begin{aligned} L_{\theta \cdot \gamma_j} &= \int_{a_{j-1}}^{a_j} \|(\theta \cdot \overline{\gamma}_j)'(t)\| \, \mathrm{d}t \\ &= \int_{a_{j-1}}^{a_j} \|\theta'(\overline{\gamma}_j(t))\| \cdot \|\dot{\overline{\gamma}}_j(t)\| \, \mathrm{d}t \\ &\leq c \cdot \int_{a_{j-1}}^{a_j} \|\dot{\overline{\gamma}}_j(t)\| \, \mathrm{d}t = c \cdot L_{\gamma_j}. \end{aligned}$$

Therefore

$$L_{\theta\cdot\gamma} = \sum_{j=1}^{k} L_{\theta\cdot\gamma_j} \leq c \cdot \sum_{j=1}^{k} L_{\gamma_j} = c \cdot L_{\gamma},$$

$$h_M(\theta\tilde{x}_n, \theta\tilde{x}_0) \leq c \cdot L_{\gamma}, \qquad \text{for every } \gamma \in \Omega_{\tilde{x}_n,\tilde{x}_0},\ \gamma([0,1]) \subset K,$$

$$h_M(\theta\tilde{x}_n, \theta\tilde{x}_0) \leq c \cdot \inf\{L_{\gamma} : \gamma \in \Omega_{\tilde{x}_n,\tilde{x}_0},\ \gamma([0,1]) \subset K\}$$
$$= c \cdot h_{\tilde{M}}(\tilde{x}_n, \tilde{x}_0), \qquad \text{for every } n \geq 1.$$

Thus $\{\theta\tilde{x}_n\}_{n=1}^{\infty} \to \theta\tilde{x}_0$. Hence we have the proof.

Now we prove that the map $\mathrm{Id} : (M, \tau_M) \to (M, \tau_h)$ is continuous. Indeed, let $\{x_n\}_{n=1}^{\infty} \to x_0$ for the topology $\tau_M$ and assume that $\{x_n\}_{n=1}^{\infty} \not\to x_0$ for the topology $\tau_h$. Then, there exists an open neighbourhood $U$ of $x_0$ (for the topology $\tau_h$) and there exists $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \notin U$ for every $k \geq 1$. In $(M, \tau_M)$ we take a compact neighbourhood $K$ of $x_0$ and we may assume

$$\{x_n\}_{n=1}^{\infty} \subset K.$$

For $k \geq 1$, choose $y_{n_k} \in \theta^{-1}(x_{n_k}) \subset \theta^{-1}(K)$, therefore

$$\{y_{n_k}\}_{k=1}^{\infty} \subset \theta^{-1}(K).$$

The sequence $\{y_{n_k}\}_{k=1}^{\infty}$ has a point of accumulation $y_0 \in \tilde{M}$. Without loss of generality, we can suppose that the sequence

$$\{y_{n_k}\}_{k=1}^{\infty} \to y_0 \in \tilde{M}.$$

Thus,

$$\{\theta(y_{n_k})\}_{k=1}^{\infty} \to \theta(y_0)$$

for the topology $\tau_M$,

$$\{\theta(y_{n_k})\}_{k=1}^{\infty} \to \theta(y_0)$$

for the topology $\tau_h$. Hence $\theta(y_0) = x_0$ and

$$\{x_{n_k}\}_{k=1}^{\infty} \to x_0$$

for the topology $\tau_h$. This is a contradiction.

Finally we prove that the map $\mathrm{Id} : (M, \tau_h) \to (M, \tau_M)$ is continuous. Indeed, let $\{x_n\}_{n=1}^{\infty} \subset M$, $\{x_n\}_{n=1}^{\infty} \to x_0$ for the topology $\tau_h$ and assume that $\{x_n\}_{n=1}^{\infty} \not\to x_0$ for the topology $\tau_M$. There exists a compact neighbourhood $U$ of $x_0$ (for the topology $\tau_M$), there exists $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \notin U$, for every $k \geq 1$. Since $h_M$ is a metric in $M$, there exists a number $c > 0$ such that

$$h_M(x_{n_k}, x_0) > c > 0,$$

for every $k \geq 1$. Hence $\{x_{n_k}\}_{k=1}^{\infty} \not\to x_0$ for the topology $\tau_h$. This is a contradiction. Q.E.D.

PROPOSITION 1.9. *Let $f : X \to Y$ be a finite proper holomorphic map between two complex spaces. Then, if $Y$ has the $D^*$-extension property, $X$ also has the $D^*$-extension property.*

PROOF. Let $g$ be any holomorphic map of $D^*$ into $X$. Consider the composition of maps $D^* \xrightarrow{g} X \xrightarrow{f} Y$; this map can be extended to a holomorphic map $G : D \to Y$. Put $G(0) = p \in Y$. Let

$$f^{-1}(p) = \{p_1, \dots, p_k\}.$$

For every $i$, $1 \leq i \leq k$, choose a relatively compact open neighbourhood $U_i$ and a hyperbolic open neighbourhood $V_i$ of $p_i$ such that

$$\overline{U}_i \subset V_i, \qquad \overline{U}_i \cap \overline{U}_j = \emptyset.$$

We prove that there exists $\varepsilon > 0$ such that

$$g(D_\varepsilon^*) = g(D_\varepsilon \backslash \{0\}) \subset \bigcup_{i=1}^{k} U_i.$$

Assume that there exists $\{z_n\}_{n=1}^{\infty} \subset D^*$, $\{z_n\}_{n=1}^{\infty} \to 0$, such that

$$g(z_n) \notin \bigcup_{i=1}^{k} U_i, \qquad \text{for every } n \geq 1.$$

Since $G$ is continuous, we have $\{G(z_n)\}_{n=1}^{\infty} \to p$, i.e. $\{f \circ g(z_n)\}_{n=1}^{\infty} \to p$. On the other hand, choose $W$, compact neighbourhood of $p$; we have $g(z_n) \in f^{-1}(W)$. By the compactness of $f^{-1}(W)$, the sequence $\{g(z_n)\}_{n=1}^{\infty}$ has a point of accumulation $x$. Without loss of generality, we can suppose that

$$\{g(z_n)\}_{n=1}^{\infty} \to x.$$

Hence $\{f \circ g(z_n)\}_{n=1}^{\infty} \to f(x)$, $f(x) = p$ and $x = p_i$. Thus $\{g(z_n)\}_{n=1}^{\infty} \to p_i$ and $g(z_n) \in U_i$, for every $n \geq N$. This is a contradiction. Thus there exists $\varepsilon > 0$ such that $g(D_\varepsilon^*) \subset \bigcup_{i=1}^{k} U_i$. Since $g(D_\varepsilon^*)$ is connected, we have

$$g(D_\varepsilon^*) \subset U_i \subset V_i,$$

$V_i$ is hyperbolic. Take any sequence $\{z_n\}_{n=1}^{\infty}$ in $D_\varepsilon^*$ that converges to 0. By the compactness of $\overline{U}_i$, the sequence $\{g(z_n)\}_{n=1}^{\infty}$ has a point of accumulation in $\overline{U}_i$. Without loss of generality, we can suppose that

$$\{g(z_n)\}_{n=1}^{\infty} \text{ converges to a point of } V_i.$$

By a Kwack's theorem (see Kobayashi [12]), the restriction of $g$ from $D_\varepsilon^*$ into $V_i$ can be extended to a holomorphic map of $D_\varepsilon$ into $V_i$. Hence the map $g$ can be extended to a holomorphic map of $D$ into $X$. Q.E.D.

COROLLARY 1.10. *If a complex space $X$ has the $D^*$-extension property, then $X$ contains no complex lines.*

PROOF. Assume that there exists a holomorphic map

$$\sigma : \mathbb{C} \to X,$$

$\sigma \neq$ constant. Consider a holomorphic map $\beta : \mathbb{C}\setminus\{0\} \to X$ defined by

$$z \longmapsto \beta(z) = \sigma\left(\frac{1}{z}\right).$$

By the mentioned condition, the map $\beta$ can be extended to a holomorphic map $\overline{\beta} : \mathbb{C} \to X$. Hence $\sigma$ can be extended to a holomorphic map $\overline{\sigma} : \mathbb{C}P^1 \to X$. Obviously $\overline{\sigma}$ is proper and finite. By the Proposition 1.9., $\mathbb{C}P^1$ has the $D^*$-extension property. This is a contradiction. Thus $X$ contains no complex lines. Q.E.D.

PROPOSITION 1.11. *Let $M$ be a Hermitian complex space. Let $Y$ be a complex subspace of $M$ and $e : Y \hookrightarrow M$ be the canonical embedding.*
*Then, $Y$ is hyperbolic if*

$$\sup\{\|(e\circ f)'(0)\| : f \in \mathrm{Hol}(D, Y)\} < +\infty.$$

*In particular, a complex space $M$ is hyperbolic if*

$$\sup\{\|f'(0)\| : f \in \mathrm{Hol}(D, M)\} < +\infty.$$

PROOF. Put

$$\sup\{\|(e\circ f)'(0)\| : f \in \mathrm{Hol}(D, Y)\} = c.$$

We prove that $\mathrm{d}_Y(p, q) > 0$, for every $p \neq q \in Y$. On $D$ we consider the Poincaré-Bergman metric $\omega$ that is defined by

$$\omega = \frac{\mathrm{d}z\cdot \mathrm{d}\overline{z}}{(1-|z|^2)^2}.$$

Hence

$$\rho_D(0, z) = \log\frac{1+|z|}{1-|z|}, \quad \text{for every } z \in D.$$

For every $a \in D$, choose a map $\sigma : \overline{D} \to \overline{D}$ defined by

$$z \longmapsto a \cdot z.$$

We have

$$\int_0^1 \left\| \dot{\sigma}(x) \left( \frac{\partial}{\partial x} \right) \right\| \mathrm{d}x$$

$$= \int_0^1 \frac{\left( \langle \mathrm{d}x + i \cdot \mathrm{d}y,\ a \cdot \left( \frac{\partial}{\partial x} \right) \rangle \cdot \langle \mathrm{d}x - i \cdot \mathrm{d}y,\ \overline{a} \cdot \left( \frac{\partial}{\partial x} \right) \rangle \right)^{1/2}}{(1 - |a|^2 \cdot x^2)} \mathrm{d}x$$

$$= |a| \cdot \int_0^1 \frac{\mathrm{d}x}{1 - |a|^2 \cdot x^2} = \log \frac{1 + |a|}{1 - |a|} = \rho_D(0, a).$$

Assume that there exist $p \neq q \in Y$ such that $\mathrm{d}_Y(p, q) = 0$. Choose $0 < \lambda < 1$ such that $a \in \overline{D}_\lambda = \{z \in D : |z| \leq \lambda < 1\}$ if

$$\rho_D(0, a) < \frac{1}{2}.$$

Assume that

$$\sup \left\{ \|(e \circ f)'(z)\| : f \in \mathrm{Hol}(D, Y),\ z \in \overline{D}_\lambda \right\} = \infty.$$

Then there exist a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathrm{Hol}(D, Y)$ and a sequence of points $\{z_n\}_{n=1}^{\infty} \subset \overline{D}_\lambda$ such that

$$\|(e \circ f_n)'(z_n)\| \to \infty, \qquad \text{when } n \to \infty.$$

For every $n \geq 1$, we choose a holomorphic map $\beta_n : D \to D$ such that $\beta_n(0) = z_n$, $\|\beta_n'(0)\| = 1 - |z_n|^2 \geq 1 - \lambda^2 > 0$, for every $n \geq 1$. We have

$$\begin{aligned} \|((e \circ f_n) \cdot \beta_n)'(0)\| &= \|(e \circ f_n)'(\beta_n(0))\| \cdot \|\beta_n'(0)\| \\ &= \|(e \circ f_n)'(z_n)\| \cdot (1 - |z_n|^2) \\ &\geq \|(e \circ f_n)'(z_n)\| \cdot (1 - \lambda^2) \to \infty, \end{aligned}$$

when $n \to \infty$. This is a contradiction. Thus

$$\sup \left\{ \|(e \circ f)'(z)\| : f \in \mathrm{Hol}(D, Y),\ z \in \overline{D}_\lambda \right\} = m < +\infty.$$

By the definition of the Kobayashi pseudodistance, there exist $a_1, \ldots, a_k \in D_{1/2}$, $f_1, \ldots, f_k \in \mathrm{Hol}(D, Y)$ such that

$$f_1(0) = p_0 = p, \quad f_i(a_i) = f_{i+1}(0) = p_i, \quad f_k(a_k) = p_k = q$$

$$\sum_{i=1}^{k} \rho_D(0, a_i) < \frac{1}{2}.$$

For every $i$, $1 \leq i \leq k$ we define the maps

$$\begin{array}{rcl} \sigma_i : [0,1] & \to & D \\ t & \mapsto & a_i \cdot t \end{array} \quad \text{and} \quad \begin{array}{rcl} \overline{\sigma}_i : \overline{D} & \to & \overline{D} \\ z & \mapsto & a_i \cdot z. \end{array}$$

Clearly $e \circ f_i \circ \sigma_i(0) = p_{i-1}$, $e \circ f_i \circ \sigma_i(1) = p_i$, $e \circ f_i \circ \sigma_i \in \Omega_{p_{i-1},p_i}$,

$$\begin{aligned} h_M(p_{i-1}, p_i) &\leq \int_0^1 \|((e \circ f_i) \cdot \overline{\sigma}_i)'(x)\| \; \mathrm{d}x \\ &= \int_0^1 \|(e \circ f_i)'(\overline{\sigma}_i(x))\| \cdot \|\dot{\overline{\sigma}}_i(x)\| \; \mathrm{d}x \\ &\leq m \cdot \int_0^1 \|\dot{\overline{\sigma}}_i(x)\| \; \mathrm{d}x = m \cdot \rho_D(0, a_i). \end{aligned}$$

Thus

$$h_M(p,q) \leq \sum_{i=1}^{k} m \cdot \rho_D(0,a_i) = m \cdot \sum_{i=1}^{k} \rho_D(0,a_i).$$

Hence

$$h_M(p,q) \leq m \cdot \inf \sum_{i=1}^{k} \rho_D(0,a_i) = m \cdot \mathrm{d}_Y(p,q).$$

Therefore $\mathrm{d}_Y(p,q) > 0$. This is a contradiction. Q.E.D.

THEOREM 1.12. (Theorem of Brody for compact complex spaces). *Let $M$ be a compact Hermitian complex space. Then the following are equivalent.*

(i) *$M$ is hyperbolic.*

(ii) *$M$ has the $D^*$-extension property.*

(iii) *$M$ contains no (nontrivial) complex lines.*

(iv) $\sup\{\|f'(0)\| : f \in \mathrm{Hol}(D,M)\} < +\infty$.

PROOF.

i $\Longrightarrow$ ii) The proposition is deduced from a Kwack's theorem (see Kwack [13]).

ii $\Longrightarrow$ iii) The proof follows immediately from Corollary 1.10.

iii $\Longrightarrow$ iv) Assume that

$$\sup\{\|f'(0)\| : f \in \mathrm{Hol}(D,M)\} = \infty.$$

Then, there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathrm{Hol}(D,M)$ such that

$$\|f_n'(0)\| = r_n \to \infty, \qquad \text{when } n \to \infty.$$

Consider a holomorphic map $g_n : D_{r_n} \to M$ defined by

$$g_n(z) = f_n\left(\frac{z}{r_n}\right), \qquad \text{for every } z \in D_{r_n}.$$

We have $\|g_n'(0)\| = 1$, for every $n \geq 1$. For every $t \in [0,1]$, put

$$s(t) = \sup_{z \in D_{r_n}} \left( \|g_n'(tz)\| \cdot \frac{r_n^2 - |z|^2}{r_n^2} \right).$$

Clearly $s(t) < \infty$, for every $t \in [0,1)$ and $s\big|_{[0,1)}$ is continuous;

$$s(1) \geq \|g_n'(0)\| = 1 \text{ and } s(0) = 0.$$

Thus there exists a number $t_0 \in (0,1]$ such that $s(t_0) = 1$. If $t_0 = 1$, we choose $\varphi_n = g_n$. If $t_0 < 1$, the supremum is actually attained at some interior point $z_0$. Let $L$ be an automorphism of $D_{r_n}$ with $L(0) = z_0$. Define

$$\varphi_n(z) = g_n(t_0 \cdot L(z)), \qquad \text{for every } z \in D_{r_n}.$$

Hence $\varphi_n \in \mathrm{Hol}(D_{r_n}, M)$ and

$$\|\varphi_n'(0)\| = 1$$

$$\|\varphi_n'(z)\| \leq \frac{r_n^2}{r_n^2 - |z|^2}, \qquad \text{for every } z \in D_{r_n/2}.$$

Therefore

$$\|\varphi_n'(z)\| \leq \frac{4}{3}, \qquad \text{for every } z \in D_{r_n/2}.$$

Hence, for every $0 < r \in \mathbb{R}$, the sequence $\{\varphi_n\}_{n=1}^{\infty}$ contains a subsequence which is equicontinuous on $D_r$. Since $M$ is compact, by Arzelà-Ascoli's theorem, there exists a subsequence of the sequence $\{\varphi_n\}_{n=1}^{\infty}$ converging on $D$ to a limit map $\varphi$. A further refinement gives a subsequence converging on $D_2$; continuing in this way allows us to extend $\varphi$ analitically to all of $\mathbb{C}$. $\varphi$ cannot be a constant map, since $\|\varphi'(0)\| = \lim_{n\to\infty} \|\varphi_n'(0)\| = 1$. The implication is proved. (Remark: on the above proof, see Brody [2], Lemma 2.1).

iv $\Longrightarrow$ i) The proof follows immediately from Proposition 1.11. Q.E.D.

## 2. - A characterization of Hartogs extension for holomorphically convex Kahler spaces

The definition of Kahler forms on complex spaces can be found in [5]. A complex space $X$ is called a Kahler space if $X$ has a Kahler form.

In [10] Ivaskowicz has given a characterization of the Hartogs extension for holomorphically convex Kahler manifolds. He has proved that a holomorphically convex Kahler manifold has the Hartogs extension property if and only if every holomorphic map $\sigma : \mathbb{C}P^1 \to X$ is constant.

The aim of this section is to generalize the result of Ivaskowicz to holomorphically convex Kahler spaces.

THEOREM 2.1. *Let $X$ be a holomorphically convex Kahler space. Then, $X$ has the Hartogs extension property if and only if every holomorphic map $\sigma : \mathbb{C}P^1 \to X$ is constant.*

PROOF. The proof is as in [10]. Let $M = \{z \in \mathcal{U} \subset \mathbb{C}^2 : \varphi(z) = 0\}$ be a strongly pseudo-convex hypersurface, where $\mathcal{U}$ is a domain in $\mathbb{C}^2$ and $\varphi$ is a $C^2$-function on $\mathcal{U}$. Put $\mathcal{U}^+ = \{z \in \mathcal{U} : \varphi(z) > 0\}$. Let $f : \mathcal{U}^+ \to X$ be a holomorphic map. We have proved that $f$ can be extended to a holomorphic map on a neighbourhood of every point belonging to $\mathcal{U}^+$. For each $n$ put

$$B_n = \{z \in \mathbb{C}^2 : \|z\| < 1/2\},$$

$\mathcal{U}_n^+ = B_n \cap \mathcal{U}^+$ and let $B = \bigcap_n f(\mathcal{U}_n^+)$. We shall prove that $B$ contains only a point.

For each $z \in \mathcal{U}^+$, let $\Delta_z^n = \{\omega : \langle \omega - z, \text{ grad } \varphi(z)\rangle = 0\} \cap \mathcal{U}_n^+$. As in [10] we have

LEMMA 2.2. *If $\mathcal{U}^+$ is sufficiently small,*

$$\sup \{\text{Vol } f(\Delta_z^n) : z \in \mathcal{U}^+/2\} < +\infty.$$

LEMMA 2.3. *For every sequence $\{\Delta_{z_j}^n\}$ ($n$ is fixed) converging to $\Delta_0^n$, we can find a subsequence $\{\Delta_{z_{j_k}}^n\}$ such that $\{f(\Delta_{z_{j_k}}^n)\}$ converges to an analytic set $A$ in $Y$ of dimension 1 with $\partial A \subset f(\partial\Delta_0^n)$.*

*Moreover the map $f|_{\Delta_0^1\setminus\{0\}}$ into $X$ can be extended to a holomorphic map on $\Delta_0^1$.*

Thus, by Lemma 2.3., $A = f(\Delta_0^1) \cup \bigcup_{j=1}^{\infty} B_j$, where $B_j$ are compact analytic sets in $X$. Observe that $\bigcup_{j=1}^{\infty} B_j$ is contained in a compact set in $X$. This implies that there exist $q \in X$ and a neighbourhood $U$ of $q$ such that $B_j \cap U \neq \emptyset$, for all $j \geq 1$. We may also assume that there exists an analytic covering map $\theta$ of $U$ into $B^k$, where $k = \dim X$ and $B^k = \{z \in \mathbb{C}^k : \|z\| < 1\}$ with $\theta(q) = 0$.

Put $\tilde{B}_j = \theta(B_j)$. By the proper mapping theorem, $\tilde{B}_j$ is an analytic subset of $B^k$ for every $j \geq 1$. By a theorem in Alexander [1], we have Vol $\tilde{B}_j \geq c \cdot \pi$, for every $j \geq 1$. This is impossible, since $\text{Vol}(\bigcup_{j=1}^{\infty} \tilde{B}_j) < +\infty$. Let $\rho$ be a metric defining the topology of $X$ and $S$ a compact complex curve in $X$. Let $\Sigma(s) = \{x_1, x_2, \dots, x_n\}$ and $\varepsilon > 0$ such that $B(x_j, \varepsilon) = \{x \in X : \rho(x, x_j) < \varepsilon\}$ are disjoint neighbourhoods of $x_j$.

Put $U_\varepsilon = \bigcup_{j=1}^{n} B_j(x_j, \varepsilon)$, $S_\varepsilon = S \backslash \overline{U}_\varepsilon$, $Y_\varepsilon = Y \backslash \overline{U}_\varepsilon$. Observe that $S_\varepsilon$ is a closed Stein manifold in $Y_\varepsilon$. Hence, by a result of Siu [17], there exists a Stein neighbourhood $W_\varepsilon$ of $S_\varepsilon$ in $Y_\varepsilon$. Let $e : W_\varepsilon \to \mathbb{C}^n$ be an embedding. As in [10] we have the following.

LEMMA 2.4. *Assume that, for every $\delta > 0$, there exists an analytic disk $f : \overline{D} \to X$ such that*

(i) *$S_\varepsilon$ is contained in the $\delta$-neighbourhood $f(\overline{D})^\delta$ of $f(\overline{D})$;*

(ii) *$\min\{\rho(x,y) : x \in f(\partial D),\ y \in S_\varepsilon\} > 2\delta$;*

(iii) *$f(\overline{D}) \cap \partial(S_\varepsilon)^\delta \subset (\partial S_\varepsilon)^\delta$.*

*Then $S \cong \mathbb{C}P^1$.*

PROOF. Let $\pi : N \to e(S_\varepsilon)$ denote the normal bundle for $e(S_\varepsilon)$ in $\mathbb{C}^n$. Then, for $\delta > 0$ sufficiently small, there exist neighbourhoods $N_\delta$ of $e(S_\varepsilon)$ in $N$ and $V_\delta$ of $e(S_\varepsilon)$ in $\mathbb{C}^n$ and a biholomorphic map $\varphi : N_\delta \to V_\delta$ such that $\varphi\big|_{e(S_\varepsilon)} = \mathrm{Id}$. Put $\tilde{\pi} = \pi \cdot \varphi^{-1} : V_\delta \to e(S_\varepsilon)$. Observe that $\tilde{\pi}^2 = \tilde{\pi}$.

From the hypothesis of the lemma, there exists an analytic disk $f : \overline{D} \to X$ such that $f(\overline{D}) \supset S_\varepsilon$, $f(\partial D) \cap e^{-1}(V_\delta) = \emptyset$ and

$$f(\overline{D}) \cap e^{-1}(\partial V_\delta) \subset e^{-1}\tilde{\pi}^{-1}(\partial e(S_\varepsilon)).$$

Then $f(D) \cap e^{-1}(V_\delta) \xrightarrow{\pi_1} S_\varepsilon$, where $\pi_1 = e^{-1} \circ \tilde{\pi} \circ e$, is proper surjective. This implies that $\pi_1 f : G = f^{-1}\left[f(D) \cap e^{-1}(V_\delta)\right] \to S_\varepsilon$ is an analytic covering map. Hence, as in the proof of Lemma 7 in [10], we have

$$H^1(S, Z) = 0, \qquad \text{i.e., } S \cong \mathbb{C}P^1.$$

From Lemma 2.4, as in [10], we have

LEMMA 2.5. *$B_j$ is a rational curve for every $j \geq 1$.*

PROOF OF THEOREM 2.1. From Lemmas 2.3 and 2.5, as in [10], $f : \mathcal{U}^+ \to X$ can be extended to a holomorphic map on a neighbourhood of $\overline{\mathcal{U}}^+$ and the theorem is proved. Q.E.D.

## 3. - $D^*$-extension and Hartogs extension

In this section, we shall prove the following

THEOREM 3.1. *Let $X$ be a holomorphically convex space having the $D^*$-extension property. Then, $X$ has the Hartogs extension property.*

PROOF. STEP 1. We shall prove that

$$\sup\{\|(e \circ f)'(0)\| : f \in \mathrm{Hol}(D, \Omega)\} < +\infty$$

for every relatively compact open subset $\Omega$ of $X$, where $e : \Omega \hookrightarrow X$ is a canonical embedding.

Indeed, assume that $\sup \{\|(e \circ f)'(0)\| : f \in \mathrm{Hol}(D, \Omega)\} = \infty$, where $\Omega$ is a relatively compact open subset of $X$ and $e : \Omega \hookrightarrow X$ is a canonical embedding. Then, there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathrm{Hol}(D, \Omega)$ such that

$$\|(e \circ f_n)'(0)\| = r_n \to \infty.$$

As in Theorem 1.12, there exists a sequence of holomorphic maps $\varphi_n : D_{r_n} \to \Omega$, $r_n \to \infty$, such that the sequence $\{e \circ \varphi_n\}_{n=1}^{\infty}$ uniformly converges, on every fixed disk, to a holomorphic map $\varphi : \mathbb{C} \to X$ and $\|(e \circ \varphi_n)'(0)\| = 1$, for all $n \geq 1$. Clearly $\varphi$ is not constant, since $\|\varphi'(0)\| = \lim_{n\to\infty} \|(e \circ \varphi_n)'(0)\| = 1$. Hence $X$ contains a nontrivial complex line. This is a contradiction. Thus $\sup \{\|(e \circ f)'(0)\| : f \in \mathrm{Hol}(D, \Omega)\} < +\infty$, for every relatively compact open subset $\Omega$ of $X$. By Proposition 1.11, $\Omega$ is hyperbolic.

STEP 2. Finally we prove that $X$ has the Hartogs extension property.

By a result of Shiffman [15], it suffices to check that $X$ satisfies the disk condition. Now assume that $\{\sigma_n\}_{n=1}^{\infty} \subset \mathrm{Hol}(D, X)$ and $\{\sigma_n\}_{n=1}^{\infty}$ converges to $\sigma$ in $H(D^*, X)$. Since $X$ is holomorphically convex, the set $\hat{K}$ is compact, where $K = \overline{\cup \sigma_n(\partial D_{1/2})}$. Let $\theta : X \to Z$ be the Remmert reduction of $X$, i.e., $\theta$ is a proper holomorphic surjection and $Z$ is a Stein space. Since $\theta \hat{K}$ is a holomorphically convex compact set, there exists a neighbourhood $\tilde{\Omega}$ of $\theta \hat{K}$ such that $\tilde{\Omega}$ is a complete $C$-space. Hence $\tilde{\Omega}$ is a complete hyperbolic space. This implies that the map $\theta : \Omega = \theta^{-1}(\tilde{\Omega}) \to \tilde{\Omega}$ can be extended to a continuous map $\hat{\theta} : \hat{\Omega} \to \tilde{\Omega}$, where $\hat{\Omega}$ is a completion of $\Omega$ for the Kobayashi metric $d_{\Omega}$ of $\Omega$.

We prove that $\hat{\Omega} = \Omega$.

Indeed, let $z \in \hat{\Omega}$ and $\{z_n\}_{n=1}^{\infty} \subset \Omega$ such that $\{z_n\}_{n=1}^{\infty} \to z$. By the compactness of the subset $\{\hat{\theta} z_n, \hat{\theta} z\}$, it follows that the subset $\{z_n\}$ is relatively compact in $\Omega$. Thus $z \in \Omega$. Therefore $\Omega$ is a complete hyperbolic space and the sequence $\{\sigma_n\}_{n=1}^{\infty}$ converges to $\sigma$ in $\mathrm{Hol}(D, \Omega)$ (see [11]). Q.E.D.

## 4. - Inverse invariance of $D^*$-extension and Hartogs extension

We have the following theorems.

THEOREM 4.1. *Let $\pi : X \to Y$ be a holomorphic map between two complex spaces satisfying the following condition.*

*For every $y \in Y$, there exists a neighbourhood $U$ of $y$ such that $\pi^{-1}(U)$ has the $D^*$-extension property (the Hartogs extension property).*

*Then, if $Y$ has the $D^*$-extension property (resp. the Hartogs extension property), so does $X$.*

PROOF. (i) Assume that $Y$ has the $D^*$-extension property. Let $\sigma : D^* \to X$

be a holomorphic map. Consider the holomorphic map $\beta = \pi\sigma : D^* \to Y$. By hypothesis, $\beta$ can be extended to a holomorphic map

$$\hat{\beta} : D \to Y.$$

Put $\hat{\beta}(0) = y_0$. By the mentioned condition, we find a neighbourhood $U$ of $y_0$ such that $\pi^{-1}(U)$ has the $D^*$-extension property. Since $\hat{\beta}$ is continuous, there exists $\varepsilon > 0$ such that $\hat{\beta}(D_\varepsilon) \subset U$. This implies that $\sigma(D_\varepsilon^*) \subset \pi^{-1}(U)$. Thus $\sigma$ can be extended to a holomorphic map $\hat{\sigma} : D \to X$.

(ii) Assume that $Y$ has the Hartogs extension property.

To prove $X$ has the Hartogs extension property, it suffices to show that the envelope to holomorphy $\Omega_f$ of every holomorphic map $f : \Omega \to X$, where $\Omega$ is a Riemann domain over a Stein manifold, is pseudoconvex.

Assume that there exists $z_0 \in \partial\Omega_f$ such that, for every neighbourhood $U$ of $z_0$ in $\Omega$, there exists a neighbourhood $V$ of $z_0$ in $U$ such that $\widehat{V \cap \Omega_f} \supset V$. By hypothesis, $g = \pi f : \Omega \to Y$ can be extended to a holomorphic map $\hat{g} : \hat{\Omega} \to Y$. Put $y_0 = \hat{g}(z_0)$. Take a neighbourhood $W$ of $y_0$ in $Y$ such that $\pi^{-1}(W)$ has the Hartogs extension property. Let $\hat{f} : \Omega_f \to X$ denote the canonical extension of $f$. Take a neighbourhood $V$ of $z_0$ in $\hat{\Omega}$ such that $\widehat{V \cap \Omega_f} \supset V$ and $\hat{g}(V) \subset W$. This yields $\hat{f}(V \cap \Omega_f) \subset \pi^{-1}(W)$. Thus $\hat{f}|_{V\cap\Omega_f}$ can be extended to a holomorphic map of $V$ into $X$. This is a contradiction. Thus the theorem is proved.

Q.E.D.

THEOREM 4.2. *Let $\theta : X \to Y$ be a proper holomorphic surjection between complex spaces, such that $\theta^{-1}(y)$ has the $D^*$-extension property (resp. has the Hartogs extension property) for all $y \in Y$.*

*Then*

(i) *If $Y$ has the $D^*$-extension property, so does $X$.*

(ii) *If $Y$ has the Hartogs extension property and $X$ is a Kahler space, then $X$ has the Hartogs extension property.*

PROOF. (i) Assume that $Y$ has the $D^*$-extension property and $\sigma : D^* \to X$ is any holomorphic map. By hypothesis, $\theta\sigma : D^* \to Y$ can be extended to a holomorphic map $\beta : D \to Y$. Put $y_0 = \beta(0)$. Take a hyperbolic neighbourhood $U$ of $y_0$ in $Y$. By hypothesis, $\theta^{-1}(U)$ contains no complex lines. This implies that $f^{-1}(V)$ is hyperbolic for every neighbourhood $V$ of $y_0$ in $U$. Take a relatively compact neighbourhood $W$ of $y_0$ in $V$, and $r > 0$ such that $\beta(D_r) \subset W$. Since $\theta$ is proper, it follows that there exists a sequence $\{z_n\}_{n=1}^\infty \subset D_r$, $\{z_n\}_{n=1}^\infty \to 0$, such that $\{\sigma(z_n)\}_{n=1}^\infty \to x_0$ in $f^{-1}(V)$ and $f^{-1}(V)$ is hyperbolic. By a Kwack's theorem [13], $\sigma$ can be extended to a holomorphic map $\hat{\sigma} : D \to X$.

(ii) Assume now that $Y$ has the Hartogs extension property.

By Theorem 4.1, it suffices to show that, for every $y \in Y$, there exists a neighbourhood $U$ of $y$ in $Y$ such that $\theta^{-1}(U)$ has the Hartogs extension property.

Let $U$ be a Stein neighbourhood of $y$ in $Y$. For every compact set $K$ in $\theta^{-1}(U)$, we have $\theta_1\hat{K} = \widehat{\theta_1 K}$, where $\theta_1 = \theta|_{\theta^{-1}(U)}$. It follows that $\theta^{-1}(U)$ is holomorphically convex. On the other hand, since $\theta^{-1}(y)$ contains no rational curves, for every $y \in U$, it follows that $\theta^{-1}(U)$ contains no rational curves. By Theorem 2.1, $\theta^{-1}(U)$ has the Hartogs extension property. Q.E.D.

## 5. - Invariance of $D^*$-extension and Hartogs extension

We have the following theorems

THEOREM 5.1. *Let $\theta : X \to Y$ be a finite proper holomorphic surjection between two complex spaces. Then*

(i) *If $H^\infty(X)$ separates points of X, then X has the $D^*$-extension property if and only if so does Y.*

(ii) *If $H(X)$ separates points of X, then X has the Hartogs extension property if and only if so does Y.*

PROOF. (i) Assume that $X$ has the $D^*$-extension property and

$$\sigma \in H(D^*, Y).$$

Consider the commutative diagram

$$\begin{array}{ccc} \sigma^*\theta & \xrightarrow{\tilde{\sigma}} & X \\ \tilde{\theta}\downarrow & & \downarrow\theta \\ D^* & \xrightarrow{\sigma} & Y. \end{array}$$

Since $D^*$ is normal and $\tilde{\theta}$ is finite proper surjective, $\tilde{\theta} : \sigma^*\theta \to D^*$ is an analytic covering map of order $n$. As in [4], we infer that $H^\infty(\sigma^*\theta)$ is an integer extension of $H^\infty(D^*)$ of order $n$ such that, for each $f \in H^\infty(\sigma^*\theta)$, there exists a polynomial $P_f \in H^\infty(D^*)[\lambda]$ of order $n_f$,

$$P_f(\lambda) = \lambda^{n_f} + a_1\lambda^{n_f-1} + \ldots + a_0,$$

$n_f \leq n$, such that $P_f(f) = 0$. Moreover

$$(*) \qquad V(P_f, \omega) = \left\{\hat{f}(\tilde{\omega}) : R\tilde{\omega} = \omega\right\}$$

for all $\omega \in SH^\infty(D^*)$, where

$$P_{f,\omega} = \lambda^{n_f} + \omega(a_1)\ \lambda^{n_f-1} + \ldots + \omega(a_0),$$

$V(P_{f,\omega}) = P_{f,\omega}^{-1}(0)$ and $R : SH^{\infty}(\sigma^*\theta) \to SH^{\infty}(D^*)$ is the restriction map. It is easy to see that $R$ is finite proper surjective. Put $Z = R^{-1}(D)$, $R_0 = R\big|_Z : Z \to D$. Observe that $D$ is an open subset of $SH^{\infty}(D) = SH^{\infty}(D^*)$ (see [9]) and $e(\sigma^*\theta) \subset Z$, where $e : \sigma^*\theta \to SH^{\infty}(\sigma^*\theta)$ denotes the canonical map. By hypothesis, $e(\sigma^*\theta)$ is open in $SH^{\infty}(\sigma^*\theta)$ and $e : \sigma^*\theta \cong e(\sigma^*\theta)$. Let $W$ be the subset of all $\omega \in D$ such that there exist neighbourhoods $G$ of $\omega$ in $D$ and $U_j$ of $\tilde{\omega}_j \in R_0^{-1}(\omega) = \{\omega_1, \ldots, \omega_p\}$ satisfying the following conditions:

$$R_0^{-1}(G) = \bigcup_{j=1}^{p} U_j \quad \text{and} \quad R_0\big|_{U_j} : U_j \cong G,$$

for every $j$, $1 \leq j \leq p$.

Obviously $W$ is open in $D$ and $R_0 : R_0^{-1}(W) \to W$ is a finite proper topological covering map. Take $\omega^0 \in D$ such that $\max \# R_0^{-1}(\omega) = \# R_0^{-1}(\omega^0)$ and $f \in H^{\infty}(\sigma^*\theta)$ such that $\hat{f}(\tilde{\omega}_j^0) \neq \hat{f}(\tilde{\omega}_i^0)$, for all $i \neq j$, where $\tilde{\omega}_j^0$, $\tilde{\omega}_i^0 \in R^{-1}(\omega^0)$. By $(*)$, we have

$$\begin{cases} \# R_0^{-1}(\omega) = \deg P_f & \text{for all } \omega \in D \backslash V(\hat{D}_f) \\ \# R_0^{-1}(\omega) < \deg P_f & \text{for all } \omega \in V(\hat{D}_f), \end{cases}$$

where $D_f$ denotes the discriminant of $P_f$. Observe $D_f \not\equiv 0$. We shall prove that $D \backslash W \subset V(\hat{D}_f)$.

Assume that there exists $\omega^0 \in D \backslash W$ such that $\hat{D}_f(\omega^0) \neq 0$. Let $R_0^{-1}(\omega^0) = \{\tilde{\omega}_1, \ldots, \tilde{\omega}_q\}$, and $U_j$ be disjoint neighbourhoods of $\tilde{\omega}_j$. Since $R_0 : Z \to D$ is proper and $\hat{D}_f(\omega^0) \neq 0$, we find a neighbourhood $G$ of $\omega^0$ such that

$$R_0^{-1}(G) \subset \bigcup_{j=1}^{q} U_j \text{ and } \hat{D}_f(\omega) \neq 0, \quad \text{for all } \omega \in G.$$

This implies that

$$R_0 : U_j \cap R_0^{-1}(G) \cong G, \quad \text{for } 1 \leq j \leq q.$$

Hence $\omega \in W$. This is a contradiction. Thus $R_0 : Z \to D$ is an analytic covering map.

By a Grauert-Remmert theorem, there exists a complex structure on $Z$ such that $H^{\infty}(\sigma^*\theta) \subset H(Z)$. On the other hand, since $\# (Z \backslash e(\sigma^*\theta)) = \# R_0^{-1}(0) < \infty$ and every 1-dimensional complex normal space is non-singular, it follows that $\tilde{\sigma} : \sigma^*\theta \to X$ can be extended to a holomorphic map $\hat{\tilde{\sigma}} : Z \to X$. Put $\pi = R_0 \times \theta : Z \times X \to D \times Y$. Obviously $\pi$ is finite proper surjective. By the direct image theorem of Grauert [6], it follows that $\pi(\Gamma_{\hat{\tilde{\sigma}}})$ is an analytic set in $D \times Y$, where $\Gamma_{\hat{\tilde{\sigma}}}$ denotes the graph of $\hat{\tilde{\sigma}}$.

Clearly $\pi(\Gamma_{\hat{\tilde{\sigma}}}) \cap (D^* \times Y) = \Gamma_{\sigma}$ and the canonical projection $p : \Gamma_{\hat{\tilde{\sigma}}} \to D$ is proper. This implies that the map $\sigma : D^* \to Y$ can be extended to a meromorphic map $\hat{\sigma} : D \to Y$. Since codim $P(\hat{\sigma}) \geq 2$ (see [15]), where $P(\hat{\sigma})$ denotes the indeterminacy locus of $\hat{\sigma}$, it follows that $\hat{\sigma}$ is holomorphic and it is an extension of $\sigma$.

Conversely, assume that $Y$ has the $D^*$-extension property. Since $\theta$ is finite for every $y \in Y$, $\theta^{-1}(y)$ has the $D^*$-extension property. Hence, by Theorem 4.2, $X$ has the $D^*$-extension property and (i) is proved.

(ii) Assume now that $X$ has the Hartogs extension property. To prove that $Y$ has also the Hartogs extension property, it suffices to show that every holomorphic map $g : H_k(r) \to Y$, where $H_k(r)$ is the Hartogs diagram given by

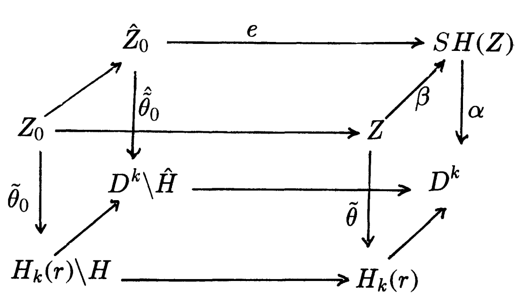
$$\begin{aligned} H_k(r) &= \left\{(z_1, \dots, z_k) \in D^k : \max \left\{|z_1|, \dots, |z_{k-1}|\right\} < r\right\} \\ &\cup \left\{(z_1, \dots, z_k) \in D^k : |z_k| > 1 - r\right\}, \end{aligned}$$

can be extended to a holomorphic map $\hat{g} : \widehat{H_k(r)} = D^k \to Y$.

Indeed, consider the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{g}} & X \\ {\scriptstyle \tilde{\theta}}\downarrow & & \downarrow{\scriptstyle \theta} \\ H_k(r) & \xrightarrow{g} & Y \end{array}$$

where $Z = g^*\theta = (H_k(r) \times_Y X)_{\text{red}}$. Then $\tilde{\theta} : Z \to H_k(r)$ is an analytic covering map. We may assume that $Z$ is normal. Then, the branch locus of $\tilde{\theta}$, $H$, has codimension 1. By a result of Dloussky [3], there exists a hypersurface $\hat{H}$ in $D^k$ such that $\widehat{H_k(r)\backslash H} = D^k \backslash \hat{H}$. Obviously $\hat{H} \cap H_k(r) \subset H$. Since $H$ can be written in the form $H = \left(\hat{H} \cap H_k(r)\right) \cup H'$, where $H'$ is a hypersurface in $H_k(r)$ such that $\widehat{H_k(r)\backslash H'} = \widehat{H_k(r)}$, without loss of generality, we may assume that $\hat{H} \cap H_k(r) = H$. Consider the commutative diagram




where $\alpha$ is induced by $\tilde{\theta}$ and $e$ by the canonical embedding $Z_0 \to Z$. As in (i), we may prove that $\alpha : SH(Z) \to D^k$ is an analytic covering map.

Without loss of generality, we may assume that $SH(Z)$ is irreducible normal. By hypothesis, it follows that $e$ and $\beta$ are open embeddings. This implies that $e(\hat{Z}_0) = SH(Z)\backslash\alpha^{-1}(\hat{H})$. Thus $\tilde{g} : Z \to X$ can be extended to a holomorphic map

$$\hat{\tilde{g}} : \left[SH(Z)\backslash\alpha^{-1}(\hat{H})\right] \cup Z \to X.$$

Consider the holomorphic map

$$\begin{aligned}\beta\hat{\tilde{g}} : \theta^{-1}\left[SH(Z)\backslash\alpha^{-1}(\hat{H})\right] \cup \theta^{-1}(Z) \\ = \left[W\backslash\beta^{-1}\alpha^{-1}(\hat{H})\right] \cup \theta^{-1}(Z) \to X,\end{aligned}$$

where $\beta : W \to SH(Z)$ is the resolution of singularities of $SH(Z)$. Since $X$ has the Hartogs extension property and $\theta^{-1}(Z)$ meets every irreducible branch of $\beta^{-1}\alpha^{-1}(\hat{H})$, $\beta\hat{\tilde{g}}$ can be extended to a holomorphic map $f : W \to X$. By hypothesis, $H(X)$ separates points of $X$, it follows that $f$ can be factorized through $\beta$. Thus $\beta\hat{\tilde{g}}$ can be extended to a holomorphic map $\tilde{f} : SH(Z) \to X$.

As in (i), $\tilde{f}$ induces a meromorphic map $\hat{g} : D^k \to Y$ which is an extension of $g$. Consider the commutative diagram

$$\begin{array}{ccccc} SH(Z) & \longleftarrow & \Gamma_{\tilde{f}} & \longrightarrow & X \\ \hat{\hat{\theta}}\big\downarrow & & \big\downarrow \hat{\hat{\theta}}\times\theta & & \big\downarrow \theta \\ D^k & \longleftarrow & \Gamma_{\hat{g}} & \longrightarrow & Y \end{array}$$

in which $\hat{\hat{\theta}}, \theta$ are finite. Thus the canonical projection $\Gamma_{\hat{g}} \to D^k$ is finite. By the meromorphicity of $\hat{g}$, it follows that $\hat{g}$ is holomorphic. Conversely, assume that $Y$ has the Hartogs extension property. For each $y \in Y$, we find a Stein neighbourhood $V$ of $y_0$. Then $\theta^{-1}(V)$ is Stein (see [4]). Hence $\theta^{-1}(V)$ has the Hartogs extension property. By Theorem 4.1, $X$ has the Hartogs extension property. Thus (ii) is proved. Q.E.D.

PROPOSITION 5.2. *Every affine algbraic variety of the type*

$$V = \left\{(z_1, z_2) \in \mathbb{C}^2 : z_1^n + a_1(z_2) \cdot z_1^{n-1} + \ldots + a_n(z_2) = 0\right\},$$

*where $a_1, \ldots, a_n$ are holomorphic functions on $\mathbb{C}$ with $a_n \not\equiv 0$, has not the $D^*$-extension property.*

PROOF. Assume that $V$ has the $D^*$-extension property. Take $r > 0$ and $z_1^0 \neq 0$ such that

$$(*)\begin{cases} a_n(z_2) \neq 0, & \text{for every } |z_2| > r, \\ (z_1^0)^n + a_1(z_2)\cdot(z_1^0)^{n-1} + \ldots + a_n(z_2) \neq 0, & \text{for every } |z_2| > r. \end{cases}$$

Put $V_1 = \{(z_1, z_2) \in V : |z_2| > r\}$. Then

$$\begin{aligned} \theta : V_1 &\to \{|z_2| > r\} \\ (z_1, z_2) &\mapsto z_2 \end{aligned}$$

is proper. By (∗), we have $\pi(V_1) \subset \mathbb{C}\backslash\{0, z_1^0\}$, where

$$\begin{aligned} \pi : \mathbb{C}^2 &\to \mathbb{C} \\ (z_1, z_2) &\mapsto z_1 \end{aligned}$$

is the canonical projection. Since $H^\infty(\mathbb{C}\backslash\{0, z_1^0\})$ and $H^\infty(\{|z_2| > r\})$ separates the points of $\mathbb{C}\backslash\{0, z_1^0\}$ and of $\{|z_2| > r\}$ respectively, it follows that $H^\infty(V_1)$ separates the points of $V_1$.

Let $\sigma : D^* \to V_1$ be a holomorphic map. Then the map $\sigma$ can be extended to a holomorphic map $\hat{\sigma} : D \to V$. Since

$$0 < \left|\frac{1}{\theta\sigma(\lambda)}\right| < \frac{1}{r}, \qquad \text{for any } \lambda \in D^*$$

we have

$$\left|\frac{1}{\theta\hat{\sigma}(\lambda)}\right| < \frac{1}{r}.$$

Therefore, $V_1$ has the $D^*$-extension property. Hence the complex subspace $\{0 < |z_2| < 1/r\}$ of $\mathbb{C}^2$ also has the $D^*$-extension property. This is a contradiction. Q.E.D.

THEOREM 5.3. *Let $\theta$ be a finite proper surjective map of a non-compact complex space $X$ into a holomorphically convex Kahler complex space $Y$. Let* $\dim X \leq 2$. *Then, $X$ has the Hartogs extension property if and only if so does $Y$.*

PROOF. Assume that $X$ has the Hartogs extension property. To prove that $Y$ has the Hartogs extension property, by Theorem 2.1, it sufficies to show that $Y$ contains no rational curves.

Let $\sigma : \mathbb{C}P^1 \to Y$ be a holomorphic map and let $\sigma \not\equiv$ constant. Consider the commutative diagram

$$\begin{array}{ccccc} \eta^*\tilde{\theta} & \xrightarrow{\tilde{\eta}} & \sigma^*\theta & \xrightarrow{\tilde{\sigma}} & X \\ \tilde{\tilde{\theta}}\big\downarrow & & \big\downarrow\tilde{\theta} & & \big\downarrow\theta \\ \mathbb{C}^2\backslash\{0\} & \xrightarrow{\eta} & \mathbb{C}P^1 & \xrightarrow{\sigma} & Y. \end{array}$$

Obviously $\tilde{\tilde{\theta}}$ is a finite proper surjective map. By the normality of $\mathbb{C}^2\backslash\{0\}$,

$\tilde{\tilde{\theta}} : \eta^*\tilde{\theta} \to \mathbb{C}^2\backslash\{0\}$ is an analytic cover. This implies that the branch locus $\Sigma(\tilde{\tilde{\theta}})$ has dimension 1. By the extension theorem of analytic sets (see [7]), we have $\overline{\Sigma(\tilde{\tilde{\theta}})} = \Sigma(\tilde{\tilde{\theta}}) \cup \{0\}$ is an analytic subset in $\mathbb{C}^2$. Thus, from a result of Stein (see [18]), we have a commutative diagram

$$\begin{array}{ccc} \eta^*\tilde{\theta} & \xrightarrow{\ \tilde{e}\ } & \widetilde{\eta^*\tilde{\theta}} \\ {\scriptstyle \tilde{\tilde{\theta}}}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\tilde{\tilde{\theta}}}} \\ \mathbb{C}^2\backslash\{0\} & \xrightarrow{\ e\ } & \mathbb{C}^2 \end{array}$$

in which $\tilde{\tilde{\tilde{\theta}}} : \widetilde{\eta^*\tilde{\theta}} \to \mathbb{C}^2$ is an analytic cover and $\tilde{e}$ is an open embedding. Since $Y$ is holomorphically convex, $X$ is also holomorphically convex. Let $\gamma : X \to Z$ be the Remmert reduction of $X$. Since $\#\,(\widetilde{\eta^*\tilde{\theta}}\backslash \mathrm{Im}\ \tilde{e}) = \#\ \tilde{\tilde{\tilde{\theta}}}^{-1}(0) < \infty$, it follows that

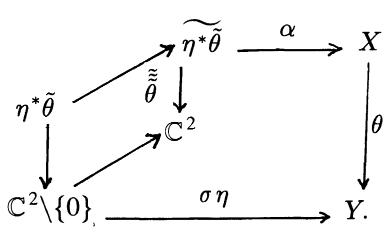
$$\gamma\tilde{\sigma}\tilde{\eta} : \eta^*\tilde{\theta} \to Z$$

can be extended to a holomorphic map

$$\widetilde{\gamma\tilde{\sigma}\tilde{\eta}} : \widetilde{\eta^*\tilde{\theta}} \to Z.$$

Consider the analytic subset $(\mathrm{id} \times \gamma)^{-1}\Gamma_\beta$, where $\beta = \widetilde{\gamma\tilde{\sigma}\tilde{\eta}}$ and $\Gamma_\beta$ denotes the graph of $\beta$. Since $X$ is not compact of dimension $\leq 2$, it follows that $\dim(0 \times \gamma^{-1}\beta(0)) < 2 = \dim\ \Gamma_{\tilde{\sigma}\tilde{\eta}}$. Thus $\overline{\Gamma_{\tilde{\sigma}\tilde{\eta}}}$ is an analytic subset of $\widetilde{\eta^*\tilde{\theta}} \times X$. Since $\gamma$ is proper and $\beta$ is holomorphic, it follows that $\overline{\Gamma_{\tilde{\sigma}\tilde{\eta}}}$ defines a meromorphic map $\alpha : \widetilde{\eta^*\tilde{\theta}} \to X$.

On the other hand, since $X$ has the Hartogs extension property and $P(\alpha)$, where $P(\alpha)$ denotes the indeterminacy locus of $\alpha$, has codimension $\geq 2$, it follows that $\alpha$ is holomorphic. Thus we have the commutative diagram




Hence $\alpha$ induces a meromorphic map $\tilde{\alpha} : \mathbb{C}^2 \to Y$ such that the canonical projection $\pi : \Gamma_{\tilde{\alpha}} \to \mathbb{C}^2$ is finite proper surjective. Since $\pi\left[\Gamma_{\tilde{\alpha}}\backslash\pi^{-1}(0)\right] \cong \mathbb{C}^2\backslash\{0\}$, by the normality of $\mathbb{C}^2$, it follows that $\pi(\Gamma_{\tilde{\alpha}}) \cong \mathbb{C}^2$ and hence $\tilde{\alpha}$ is holomorphic. Thus we have

$$\infty > \#\ (\sigma^{-1}(\tilde{\alpha}(0)) = \#\ \{\lim_{x\to 0} \eta x\} = \infty.$$

Therefore $\sigma$ is a constant map. This is a contradiction.

Assume now that $Y$ has the Hartogs extension property. To prove the Hartogs extension of $X$, by Theorem 4.1, it suffices to show that for every $y \in Y$ there exists a neighbourhood $U$ of $y$ such that $\theta^{-1}(U)$ has the Hartogs extension property.

Let $U$ be a Stein neighbourhood of $y$. Since the Stein property is invariant under finite proper surjections, it follows that $\theta^{-1}(U)$ is Stein. This yields $\theta^{-1}(U)$ has the Hartogs extension property. Q.E.D.

THEOREM 5.4. *Let $\theta : X \to Y$ be a holomorphic map between two complex spaces satisfying the following condition*:

*For every $y \in Y$, there exists a neighbourhood $V$ of $y$ such that the restriction of $\theta$ on every* (*topological*) *connected component $U_j$ of $\theta^{-1}(V)$*

$$\theta\big|_{U_j} : U_j \to V$$

*is a holomorphic bijection.*

*Then*

(i) *$X$ has the Hartogs extension property if and only if so does $Y$.*

(ii) *If $Y$ is compact, then $X$ has the $D^*$-extension property if and only if so does $Y$.*

PROOF. (i) Assume that $X$ has the Hartogs extension property. To prove that $Y$ has the Hartogs extension property, it suffices to show that every holomorphic map $g : H_k(r) \to Y$ can be extended to a holomorphic map $\hat{g} : D^k \to Y$.

Consider the commutative diagram

$$\begin{array}{ccc} g^*\theta & \xrightarrow{\tilde{g}} & X \\ \tilde{\theta}\big\downarrow & & \big\downarrow\theta \\ H_k(r) & \xrightarrow{g} & Y. \end{array}$$

By hypothesis and by openess of $\tilde{\theta}$ (see [4]), it follows that $\tilde{\theta}$ is an unbranch covering map. This implies that there exists a holomorphic map $f : H_k(r) \to X$ such that $\theta \circ f = g$. Clearly $f$ can be extended to a holomorphic map $\hat{f} : D^k \to X$. Thus $\hat{g} = \theta \circ \hat{f}$ is an extension of $g$.

Conversely, assume that $Y$ has the Hartogs extension property. By hypothesis, for every $y_0 \in Y$, there exists a Stein neighbourhood $V$ of $y_0$ such that the restriction of $\theta$ on every connected component of $\theta^{-1}(V)$

$$\theta\big|_{U_j} : U_j \to V$$

is a holomorphic bijection. By Theorem 4.1, it remains to show that $U_j$ has the Hartogs extension property. Given $f : H_k(r) \to U_j$, a holomorphic map, by the Stein property of $V$, $\theta \circ f$ can be extended to a holomorphic map

$$g = \widehat{\theta \circ f} : D^k \to V.$$

By considering the commutative diagram

$$\begin{array}{ccc} g^*\theta & \xrightarrow{\tilde{g}} & U_j \\ \tilde{\theta}\big\downarrow & & \big\downarrow\theta \\ D^k & \xrightarrow{g} & V \end{array}$$

it follows that $(\theta|_{U_j})^{-1} \circ g = \tilde{g} \circ \tilde{\theta}^{-1}$ is a holomorphic extension of $f$. Hence (i) is proved.

(ii) As in (i) and by Theorem 4.1, it suffices to show that $Y$ has the $D^*$-extension property when $X$ has the $D^*$-extension property.

Since $X$ has the $D^*$-extension property, it follows that $X$ contains no complex lines. This implies that $Y$ contains no complex lines. By the compactness of $Y$ and by Theorem 2.1, we infer that $Y$ has the $D^*$-extension property. Q.E.D.

## 6. - $D^*$-extension and Hartogs extension for 1-convex spaces

We recall that a complex space is called 1-convex if there exists a proper holomorphic map $\theta$ of $X$ onto a Stein space $Z$ such that $\theta\left[X\backslash\theta^{-1}(A)\right] \cong Z\backslash A$ for some finite subset $A$ of $Z$. Put $\mathrm{Ext}(X) = \theta^{-1}(A)$. First we prove the following

THEOREM 6.1. *Let $X$ be a 1-convex space. Then $X$ is Stein if and only if* $\mathrm{Ext}(X)$ *is analytically rare and $X$ has the Hartogs extension property.*

PROOF. It suffices to prove the sufficiency of the theorem. Assume that $\theta : X \to Z$ is proper surjective such that $\theta\left[X\backslash \mathrm{Ext}(X)\right] \cong Z\backslash A$, where $Z$ is a Stein space and $A$ is a finite subset of $Z$. Since $\mathrm{Ext}(X) = \theta^{-1}(\theta(\mathrm{Ext}(X))$ is analytically rare and $\theta\left[X\backslash \mathrm{Ext}(X)\right] \cong Z\backslash\theta(\mathrm{Ext}(X))$, it follows that $\theta^{-1} : Z \to X$ is a meromorphic map. By a Hironaka's theorem (see [8]), there exists a finite sequence of monoidal transformations $\sigma_i : Q_i \to Q_{i-1}$, where $Q_0$ is a relatively compact open subset of $Z$, such that $Q_0 \cap P(\theta^{-1}) \neq \emptyset$, when $P(\theta^{-1}) \neq \emptyset$, with non-singular center $C_{i-1} \subset Q_{i-1}$, such that $h = \theta^{-1} \circ \sigma_1 \circ \ldots \circ \sigma_m : Q_m \to X$ is holomorphic. Take such a $\{\sigma_i\}_{i=1}^m$ where $m$ is minimal. (Since $Q_0 \cap P(\theta^{-1}) \neq \emptyset$, $m > 0$). For each $y \in C_{m-1}$, $\sigma_m^{-1}(y) \cong \mathbb{C}P^k$, for some $k > 0$. Since $X$ contains no rational curves, for every complex line $l \subset \sigma_m^{-1}(y)$, we have $h(l) = $ constant. Thus $h\big|_{\sigma_m^{-1}(y)} = $ const. for every $y \in C_{m-1}$. Hence $\theta^{-1} \circ \sigma_1 \circ \ldots \circ \sigma_{m-1} : Q_{m-1} \to X$ is holomorphic, contradicting the minimality of $m$. Hence $P(\theta^{-1}) = \emptyset$ and $\theta^{-1}$ is holomorphic. Thus $X \cong Z$ is a Stein space.

Q.E.D.

THEOREM 6.2. *Let $X$ be a 1-convex space and $\theta : X \to Z$ be the holomorphic map as in Theorem* 6.1. *Then*

(i) *$X$ has the $D^*$-extension property if and only if* $\mathrm{Ext}(X)$ *and $Z$ have the $D^*$-extension property;*

(ii) *$X$ has the Hartogs extension property if and only if so does* Ext($X$).

PROOF. (i) If Ext($X$) and $Z$ have the $D^*$-extension property, by Theorem 4.2, so does $X$.

Assume now that $X$ has the $D^*$-extension property and let $\sigma : D^* \to Z$ be a non-constant holomorphic map. Without loss of generality, we may suppose that $A$ consists of a single point, namely $A = \{z_0\}$. If $\sigma(D^*) \not\ni z_0$, then the map $(\theta|_{Z\setminus\{z_0\}})^{-1} \circ \sigma : D^* \to X$ can be extended to a holomorphic map of $D$ into $X$. Hence the map $\sigma$ can be extended to a holomorphic map of $D$ into $Z$. If $z_0 \in \sigma(D^*)$, then $\sigma^{-1}(z_0)$ is a discrete subset of $D^*$. By $D^*$-extension of $X$, the map

$$(\theta|_{Z\setminus\{z_0\}})^{-1} \circ (\sigma|_{D^*\setminus\sigma^{-1}(z_0)}) : D^*\setminus\sigma^{-1}(z_0) \to X$$

can be extended to a holomorphic map of $D$ into $X$. Thus the map $\sigma$ can be extended to a holomorphic map from $D$ into $Z$. Hence $Z$ has the $D^*$-extension property. Obviously Ext($X$) has the $D^*$-extension property. Thus (i) is proved.

(ii) We write $X = X' \cup \text{Ext}(X)$, where $X'$ is a subspace of $X$ such that Ext($X'$) is analytically rare in $X'$. By Theorem 6.1, $X'$ is a Stein space. Assume now that $f : \Omega \to X$ is a holomorphic map, where $\Omega$ is a Riemann domain over a Stein manifold. If Int $f^{-1}(\text{Ext}(X)) = \emptyset$, then $f(\Omega) \subset X'$ and hence $f$ can be extended to a holomorphic map $\hat{f} : \hat{\Omega} \to X'$. Let Int $f^{-1}(\text{Ext}(X)) \neq \emptyset$, then $f(\Omega) \subset \text{Ext}(X)$. By hypothesis, $f$ can be extended to a holomorphic map

$$\hat{f} : \hat{\Omega} \to X.$$

Q.E.D.

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