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Highly Degenerate Quasilinear Parabolic Systems

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Introduction

In interesting recent investigations concerning diffusion and viscoelastic relaxation in polymers, D.S. Cohen introduced a class of highly degenerate and nonstandard reaction-diffusion models. Numerical calculations and formal asymptotic methods, which he and his collaborators carried out, show that these models exhibit phenomena which have been observed experimentally in controlled release technology of the pharmaceutical industry.

Many of the problems, which have been considered by Cohen and his coworkers, are particular cases of the following general system:

$$\left.\begin{aligned} &c_t + [j(c,\sigma)]_x = 0 && (1)\\ &\sigma_t - \varphi(c)c_t = \rho(c)c - \beta(c)\sigma && (2)\end{aligned}\right\} \quad \text{in } \Omega(0,\infty),$$

where Ω is a bounded open interval in $\mathbb{R}$ and

$$j(c,\sigma) := -D(c)c_x - E(c)\sigma_x + M(c,\sigma)c. \tag{3}$$

These equations are complemented by the Dirichlet boundary condition:

$$c = \psi_0 \qquad \text{on } \partial\Omega \times (0,\infty) \tag{4}$$

and the initial condition:

$$c(\,\cdot\,,0) = c_0, \quad \sigma(\,\cdot\,,0) = \sigma_0 \qquad \text{on } \Omega \tag{5}$$

Other boundary conditions, in particular nonlinear Neumann type conditions, are of interest too.

In the above system, which basically describes the diffusion of a penetrant on a polymer entanglement network, c is the density of the penetrant and σ the

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stress induced by the penetrating molecules. The 'diffusion coefficient' $D(\cdot)$ is a strictly positive, and E, φ, ρ, and β are nonnegative smooth functions on $\mathbb{R}$. The function $\psi_0 : \partial\Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is smooth too.

Problem (1)–(5) has been investigated—numerically and by formal asymptotic methods—in [14], [15] under the assumption that D, E, and ρ are constant, and M and φ are identically zero. In [8] we have shown that this particular problem is well posed and that it possesses a global solution if β is uniformly bounded.

The full problem (1)–(5) has been analyzed—again numerically and by formal techniques—by Cohen and Cox [13] and by Cox [17] (even in the case of a moving boundary).

In this paper we shall show that problem (1)–(5)—in fact, much more general problems of this type—is well posed in the sense that it possesses a unique maximal solution, provided suitable compatibility conditions are satisfied.

Since c represents a density, the above problem should, of course, have the property that $c(\cdot, t) \geq 0$, if this is true for $t = 0$ and on the boundary. Numerical computations show that this 'positivity preserving property' does not hold if $E(0) > 0$ (cf. corresponding remarks in [13], [16]). However, it has been conjectured by Cohen and Cox that (1)–(5) possesses the desired 'positivity preserving property' if $E(0) = 0$. Below we shall show—again in much greater generality—that this conjecture is true.

Our approach to the above problems consists in showing that it can be reduced to be a concrete realization of an abstract system of the form

$$\begin{aligned} \dot{u}_1 + A_1(u)u_1 + \mathcal{A}_2(u)u_2 &= f_1(u), \\ \dot{u}_2 &= f_2(u), \end{aligned} \qquad t > 0, \tag{6}$$

where $u := (u_1, u_2)$, and $A_1(u)$ and $\mathcal{A}_2(u)$ are linear operators on suitable Banach spaces. The basic property of this system is the fact that $A_1(u)$ is the negative generator of an analytic semigroup. This is the justification for calling (6) a 'highly degenerate quasilinear parabolic problem'. Thus the main results of this paper, which are of independent interest, concern abstract highly degenerate quasilinear parabolic systems.

Below, in Section 1, we collect some results about generators of analytic semigroups. Although these results are more or less known, it is hoped that our presentation contains aspects which are new even for specialists. In Section 2 we study a class of 'matrix generators' on product spaces. Section 3 contains the basic existence theorem for system (6). In Section 4 we collect some facts about elliptic systems. Section 5 contains a general existence result for highly degenerate quasilinear parabolic systems under 'constant' boundary conditions (e.g., Dirichlet boundary conditions). In Section 6 we discuss the much more complicated case of nonlinear boundary conditions, establishing the existence of weak solutions. In the last section we apply our general result to show that problem (1)–(5) is well posed and that it possesses the positivity preserving property mentioned above.

1. - Generators of Analytic Semigroups

Let E and F be Banach spaces over $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$. Then $\mathcal{L}(E,F)$ is the Banach space of all bounded linear operators from E into F, and $\mathcal{L}(E) := \mathcal{L}(E,E)$. We denote by $\mathcal{L}is(E,F)$ the set of all isomorphisms in $\mathcal{L}(E,F)$. We write $E \hookrightarrow F$ if E is continuously injected in F, and $E \overset{d}{\hookrightarrow} F$ means that E is also dense in F.

We use the convention that all formulas, in which complex numbers occur explicitly, refer to the corresponding complexifications of spaces and operators if $\mathbb{K} = \mathbb{R}$. Finally, we denote by $\overline{\mathbf{B}}_E(x,r)$ the closed ball in E with center at x and radius r, by $\rho(A)$ the resolvent set of a linear operator A in E, and by $\sigma(A) := \mathbb{C}\backslash\rho(A)$ its spectrum.

Let $E_j := \big(E_j, \|\cdot\|_j\big)$, $j = 0,1$, be $\mathbb{K}$-Banach spaces with $E_1 \hookrightarrow E_0$. Given $\kappa \geq 1$ and $\omega > 0$, we denote by

$$\mathcal{H}(E_1, E_0, \kappa, \omega)$$

the set of all $A \in \mathcal{L}(E_1, E_0)$ such that

$$\kappa^{-1} \leq \frac{\|(\lambda + A)x\|_0}{|\lambda|\,\|x\|_0 + \|x\|_1} \leq \kappa, \quad x \in E_1\backslash\{0\}, \quad \lambda \in [\operatorname{Re} z \geq \omega], \tag{1.1}$$

and such that $\lambda_0 + A \in \mathcal{L}is(E_1, E_0)$ for some

$$\lambda_0 \in [\operatorname{Re} z \geq \omega] := \{z \in \mathbb{C};\ \operatorname{Re} z \geq \omega\},$$

where $\lambda + A := \lambda i + A$ and $i : E_1 \hookrightarrow E_0$ is the injection. Moreover,

$$\mathcal{H}(E_1, E_0) := \bigcup_{\substack{\kappa\geq 1\\ \omega>0}} \mathcal{H}(E_1, E_0, \kappa, \omega).$$

In the following proposition we collect some of the basic properties of these sets.

PROPOSITION 1.1.

(i) $A \in \mathcal{H}(E_1, E_0, \kappa, \omega)$ *iff* $A \in \mathcal{L}(E_1, E_0)$, (1.1) *is satisfied, and*

$$\lambda + A \in \mathcal{L}is(E_1, E_0) \ \textit{for}\ \lambda \in [\operatorname{Re} z \geq \omega].$$

(ii) $\mathcal{H}(E_1, E_0)$ *is open in* $\mathcal{L}(E_1, E_0)$. *More precisely, given* $A \in \mathcal{H}(E_1, E_0, \kappa, \omega)$ *and* $r \in (0, 1/\kappa)$,

$$\overline{\mathbf{B}}_{\mathcal{L}(E_1,E_0)}(A, r) \subset \mathcal{H}\big(E_1, E_0, \kappa/(1 - r\kappa), \omega\big).$$

(iii) *If* $A \in \mathcal{H}(E_1, E_0, \kappa, \omega)$, $0 < \alpha < 1/\kappa$, *and* $\beta \geq 0$, *and if* $B \in \mathcal{L}(E_1, E_0)$ *satisfies*

$$\|Bx\|_0 \leq \alpha\|x\|_1 + \beta\|x\|_0, \quad x \in E_1,$$

then $A + B \in \mathcal{H}\big(E_1, E_0, \kappa/(1-\alpha\kappa), \omega \vee (\beta/\alpha)\big)$.

(iv) *If* $A \in \mathcal{L}(E)$ *then* $A \in \mathcal{H}(E, E, \kappa, \omega)$, *where* $\omega > \|A\|$ *and*

$$\kappa := (1 + \|A\|) \vee ((1+\omega)/(\omega - \|A\|)).$$

PROOF. The assertions follow by simple calculations based on (1.1), the fact that $\mathcal{L}is(E_1, E_0)$ is open in $\mathcal{L}(E_1, E_0)$, and a well known stability theorem for bounded invertibility (e.g., [23, Theorem 5.22]). Details can be left to the reader. ■

Given a linear operator $A : \mathrm{dom}(A) \subset E \to E$, we denote by

$$\|\cdot\|_A := \|\cdot\| + \|A\cdot\|$$

its graph norm, and $D(A) := (\mathrm{dom}(A), \|\cdot\|_A)$. Recall that $D(A)$ is a Banach space iff $A \in \mathcal{C}(E)$, where $\mathcal{C}(E)$ is the set of all closed linear operators in E.

If $A : E_1 \to E_0$ is a linear operator, we can interpret it as a linear operator, A_0, in E_0 with domain E_1. If no confusion seems possible, we write again A for A_0. Of course, $\lambda + A$ means now $1_{E_0} + A_0$, where $1_{E_0} := 1$ is the identity in $\mathcal{L}(E_0)$.

LEMMA 1.2. $\mathcal{H}(E_1, E_0) \subset \mathcal{C}(E_0)$.

PROOF. Since $E_1 \hookrightarrow E_0$, it follows that, given any $A \in \mathcal{H}(E_1, E_0)$, there exists $\omega > 0$ such that $(\omega + A)^{-1} \in \mathcal{L}(E_0) \subset \mathcal{C}(E_0)$. Now the assertion follows from standard results about closed linear operators. ■

LEMMA 1.3. *Let* $A : E_1 \to E_0$ *be linear. Then* $A \in \mathcal{L}(E_1, E_0) \cap \mathcal{C}(E_0)$ *iff* $\|\cdot\|_1$ *and* $\|\cdot\|_A$ *are equivalent.*

PROOF. "$\Rightarrow$": Since $\|x\|_A \le (\|i\| + \|A\|)\|x\|_1$, it follows that $E_1 \hookrightarrow D(A)$. Since $D(A)$ is complete, the assertion follows from the open mapping theorem.

"$\Leftarrow$": Since there exists a constant α such that $\|Ax\| \le \|x\|_A \le \alpha\|x\|_1$ for $x \in E_1$, we see that $A \in \mathcal{L}(E_1, E_0)$. Since $E_1 = D(A)$ except for equivalent norms, $D(A)$ is complete. Hence $A \in \mathcal{C}(E_0)$. ■

THEOREM 1.4.

(i) *If* $A \in \mathcal{H}(E_1, E_0)$ *then* $-A$ *generates an analytic semigroup in* $\mathcal{L}(E_0)$.

(ii) *Let* $E_1 \overset{d}{\hookrightarrow} E_0$. *Then* $A \in \mathcal{H}(E_1, E_0)$ *iff* $A \in \mathcal{L}(E_1, E_0)$ *and* $-A$ *generates a strongly continuous analytic semigroup in* $\mathcal{L}(E_0)$.

PROOF. (i) Suppose that $A \in \mathcal{H}(E_1, E_0, \kappa, \omega)$. Then $\rho(-A) \supset [\mathrm{Re}\, z \ge \omega]$, and (1.1) implies

$$\|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(E_0)} \le \kappa, \quad \mathrm{Re}\,\lambda \ge \omega. \tag{1.2}$$

From this we deduce by standard arguments (e.g., [24]) the existence of constants $M := M(\kappa, \omega) > 0$ and $\vartheta := \vartheta(\kappa, \omega) \in (\pi/2, \pi)$ such that

$$\rho(-A) \supset [\,|\arg(z-\omega)| \leq \vartheta] \cup \{\omega\} \tag{1.3}$$

and

$$\|(\lambda-\omega)(\lambda+A)^{-1}\|_{\mathcal{L}(E_0)} \leq M, \quad \lambda \in [\,|\arg(z-\omega)| \leq \vartheta]. \tag{1.4}$$

Now the assertion follows from a result of Sinestrari [29], which extends the well known classical case of operators with dense domains to the case where the domain is not necessarily dense.

(ii) If $A \in \mathcal{H}(E_1, E_0)$, the semigroup generated by $-A$ is strongly continuous, thanks to the fact that E_1 is dense in E_0.

Suppose that $A \in \mathcal{L}(E_1, E_0)$ and that $-A$ generates a strongly continuous analytic semigroup on E_0. Then it is well known that $A \in \mathcal{C}(E_0)$ and that there exist constants $M > 0$ and $\vartheta \in (\pi/2, \pi)$ such that (1.3) and (1.4) are true. Since Lemma 1.3 guarantees the equivalence of $\|\cdot\|_1$ and $\|\cdot\|_A$, it follows that $\lambda + A \in \mathcal{L}is(E_1, E_0)$ for $\lambda \in [\operatorname{Re} z \geq \omega]$.

Observe that (1.4) implies

$$\|(\lambda+A)^{-1}\|_{\mathcal{L}(E_0)} \leq \frac{M|\lambda|}{|\lambda-\omega|}\frac{1}{|\lambda|} \leq \kappa_0/|\lambda|, \quad \operatorname{Re}\lambda \geq \omega+1,$$

where $\kappa_0 := \kappa_0(M, \omega) \geq 1$. Thus

$$|\lambda|\,\|x\|_0 \leq \kappa_0\|(\lambda+A)x\|_0, \quad x \in E_1, \quad \operatorname{Re}\lambda \geq \omega+1, \tag{1.5}$$

and, consequently,

$$\begin{aligned}\|x\|_1 &\leq \|(\omega+A)^{-1}\|_{\mathcal{L}(E_0,E_1)}(\|(\lambda+A)x\|_0 + |\omega-\lambda|\,\|x\|_0) \\ &\leq 3\kappa_0\|(\omega+A)^{-1}\|_{\mathcal{L}(E_0,E_1)}\|(\lambda+A)x\|_0\end{aligned} \tag{1.6}$$

for $x \in E_1$ and $\operatorname{Re}\lambda \geq \omega+1$. Now the assertion follows from (1.5), (1.6), and the fact that $A \in \mathcal{L}(E_1, E_0)$. ∎

REMARKS 1.5.

(a) If $A \in \mathcal{H}(E_1, E_0, \kappa, \omega)$ then

$$A \in \mathcal{L}(E_1, E_0) \quad \text{and} \quad \omega + A \in \mathcal{L}is(E_1, E_0). \tag{1.7}$$

Moreover,

$$|\lambda|\,\|x\|_0 \leq \kappa\|(\lambda+A)x\|_0, \quad x \in E_1, \quad \operatorname{Re}\lambda \geq \omega, \tag{1.8}$$

and

$$\|x\|_1 \leq \kappa\|(\lambda + A)x\|_0, \quad x \in E_1, \quad \operatorname{Re}\lambda \geq \omega. \tag{1.9}$$

Conversely, if A satisfies (1.7) and (1.8) for some $\kappa \geq 1$ and $\omega > 0$, then

$$A \in \mathcal{H}\left(E_1, E_0, \kappa \vee \|A\|_{\mathcal{L}(E_1,E_0)} \vee 3\kappa\|(\omega + A)^{-1}\|_{\mathcal{L}(E_0,E_1)}, \omega\right).$$

If A satisfies (1.7) and (1.9) for some $\kappa \geq 1$ and $\omega > 0$, then

$$A \in \mathcal{H}\left(E_1, E_0, 1 + \kappa\|A\|_{\mathcal{L}(E_1,E_0)}, \omega\right).$$

This follows from the fact that

$$|\lambda|\,\|x\|_0 = \|(\lambda + A)x - Ax\|_0 \leq \|(\lambda + A)x\| + \|A\|\,\|x\|_1,$$

the estimate (1.6), and from Proposition 1.1(i).

(b) The set $\mathcal{H}(E_1, E_0)$ has been introduced, in the case that $E_1 \overset{d}{\hookrightarrow} E_0$, in [3] and, independently, in [12, Chapter 5] (where it is called $\mathrm{Hol}(E_1, E_0)$). It is easily verified that a subset $\mathcal{U}$ of $\mathcal{H}(E_1, E_0)$ is regularly bounded in the sense of [3] iff it is bounded in $\mathcal{L}(E_1, E_0)$ and there exist $\kappa \geq 1$ and $\omega > 0$ such that $\mathcal{U} \subset \mathcal{H}(E_1, E_0, \kappa, \omega)$. Moreover, the assertions (ii) and (iii) of Proposition 1.1 are quantitative formulations of well known perturbation theorems for generators of analytic semigroups (if E_1 is dense in E_0).

(c) It should be observed that E_1 is dense in E_0 if either E_1 or E_0 is reflexive and $\mathcal{H}(E_1, E_0) \neq \emptyset$. This is a consequence of (1.2) and a result of Kato [22], since the reflexivity of one of the spaces implies the reflexivity of the other one, thus in particular of E_0, due to the fact that $\mathcal{L}is(E_1, E_0) \neq \emptyset$ if $\mathcal{H}(E_1, E_0) \neq \emptyset$.

2. - A Class of Matrix Generators

In the following we endow the product space $E \times F$ with the norm $\|\cdot\|_E + \|\cdot\|_F$.

Let X_0, X_1, and Y be $\mathbb{K}$-Banach spaces such that $X_1 \hookrightarrow X_0$. Then

$$E_1 := X_1 \times Y \hookrightarrow E_0 := X_0 \times Y.$$

Suppose that, using obvious matrix notation,

$$A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathcal{L}(E_1, E_0).$$

THEOREM 2.1. *If* $A_{11} \in \mathcal{H}(X_1, X_0)$, *then* $A \in \mathcal{H}(E_1, E_0)$.

PROOF. Suppose that $A_{11} \in \mathcal{H}(X_1, X_0, \kappa_1, \omega_1)$. Put $\omega_2 := 1 + \|A_{22}\|$ and observe that $\kappa_2 := 1 + \omega_2 = \big(1 + \|A_{22}\|\big) \vee (1 + \omega_2) / \big(\omega_2 - \|A_{22}\|\big)$. Hence $A_{22} \in \mathcal{H}(Y, Y, \kappa_2, \omega_2)$ by Proposition 1.1(iv).

Put $\omega := \omega_1 \vee \omega_2 \vee 2\kappa_1\kappa_2\|A_{12}\|\,\|A_{21}\|$. Given $\xi = (\xi_1, \xi_2) \in E_0$ and $\lambda \in [\operatorname{Re} z \geq \omega]$, the equation $(\lambda + A)x = \xi$ is equivalent to

$$x_1 + (\lambda + A_{11})^{-1} A_{12} x_2 = (\lambda + A_{11})^{-1} \xi_1, \tag{2.1}$$

$$x_2 + (\lambda + A_{22})^{-1} A_{21} x_1 = (\lambda + A_{22})^{-1} \xi_2. \tag{2.2}$$

Hence

$$\begin{aligned}&[1 - (\lambda + A_{22})^{-1} A_{21} (\lambda + A_{11})^{-1} A_{12}]x_2 \\ &= (\lambda + A_{22})^{-1} [\xi_2 - A_{21} (\lambda + A_{11})^{-1} \xi_1].\end{aligned}$$

Since, thanks to (1.1),

$$\| (\lambda + A_{22})^{-1} A_{21} (\lambda + A_{11})^{-1} A_{12} \|_{\mathcal{L}(Y)} \leq \kappa_1\kappa_2\|A_{21}\|\,\|A_{12}\|/|\lambda| \leq 1/2,$$

the Neumann series shows that

$$B := [1 - (\lambda + A_{22})^{-1} A_{21} (\lambda + A_{11})^{-1} A_{12}]^{-1} \in \mathcal{L}(Y)$$

and that $\|B\| \leq 2$. Hence

$$x_2 = B (\lambda + A_{22})^{-1} [\xi_2 - A_{21} (\lambda + A_{11})^{-1} \xi_1] \in Y$$

and

$$|\lambda|\,\|x_2\|_Y + \|x_2\|_Y \leq 4\kappa_2 \big(1 \vee \kappa_1\|A_{21}\|\big)\,\|\xi\|_0.$$

Now it follows from (2.1) that

$$x_1 = (\lambda + A_{11})^{-1} [\xi_1 - A_{12} x_2] \in X_1,$$

and, by again using (1.1), we see that

$$\begin{aligned}|\lambda|\,\|x_1\|_{X_0} + \|x_1\|_{X_1} &\leq \kappa_1 \big(1 \vee \|A_{12}\|\big) \big(\|\xi_1\|_{X_0} + \|x_2\|_Y\big) \\ &\leq 8\kappa_1\kappa_2 \big(1 \vee \|A_{12}\|\big) \big(1 \vee \kappa_1\|A_{21}\|\big)\,\|\xi\|_0.\end{aligned}$$

This shows that, given $\xi \in E_0$ and $\lambda \in [\operatorname{Re} z \geq \omega]$, the equation $(\lambda + A)x = \xi$ has a unique solution $x \in E_1$ and that

$$|\lambda|\,\|x\|_0 + \|x\|_1 \leq 8\kappa_1\kappa_2 \big(1 \vee \|A_{12}\|\big) \big(1 \vee \kappa_1\|A_{21}\|\big)\,\|(\lambda + A)x\|_0.$$

Hence $\lambda + A \in \mathcal{L}is(E_1, E_0)$ for $\lambda \in [\operatorname{Re} z \geq \omega]$ by the open mapping theorem.

Since

$$\|(\lambda + A)x\|_0 \le |\lambda|\,\|x\|_0 + \|A\|\,\|x\|_1 \le (1 \vee \|A\|)\big(|\lambda|\,\|x\|_0 + \|x\|_1\big)\,,$$

the assertion follows. ∎

Recently, the question, under which conditions matrices of operators generate (strongly continuous) semigroups, has been studied by Nagel and his students (cf. [18], [26]). However, our result—which is not contained in their work—is much more simple minded than their investigations, since they consider matrices of unbounded linear operators, whereas in our case we can deal with bounded ones.

3. - Abstract Highly Degenerate Quasilinear Parabolic Systems

In the following we denote by J a subinterval of $\mathbb{R}^+$ containing 0 such that $\dot{J} := J\backslash\{0\} \neq \emptyset$.

Given $\rho \in (0,1)$, we write $u \in C_\rho^\rho(J, E_1)$ if $u : \dot{J} \to E_1$ is locally ρ-Hölder continuous and if

$$\lim_{\varepsilon\to 0} \varepsilon^\rho \sup_{\varepsilon \le s < t \le 2\varepsilon} \frac{\|u(s) - u(t)\|_1}{|s-t|^\rho} = 0.$$

Using these notations, we can formulate a general existence theorem for abstract fully nonlinear parabolic equations, due to Lunardi.

PROPOSITION 3.1. *Suppose that $E_1 \overset{d}{\hookrightarrow} E_0$ and V is open in E_1. Suppose also that $\varphi \in C^2(J \times V, E_0)$ such that*

$$-\partial_2\varphi(t,y) \in \mathcal{H}(E_1, E_0)\,, \quad (t,y) \in J \times V.$$

Then, given $\rho \in (0,1)$, the Cauchy problem

$$\dot{u} = \varphi(t,u) \quad in\ \dot{J}, \quad u(0) = x$$

possesses for each $x \in V$ a unique maximal solution

$$u_\rho(\cdot\,, x) \in C\big(J_\rho(x), V\big) \cap C^1\big(J_\rho(x), E_0\big) \cap C_\rho^\rho\big(J_\rho(x), E_1\big)\,, \tag{3.1}$$

where $J_\rho(x) := \operatorname{dom}\big(u_\rho(\cdot\,, x)\big)$.

PROOF. Thanks to Lemmas 1.2 and 1.3 and Theorem 1.4 this follows directly from (an obvious localization of) [25, Theorem 2]. ∎

REMARKS 3.2.

(a) Lunardi's theorem is more precise. In fact, Proposition 3.1 is true if $\varphi(t,\cdot) \in C^2(V, E_0)$ for each $t \in J$ and if $\varphi(\cdot, y)$ and $\partial_2\varphi(\cdot, y)$ are locally ρ-Hölder continuous, locally uniformly with respect to $y \in V$. In addition, the assumption that E_1 be dense in E_0 can be dropped provided $\varphi(0,y) \in \mathrm{cl}_{E_0}(E_1)$.

(b) It should be observed that Proposition 3.1 guarantees the uniqueness of $u_\rho(\cdot, x)$ only within the class (3.1).

Now we suppose that

$$X_0, X_1, \text{ and } Y \text{ are } \mathbb{K}\text{-Banach spaces such that } X_1 \overset{d}{\hookrightarrow} X_0. \tag{A1}$$

Given $\theta \in (0,1)$, we fix an interpolation functor, $(\cdot,\cdot)_\theta$, of exponent θ such that X_1 is dense in $X_\theta := (X_0, X_1)_\theta$. Then

$$E_1 := X_1 \times Y \overset{d}{\hookrightarrow} E_\theta := X_\theta \times Y \overset{d}{\hookrightarrow} E_0 := X_0 \times Y. \tag{3.2}$$

Given $0 < \beta < \alpha \leq 1$ and a subset M_β of E_β, we put $M_\alpha := M_\beta \cap E_\alpha$ equipped with the topology induced by E_α. Then we assume that

$$\begin{aligned} &V_\beta \text{ is an open subset of } E_\beta \text{ for some } \beta \in (0,1);\\ &(A_{11}, A_{12}, f_1) \in C^2\left(J \times V_\beta, \mathcal{H}(X_1, X_0) \times \mathcal{L}(Y, X_0) \times X_0\right);\\ &f_2 \in C^2(J \times V_1, Y). \end{aligned} \tag{A2}$$

Letting $u := (u_1, u_2)$, we consider the abstract degenerate quasilinear parabolic system

$$\left.\begin{aligned} \dot u_1 + A_{11}(t,u)u_1 + A_{12}(t,u)u_2 &= f_1(t,u),\\ \dot u_2 &= f_2(t,u), \end{aligned}\right. \qquad t \in \dot J. \tag{3.3}$$

It is now easy to prove the following

PROPOSITION 3.3. *Given $\rho \in (0,1)$ and $x := (x_1, x_2) \in V_1$, problem* (3.3) *possesses a unique maximal solution*

$$u_\rho(\cdot, x) \in C\left(J_\rho(x), V_1\right) \cap C^1\left(J_\rho(x), E_0\right) \cap C_\rho^\rho\left(J_\rho(x), E_1\right)$$

satisfying $u_\rho(0,x) = x$.

PROOF. Since $E_1 \hookrightarrow E_\beta$, it follows that

$$A := \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \in C^2(J \times V_1, \mathcal{L}(E_1, E_0))$$

and that $f := (f_1, f_2) \in C^2(J \times V_1, E_0)$. Hence, letting

$$\varphi(t, y) := -A(t, y)y + f(t, y),$$

we see that

$$\varphi = (\varphi_1, \varphi_2) \in C^2(J \times V_1, E_0).$$

Thus the assertion is a consequence of Theorem 3.1, provided

$$-\partial_2\varphi(t, x) = -\begin{bmatrix} \partial_2\varphi_1(t, x_1, x_2) & \partial_3\varphi_1(t, x_1, x_2) \\ \partial_2\varphi_2(t, x_1, x_2) & \partial_3\varphi_2(t, x_1, x_2) \end{bmatrix} \in \mathcal{H}(E_1, E_0) \tag{3.4}$$

for $(t, x) \in J \times V_1$.

Observe that

$$-\partial_2\varphi_1(t, x_1, x_2) = A_{11}(t, x_1, x_2) + B(t, x_1, x_2), \tag{3.5}$$

where

$$B(t, x_1, x_2) := \partial_2 A_{11}(t, x_1, x_2)[\,\cdot\,, x_1] + \partial_2 A_{12}(t, x_1, x_2)[\,\cdot\,, x_2] \\ - \partial_2 f_1(t, x_1, x_2).$$

Let $(t, x) \in J \times V_1$ be fixed and put $B := B(t, x_1, x_2)$. Then it follows from (A2) that $B \in \mathcal{L}(X_\beta, X_0)$. Since there exists a constant c_β such that

$$\|y\|_{X_\beta} \le c_\beta \|y\|_{X_0}^{1-\beta} \|y\|_{X_1}^{\beta}, \quad y \in X_1, \tag{3.6}$$

thanks to the fact that X_β is an interpolation space of exponent β between X_0 and X_1, we deduce the existence of a constant c such that

$$\|By\|_{X_0} \le c\|y\|_{X_0}^{1-\beta} \|y\|_{X_1}^{\beta}, \quad y \in X_1.$$

Hence, given any $\varepsilon > 0$, Young's inequality implies the existence of a constant c such that

$$\|By\|_{X_0} \le \varepsilon \|y\|_{X_1} + c\|y\|_{X_0}, \quad y \in X_1.$$

We now deduce from $A_{11}(t, x) \in \mathcal{H}(X_1, X_0)$, Proposition 1.1(iii), and (3.5) that $-\partial_2\varphi_1(t, x_1, x_2) \in \mathcal{H}(X_1, X_0)$ for $(t, x) \in J \times V_1$. Thus (3.4) is a consequence of Theorem 2.1. ∎

As indicated by the index ρ, the solution $u_\rho(\,\cdot\,, x)$ depends on $\rho \in (0, 1)$, in general, and it is not clear at all what the relations are between $u_{\rho_1}(\,\cdot\,, x)$ and $u_{\rho_2}(\,\cdot\,, x)$ for $\rho_1 \ne \rho_2$.

However, we shall now show that, given the additional assumption (A3) below, which is satisfied in our applications, (3.3) possesses a unique maximal solution

$$u(\,\cdot\,, x) \in C(J(x), E_1) \cap C^1(J(x), E_0).$$

Consequently, $u_\rho(\cdot, x) \subset u(\cdot, x)$ for $0 < \rho < 1$.

Assume that

(A3) there exist $\alpha \in (\beta, 1)$ and $\gamma \in (0,1)$ and Banach spaces $X_{\alpha-1}$ and Y_α such that

$$X_0 \hookrightarrow X_{\alpha-1}, \quad Y \hookrightarrow Y_\alpha, \quad (X_{\alpha-1}, X_\alpha)_\gamma \hookrightarrow X_\beta.$$

There exists an open subset $\tilde{V}_\beta$ of $X_\beta \times Y_\alpha$ such that $V_\beta = \tilde{V}_\beta \cap \left(X_\beta \times Y\right)$ and

$$(A_{11}, A_{12}, f_1) \in C^2\left(J \times \tilde{V}_\beta, \mathcal{H}(X_\alpha, X_{\alpha-1}) \times \mathcal{L}(Y_\alpha, X_{\alpha-1}) \times X_{\alpha-1}\right),$$

$$f_2 \in C^2\left(J \times \tilde{V}_\alpha, Y_\alpha\right),$$

where $\tilde{V}_\alpha = \tilde{V}_\beta \cap (X_\alpha \times Y_\alpha)$, with the topology induced by $X_\alpha \times Y_\alpha$.

Here and in the following we do not distinguish between a map and its restrictions or extensions to various sub- or superspaces, respectively, if no confusion seems likely.

After these preparations we can prove the main result of this section.

THEOREM 3.4. *Let assumptions* (A1)–(A3) *be satisfied. Then problem* (3.3) *possesses for each* $x \in V_1$ *a unique maximal solution*

$$u(\cdot, x) \in C(J(x), V_1) \cap C^1(J(x), E_0) \tag{3.7}$$

satisfying $u(0,x) = x$. *The maximal interval of existence,* $J(x)$, *of* $u(\cdot, x)$ *is open in* J.

PROOF. Fix any $\rho \in (0,1)$. Then Proposition 3.3 guarantees the existence of a solution

$$u_\rho(\cdot, x) \in C\left(J_\rho(x), V_1\right) \cap C^1\left(J_\rho(x), E_0\right)$$

of (3.3) satisfying $u_\rho(\cdot, x) = x$. Hence, by a standard argument based upon Zorn's lemma, we deduce the existence of a maximal solution $u(\cdot, x)$ of (3.3) satisfying (3.7) and $u(0,x) = x$. It is obvious that $J(x)$ is open in J since, otherwise, $u(\cdot, x)$ could be extended to a solution on a larger interval. It remains to show that $u(\cdot, x)$ is unique.

Fix $\sigma \in (0, 1-\alpha)$ and put

$$\tilde{E}_1 := X_\alpha \times Y_\alpha, \quad \tilde{E}_0 := X_{\alpha-1} \times Y_\alpha, \quad \tilde{E}_\beta := X_\beta \times Y_\alpha.$$

Observe that assumption (A3) implies the existence of a constant c such that

$$\|y\|_{X_\beta} \le c\|y\|_{X_{\alpha-1}}^{1-\gamma} \|y\|_{X_\alpha}^{\gamma}, \quad y \in X_\alpha. \tag{3.8}$$

Hence, by replacing (3.6) by (3.8) and E_j by $\widetilde{E}_j$ for $j \in \{0, 1, \beta\}$, it follows from Proposition 3.3 that the extension of problem (3.3) possesses a unique maximal solution

$$\widetilde{u}(\cdot, x) \in C\left(\widetilde{J}(x), \widetilde{V}_\alpha\right) \cap C^1\left(\widetilde{J}(x), \widetilde{E}_0\right) \cap C_\sigma^\sigma\left(\widetilde{J}(x), \widetilde{E}_1\right)$$

satisfying $\widetilde{u}(0, x) = x$.

Let now

$$u(\cdot, x) \in C\left(J(x), V_1\right) \cap C^1\left(J(x), E_0\right)$$

be any maximal solution of (3.3) satisfying $u(0, x) = x$. Then we obtain by a standard interpolation argument that

$$u(\cdot, x) \in C^{1-\alpha}\left(J(x), E_\alpha\right).$$

This implies, thanks to assumption (A3), that

$$u(\cdot, x) \in C\left(J(x), \widetilde{V}_\alpha\right) \cap C^1\left(J(x), \widetilde{E}_0\right) \cap C_\sigma^\sigma\left(\dot{J}(x), \widetilde{E}_1\right).$$

Since $u(\cdot, x)$ is a solution of the extended problem (3.3) satisfying $u(0, x) = x$, it follows from the uniqueness and maximality of $\widetilde{u}(\cdot, x)$ that $u(\cdot, x) \subset \widetilde{u}(\cdot, x)$. This implies the uniqueness assertion. ∎

COROLLARY 3.5. *Suppose that, given any $T \in \dot{J}$,*

$$\operatorname{dist}\left(u([0, T] \cap J(x), x), \partial V_1\right) > 0$$

and

$$u(\cdot, x) \in BUC\left([0, T] \cap J(x), E_1\right). \tag{3.9}$$

Then $u(\cdot, x)$ is a global solution, that is, $J(x) = J$.

PROOF. Suppose that $J(x) \neq J$ and fix any $T \in \dot{J}$ with

$$T > t^+(x) := \sup J(x).$$

Then the existence of $\widetilde{u} \in C\left([0, t^+(x)], V_1\right)$ extending $u(\cdot, x)$ is implied by the hypotheses. It is clear that $\widetilde{u}$ is a solution of (3.3) on $[0, t^+(x)]$ satisfying $\widetilde{u}(0) = x$. But this contradicts the maximality of $u(\cdot, x)$. ∎

REMARKS 3.6.

(a) The difficulty in dealing with problem (3.3) stems from the fact that f_2 maps $J \times V_1$ into Y, but, in general, not a set of the form $J \times W$, where W is an open subset of E_β. If the latter were the case, we could prove Theorem 3.4, without imposing condition (A3), by using the fact that analytic semigroups possess smoothing properties. In fact, we could directly apply the general existence theorems for abstract quasilinear parabolic equations given in [3] (cf. [9, Remarks] for a correction). In that case we could solve (3.3) for initial values in V_α and still get solutions in $C(\dot{J}(x), V_1) \cap C^1(\dot{J}(x), E_0)$. This regularizing effect would guarantee global existence, provided there are a priori bounds in intermediate spaces between E_0 and E_1, a condition which is much more flexible and easier to verify in practical applications than (3.9) (cf. [6]).

(b) By using the sharp form of Lunardi's theorem the regularity requirements with repect to $t \in J$ can be weakened.

4. - Normally Elliptic Boundary Value Problems

We denote by Ω a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, that is, we assume that $\overline{\Omega}$ is an n-dimensional C^∞-submanifold of $\mathbb{R}^n$.

In the following, we use the summation convention for j and k, where these indices always run from 1 to n. If M is any smooth manifold, we denote by $T(M)$ its tangent bundle.

Suppose that $a_{jk} \in C\left(\overline{\Omega}, \mathcal{L}\left(\mathbb{K}^N\right)\right)$ and consider the differential operator

$$\mathcal{A} := -a_{jk}\partial_j\partial_k$$

operating on $\mathbb{K}^N$-valued functions on Ω. We denote the symbol of $\mathcal{A}$ by $a_\pi \in C\left(\overline{\Omega} \times \mathbb{R}^n, \mathcal{L}\left(\mathbb{K}^N\right)\right)$, that is,

$$a_\pi(x, \xi) := a_{jk}(x)\xi^j\xi^k, \quad (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n.$$

Then $\mathcal{A}$ is *normally elliptic* if

$$\sigma\left(a_\pi(x, \xi)\right) \subset [\operatorname{Re} z > 0], \quad (x, \xi) \in \overline{\Omega} \times \left(\mathbb{R}^n \backslash \{0\}\right).$$

Suppose also that b_0, b_j, $c \in C\left(\partial\Omega, \mathcal{L}\left(\mathbb{K}^N\right)\right)$, that $\delta_r \in C\left(\partial\Omega, \{0, 1\}\right)$, $1 \leq r \leq N$, and put $\delta := \operatorname{diag}[\delta_1, \ldots, \delta_N]$. Then, denoting by γ_∂ the trace operator on $\partial\Omega$, we define a boundary operator, $\mathcal{B}$, by

$$\mathcal{B} := \delta\left(b_j\gamma_\partial\partial_j + b_0\gamma_\partial\right) + (1 - \delta)c\gamma_\partial. \tag{4.1}$$

Observe that every linear boundary operator possessing N rows and having order at most 1 can be written in the form (4.1), provided the order of each row is constant on each component of $\partial\Omega$.

Denote by b_π the principal boundary symbol, that is,

$$b_\pi(x, \xi) := \delta(x)b_j(x)\xi^j + (1 - \delta(x))c(x), \quad (x, \xi) \in \partial\Omega \times \mathbb{R}^n.$$

Then $\mathcal{B}$ is said to satisfy the *normal complementing condition with respect to* $\mathcal{A}$ if, given any $(x, \xi) \in T(\partial\Omega)$ and any $\lambda \in [\operatorname{Re} z \geq 0]$ with $(\xi, \lambda) \neq (0,0)$, zero is the only exponentially decaying solution of the initial value problem on $\mathbb{R}^+$:

$$[\lambda + a_\pi(x, \xi + \nu(x) i\partial_t)]u = 0 \quad \text{on } (0,\infty),$$
$$b_\pi(x, \xi + \nu(x) i\partial_t)\, u(0) = 0,$$

where ν is the outer unit normal vector field on $\partial\Omega$. Finally, $(\mathcal{A}, \mathcal{B})$ is said to be [a] *normally elliptic* [boundary value problem] if $\mathcal{A}$ is normally elliptic and $\mathcal{B}$ satisfies the normal complementing condition with respect to $\mathcal{A}$.

REMARKS 4.1.

(a) We refer to [5, Section 4] for sufficient conditions guaranteeing that $(\mathcal{A}, \mathcal{B})$ is normally elliptic. In particular, $(\mathcal{A}, \mathcal{B})$ is normally elliptic if one of the following conditions is satisfied:

(i) $\mathcal{A}$ is *strongly uniformly elliptic*, that is,

$$\operatorname{Re}\big(a_\pi(x,\xi)\eta \,|\, \eta\big) > 0, \quad (x,\xi,\eta) \in \overline{\Omega} \times \mathbb{R}^n \times \mathbb{C}^N, \quad \xi \neq 0, \quad \eta \neq 0,$$

where $(\cdot\,|\,\cdot)$ is the standard inner product in $\mathbb{C}^N$, and $\mathcal{B} := \gamma_\partial$, the Dirichlet boundary operator, that is, $c = 1$ and $\delta = 0$.

(ii) $\mathcal{A}$ is *very strongly uniformly elliptic*, that is,

$$\operatorname{Re} \sum_{r,s=1}^{N} a_{jk}^{rs}(x) \varsigma_r^j \overline{\varsigma_s^k} > 0, \quad x \in \overline{\Omega}, \quad \varsigma = \big(\varsigma_r^j\big) \in \mathbb{C}^{nN} \setminus \{0\},$$

and

$$b_j := a_{jk}\nu^k, \quad c = 1.$$

(iii) There exist $A \in C\big(\overline{\Omega}, \mathcal{L}\big(\mathbb{K}^N\big)\big)$ and a symmetric uniformly positive definite $\alpha := [\alpha_{jk}] \in C\big(\overline{\Omega}, \mathcal{L}\big(\mathbb{K}^N\big)\big)$ such that $\sigma(A(x)) \subset [\operatorname{Re} z > 0]$ for $x \in \overline{\Omega}$, and

$$a_{jk} = A\alpha_{jk}, \quad b_j = A\nu^j, \quad c = 1, \quad (1-\delta)A\delta = 0.$$

(b) If $N = 1$, the concepts of normal ellipticity for $\mathcal{A}$, strong uniform ellipticity, and very strong uniform ellipticity coincide.

(c) If $N = 1$ and $b_j = a_{jk}\nu^k$, then $b_j \gamma_\partial \partial_j$ is the conormal derivative ∂_{ν_a} (with respect to $a := [a_{jk}]$), where $\nu_a := a\nu$ is the outer conormal vector field on $\partial\Omega$ (with respect to a).

(d) If $N = 1$ and $\mathcal{A}$ is normally elliptic, that is, strongly uniformly elliptic, then $\mathcal{B}$ satisfies the normal complementing condition with respect to $\mathcal{A}$, if $b_j(x)\nu^j(x) \neq 0$ for $x \in \partial\Omega$, that is, if the vector field

$$b := (b_1, \ldots, b_n) \in T(\mathbb{R}^n)$$

is nowhere tangent on $\partial\Omega$, and if $c = 1$.

(e) Let the conditions of (a)(iii) be satisfied and denote by Γ a component of $\partial\Omega$. Then, on Γ, the boundary condition $\mathcal{B}u$ is the Dirichlet boundary condition $u|\Gamma = 0$ if $\delta|\Gamma = 0$, whereas it is equivalent to the Neumann (or Robin) boundary condition

$$\partial_{\nu_\alpha} u + bu = 0,$$

if $\delta|\Gamma = 1$, where $b := A^{-1}b_0$.

5. - Highly Degenerate Parabolic Systems: Constant Boundary Conditions

We denote by G an open subset of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and suppose that

$$\left(a_{jk}^1, a_{jk}^2\right) \in C^\infty\left(\overline{\Omega} \times \mathbb{R}^+ \times G \times \mathbb{R}^{n(N_1+N_2)}, \mathcal{L}\left(\mathbb{R}^{N_1}\right) \times \mathcal{L}\left(\mathbb{R}^{N_2}, \mathbb{R}^{N_1}\right)\right)$$

and

$$b_0, b_j, c \in C^\infty\left(\partial\Omega, \mathcal{L}\left(\mathbb{R}^{N_1}\right)\right).$$

Given $(t, v) \in \mathbb{R}^+ \times C^1\left(\overline{\Omega}, G\right)$, we define differential operators by

$$\mathcal{A}_r(t, v, \partial v)u_r := -a_{jk}^r(\cdot\,, t, v, \partial v)\partial_j\partial_k u_r, \quad r = 1, 2,$$

for $u_r \in C^2\left(\overline{\Omega}, \mathbb{R}^{N_r}\right)$, and a boundary operator, $\mathcal{B}$, acting on $\mathbb{R}^{N_1}$-valued functions, by (4.1). We assume that

$$\begin{aligned} &(\mathcal{A}_1(t, v, \partial v), \mathcal{B}) \text{ is normally elliptic} \\ &\text{for } (t, v) \in \mathbb{R}^+ \times C^1\left(\overline{\Omega}, G\right). \end{aligned} \tag{5.1}$$

We also assume that

$$f_1 \in C^\infty\left(\overline{\Omega} \times \mathbb{R}^+ \times G \times \mathbb{R}^{n(N_1+N_2)}, \mathbb{R}^{N_1}\right)$$

and

$$f_2 \in C^\infty\left(\overline{\Omega} \times \mathbb{R}^+ \times G, \mathbb{R}^{N_2}\right).$$

We then consider the *highly degenerate quasilinear parabolic system with 'constant' boundary conditions*:

$$\begin{aligned} \partial_t u_1 + \sum_{j=1}^{2} \mathcal{A}_j(t, u, \partial u)u_j &= f_1(\cdot\,, t, u, \partial u) && \\ && &\text{in } \Omega \times (0, \infty), \\ \partial_t u_2 &= f_2(\cdot\,, t, u) && \\ \mathcal{B}u_1 &= 0 && \text{on } \partial\Omega \times (0, \infty), \end{aligned} \tag{5.2}$$

where $u := (u_1, u_2)$.

We fix any $p > n$ and put

$$W^s_{p,\mathcal{B}} := W^s_{p,\mathcal{B}}\left(\Omega, \mathbb{R}^{N_1}\right) := \begin{cases} \{u \in W^s_p;\ \mathcal{B}u = 0\}, & 1 + 1/p < s \le 2, \\ W^s_p, & -1 + 1/p < s < 1/p, \end{cases}$$

$W^\sigma_p := W^\sigma_p\left(\Omega, \mathbb{R}^{N_1}\right)$ being the usual Sobolev-Slobodeckii spaces for $\sigma \in \mathbb{R}^+$, and $W^\sigma_p := (\overset{\circ}{W}{}^{-\sigma}_{p'})'$ for $\sigma \in -\mathbb{R}^+$, where $\overset{\circ}{W}{}^s_p$ is the closure of $\mathcal{D} := \mathcal{D}\left(\Omega, \mathbb{R}^{N_1}\right)$, the space of test functions in Ω, in W^s_p for $s \in \mathbb{R}^+$.

We denote by $(\cdot\,,\cdot)_{\theta,p}$, $0 < \theta < 1$, the standard real interpolation functor. Moreover, in the following we identify spaces differing by equivalent norms only.

LEMMA 5.1. *Suppose that* $1 + 1/p < 2\beta < 2\alpha < 2$. *Then*

$$\left(W^{2\alpha-2}_{p,\mathcal{B}}, W^{2\alpha}_{p,\mathcal{B}}\right)_{\gamma,p} \hookrightarrow W^{2\beta}_{p,\mathcal{B}} = \left(L_p, W^2_{p,\mathcal{B}}\right)_{\beta,p}$$

provided $1 + \beta - \alpha < \gamma < 1$.

PROOF. It is an easy consequence of (5.1) that $\mathcal{B}$ is a normal (system of) boundary operator(s). Hence $W^{2\theta}_{p,\mathcal{B}} = \left(L_p, W^2_{p,\mathcal{B}}\right)_{\theta,p}$ for $2\theta \in (0, 1/p) \cup (1 + 1/p, 2)$ follows from results of Grisvard [20] and Seeley [28] (cf. the latter paper for the case of systems, cf. also [21]).

Fix any $v \in C^\infty\left(\overline{\Omega}, G\right)$ and put $\mathcal{A} := \mathcal{A}_1(0, v, \partial v)$. Moreover, let A be the L_p-realization of $\mathcal{A}$, that is, $A := \mathcal{A}|W^2_{p,\mathcal{B}}$, considered as an unbounded linear operator in L_p. Then $A \in \mathcal{H}\left(W^2_{p,\mathcal{B}}, L_p\right)$ by [5, Theorem 2.3]. Thus, putting $(\cdot\,,\cdot)_\theta := (\cdot\,,\cdot)_{\theta,p}$, $0 < \theta < 1$, the scale of Banach spaces E_ξ, $\xi \in \mathbb{R}$, introduced in [1, Appendix], is well defined. Since $W^s_{p,\mathcal{B}} = W^s_p = \left(W^{-s}_{p'}\right)'$ for $-1 + 1/p < s < 1/p$, thanks to the fact that $\mathcal{D}$ is dense in W^s_p for $0 \le s < 1/p$, it follows that $E_\xi = W^{2\xi}_{p,\mathcal{B}}$ for $2\xi \in (-1 + 1/p, 1/p) \cup (1 + 1/p, 2]$. Now the stated continuous inclusion is an easy consequence of the 'almost reiteration theorem' proven in [4, Theorem 8.3]. ∎

We put

$$W^s_{p,\mathcal{B}} \times W^t_p := W^s_{p,\mathcal{B}}\left(\Omega, \mathbb{R}^{N_1}\right) \times W^t_p\left(\Omega, \mathbb{R}^{N_2}\right)$$

for $s \in (-1 + 1/p, 1/p) \cup (1 + 1/p, 2]$ and $t \in \mathbb{R}$. Moreover,

$$V_{s,t} := \left\{u := (u_1, u_2) \in W^s_{p,\mathcal{B}} \times W^t_p;\ u\left(\overline{\Omega}\right) \in G\right\}$$

for $s, t \in (1 + n/p, 2]$. Observe that

$$W^s_{p,\mathcal{B}} \times W^t_p \hookrightarrow C^\mu\left(\overline{\Omega}, \mathbb{R}^{N_1+N_2}\right), \quad 1 + n/p < s \wedge t \le 2, \tag{5.3}$$

where $\mu := s \wedge t - n/p$, thanks to Sobolev's imbedding theorem. Consequently,

$$V_{s,t} \text{ is open in } W^s_{p,\mathcal{B}} \times W^t_p \quad \text{for } 1 + n/p < s \wedge t \le 2. \tag{5.4}$$

After these preparations we can prove the main result of this section.

THEOREM 5.2. *Given* $u_0 \in V_1 := V_{2,2}$, *the highly degenerate quasilinear system* (5.2) *possesses a unique maximal solution*

$$u(\cdot, u_0) \in C(J(u_0), V_1) \cap C^1\left(J(u_0), L_p \times W_p^2\right).$$

The maximal interval of existence, $J(u_0)$, *is open in* $\mathbb{R}^+$. *If*

$$\operatorname{dist}(u([0,T] \cap J(u_0), u_0), \partial V_1) > 0$$

and

$$u(\cdot, u_0) \in BUC\left([0,T] \cap J(u_0), W_p^2\left(\Omega, \mathbb{R}^{N_1+N_2}\right)\right),$$

$u(\cdot, u_0)$ *is a global solution, that is,* $J(u_0) = \mathbb{R}^+$.

PROOF. Put $X_1 := W^2_{p,\mathcal{B}}$, $X_0 := L_p\left(\Omega, \mathbb{R}^{N_1}\right)$, and $Y := W_p^2\left(\Omega, \mathbb{R}^{N_2}\right)$. Then condition (A1) is satisfied. Fix α and β with $1 + n/p < 2\beta < 2\alpha < 2$ and $2\alpha + 2\beta > 3 + n/p$, and put $X_{\alpha-1} := W^{2\alpha-2}_{p,\mathcal{B}}$ and $Y_\alpha := W_p^{2\alpha}\left(\Omega, \mathbb{R}^{N_2}\right)$. Then, letting $(\cdot\,,\cdot)_\theta := (\cdot\,,\cdot)_{\theta,p}$, $0 < \theta < 1$, Lemma 5.1 implies that $X_\theta := (X_0, X_1)_\theta = W^{2\theta}_{p,\mathcal{B}}$, $\theta \in \{\alpha, \beta\}$, and that the first part of condition (A3) is satisfied.

Put $V_\beta := V_{2\beta,2}$ and $\widetilde{V}_\beta := V_{2\beta,2\alpha}$ and observe that V_β is open in $E_\beta := X_\beta \times Y$ and $\widetilde{V}_\beta$ is open in $X_\beta \times Y_\alpha$, thanks to (5.4). Moreover, $V_\beta = \widetilde{V}_\beta \cap E_\beta$.

Given $(t, v) \in \mathbb{R}^+ \times \widetilde{V}_\beta$, put

$$A_{11}(t, v) := \mathcal{A}_1(t, v, \partial v) | W^2_{p,\mathcal{B}}$$

and

$$A_{12}(t, v) := \mathcal{A}_2(t, v, \partial v).$$

Moreover, let

$$(f_1(t, v), f_2(t, v)) := (f_1(\cdot\,, t, v, \partial v), f_2(\cdot\,, t, v)).$$

It follows from our regularity hypotheses that

$$f := (f_1, f_2) \in C^\infty\left(\mathbb{R}^+ \times V_1, X_0 \times Y\right) \cap C^\infty\left(\mathbb{R}^+ \times \widetilde{V}_\beta, X_0 \times Y_\alpha\right)$$

(cf. [10]). Thanks to (5.3) it is easily verified that

$$(A_{11}, A_{12}) \in C^\infty(J \times V_1, \mathcal{L}(X_1, X_0) \times \mathcal{L}(Y, X_0)).$$

Observe that $\partial_j \partial_k \in \mathcal{L}(X_\alpha, X_{\alpha-1}) \cap \mathcal{L}(Y_\alpha, X_{\alpha-1})$ and that (5.3) implies

$$[(t, v) \mapsto a^r_{jk}(\cdot\,, t, v, \partial v)] \in C^\infty\left(J \times \widetilde{V}_\beta, C^\mu\left(\overline{\Omega}, \mathcal{L}\left(\mathbb{R}^{N_r}, \mathbb{R}^{N_1}\right)\right)\right)$$

for $r = 1, 2$, where $\mu := 2\beta - 1 - n/p > 2 - 2\alpha$. Hence

$$(A_{11}, A_{12}) \in C^\infty \left(J \times \tilde{V}_\beta, \mathcal{L}(X_\alpha, X_{\alpha-1}) \times \mathcal{L}(Y_\alpha, X_{\alpha-1})\right),$$

(cf. [10]).

Lastly, $A_{11}(t,v) \in \mathcal{H}(X_1, X_0)$ for $(t,v) \in J \times \tilde{V}_\alpha$ is implied by [5, Theorem 2.3]. Now we deduce from [1, Theorem 6] that $A_{11}(t,v) \in \mathcal{H}(X_\alpha, X_{\alpha-1})$ for $(t,v) \in J \times \tilde{V}_\alpha$.

These considerations imply that condition (A2) and the second part of condition (A3) are satisfied too. Hence the assertion follows from Theorem 3.4 and Corollary 3.5. ∎

REMARKS 5.3.

(a) The regularity hypotheses of Theorem 5.2 can be considerably weakened.

(b) It is obvious that Theorem 5.2 can be generalized to systems in which $\mathcal{A}_1$, $\mathcal{A}_2$, f_1, and f_2 are nonlocal functions of $u \in V_1$.

6. - Highly Degenerate Parabolic Systems: Nonlinear Boundary Conditions

We now assume that

$$\left(a^1_{jk}, a^2_{jk}\right), \left(a^1_j, a^2_j\right) \in C^\infty\left(\overline{\Omega} \times \mathbb{R}^+ \times G, \mathcal{L}\left(\mathbb{R}^{N_1}\right) \times \mathcal{L}\left(\mathbb{R}^{N_2}, \mathbb{R}^{N_1}\right)\right)$$

and, given $(t,v) \in \mathbb{R}^+ \times C\left(\overline{\Omega}, G\right)$, we define (formal) differential operators by

$$\mathcal{A}_r(t,v)u_r := -\partial_j\left(a^r_{jk}(\cdot, t, v)\partial_k u_r\right) + a^r_j(\cdot, t, v)\partial_j u_r$$

for $u_r \in C^2\left(\overline{\Omega}, \mathbb{R}^{N_r}\right)$. We also put

$$\mathcal{B}_1(t,v)u_1 := \delta\gamma_\partial a^1_{jk}(\cdot, t, v)\nu^j\gamma_\partial\partial_k u_1 + (1-\delta)\gamma_\partial u_1$$

and

$$\mathcal{B}_2(t,v)u_2 := \delta\gamma_\partial a^2_{jk}(\cdot, t, v)\nu^j\gamma_\partial\partial_k u_2$$

for some fixed $\delta \in C(\partial\Omega, \{0,1\})$.

We denote by $\mathcal{A}_{1,\pi}(t,v) := -a^1_{jk}(\cdot, t, v)\partial_j\partial_k$ the principal part of $\mathcal{A}_1(t,v)$ and, given any $x_0 \in \overline{\Omega}$,

$$\begin{aligned}&\left(\mathcal{A}_{1,\pi}(x_0, t, v(x_0)), \mathcal{B}_1(x_0, t, v(x_0))\right)\\&:= \left(-a^1_{jk}(x_0, t, v(x_0))\partial_j\partial_k, \delta\gamma_\partial a^1_{jk}(x_0, t, v(x_0))\nu^j\gamma_\partial\partial_k + (1-\delta)\gamma_\partial\right).\end{aligned}$$

We then assume that

$$\begin{aligned}&\text{given } (t,v) \in \mathbb{R}^+ \times C\left(\overline{\Omega}, G\right) \text{ and } x_0 \in \overline{\Omega},\\ &\text{the boundary value problems } \left(\mathcal{A}_{1,\pi}(t,v), \mathcal{B}_1(t,v)\right)\\ &\text{and } \left(\mathcal{A}_{1,\pi}\left(x_0, t, v\left(x_0\right)\right), \mathcal{B}\left(x_0, t, v\left(x_0\right)\right)\right)\\ &\text{are normally elliptic.}\end{aligned} \tag{6.1}$$

REMARK 6.1. Condition (6.1) is satisfied if, given any

$$(t,v) \in \mathbb{R}^+ \times C\left(\overline{\Omega}, G\right),$$

the boundary value problem $\left(\mathcal{A}_{1,\pi}(t,v), \mathcal{B}(t,v)\right)$ satisfies any one of the conditions (i)-(iii) of Remark 4.1(a).

Finally, we assume that

$$f_r \in C^\infty\left(\overline{\Omega} \times \mathbb{R}^+ \times G, \mathbb{R}^{N_r}\right), \quad r = 1,2,$$

and that

$$g \in C^\infty\left(\partial\Omega \times \mathbb{R}^+ \times G, \mathbb{R}^{N_1}\right).$$

Then we consider the *highly degenerate quasilinear parabolic system with nonlinear boundary conditions*:

$$\begin{aligned}\partial_t u_1 + \mathcal{A}_1(t,u)u_1 + \mathcal{A}_2(t,u)u_2 &= f_1(\cdot, t, u) && \text{in } \Omega \times (0,\infty),\\ \partial_t u_2 &= f_2(\cdot, t, u) && \\ \mathcal{B}_1(t,u)u_1 + \mathcal{B}_2(t,u)u_2 &= \delta g(\cdot, t, u) && \text{on } \partial\Omega \times (0,\infty),\end{aligned} \tag{6.2}$$

where $u := (u_1, u_2)$.

Given $q \in (1,\infty)$ and s with $1/q < s < 1 + 1/q$, we put

$$W^s_{q,\mathcal{B}} := W^s_{q,\mathcal{B}}\left(\Omega, \mathbb{R}^{N_1}\right) := \left\{u_1 \in W^s_q\left(\Omega, \mathbb{R}^{N_1}\right);\, (1-\delta)\gamma_\partial u_1 = 0\right\}.$$

We denote by $\mathcal{L}(E, F; \mathbb{K})$ the Banach space of all continuous bilinear forms on $E \times F$, and $\langle\cdot,\cdot\rangle \in \mathcal{L}\left(L_{q'}, L_q; \mathbb{R}\right)$ is the duality pairing between

$$L_q := L_q\left(\Omega, \mathbb{R}^M\right)$$

and $L_{q'}$, where $q' := q/(q-1)$, induced by

$$\langle u, v\rangle = \int_\Omega \langle u(x), v(x)\rangle_{\mathbb{R}^M}\, dx.$$

We fix $p > n$ and, given $(t,v) \in \mathbb{R}^+ \times C\left(\overline{\Omega}, G\right)$, we define

$$\boldsymbol{a}_1(t,v) \in \mathcal{L}\left(W^1_{p,\mathcal{B}}, W^1_{p',\mathcal{B}}; \mathbb{R}\right)$$

by

$$\boldsymbol{a}_1(t,v)\,(u_1,w_1) := \langle a^1_{jk}(\,\cdot\,,t,v)\partial_k u_1, \partial_j w_1\rangle + \langle a^1_j(\,\cdot\,,t,v)\partial_j u_1, w_1\rangle.$$

Similarly,

$$\boldsymbol{a}_2(t,v) \in \mathcal{L}\left(W^1_p\left(\Omega,\mathbb{R}^{N_2}\right), W^1_{p'}\left(\Omega,\mathbb{R}^{N_1}\right); \mathbb{R}\right)$$

is defined by

$$\boldsymbol{a}_2(t,v)\,(u_2,w_1) := \langle a^2_{jk}(\,\cdot\,,t,v)\partial_k u_2, \partial_j w_1\rangle + \langle a^2_j(\,\cdot\,,t,v)\partial_j u_2, w_1\rangle.$$

Lastly,

$$W^s_{p,\mathcal{B}} := [W^{-s}_{p',\mathcal{B}}]', \qquad -2+1/p < s < -1+1/p,$$

with respect to the duality pairing induced by $\langle\cdot\,,\,\cdot\rangle$. Observe that

$$W^s_{p,\mathcal{B}} \overset{d}{\hookrightarrow} L_p \overset{d}{\hookrightarrow} W^{s-2}_{p,\mathcal{B}}$$

for $1/p < s < 1+1/p$.

In the following

$$W^s_{p,\mathcal{B}} \times W^t_p := W^s_{p,\mathcal{B}}\left(\Omega,\mathbb{R}^{N_1}\right) \times W^t_p\left(\Omega,\mathbb{R}^{N_2}\right)$$

for $s \in (-2+1/p, -1+1/p) \cup (1/p, 1+1/p)$ and $t \in \mathbb{R}$.

Let J be a nontrivial subinterval of $\mathbb{R}^+$ containing 0. Then a function $u : J \to \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ is said to be *a weak $W^1_{p,\mathcal{B}} \times W^1_p$-solution* of (6.2) on J provided

$$u \in C\left(J, W^1_{p,\mathcal{B}} \times W^1_p\right) \cap C^1\left(J, W^{-1}_{p,\mathcal{B}} \times W^1_p\right)$$

and u satisfies $u\left(\overline{\Omega}\right) \subset G$,

$$\tag{6.3} \begin{aligned} &\langle \dot{u}_1(t), v_1\rangle + \boldsymbol{a}_1(t,u(t))\,(u_1(t),v_1) + \boldsymbol{a}_2(t,u(t))\,(u_2(t),v_1) \\ &\qquad = \langle f_1(\,\cdot\,,t,u(t)), v_1\rangle + \langle g(\,\cdot\,,t,u(t)), \gamma_\partial v_1\rangle_\partial \end{aligned}$$

for $v_1 \in W^1_{p',\mathcal{B}}\left(\Omega,\mathbb{R}^{N_1}\right)$, as well as

$$\dot{u}_2(t) = f_2(\,\cdot\,,t,u(t))$$

for $t \in J$, where

$$\langle v,w\rangle_\partial := \int_{\partial\Omega} \langle v(x), w(x)\rangle_{\mathbb{R}^{N_1}}\, d\sigma(x).$$

REMARKS 6.2

(a) Observe that (6.3) is—formally—obtained from (6.2) by multiplying the first equation in (6.2) by $v_1 \in W^1_{p',\mathcal{B}}\left(\Omega,\mathbb{R}^{N_1}\right)$, integrating over Ω, using Gauss' theorem, and taking into consideration the boundary condition in

(6.2) as well as the fact that v_1 vanishes on those components Γ of $\partial\Omega$ for which $\delta|\Gamma = 0$.

(b) It should be observed that the term $\mathcal{B}_2(t,u)u_2$ in (6.2) occurs on those components Γ of $\partial\Omega$ only on which $\delta|\Gamma = 1$. On the component Γ with $\delta|\Gamma = 0$, (6.2) reduces to the Dirichlet boundary condition $u_1|\Gamma = 0$ for u_1 alone. Thus, if $\delta = 0$, problem (6.2) is a special case of problem (5.1).

The following lemma shows that the spaces $W^s_{p,\mathcal{B}}$ are to a certain extent stable under the $(\cdot\,,\,\cdot)_{\theta,p}$ interpolation method.

LEMMA 6.3. *Suppose that* $s, s_0, s_1 \in (-2+1/p, -1+1/p) \cup (1/p, 1+1/p)$ *satisfy* $s_0 < s < s_1$ *and* $s \notin \mathbb{Z}$. *Then, letting* $\theta := (s-s_0)/(s_1-s_0)$,

$$\left(W^{s_0}_{p,\mathcal{B}}, W^{s_1}_{p,\mathcal{B}}\right)_{\theta,p} = W^s_{p,\mathcal{B}}.$$

PROOF. Put $\mathcal{B}_0 := \delta\partial_\nu + (1-\delta)\gamma_\partial$ and

$$E_\alpha := \begin{cases} \{u \in W^{2\alpha}_p;\ \mathcal{B}_0 u = 0\} & 1+\frac{1}{p} < 2\alpha \le 2, \\ W^{2\alpha}_{p,\mathcal{B}} & \frac{1}{p} < 2\alpha < 1+\frac{1}{p}, \\ W^{2\alpha}_p & -1+\frac{1}{p} < 2\alpha < \frac{1}{p}, \\ W^{2\alpha}_{p,\mathcal{B}} & -2+\frac{1}{p} < 2\alpha < -1+\frac{1}{p}, \\ \{u \in W^{-2\alpha}_{p'};\ \mathcal{B}_0 u = 0\}' & -2 \le 2\alpha < -2+\frac{1}{p}, \end{cases}$$

where the dual space is taken with respect to the duality pairing induced by the L_p-duality pairing.

By means of the results of Grisvard [20] and Seeley [28] we find that, letting $(\cdot\,,\,\cdot)_\theta := (\cdot\,,\,\cdot)_{\theta,p}$,

$$E_\alpha = (E_0, E_1)_\alpha, \quad 0 < \alpha < 1, \quad 2\alpha \notin \{1/p, 1, 1+1/p\}. \tag{6.4}$$

Thus the reiteration theorem (e.g., [30, Theorem 1.10.2]) implies

$$\left(E_\alpha, E_\beta\right)_\theta = (E_0, E_1)_{\alpha(1-\theta)+\beta\theta} = E_\gamma, \tag{6.5}$$

provided

$$2\alpha, 2\beta, 2\gamma := 2\alpha(1-\theta) + 2\beta\theta \in [0,2] \setminus \{1/p, 1, 1+1/p\} \tag{6.6}$$

satisfy $\alpha < \gamma < \beta$.

Let F_α, $\alpha \in [-2,2] \backslash (\mathbb{Z} + 1/p')$, be defined in the same way as E_α, except that p is being replaced by p'. Then (6.4)–(6.6) hold for the spaces F_α, provided p is being replaced by p' and $(\cdot\,,\,\cdot)_{\theta,p}$ by $(\cdot\,,\,\cdot)_{\theta,p'}$, respectively. Observe that

$E_\alpha = (F_{-\alpha})'$ for $2\alpha \in [-2,2]\backslash(\mathbb{Z} + 1/p)$. Thus, given $\alpha, \beta, \gamma \in [-1,0]$ with $2\alpha, 2\beta, 2\gamma \notin \{-2+1/p, -1, -1+1/p\}$ and $\alpha < \gamma < \beta$, it follows that

$$E_\gamma = \left(F_{-\gamma}\right)' = \left(F_{-\beta}, F_{-\alpha}\right)'_{1-\theta} = \left(E_\alpha, E_\beta\right)_\theta, \tag{6.7}$$

where $\theta := (\gamma - \alpha)/(\beta - \alpha)$ (cf. [30, Theorems 1.11.2 and 1.3.3(b)]).

Put $\mathcal{A}_0 := -\Delta$ and $A_0 := \mathcal{A}_0|W^2_{p,\mathcal{B}_0}$ considered as an unbounded linear operator in E_0. Then $A_0 \in \mathcal{H}(E_1, E_0)$ and it is known (e.g., [19], [27]) that the purely imaginary powers A_0^{it} are uniformly bounded for t in a neighbourhood of zero in $\mathbb{R}$. Thus, denoting by $[\cdot\,,\cdot]_\theta$, $0 < \theta < 1$, the complex interpolation functor, we deduce from [2, Theorems 3.3 and 1.3] that

$$[E_{-1}, E_1]_{1/2} = E_0. \tag{6.8}$$

Hence the reiteration theorem (cf. [30, Theorem 1.10.3.2]) implies

$$E_\alpha = (E_{-1}, E_0)_{1+\alpha} = (E_{-1}, E_1)_{(1+\alpha)/2} \tag{6.9}$$

for $2\alpha \in [-2,0]\backslash\{-2+1/p, -1, -1+1/p\}$ and

$$E_\beta = (E_0, E_1)_\beta = (E_{-1}, E_1)_{(1+\beta)/2} \tag{6.10}$$

for $2\beta \in [0,2]\backslash\{1/p, 1, 1+1/p\}$. Consequently, by employing the reiteration theorem once more,

$$\left(E_\alpha, E_\beta\right)_\theta = (E_{-1}, E_1)_{(1+\alpha(1-\theta)+\beta\theta)/2} = E_\gamma \tag{6.11}$$

for the above values of α and β and for $\gamma := \alpha(1-\theta) + \beta\theta \in (-2,2)$ with $2\gamma \notin \mathbb{Z} \cup (\mathbb{Z} + 1/p)$.

By using again Seeley's [28] results,

$$E_{1/2} = [E_0, E_1]_{1/2}, \quad F_{1/2} = [F_0, F_1]_{1/2}.$$

Hence

$$E_{-1/2} = F'_{1/2} = [F_0, F_1]'_{1/2} = [E_{-1}, E_0]_{1/2},$$

thanks to the duality theorem for the complex interpolation functor (e.g., [11, Corollary 4.5.2]). Thus (6.8) and the reiteration theorems for the complex method (e.g., [11, Theorem 4.6.1]) imply

$$E_{1/2} = [[E_{-1}, E_1]_{1/2}, E_1]_{1/2} = [E_{-1}, E_1]_{3/4} \tag{6.12}$$

and

$$E_{-1/2} = [E_{-1}, [E_{-1}, E_1]_{1/2}]_{1/2} = [E_{-1}, E_1]_{1/4}. \tag{6.13}$$

Suppose now that $2\gamma \in (-1,1)\backslash\mathbb{Z} + 1/p$ and $\gamma \neq 0$. Then (6.12), (6.13), and the reiteration theorem for the real method show that

$$E_\gamma = (E_{-1}, E_1)_{(1+\gamma)/2} = \left(E_{-1/2}, E_{1/2}\right)_\theta, \tag{6.14}$$

where $(1-\theta)(1/4)+\theta(3/4) = (1+\gamma)/2$, that is, $\gamma = (1-\theta)(-1/2)+\theta/2$. Similarly, by using (6.9), (6.10), (6.12), and the reiteration theorem,

$$E_\gamma = (E_{-1}, E_1)_{(1+\gamma)/2} = \left(E_\alpha, E_{1/2}\right)_\theta, \tag{6.15}$$

where $(1-\theta)[(1+\alpha)/2] + 3\theta/4 = (1+\gamma)/2$, that is, $\gamma = (1-\theta)\alpha + \theta/2$, provided $2\alpha \in [-2,1)\backslash(\mathbb{Z} + 1/p) \cup \{-1\}$ and $2\gamma \in (2\alpha, 1)\backslash(\mathbb{Z} + 1/p) \cup \mathbb{Z}$. Lastly, by an analogous argument we find that

$$E_\gamma = \left(E_{-1/2}, E_\beta\right)_\theta, \quad \gamma = (1-\theta)(-1/2) + \theta\beta, \tag{6.16}$$

provided $2\beta \in (-1,2]\backslash(\mathbb{Z} + 1/p) \cup \mathbb{Z}$ and $2\alpha \in (-1/2, 2\beta)\backslash(\mathbb{Z} + 1/p) \cup \mathbb{Z}$.

Now the assertion follows from (6.5), (6.7), (6.11), and (6.14)–(6.16) together with the fact that $E_\alpha = W^{2\alpha}_{p,\mathcal{B}}$ for

$$2\alpha \in (-2 + 1/p, -1 + 1/p) \cup (1/p, 1 + 1/p).$$

∎

It should be noted that the assumption $p > n$ has not been used in the proof of the above lemma.

Observe that

$$W^s_{p,\mathcal{B}} \times W^t_p \hookrightarrow C^\mu\left(\overline{\Omega}, \mathbb{R}^{N_1 \times N_2}\right), \quad n/p < s \wedge t < 1 + 1/p, \tag{6.17}$$

where $\mu := s \wedge t - n/p$. Hence

$$\begin{aligned} &V_{s,t} := \left\{u \in W^s_{p,\mathcal{B}} \times W^t_p;\ u\left(\overline{\Omega}\right) \subset G\right\} \\ &\text{is open in } W^s_{p,\mathcal{B}} \times W^t_p \quad \text{for } s,t \in (n/p, 1]. \end{aligned} \tag{6.18}$$

After these preparations we can prove the main result of this section.

THEOREM 6.4. *Given $u_0 \in V_1 := V_{1,1}$, system* (6.2) *possesses a unique maximal weak $W^1_{p,\mathcal{B}} \times W^1_p$-solution $u(\cdot, u_0)$ satisfying $u(\cdot, u_0) = u_0$. The maximal interval of existence, $J(u_0)$, is open in $\mathbb{R}^+$. If*

$$\operatorname{dist}(u([0,T] \cap J(u_0), u_0), \partial V_1) = 0$$

and

$$u(\cdot, u_0) \in BUC\left([0,T] \cap J(u_0), W^1_p\left(\Omega, \mathbb{R}^{N_1 \times N_2}\right)\right)$$

for each $T > 0$, then $J(u_0) = \mathbb{R}^+$.

PROOF. Put $X_1 := W^1_{p,\mathcal{B}}$, $X_0 := W^{-1}_{p,\mathcal{B}}$, and $Y := W^1_p\left(\Omega, \mathbb{R}^{N_2}\right)$. Then condition (A1) is satisfied. Fix α and β with $1 + n/p < 2\beta < 2\alpha < 2$ and $2\alpha + 2\beta > 3 + n/p$, and put $X_{\alpha-1} := W^{2\alpha-3}_{p,\mathcal{B}}$ and $Y_\alpha := W^{2\alpha-1}_p\left(\Omega, \mathbb{R}^{N_2}\right)$. Letting $(\cdot\,,\cdot)_\theta := (\cdot\,,\cdot)_{\theta,p}$ for $0 < \theta < 1$, Lemma 6.3 implies that

$$X_\theta := (X_0, X_1)_\theta = W^{2\theta-1}_{p,\mathcal{B}}$$

for $\theta \in \{\alpha, \beta\}$ and that the first part of condition (A3) is satisfied.

Put $V_\beta := V_{2\beta-1,1}$ and $\widetilde{V}_\beta := V_{2\beta-1,2\alpha-1}$, and observe that V_β is open in $E_\beta := X_\beta \times Y$ and $\widetilde{V}_\beta$ is open in $X_\beta \times Y_\alpha$, thanks to (6.18). Moreover, $V_\beta = \widetilde{V}_\beta \cap E_\beta$.

It follows from (6.17) that

$$V_\beta \hookrightarrow \widetilde{V}_\beta \hookrightarrow C^{2\beta-1-n/p}\left(\overline{\Omega}, G\right).$$

Thus we deduce from the results in [10] that

$$\boldsymbol{a}_1 \in C^\infty\left(\mathbb{R}^+\times V_\beta, \mathcal{L}\left(X_1, X_0'; \mathbb{R}\right)\right) \cap C^\infty\left(\mathbb{R}^+\times \widetilde{V}_\beta,\ \mathcal{L}\left(X_\alpha, X_{\alpha-1}'; \mathbb{R}\right)\right),$$

thanks to $2\beta - n/p > 1 - 2\alpha$. Similarly,

$$\boldsymbol{a}_2 \in C^\infty\left(\mathbb{R}^+\times V_\beta, \mathcal{L}\left(Y, X_0'; \mathbb{R}\right)\right) \cap C^\infty\left(\mathbb{R}^+\times \widetilde{V}_\beta,\ \mathcal{L}\left(Y_\alpha, X_{\alpha-1}'; \mathbb{R}\right)\right).$$

Consequently, there exist

$$A_{11} \in C^\infty\left(\mathbb{R}^+\times V_\beta, \mathcal{L}\left(X_1, X_0\right)\right) \cap C^\infty\left(\mathbb{R}^+\times \widetilde{V}_\beta,\ \mathcal{L}(X_\alpha, X_{\alpha-1})\right),$$

and

$$A_{12} \in C^\infty\left(\mathbb{R}^+\times V_\beta, \mathcal{L}\left(Y, X_0\right)\right) \cap C^\infty\left(\mathbb{R}^+\times \widetilde{V}_\beta,\ \mathcal{L}(Y_\alpha, X_{\alpha-1})\right)$$

such that

$$\boldsymbol{a}_1(t,v)(u_1,w_1) = \langle A_{11}(t,v)u_1, w_1\rangle, \quad (u_1,w_1) \in X_\alpha \times X_{\alpha-1}',$$

and

$$\boldsymbol{a}_2(t,v)(u_2,w_1) = \langle A_{12}(t,v)u_2, w_1\rangle, \quad (u_2,w_1) \in Y_\alpha \times X_{\alpha-1}',$$

for $(t,v) \in \mathbb{R}^+\times \widetilde{V}_\beta$. Moreover, [6, Theorem 2.1] (cf. also [7]) guarantees that

$$A_{11}(t,v) \in \mathcal{H}\left(X_1, X_0\right) \cap \mathcal{H}\left(X_\alpha, X_{\alpha-1}\right), \quad (t,v) \in \mathbb{R}^+\times \widetilde{V}_\beta.$$

We put

$$f_1(t,v) := f_1(\cdot\,,t,v) + \gamma_\partial' g(\cdot\,,t,v), \quad f_2(t,v) := f_2(\cdot,t,v),$$

where

$$\gamma_\partial' \in \mathcal{L}\left(B^{-1/p}_{p,p}, W^{-1}_{p,\mathcal{B}}\right) \cap \mathcal{L}\left(B^{2\alpha-2-1/p}_{p,p}, W^{2\alpha-3}_{p,\mathcal{B}}\right)$$

is the dual of the trace operator

$$\gamma_\partial \in \mathcal{L}\left(W^1_{p',\mathcal{B}}, B^{1-1/p'}_{p',p'}\right) \cap \mathcal{L}\left(W^{3-2\alpha}_{p',\mathcal{B}}, B^{3-2\alpha-1/p'}_{p',p'}\right),$$

and where $B^\sigma_{q,q} = B^\sigma_{q,q}\left(\partial\Omega, \mathbb{R}^{N_1}\right)$, $\sigma \in \mathbb{R}$, $1 < q < \infty$, are Besov spaces on $\partial\Omega$. Since

$$C\left(\partial\Omega, \mathbb{R}^{N_1}\right) \hookrightarrow L_p\left(\partial\Omega, \mathbb{R}^{N_1}\right) \hookrightarrow B^{-1/p}_{p,p} \hookrightarrow B^{2\alpha-2-1/p}_{p,p}$$

and

$$X_1 \times Y \hookrightarrow X_\beta \times X_\alpha \xrightarrow{\gamma_\partial} B^{2\beta-1-1/p}_{p,p}\left(\partial\Omega, \mathbb{R}^{N_1+N_2}\right) \hookrightarrow C\left(\partial\Omega, \mathbb{R}^{N_1+N_2}\right),$$

where $\xrightarrow{\gamma_\partial}$ stands for 'γ_∂ is a continuous linear map', it is not difficult to verify that

$$f = (f_1, f_2) \in C^\infty\left(\mathbb{R}^+ \times \tilde{V}_\beta, X_0 \times Y\right).$$

Since $V_\beta \hookrightarrow \tilde{V}_\beta$ and $X_0 \times Y \hookrightarrow X_{\alpha-1} \times Y_\alpha$, we see, finally, that condition (A2) and the second part of condition (A3) are satisfied. Hence the assertion follows from Theorem 3.4 and Corollary 3.5 and the obvious fact that—in the present situation—a solution of (3.3) is a weak $W^1_{p,\mathcal{B}} \times W^1_p$-solution and vice versa. ∎

It is obvious that Remarks 5.3(a) and (b) apply here too.

7. - Diffusion in Polymers

We denote by $\mathcal{L}_{\text{sym}}(\mathbb{R}^n)$ the linear subspace of $\mathcal{L}(\mathbb{R}^n)$ consisting of all symmetric $n \times n$-matrices. Given $B \in \mathcal{L}_{\text{sym}}(\mathbb{R}^n)$, we write $B > 0$ if B is positive definite, and $B \geq 0$ if it is positive semidefinite.

We assume that

$$D, E \in C^\infty\left(\overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^2, \mathcal{L}_{\text{sym}}(\mathbb{R}^n)\right), \quad M \in C^\infty\left(\overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R}^n\right),$$

and that

$$D(x,t,\eta) > 0, \quad E(x,t,\eta) \geq 0, \quad (x,t,\eta) \in \overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^2. \tag{7.1}$$

We put

$$j(t,c,\sigma) := -D(\cdot\,,t,c,\sigma)\nabla c - E(\cdot\,,t,c,\sigma)\nabla\sigma + M(\cdot\,,t,c,\sigma)c \tag{7.2}$$

for $(t,(c,\sigma)) \in \mathbb{R}^+ \times C\left(\overline{\Omega}, \mathbb{R}^2\right)$.

We also assume that

$$\varphi \in C^\infty\left(\overline{\Omega} \times \mathbb{R}, \mathbb{R}^+\right) \tag{7.3}$$

and that

$$h_r \in C^\infty\left(\overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R}\right), \quad r = 1, 2.$$

We then consider the following obvious n-dimensional generalization of (1)–(3):

$$\begin{aligned} \partial_t c + \operatorname{div} j(t, c, \sigma) &= h_1(\cdot\,, t, c, \sigma) \\ \partial_t \sigma - \varphi(\cdot\,, c)\partial_t c &= h_2(\cdot\,, t, c, \sigma) \end{aligned} \quad \text{in } \Omega \times (0, \infty), \tag{7.4}$$

subject to the Dirichlet boundary condition

$$c = \psi \quad \text{on } \partial\Omega \times (0, \infty), \tag{7.5}$$

where $\psi \in C^\infty\left(\partial\Omega \times \mathbb{R}^+, \mathbb{R}\right)$.

THEOREM 7.1. *Suppose that $n < p < \infty$ and that $(c_0, \sigma_0) \in W_p^2\left(\Omega, \mathbb{R}^2\right)$ satisfies*

$$c_0|_{\partial\Omega} = \psi(\cdot\,, 0). \tag{7.6}$$

Then (7.1)–(7.5) *possesses a unique maximal solution*

$$(c, \sigma) \in C\left(J, W_p^2\left(\Omega, \mathbb{R}^2\right)\right) \cap C^1\left(J, L_p(\Omega, \mathbb{R}) \times W_p^2(\Omega, \mathbb{R})\right) \tag{7.7}$$

satisfying $(c(0), \sigma(0)) = (c_0, \sigma_0)$, where $J := J(c_0, \sigma_0)$, the maximal interval of existence, is open in $\mathbb{R}^+$. If $(c, \sigma) \in BUC\left([0, T] \cap J, W_p^2\left(\Omega, \mathbb{R}^2\right)\right)$ for each $T > 0$, then $J = \mathbb{R}^+$.

PROOF. Choose any $\widetilde{\psi} \in C^\infty\left(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R}\right)$ satisfying $\gamma_\partial \widetilde{\psi}(\cdot\,, t) = \psi(\cdot\,, t)$ for $t \geq 0$. Define $\Phi \in C^\infty\left(\overline{\Omega} \times \mathbb{R}, \mathbb{R}\right)$ by

$$\Phi(x, \xi) := \int_0^\xi \varphi(x, \eta)\, d\eta.$$

Suppose that (c, σ) satisfies (7.7) and is a solution of (7.4), (7.5) on J. Then, letting

$$u_1 := c - \widetilde{\psi}, \quad u_2 := \sigma - \Phi(\cdot\,, c), \tag{7.8}$$

it is easily verified that $u := (u_1, u_2)$ is a solution of a system of the form (5.1) on J and

$$u \in C\left(J, W_{p,\mathcal{B}}^2 \times W_p^2\right) \cap C^1\left(J, L_p \times W_p^2\right), \tag{7.9}$$

where $\mathcal{B} := \gamma_\partial$ and

$$
\begin{aligned}
&a^1_{jk}(\cdot\,, t, u, \partial u) \\
(7.10) \qquad &:= \left(\mathrm{D}_{jk} + \varphi E_{jk}\right)\left(\cdot\,, t, u_1 + \widetilde{\psi}(\cdot\,, t), u_2 + \Phi\big(\cdot\,, t, u_1 + \widetilde{\psi}(\cdot\,, t)\big)\right), \\
&a^2_{jk}(\cdot\,, t, u, \partial u) := E_{jk}\left(\cdot\,, t, u_1 + \widetilde{\psi}(\cdot\,, t), u_2 + \Phi\big(\cdot\,, t, u_1 + \widetilde{\psi}(\cdot\,, t)\big)\right).
\end{aligned}
$$

More precisely, the transformation (7.8) is a bijection between the solutions (c, σ) on J of (7.4), (7.5) satisfying (7.7), and the solutions u on J of an appropriate system of the form (5.1), which satisfy (7.9). Moreover, the differential operators $\mathcal{A}_1$ and $\mathcal{A}_2$ of (5.1) are defined by means of (7.10). Since (7.1) and the fact that $\varphi \geq 0$ imply that $(\mathcal{A}_1(t, u, \partial u), \mathcal{B})$ is normally elliptic for $(t, u) \in \mathbb{R}^+ \times \left(W^2_{p,\mathcal{B}} \times W^2_p\right)$, thanks to Remarks 4.1(a) and (b), the assertion follows from Theorem 5.2. ∎

We fix $\delta \in C(\partial\Omega, \{0, 1\})$ and assume that

$$k \in C^\infty\left(\partial\Omega \times \mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R}\right).$$

We then consider the system (7.1)–(7.4) under the nonlinear boundary condition

$$(7.11) \qquad -\delta(j(t, c, \sigma)|\nu) + (1 - \delta)c = \delta k(\cdot\,, t, c, \sigma) + (1 - \delta)\psi \quad \text{on } \partial\Omega \times (0, \infty),$$

where $(\cdot\,|\,\cdot)$ is the inner product in $\mathbb{R}^n$. Observe that (7.11) reduces to the 'flux boundary condition'

$$-(j(t, c, \sigma)|\nu) = k(\cdot\,, t, c, \sigma) \quad \text{on } \partial\Omega \times (0, \infty)$$

if $\delta = 1$.

A function $(c, \sigma) : J \to W^1_p\left(\Omega, \mathbb{R}^2\right)$ is said to be a *weak W^1_p-solution* of (7.1)–(7.4), (7.11) on J, if, given any $\widetilde{\psi} \in C^\infty\left(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R}\right)$ with

$$\gamma_\partial \widetilde{\psi}(\cdot\,, t) = \psi(\cdot\,, t) \text{ on } \partial\Omega,$$

$$
\begin{aligned}
&\left(c - \widetilde{\psi}, \sigma\right) \in C\left(J, W^1_{p,\mathcal{B}} \times W^1_p\right) \cap C^1\left(J, W^{-1}_{p,\mathcal{B}} \times W^1_p\right), \\
(7.2) \qquad &\langle \dot{c}(t), v_1 \rangle - \langle j(t, c(t), \sigma(t)), \nabla v_1 \rangle \\
&= \langle h_1(t, c(t), \sigma(t)), v_1 \rangle + \langle k(t, c(t), \sigma(t)), \gamma_\partial v_1 \rangle_\partial
\end{aligned}
$$

for $t \in J$ and $v_1 \in W^1_{p',\mathcal{B}}$, and (c, σ) satisfies the second equation in (7.4) on J.

THEOREM 7.2. *Suppose that* $n < p < \infty$ *and that* $(c_0, \sigma_0) \in W_p^1(\Omega, \mathbb{R}^2)$ *satisfies*

$$(1-\delta)c_0|_{\partial\Omega} = (1-\delta)\psi_0(\,\cdot\,, 0).$$

Then problem (7.1)–(7.4), (7.11) *possesses a unique maximal weak* W_p^1-*solution* (c, σ) *satisfying* $(c(0), \sigma(0)) = (c_0, \sigma_0)$. *The maximal interval of existence,* $J := J(c_0, \sigma_0)$, *is open in* $\mathbb{R}^+$. *If*

$$(c, \sigma) \in BUC([0, T] \cap J, W_p^1(\Omega, \mathbb{R}^2))$$

for each $T > 0$, *then* $J = \mathbb{R}^+$.

PROOF. By means of the transformation (7.8), problem (7.1)–(7.4), (7.11) is reduced to an equivalent problem of the form (6.2), where a_{jk}^r, $r = 1, 2$, are given by (7.10). (Observe that these quantities are independent of ∂u.) Now the assertion follows easily from Remark 6.1 and Theorem 6.4. ∎

REMARKS 7.3.

(a) It is obvious that in Theorem 7.1 we can consider more general 'constant' boundary conditions, the Neumann boundary condition $\partial_\nu c = 0$, for example. It is also clear that—again in Theorem 7.1—h_1 can depend smoothly upon the spacial derivatives ∂c, $\partial\sigma$ in an arbitrary way.

(b) Of course, we can add a term $N(\,\cdot\,, t, c, \sigma)\sigma$ to the 'flux vector' (7.2), where $N \in C^\infty(\overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R}^n)$, without changing the validity of Theorems 7.1 and 7.2.

(c) The smoothness assumptions in the above theorems can be considerably weakened.

(d) If $\delta = 0$, the boundary condition (7.11) reduces to the Dirichlet boundary condition (7.5). Hence, if $(c_0, \sigma_0) \in W_p^1(\Omega, \mathbb{R}^2)$ satisfies the compatibility condition $\sigma_0|_{\partial\Omega} = \psi(\,\cdot\,, 0)$, Theorem 7.2 guarantees the existence of a unique maximal weak W_p^1-solution of (7.1)–(7.5), defined on the maximal interval of existence $J_0 := J_0(c_0, \sigma_0)$. If, in addition, $(c_0, \sigma_0) \in W_p^2(\Omega, \mathbb{R}^n)$, Theorem 7.1 guarantees the existence of a unique maximal solution to (7.1)–(7.5) satisfying (7.7). It is clear that the latter solution is also a weak W_p^1-solution on J of (7.1)–(7.5). Hence $J \subset J_0$. However, due to the lack of a regularizing property of problem (7.1)–(7.5), it cannot be shown that $J = J_0$. Thus, if $(c_0, \sigma_0) \in W_p^2(\Omega, \mathbb{R}^n)$, the maximal weak W_p^1-solution may exist on a larger interval than the W_p^2-solution of Theorem 7.1.

Finally, we shall show that the system (1)–(4) has the positivity preserving property for c mentioned in the Introduction. In fact, we shall prove the following more general

THEOREM 7.4. *Suppose that* E *is independent of* σ, *that*

$$E(x, t, 0) = 0, \quad h_1(x, t, 0, \sigma) \geq 0, \quad (x, t, \sigma) \in \overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R},$$

and that

$$k(x,t,0,\sigma) \geq 0, \quad (x,t,\sigma) \in \partial\Omega \times \mathbb{R}^+ \times \mathbb{R}.$$

Assume that $n < p < \infty$, *that* $(c_0, \sigma_0) \in W_p^1(\Omega, \mathbb{R}^2)$ *satisfies*

$$(1-\delta)c_0|_{\partial\Omega} = (1-\delta)\psi(\cdot, 0),$$

and that (c,σ) *is the unique maximal weak* W_p^1*-solution of* (7.1)–(7.4), (7.11) *satisfying* $(c(0), \sigma(0)) = (c_0, \sigma_0)$. *Then* $c_0 \geq 0$ *implies*

$$c(t) \geq 0 \textit{ for } t \in J := J(c_0, \sigma_0).$$

PROOF. There exist

$$E_1 \in C^\infty\left(\overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}\right), \quad h_{11} \in C^\infty\left(\overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R}\right)$$

and

$$k_1 \in C^\infty\left(\partial\Omega \times \mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R}\right)$$

such that

$$E(\cdot, t, c) = E_1(\cdot, t, c)c, \quad h_1(\cdot, t, c, \sigma) = h_1(\cdot, t, 0, \sigma) + h_{11}(\cdot, t, c, \sigma)c$$

and

$$k(\cdot, t, c, \sigma) = k(\cdot, t, 0, \sigma) + k_1(\cdot, t, c, \sigma)c$$

for $(t, c, \sigma) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$. Hence we see that $c = c(t)$ satisfies

$$\begin{aligned} &\langle \dot{c}(t), v_1 \rangle + \langle D(t)\nabla c(t) + E(t)c(t), \nabla v_1 \rangle \\ &\qquad - \langle h_{11}(t)c(t), v_1 \rangle - \langle k_1(t)\gamma_\partial c(t), \gamma_\partial v_1 \rangle_\partial = \langle h_0(t), v_1 \rangle + \langle k_0(t), \gamma_\partial v_1 \rangle_\partial \end{aligned}$$

for $t \in J$ and $v_1 \in W^1_{p',\mathcal{B}}(\Omega, \mathbb{R})$, as well as

$$c(\cdot, t) - \widetilde{\psi}(\cdot, t) \in W^1_{p,\mathcal{B}}(\Omega, \mathbb{R}), \quad t \in J,$$

where

$$\begin{aligned} D(t) &:= D(\cdot, t, c(t), \sigma(t)), \\ E(t) &:= E_1(\cdot, t, c(t))\nabla\sigma(t) - M(\cdot, t, c(t), \sigma(t)), \\ h_{11}(t) &:= h_{11}(\cdot, t, c(t), \sigma(t)), \quad k_1(t) := k_1(\cdot, t, \gamma_\partial c(t), \gamma_\partial \sigma(t)), \end{aligned}$$

and

$$h_0(t) := h_1(\cdot, t, 0, \sigma(t)), \quad k_0(t) := k(\cdot, t, 0, \gamma_\partial \sigma(t)).$$

Since (c, σ) has the regularity properties specified in (7.12) and since

$$W^{2\gamma}_{p,\mathcal{B}} \times W^1_p = \left(W^{-1}_{p,\mathcal{B}} \times W^1_p, W^1_{p,\mathcal{B}} \times W^1_p\right)_{\gamma-1/2,p}, \quad 1/p < 2\gamma < 1,$$

by Lemma 6.3, it follows that

$$\left(c-\tilde{\psi},\sigma\right)\in C^{(1-2\gamma)/2}\left(J,W^{2\gamma}_{p,\mathcal{B}}\times W^1_p\right),\quad 1/p<2\gamma<1,$$

(cf., for example, [5, formulas (8.7)–(8.9)]). Hence (6.17) implies

$$(c,\sigma)\in C^{(1-2\gamma)/2}\left(J,C\left(\overline{\Omega},\mathbb{R}^2\right)\right),\quad n/p<2\gamma<1.$$

From this we deduce that

$$D,h_{11},h_0\in C^{(1-2\gamma)/2}\left(J,C\left(\overline{\Omega},\mathbb{R}\right)\right),$$
$$k_0,k_1\in C^{(1-2\gamma)/2}(J,C(\partial\Omega,\mathbb{R})),$$

for $n/p<2\gamma<1$ and that

$$E\in C^1\left(J,L_p(\Omega,\mathbb{R})\right).$$

Now the assertion is a consequence of an obvious generalization of the maximum principle. ∎

REMARK 7.5. If $\delta=0$ (Dirichlet boundary condition), Theorem 7.4 is also true for the W^2_p-solution of Theorem 7.1, provided $(c_0,\sigma_0)\in W^2_p\left(\Omega,\mathbb{R}^2\right)$, of course. This follows from Remark 7.3(d).

REFERENCES

[1] H. AMANN, *Semigroups and nonlinear evolution equations*, Linear Algebra Appl. **84** (1986), 3–32.

[2] H. AMANN, *On abstract parabolic fundamental solutions*, J. Math. Soc. Japan **39** (1987), 93–116.

[3] H. AMANN, *Dynamic theory of quasilinear parabolic equations - I. Abstract evolution equations*, Nonlinear Anal., Theory, Methods & Appl. **12** (1988), 895–919.

[4] H. AMANN, *Parabolic evolution equations and nonlinear boundary conditions*, J. Differential Equations **72** (1988), 201–270.

[5] H. AMANN, *Dynamic theory of quasilinear parabolic equations - II. Reaction-diffusion systems*, Differential Integral Equations **3** (1990), 13–75.

[6] H. AMANN, *Dynamic theory of quasilinear parabolic systems - III. Global existence*, Math. Z. **202** (1989), 219–250.

[7] H. AMANN, *Erratum*, Math. Z. **205** (1990), 331.

[8] H. AMANN, *Global existence for a class of highly degenerate parabolic systems*, Japan J. Industrial Appl. Math. (to appear).

[9] H. AMANN, *Hopf bifurcation in quasilinear reaction-diffusion systems*, in "Proceedings of the Claremont Conference on Differential Equations and Applications to Biology and Population Dynamics" (to appear).

[10] H. AMANN, *Substitution operators in Besov spaces*, (to appear).

[11] J. BERGH - J. LÖFSTRÖM, *Interpolation Spaces, An Introduction*, Springer Verlag, Berlin, 1976.

[12] PH. CLÉMENT - H.J.A.M. HEIJMANS et al., *One-Parameter Semigroups*, CWI Monographs **5**, North-Holland, Amsterdam, 1987.

[13] D.S. COHEN - R.W. COX, *A mathematical model for stress-driven diffusion in polymers*, J. Polymer Science, Part B: Polymer Physics **27** (1989), 589–602.

[14] D.S. COHEN - A.B. WHITE Jr., *Sharp fronts due to diffusion and stress at the glass transition in polymers*, J. Polymer Science, Part B: Polymer Physics **27** (1989), 1731–1747.

[15] D.S. COHEN - A.B. WHITE Jr., *Sharp fronts due to diffusion and viscoelastic relaxation in polymers*, SIAM J. Appl. Math. (to appear).

[16] R.W. COX, *A model for stress-driven diffusion in polymers*, Ph.D. thesis, Dept. Appl. Math., California Institute of Technology, (1988).

[17] R.W. COX, *Stress-assisted diffusion: a free boundary problem*, (to appear).

[18] K.-J. ENGEL, *Polynomial operator matrices as semigroup generators: the 2×2 case*, Math. Ann. **284** (1989), 563–576.

[19] D. FUJIWARA, *Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order*, Proc. Japan. Acad. **43** (1967), 82–86.

[20] P. GRISVARD, *Équations différentielles abstraites*, Ann. Sci. École Norm. Sup., (4), **2** (1969), 311–395.

[21] D. GUIDETTI, *On interpolation with boundary conditions*, (to appear).

[22] T. KATO, *Remarks on pseudo-resolvents and infinitesimal generators of semigroups*, Proc. Japan Acad. **35** (1959), 467–468.

[23] T. KATO, *Perturbation Theory for Linear Operators*, Springer Verlag, New York, 1966.

[24] M.A. KRASNOSELSKII - P.P. ZABREIKO - E.I. PUSTYLNIK - P.E. SOBOLEVSKI, *Integral Operators in Spaces of Summable Functions*, Noordhoff, Leyden, 1976.

[25] A. LUNARDI, *On the local dynamical system associated to a fully nonlinear abstract parabolic equation*, in "Nonl. Anal. and Appl.", ed. V. Lakshmikantham, M. Decker, New York, 1987, 319–326.

[26] R. NAGEL, *Towards a "matrix theory" for unbounded operator matrices*, Math. Z. **201** (1989), 57–68.

[27] R. SEELEY, *Norms and domains of the complex powers A_B^z*, Amer. J. Math. **93** (1971), 299–309.

[28] R. SEELEY, *Interpolation in L^p with boundary conditions*, Stud. Math. Appl. **XLIV** (1972), 47–60.

[29] E. Sinestrari, *On the abstract Cauchy problems of parabolic type in spaces of continuous functions*, J. Math. Anal. Appl. **107** (1985), 16–66.

[30] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North Holland, Amsterdam, 1978.

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