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1. - Introduction

As is well known, the qualitative properties of the motions of a linearly hyperelastic body $\mathcal{B}$ (such as Uniqueness, Domain of Influence Theorem, Work and Energy Theorem, Reciprocity Relation, etc. ...) are strictly linked with the behaviour of the fields that express the material features of $\mathcal{B}$ (the density ρ and the elasticity tensor $\mathbb{C}$ (cf. Section 2)). If $\mathcal{B}$ is bounded, the above properties may be all proved by only requiring that ρ and $\mathbb{C}$ are regular and — as far as the domain of influence theorem is concerned — $\mathbb{C}$ is positive definite [1]. Under this last assumption on $\mathbb{C}$, and the hypothesis that the initial and boundary data have a compact support, the extension of these theorems to unbounded bodies, has been performed in [2] for homogeneous and isotropic materials, in [3] for homogeneous materials, and in [1] by assuming $\mathbb{C}$ to be regular and bounded, and ρ to be continuous and positive. Moreover, in [4] the uniqueness of the displacement problem has been proved by assuming $\mathcal{B}$ to be homogeneous and $\mathbb{C}$ to be semi-strongly elliptic.

The problem of extending the above results to unbounded bodies whose acoustic tensor $\mathbf{A}$ (cf. Section 2) is not necessarily bounded, has been tackled in [5-7]. Such extension turned out to be possible provided ρ is positive, $\mathbb{C}$ is positive semidefinite and $\mathbf{A}$ is regular and satisfies a suitable growth condition at infinity, the so-called *hyperbolicity condition* (cf. Section 2).

When this last condition and/or the regularity assumption on $\mathbf{A}$ are given up, then the above theorems loose their validity [7, 8]. *E.g.*, when the body $\mathcal{B}$ *stiffens* too rapidly at large spatial distance, then any perturbation initially confined in a bounded subset of $\mathcal{B}$, invades the whole of $\mathcal{B}$ in a finite time: as a consequence, the motion of the body is not uniquely determined by the initial and boundary conditions and the body forces acting on $\mathcal{B}$ [7]. The same can be stated when $\mathcal{B}$ stiffens too rapidly at a point $\mathbf{o}$ [8, 9]. Also interesting is the case in which the acoustic tensor $\mathbf{A}$ decays too rapidly at $\mathbf{o}$. In this case, the motions of the body are still uniquely determined by the data, but the behaviour of $\mathcal{B}$ at $\mathbf{o}$ becomes quite paradoxical: the perturbations initially

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located at **o** *cannot reach* the other points of $\mathcal{B}$; conversely, no perturbation initially confined in a region which does not contain **o** can reach **o** at any time [8].

As they have been stated here, all these results seem to be rather disconnected from each other. Moreover, the counter-examples to uniqueness, given in [9], apply to one-dimensional and two-dimensional bodies, and cannot be extended to three dimensions. According to these remarks, the current paper essentially aims at giving a more general and comprehensive view of the qualitative properties of the motions of a linearly elastic body and of their link with the behaviour of the acoustic tensor **A**. To this aim, we considcr hcrc: (a) an unbounded three-dimensional body $\mathcal{B}$, "crossed' by a curve Γ, whose points are *singularity points* for **A**. This means that **A** either "grows up" to infinity when approaching the points of Γ, or vanishes on Γ; (b) an unbounded three-dimensional body $\mathcal{B}$, such that **A** does not satisfy the *hyperbolicity condition.* Thus we are in a position to obtain the desired general picture and, what is more, to *test* the loss of uniqueness of three-dimensional motions.

The plan of the work is as follows: Section 2 is devoted to a general statement of the problem to be studied, and to the proof of some very useful energy inequalities; in Section 3, we study the case in which $|\mathbf{A}|$ decays at the points of Γ: it is shown, in particular, that the paradoxical behaviour found in the case of a singularity point **o**, is completely reproduced for Γ, which could be referred to as an "unperturbable curve": perturbations arising at the points of Γ remain confined at Γ, while conversely Γ cannot "feel" any perturbation on $\mathcal{B}\backslash\Gamma$; Section 4 is mainly concerned with some examples of loss of uniqueness of the motion when $|\mathbf{A}|$ "grows up" to infinity at Γ, while Section 5 treats the same problem when **A** violates the so-called *hyperbolicity condition.* A way to *restore* uniqueness, in view of the physical meaning of the mathematical notion of "motion", is given at the end of Section 4.

Notation. Scalars are denoted by light-face letters; vectors (on $\mathbb{R}^3$) are indicated by bold-face lower-case letters; the symbols **o**, **x** and **y** are reserved to denote respectively the origin of an assigned reference frame $\{\mathbf{o};\mathbf{e}_i\}_{1\le i\le 3}$ on $\mathbb{R}^3$ and generic points of $\mathbb{R}^3$; bold-face upper-case letters stand for second-order tensors (linear transformations from $\mathbb{R}^3$ into $\mathbb{R}^3$); $\nabla\mathbf{u}$ is the second-order tensor with components $(\nabla\mathbf{u})_{ij} = \partial_j u_i$ $(\partial_j = \partial/\partial x_j)$; $\operatorname{div}\mathbf{S}$ is the vector with components $\partial_j S_{ij}$ (here and in the sequel, the sum over repeated indexes is implied); a superimposed dot means partial differentiation with respect to time; finally,

$$\forall\mathbf{x}\in\mathbb{R}^3,\ \forall r>0,\quad r=r(\mathbf{x})=|\mathbf{x}-\mathbf{o}|=\left(\sum_{i=1}^{3}x_i^2\right)^{1/2}$$

$$\forall\mathbf{x}_0\in\mathbb{R}^3,\ \forall R>0,\quad S_R(\mathbf{x}_0)=\{\mathbf{x}\in\mathbb{R}^3:|\mathbf{x}-\mathbf{x}_0|<R\},\ S_R(\mathbf{o})=S_R$$

$$\forall\mathbf{x}\neq\mathbf{x}_0,\quad \mathbf{e}_r^0=\mathbf{e}_r^0(\mathbf{x})=|\mathbf{x}-\mathbf{x}_0|^{-1}(\mathbf{x}-\mathbf{x}_0);$$

$$\forall\mathbf{x}\neq\mathbf{o},\quad \mathbf{e}_r=\mathbf{e}_r(\mathbf{x})=r^{-1}(\mathbf{x}-\mathbf{o}).$$

2. - Basic concepts and tools

This Section is devoted to give a general statement of the problem to be studied in the paper, and to outline the basic concepts and tools that will help to tackle it. We shall first define the class of the *singular motions* of an elastic body $\mathcal{B}$; then, we shall derive some energy inequalities that, beyond their intrinsic interest, will play a fundamental rôle in the proof of our main results.

2.1. - *Basic equations. Singular motions*

Let $\mathcal{B}$ be a *linearly elastic body*, identified with the open connected set of $\mathbb{R}^3$ it occupies in an assigned reference configuration. We assume that $\mathcal{B}$ is *unbounded*, and that the boundary $\partial\mathcal{B}$ is so regular as to allow the divergence theorem to be applied.

Let Γ be any smooth curve contained in $\mathcal{B}$, or $\Gamma = \emptyset$. Let us assign

i) a continuous and *a.e.*-positive scalar field ρ on $\overline{\mathcal{B}}\backslash\Gamma$ (*mass density*);

ii) a fourth-order tensor field $\mathbb{C}$ (*elasticity tensor*), continuous on $\overline{\mathcal{B}}\backslash\Gamma$ and smooth on $\mathcal{B}\backslash\Gamma$;

iii) a continuous vector field $\mathbf{b}$ on $\overline{\mathcal{B}}\times[0,+\infty)$ (*body force per unit volume*).

Here, for any $\mathbf{x}\in\overline{\mathcal{B}}\backslash\Gamma, \mathbb{C}$ is a linear transformation from the space of all second-order tensors into the space of all symmetric second-order tensors, such that $\mathbb{C}[\mathbf{W}] = \mathbf{O}$ for any skew $\mathbf{W}$. Throughout the paper, it will be assumed to be *symmetric*, *i.e.* such that

$$\mathbf{L}\cdot\mathbb{C}[\mathbf{M}] = \mathbf{M}\cdot\mathbb{C}[\mathbf{L}], \qquad \forall\,\mathbf{L},\mathbf{M},$$

and *positive semi-definite, i.e.* such that

$$\mathbf{M}\cdot\mathbb{C}[\mathbf{M}] \geq 0, \qquad \forall\,\mathbf{M}.$$

For any $\mathbf{x}\in\overline{\mathcal{B}}\backslash\Gamma$, and any assigned unit vector $\mathbf{m}$, the *acoustic tensor* $\mathbf{A}(\mathbf{x},\mathbf{m})$ *in the direction* $\mathbf{m}$ is defined by the relation

$$\mathbf{A}(\mathbf{x},\mathbf{m})\mathbf{a} = \rho^{-1}\mathbb{C}[\mathbf{a}\otimes\mathbf{m}]\mathbf{m}, \qquad \forall\,\mathbf{a}.$$

As is well-known, a motion of $\mathcal{B}$ in the time interval $(0,+\infty)$ is a solution $\mathbf{u}(\mathbf{x},t)$ to the System

$$\rho\ddot{\mathbf{u}} = \operatorname{div}\mathbb{C}[\nabla\mathbf{u}] + \mathbf{b}, \qquad \text{on } (\mathcal{B}\backslash\Gamma)\times(0,+\infty). \tag{2.1}$$

Throughout this paper, we shall only consider solutions to System (2.1) which are twice continuously differentiable on $(\overline{\mathcal{B}}\backslash\Gamma)\times[0,+\infty)$.

Assume now $\Gamma = \emptyset$, so that ρ and $\mathbb{C}$ are continuous on $\overline{\mathcal{B}}$. Let

$$\alpha(\xi) = \max\{c_M^{-2}|\mathbf{A}(\mathbf{x},\mathbf{m})|,\ \mathbf{x}\in S_\xi,\ \mathbf{m}:|\mathbf{m}|=1\}, \qquad \forall\xi\geq 0,$$

for some positive c_M and

$$\varphi(\xi) = \int_0^{\xi} [\alpha(s)]^{-1/2} ds.$$

Then $\varphi(\xi)$ is of course a positive, increasing and convex function on $[0,+\infty)$, and

$$|\mathbf{A}(\mathbf{x},\mathbf{m})| \leq c_M^2 [\varphi'(r(\mathbf{x}))]^{-2}, \qquad \forall \mathbf{x} \in \mathcal{B}. \tag{2.2}$$

We have already pointed out in some previous papers [5, 7] the link between the growth of the initial support of a solution $\mathbf{u}$ to System (2.1) and the limit

$$\varphi_\infty = \lim_{r\to+\infty} \varphi(r),$$

which certainly exists by virtue of the monotonicity of φ. In particular, we showed [5] that, if $\varphi_\infty = +\infty$, then any solution $\mathbf{u}$ to System (2.1) identically vanishing outside a bounded subset of $\mathcal{B}$ at $t = 0$, has a compact support on $\mathcal{B}$ at each instant $t > 0$. According to this property, we give the following

DEFINITION 2.1. The acoustic tensor $\mathbf{A}$ is said *to satisfy the hyperbolicity condition* if and only if $\varphi_\infty = +\infty$.

We have already shown ([7], cf. also Section 4) that, when the hyperbolicity condition is violated, then System (2.1) admits solutions corresponding to zero data, which are different from zero on the whole of $\mathcal{B}$ in the time interval $[c_M^{-1}\varphi_\infty, +\infty)$. This naturally leads to the following

DEFINITION 2.2. If $\Gamma = \emptyset$, then any motion $\mathbf{u}$ of $\mathcal{B}$ corresponding to material data ρ and $\mathbb{C}$ such that $\varphi_\infty < +\infty$ is said to be *singular at infinity*.

The class of all motions of $\mathcal{B}$ that are singular at infinity, will be denoted by $\mathcal{S}_\infty$.

Assume now $\Gamma \neq \emptyset$, and $\mathbf{A}$ to satisfy the hyperbolicity condition. As far as the behaviour of $\mathbf{A}$ at Γ is concerned, denoting by

$$\delta(\mathbf{x}) = \min\{|\mathbf{x} - \mathbf{y}|, \quad \mathbf{y} \in \Gamma\}$$

the distance of $\mathbf{x}$ from Γ, we give the following definitions.

DEFINITION 2.3. If $\Gamma \neq \emptyset$, then any motion $\mathbf{u}$ of $\mathcal{B}$ corresponding to material data ρ and $\mathbb{C}$ such that

(a) a smooth, positive and decreasing function p on $(0,+\infty)$ exists such that

$$\lim_{\xi\to 0} p(\xi) = +\infty \quad , \quad \lim_{\xi\to 0} p'(\xi) = -\infty,$$

$$[p'(\delta(\mathbf{x}))]^2 |\mathbf{A}(\mathbf{x},\mathbf{m})| \leq c_1^2 \quad , \quad \forall \mathbf{x} \in \mathcal{B}\backslash\Gamma, \ \forall \mathbf{m} : |\mathbf{m}| = 1,$$

for some positive constant c_1, is said to be *weakly singular at* Γ.

The class of all motions of $\mathcal{B}$ which are weakly singular at Γ, will be denoted by $\mathcal{S}_{w,\Gamma}$.

DEFINITION 2.4. If $\Gamma \neq \emptyset$, then any motion $\mathbf{u}$ of $\mathcal{B}$ corresponding to material data ρ and $\mathbb{C}$ such that

(b) a smooth, positive and incresing function q on $[0, +\infty)$ exists such that

$$q(0) = 0 \qquad , \qquad q'(0) = 0,$$
$$[q'(\delta(\mathbf{x}))]^2 |\mathbf{A}(\mathbf{x}, \mathbf{m})| \leq c_2^2 \quad , \quad \forall \mathbf{x} \in \mathcal{B} \backslash \Gamma, \ \forall \mathbf{m} : |\mathbf{m}| = 1,$$

for some positive constant c_2, is said to be *singular at* Γ.

The class of all motions of $\mathcal{B}$ which are singular at Γ, will be denoted by $\mathcal{S}_\Gamma$.

REMARK 2.1. The solutions to System (2.1) under condition (a) have been called "*weakly* singular" because, as can be seen by using the same methods employed in [6-8], in the class $\mathcal{S}_{w,\Gamma}$ all the main qualitative properties of classical solutions (such as Uniqueness for the boundary-initial value problems, Work and Energy Theorem, Reciprocity Relation) can be still proved. This is no more true in the class $\mathcal{S}_\Gamma$, unless we impose some restrictions on the behaviour of the motions near to Γ.

Let $\{\partial_1 \mathcal{B}, \partial_2 \mathcal{B}\}$ be a partition of $\partial \mathcal{B}$ and assign

iv) two smooth fields $\hat{\mathbf{u}}$ (*surface displacement*) on $\partial_1 \mathcal{B} \times [0, +\infty)$ and $\hat{\mathbf{s}}$ (*surface traction*) on $\partial_2 \mathcal{B} \times [0, +\infty)$;

v) two smooth fields $\mathbf{u}_0$ (*initial displacement*) and $\dot{\mathbf{u}}_0$ (*initial velocity*) on $\overline{\mathcal{B}} \backslash \Gamma$.

Then the boundary-initial value problem corresponding to the above data consists in finding a motion $\mathbf{u}$ of $\mathcal{B}$ which satisfies the *boundary conditions*

$$\text{(2.3)} \qquad \begin{cases} \mathbf{u} = \hat{\mathbf{u}} & \text{on } \partial_1 \mathcal{B} \times [0, +\infty), \\ \mathbb{C}[\nabla \mathbf{u}]\mathbf{n} = \hat{\mathbf{s}} & \text{on } \partial_2 \mathcal{B} \times [0, +\infty), \end{cases}$$

where $\mathbf{n}$ is the outward unit normal to $\partial \mathcal{B}$, and the *initial conditions*

$$\text{(2.4)} \qquad \mathbf{u} = \mathbf{u}_0, \qquad \dot{\mathbf{u}} = \dot{\mathbf{u}}_0, \qquad \text{on } (\mathcal{B} \backslash \Gamma) \times \{0\}.$$

Let $\mathbf{u}$ be a motion of $\mathcal{B}$ and let f be any smooth function on $\mathbb{R}^3 \times [0, +\infty)$ which, $\forall s \geq 0$, has a compact support on $\mathbb{R}^3$ and identically vanishes in a region containing Γ (if $\Gamma \neq \emptyset$). By multiplying both sides of (2.1) by $f\dot{\mathbf{u}}$, and integrating over $\mathcal{D} \times [0, t]$, where $\mathcal{D}$ is a regular subset of $\mathcal{B}$, an integration by parts leads to the relation

$$\int_D (f\eta(\mathbf{u}))(\mathbf{x},t)\mathrm{d}v = \int_D (f\eta(\mathbf{u}))(\mathbf{x},0)\mathrm{d}v$$

$$+\int_0^t \mathrm{d}s\left\{\int_D \dot{f}\eta(\mathbf{u})\mathrm{d}v - \int_D \dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\nabla f\mathrm{d}v \right. \tag{2.5}$$

$$\left. +\int_D f\dot{\mathbf{u}}\cdot\mathbf{b}\ \mathrm{d}v + \int_{\partial D} f\dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{n}\ \mathrm{d}a\right\}, \qquad \forall t \geq 0,$$

where

$$\eta(\mathbf{u}) = \frac{1}{2}(\rho\dot{\mathbf{u}}^2 + \nabla\mathbf{u}\cdot\mathbb{C}[\nabla\mathbf{u}]), \qquad \text{on } (\mathcal{B}\backslash\Gamma)\times[0,+\infty),$$

denotes the total mechanical energy density of $\mathcal{B}$. Relation (2.5) will be useful in the sequel.

It is convenient for our purposes to introduce the following notation:

$$\mathcal{B}_R(\mathbf{x}_0) = \mathcal{B}\cap S_R(\mathbf{x}_0),$$

$$\forall R' > 0, \quad P_{R'} = \{\mathbf{x} : \delta(\mathbf{x}) < R'\};\ P^c_{R'} = \mathbb{R}^3\backslash P_{R'}.$$

We must note that, if $\Gamma \neq \emptyset$, then the set

$$\Delta_\Gamma(\mathbf{x}) = \{\mathbf{y}\in\Gamma |\ |\mathbf{x}-\mathbf{y}| = \delta(\mathbf{x})\}$$

is *not necessarily a singleton*. Therefore, the set of the points $\mathbf{x}\in\mathbb{R}^3\backslash\Gamma$, such that $\Delta_\Gamma(\mathbf{x})$ is not a singleton, will be denoted by the symbol Δ_Γ, and we put

$$\mathcal{B}_\omega = \{\mathbf{x}\in\mathcal{B} |\ \inf_{\mathbf{z}\in\Delta_\Gamma}|\mathbf{x}-\mathbf{z}| > \omega\}.$$

As a consequence, if $\mathbf{x}\in\mathcal{B}_\omega$, we may write

$$\delta(\mathbf{x}) = |\mathbf{x}-\mathbf{x}_\Gamma|,$$

where $\mathbf{x}_\Gamma \in \Gamma$ *is uniquely determined.* We shall often use the notation

$$\mathbf{e}_\Gamma(\mathbf{x}) = |\mathbf{x}-\mathbf{x}_\Gamma|^{-1}(\mathbf{x}-\mathbf{x}_\Gamma), \qquad \forall\mathbf{x}\in\mathcal{B}_\omega.$$

Finally, it is worth remarking that, in a three-dimensional point space, the set Δ_Γ is in general the join of a family of surfaces.

2.2. - *Energy inequalities*

We want now to derive some estimates involving the total mechanical energy density $\eta(\mathbf{u})$ over cylindrical shells surrounding finite arcs of Γ. These will allow us to deduce a number of inequalities either over subsets of a cylindrical pipe P_R containing Γ (*internal energy inequalities*) or over subsets of $\mathcal{B}\backslash P_R$ (*external energy inequalities*) in both cases (a) and (b); from now on to the end of this Section, the tensor $\mathbf{A}$ is assumed to satisfy the hyperbolicity condition.

In order to write the formulae in a simpler and more compact way, we set

$$
\begin{aligned}
R_\xi &= p^{-1}(p(R)+c\xi) \\
\tilde{R}_\xi &= q^{-1}(q(R)+c\xi) && \forall R>0, \qquad \forall \xi \geq 0. \\
R^*_\xi &= \varphi^{-1}(\varphi(R+r_0)+c\xi)-r_0, && r_0 = r(\mathbf{x}_0).
\end{aligned}
$$

The following theorems hold.

THEOREM 2.1. *Let* $\mathbf{u} \in \mathcal{S}_{w,\Gamma}$. *Then*

$$
\begin{aligned}
&\int\limits_{(P_{R'_t}\backslash P_{R''})\cap \mathcal{B}_R(\mathbf{x}_0)} (\eta(\mathbf{u}))(\mathbf{x},t)\mathrm{d}v \\
&\leq \exp[t/t_0]\Bigg\{ \int\limits_{(P_{R'}\backslash P_{R''_t})\cap \mathcal{B}_{R^*_t}(\mathbf{x}_0)} (\eta(\mathbf{u}))(\mathbf{x},0)\mathrm{d}v \\
&+\int_0^t \mathrm{d}s \Bigg\{ t_0 \int\limits_{(P_{R'_s}\backslash P_{R''_{t-s}})\cap \mathcal{B}_{R^*_{t-s}}(\mathbf{x}_0)} \rho^{-1}\mathbf{b}^2 \mathrm{d}v \\
&+ \int\limits_{\partial\mathcal{B}\cap(P_{R'_s}\backslash P_{R''_{t-s}})\cap \mathcal{S}_{R^*_{t-s}}(\mathbf{x}_0)} |\dot{\mathbf{u}}\cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{n}|\mathrm{d}a \Bigg\}\Bigg\},
\end{aligned}
\tag{2.6}
$$

for any $\mathbf{x}_0 \in \mathcal{B}$, *for any* $R, t > 0$ *and for any* $R', R'' > 0$ *such that* $R'' < R'_t$.

THEOREM 2.2. *Let* $\mathbf{u} \in \mathcal{S}_\Gamma$. *Then*

$$
\begin{aligned}
&\int\limits_{(P_{R'_t}\backslash P_{\tilde{R}''})\cap \mathcal{B}_R(\mathbf{x}_0)} (\eta(\mathbf{u}))(\mathbf{x},t)\mathrm{d}v \\
&\leq \exp[t/t_0]\Bigg\{ \int\limits_{(P_{\tilde{R}'_t}\backslash P_{R''})\cap \mathcal{B}_{R^*_t}(\mathbf{x}_0)} (\eta(\mathbf{u}))(\mathbf{x},0)\mathrm{d}v
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
&+\int_0^t \mathrm{d}s\Bigg\{t_0 \int_{(P_{\tilde{R}'_t}\backslash P_{\tilde{R}''_s})\cap\mathcal{B}_{R^*_{t-s}}(\mathbf{x}_0)} \rho^{-1}\mathbf{b}^2\mathrm{d}v \\
&+ \int_{\partial\mathcal{B}\cap(P_{\tilde{R}'_{t-s}}\backslash P_{\tilde{R}''_s})\cap S_{R^*_{t-s}}(\mathbf{x}_0)} |\dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{n}|\mathrm{d}a\Bigg\}\Bigg\},
\end{aligned} \tag{2.7}
$$

for any $\mathbf{x}_0 \in \mathcal{B}$, *for any* $R > 0$ *and for any* $t, R', R'' > 0$ *such that* $R'' < q^{-1}(q(R') - ct)$ *and*

$$t < c^{-1}q(R').$$

Proof of Theorem 2.1. Consider the function

$$g = w(\lambda_1)w(\lambda_2)w(\lambda_3) \qquad \text{on } \mathbb{R}^3 \times [0, T],$$

where

$$
\begin{aligned}
\lambda_1 &= \left(\frac{1}{c\sigma}\right)(\varphi(R^*_{t-s} + r_0) - \varphi(|\mathbf{x} - \mathbf{x}_0| + r_0)),\\
\lambda_2 &= \left(\frac{1}{c\sigma}\right)(p(\delta(\mathbf{x})) - p(R'_s)),\\
\lambda_3 &= \left(\frac{1}{c\sigma}\right)(p(R''_{t-s}) - p(\delta(\mathbf{x}))).
\end{aligned}
$$

Here w is a smooth increasing function on $\mathbb{R}$, vanishing on $(-\infty, 0]$ and equal to 1 on $[1, +\infty)$, and σ is an arbitrarily fixed positive constant.

The spatial support of g at instant s is the set

$$\Sigma_s = S_{R^*_{t-s}}(\mathbf{x}_0) \cap (P_{R'_s}\backslash P_{R''_{t-s}}).$$

In spite of the fact that ∇g is not defined along the space-time axis $\mathbf{x} = \mathbf{x}_0$, it is easily verified that, by choosing σ suitably small, g is smooth on $\mathbb{R}^3 \times [0, t]$. As a consequence, g satisfies the conditions imposed on f for the validity of (2.5). Then, setting

$$w^+ = w'(\lambda_2)w(\lambda_3) + w'(\lambda_3)w(\lambda_2) \qquad \left(w'(y) = \frac{\mathrm{d}w}{\mathrm{d}y}\right) \qquad \text{on } \mathbb{R}^3 \times [0, t],$$

and

$$w^- = w'(\lambda_2)w(\lambda_3) - w'(\lambda_3)w(\lambda_2),$$

we have

$$
\begin{aligned}
\int_{B_\omega} (g\eta(\mathbf{u}))(\mathbf{x},t)\mathrm{d}v &= \int_{B_\omega} (g\eta(\mathbf{u}))(\mathbf{x},0)\mathrm{d}v \\
&- \frac{1}{c\sigma}\int_0^t \mathrm{d}s \int_{B_\omega} w(\lambda_1)\{cw^+\eta(\mathbf{u}) - w^- p'(\delta(\mathbf{x}))\dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\nabla\delta\}\mathrm{d}v \\
&- \frac{1}{c\sigma}\int_0^t \mathrm{d}s \int_{B_\omega} w(\lambda_2)w(\lambda_3)w'(\lambda_1)\{c\eta(\mathbf{u}) \\
&- \varphi'(|\mathbf{x}-\mathbf{x}_0| + r_0)\dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_r^0\}\mathrm{d}v \\
&+ \int_0^t \mathrm{d}s \left\{ \int_{B_\omega} g\mathbf{b}\cdot\dot{\mathbf{u}}\ \mathrm{d}v + \int_{\partial B_\omega} g\dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{n}\ \mathrm{d}a \right\}.
\end{aligned} \tag{2.8}
$$

Since, in B_ω,

$$
\begin{aligned}
(\nabla\delta)(\mathbf{x}) = (\nabla(|\mathbf{x}-\mathbf{x}_\Gamma|))(\mathbf{x}) &= \frac{\partial}{\partial x^h}\left(\sqrt{\sum_i (x^i - x^i_\Gamma)^2}\right)\mathbf{e}_h \\
&= \mathbf{e}_\Gamma(\mathbf{x}) - \left[\mathbf{e}_\Gamma\cdot\left(\frac{\partial \mathbf{x}_\Gamma}{\partial x^h}\right)\right]\mathbf{e}_h = \mathbf{e}_\Gamma,
\end{aligned}
$$

we may now use the inequality

$$
2\mathbf{L}\cdot\mathbb{C}[\mathbf{M}] \leq \xi^{-1}\mathbf{L}\cdot\mathbb{C}[\mathbf{L}] + \xi\mathbf{M}\cdot\mathbb{C}[\mathbf{M}], \qquad \forall\,\mathbf{L},\mathbf{M}, \quad \forall\xi > 0,
$$

and the arithmetic-geometric mean inequality to majorize respectively the second and third integrals and the fourth integral at RHS of (2.8). Then, by taking into account assumption (a),

$$
\begin{aligned}
\frac{1}{c}\, w^- p'(\delta(\mathbf{x}))\dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_\Gamma &\leq \frac{1}{2}\, w^+\Big\{\nabla\mathbf{u}\cdot\mathbb{C}[\nabla\mathbf{u}] \\
&+ \frac{1}{c^2}\,[p'(\delta(\mathbf{x}))]^2\dot{\mathbf{u}}\cdot\mathbb{C}[\dot{\mathbf{u}}\otimes\mathbf{e}_\Gamma(\mathbf{x})]\mathbf{e}_\Gamma(\mathbf{x})\Big\} \\
&= \frac{1}{2}\, w^+\Big\{\nabla\mathbf{u}\cdot\mathbb{C}[\nabla\mathbf{u}] \\
&+ \frac{1}{c^2}\,[p'(\delta(\mathbf{x}))]^2|\mathbf{A}(\mathbf{x},\mathbf{e}_\Gamma(\mathbf{x}))|\rho\dot{\mathbf{u}}^2\Big\} \leq w^+\eta(\mathbf{u}),
\end{aligned} \tag{2.9}
$$

and, by virtue of the convexity of φ,

$$
\begin{aligned}
&\frac{1}{c}\,\varphi'(|\mathbf{x}-\mathbf{x}_0|+r_0)\dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_r \le \frac{1}{2}\Big\{\nabla\mathbf{u}\cdot\mathbb{C}[\nabla\mathbf{u}] \\
&+\frac{1}{c^2}\,[\varphi'(|\mathbf{x}-\mathbf{x}_0|+r_0)]^2|\mathbf{A}(\mathbf{x},\mathbf{e}_r^0(\mathbf{x}))|\rho\dot{\mathbf{u}}^2\Big\} \qquad (2.10)\\
&\le \frac{1}{2}\Big\{\nabla\mathbf{u}\cdot\mathbb{C}[\nabla\mathbf{u}]+\frac{1}{c^2}\,[\varphi'(r(\mathbf{x}))]^2|\mathbf{A}(\mathbf{x},\mathbf{e}_r^0(\mathbf{x}))|\rho\dot{\mathbf{u}}^2\Big\}\le\eta(\mathbf{u}).
\end{aligned}
$$

Finally,

$$
\mathbf{b}\cdot\dot{\mathbf{u}}\le\frac{1}{2}\,(t_0\rho^{-1}\mathbf{b}^2+t_0^{-1}\eta(\mathbf{u})), \qquad (2.11)
$$

where t_0 is a reference time.

According to inequalities (2.9)-(2.10)-(2.11), the second and third integral at RHS of (2.8) are nonpositive, so that (2.8) yields

$$
\begin{aligned}
&\int_{B_\omega}(g\eta(\mathbf{u}))(\mathbf{x},t)\mathrm{d}v\le\int_{B_\omega}(g\eta(\mathbf{u}))(\mathbf{x},0)\mathrm{d}v \\
&+\frac{1}{t_0}\int_0^t\mathrm{d}s\int_{B_\omega}(g\eta(\mathbf{u}))(\mathbf{x},s)\mathrm{d}v \qquad (2.12)\\
&+\int_0^t\mathrm{d}s\Big\{\frac{t_0}{2}\int_{B_\omega}(g\rho^{-1}\mathbf{b}^2)(\mathbf{x},s)\mathrm{d}v \\
&+\int_{\partial B}g\dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{n}\;\mathrm{d}a+\int_{\partial B_\omega\setminus\partial B}g\dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{n}_\omega\mathrm{d}a\Big\},
\end{aligned}
$$

where $\partial B_\omega\setminus\partial B$ is the set

$$
\{\mathbf{x}\in B|\inf_{\mathbf{z}\in\Delta_\Gamma}|\mathbf{x}-\mathbf{z}|=\omega\}.
$$

This set is in turn a join of surfaces, and $\mathbf{n}_\omega$ stands for the normal unit vector to $\partial B_\omega\setminus\partial B$, directed *outside* B_ω. Since all the fields in (2.12) are *smooth* across

$\partial B_\omega \backslash \partial B$, we are allowed to take the limit $\omega \to 0$ of (2.12), to get

$$\begin{aligned}\int_B (g\eta(\mathbf{u}))(\mathbf{x},t)\mathrm{d}v \leq \int_B (g\eta(\mathbf{u}))(\mathbf{x},0)\mathrm{d}v \\ + \frac{1}{t_0}\int_0^t \mathrm{d}s \int_B (g\eta(\mathbf{u}))(\mathbf{x},s)\mathrm{d}v \\ + \int_0^t \mathrm{d}s \left\{ \frac{t_0}{2} \int_B (g\rho^{-1}\mathbf{b}^2)(\mathbf{x},s)\mathrm{d}v \right. \\ \left. + \int_{\partial B} g\dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{n}\ \mathrm{d}a \right\},\end{aligned} \tag{2.13}$$

whence, by virtue of Grönwall's lemma, it follows that

$$\begin{aligned}\int_B (g\eta(\mathbf{u}))(\mathbf{x},t)\mathrm{d}v \leq \exp\left[\frac{t}{t_0}\right] \left\{ \int_B (g\eta(\mathbf{u}))(\mathbf{x},0)\mathrm{d}v \right. \\ + \int_0^t \exp\left[-\frac{s}{t_0}\right] \left\{ \frac{t_0}{2} \int_B (g\rho^{-1}\mathbf{b}^2)(\mathbf{x},s)\mathrm{d}v \right. \\ \left.\left. + \int_{\partial B} g\dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{n}\ \mathrm{d}a \right\} \mathrm{d}s \right\}.\end{aligned} \tag{2.14}$$

Since, as $\sigma \to 0$, g tends boundedly to the characteristic function of the set $\bigcup_{s=0}^{t} \Sigma_s$, the passage to the limit $\sigma \to 0$ is permissible in (2.14) by virtue of Lebesgue's dominated convergence theorem. Hence, (2.6) follows by letting $\sigma \to 0$ in (2.14). □

REMARK 2.2. Observe, by the way, that if $\mathbf{b} = \mathbf{0}$, then the energy term arising from (2.11) disappears, so that the term "$\exp[t/t_0]$" in (2.6) is replaced by 1.

PROOF OF THEOREM 2.2. Consider the function

$$\tilde{g}(\mathbf{x},t) = w(\lambda_1(\mathbf{x},s))w\left(\frac{1}{c\sigma}[q(\tilde{R}'_{t-s}) - q(\delta(\mathbf{x}))]\right) w\left(\frac{1}{c\delta}[q(\delta(\mathbf{x})) - q(\tilde{R}''_s)]\right),$$
$$\forall(\mathbf{x},s) \in \mathbb{R}^3 \times [0,t],$$

where σ is again an arbitrarily fixed positive constant. Estimate (2.7) may be derived by repeating step by step reasoning which led to (2.6), with g replaced by $\tilde{g}$. □

3. - Qualitative properties of motions which are weakly singular at a curve Γ

Throughout this Section, we assume that the acoustic tensor $\mathbf{A}$ satisfies assumption (a) and the hyperbolicity condition. We prove a general domain of influence theorem, from which we deduce, as an immediate consequence, the uniqueness of solutions to the boundary-initial value problem (2.1)-(2.3)-(2.4). In this connection, we shall also point out the paradoxical behaviour of perturbations at the points of Γ, previously laid out in the Introduction.

3.1. - *The Domain of Influence Theorem. Uniqueness*

Let $\hat{D}_t$ be the set of all points $\mathbf{x} \in \mathcal{B}\backslash\Gamma$ such that

$$\begin{aligned} &\mathbf{x} \in \mathcal{B}\backslash\Gamma \;\Rightarrow \mathbf{u}(\mathbf{x},0) \neq \mathbf{0} \text{ or } \dot{\mathbf{u}}(\mathbf{x},0) \neq \mathbf{0} \text{ or } \exists\tau \in [0,t] : \mathbf{b}(\mathbf{x},\tau) \neq \mathbf{0};\\ &\mathbf{x} \in \partial\mathcal{B} \quad\Rightarrow \exists\tau \in [0,t] : (\dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{n})(\mathbf{x},\tau) \neq 0, \end{aligned}$$

and let

$$D_\varphi(t) = \{\mathbf{x}_0 \in \overline{\mathcal{B}}\backslash\Gamma : \hat{D}_t \cap S_{\varphi^{-1}(\varphi(r_0)+ct)-r_0}(\mathbf{x}_0) \neq \emptyset\}.$$

THEOREM 3.1 (Domain of Influence Theorem). *Let* $\mathbf{u}$ *be a solution to System* (2.1). *Then,* $\forall t > 0$,

$$\mathbf{u} = \mathbf{0} \quad on \quad \{(\overline{\mathcal{B}}\backslash\Gamma)\backslash\overline{D}_\varphi(t)\} \times [0,t].$$

PROOF. Let $(\mathbf{x}_0,\lambda) \in \{(\mathcal{B}\backslash\Gamma)\backslash\overline{D}_\varphi(t)\} \times (0,t)$. Then, writing (2.6) with $t = \lambda$ and $R = \varphi^{-1}(\varphi(r_0) + c(t-\lambda)) - r_0$, choosing R' and R'' in such a way that $\mathbf{x}_0 \in (P_{R'_\lambda}\backslash P_{R''}) \cap \mathcal{B}_R(\mathbf{x}_0)$, and setting

$$\begin{aligned} D(\lambda) &= (P_{R'_\lambda}\backslash P_{R''}) \cap \mathcal{B}_R(\mathbf{x}_0),\\ D(0) &= (P_{R'}\backslash P_{R''_\lambda}) \cap \mathcal{B}_{\varphi^{-1}(\varphi(r_0)+ct)-r_0}(\mathbf{x}_0),\\ D(\lambda-s) &= (P_{R'_s}\backslash P_{R''_{\lambda-s}}) \cap \mathcal{B}_{\varphi^{-1}(\varphi(r_0)+c(t-s))-r_0}(\mathbf{x}_0),\\ \Sigma(\lambda-s) &= \partial\mathcal{B} \cap (P_{R'_s}\backslash P_{R''_{\lambda-s}}) \cap S_{R^*_{\lambda-s}}(\mathbf{x}_0) \end{aligned}$$

we have

$$\begin{aligned} \int_{D(\lambda)} (\eta(\mathbf{u}))(\mathbf{x},\lambda)\mathrm{d}v \leq& \int_{D(0)} (\eta(\mathbf{u}))(\mathbf{x},0)\mathrm{d}v\\ &+\int_0^\lambda \left\{ t_0 \int_{D(\lambda-s)} \rho^{-1}\mathbf{b}^2\mathrm{d}v + \int_{\Sigma(\lambda-s)} |\dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{n}|\mathrm{d}a \right\}\mathrm{d}s. \end{aligned} \tag{3.1}$$

Bearing in mind the definition of $D_\varphi(t)$ and our choice of the couple $(\mathbf{x}_0,\lambda)$, we see that all the integrals at RHS of (3.1) are zero, so that, by virtue of definite positiveness of $\mathbb{C}$, (3.1) yields

$$\int_{(P_{R'_\lambda}\setminus P_{R''})\cap \mathcal{B}_R(\mathbf{x}_0)} (\rho\dot{\mathbf{u}}^2)(\mathbf{x}_0,\lambda)dv \le 0. \tag{3.2}$$

Since $(\mathbf{x}_0,\lambda)$ is arbitrarily chosen in $[(\mathcal{B}\setminus\Gamma)\setminus\overline{D}_\varphi(t)]\times(0,t)$, (3.2) implies $\dot{\mathbf{u}}=\mathbf{0}$ on $[(\mathcal{B}\setminus\Gamma)\setminus\overline{D}_\varphi(t)]\times[0,t]$. Hence, taking into account that $\mathbf{u}=\mathbf{0}$ on $[(\mathcal{B}\setminus\Gamma)\setminus\hat{D}_t]\times\{0\}\supseteq[(\mathcal{B}\setminus\Gamma)\setminus\overline{D}_\varphi(t)]\times\{0\}$, the desired result follows at once. □

A simple consequence of Theorem 3.1 is the following Uniqueness Theorem:

THEOREM 3.2. *System* (2.1)-(2.3)-(2.4) *has at most one solution.*

PROOF. Since System (2.1)-(2.3)-(2.4) is linear, it is sufficient to show that, if $\mathbf{b}$, $\hat{\mathbf{u}}$, $\hat{\mathbf{s}}$, $\mathbf{u}_0$ and $\dot{\mathbf{u}}_0$ identically vanish in their domains of definition, then $\mathbf{u}=\mathbf{0}$ on $(\overline{\mathcal{B}}\setminus\Gamma)\times[0,+\infty)$. To this aim, one needs nothing more than remarking that, in this case, $D_\varphi(t)=\emptyset$, $\forall t\in[0,+\infty)$. □

3.2. - *A paradoxical behaviour: the unperturbable line*

As far as the propagation of perturbations in $\mathcal{B}$ is concerned, the singularity line Γ behaves in a rather unexpected way: we shall now show that it behaves as an *unperturbable line*, namely, a line which is uncapable of receiving as well as of transmitting signals.

In order to make the discussion as simple as possible, we assume that the body force field and the boundary data are identically zero, and that $(\eta(\mathbf{u}))(\mathbf{x},0)$ is locally integrable over $\mathcal{B}$. Then, for any bounded subset Γ_0 of Γ, by choosing $\mathbf{x}_0\in\Gamma_0$ and R in such a way that Γ_0 is completely contained in $S_R(\mathbf{x}_0)$, and letting $R''\to 0$, (2.6) yields (cf. Remark 2.2)

$$\int_{P_{R'_t}\cap\mathcal{B}_R(\mathbf{x}_0)} (\eta(\mathbf{u}))(\mathbf{x},t)dv \le \int_{P_{R'}\cap\mathcal{B}_{R_0^*}(\mathbf{x}_0)} (\eta(\mathbf{u}))(\mathbf{x},0)dv. \tag{3.3}$$

Thus, if we assume that a cylindrical pipe $P_{R'}$ surrounding Γ exists such that $\mathbf{u}_0=\dot{\mathbf{u}}_0=\mathbf{0}$ on $P_{R'}$, then (3.3) implies that, for any instant $t>0$, there exists a nonempty neighbourhood of Γ, namely, $P_{R'_t}$, where $\mathbf{u}$ identically vanishes. In physical terms, this obviously means that Γ cannot be reached at any finite time by any perturbation initially located outside a neighbourhood of Γ.

On the other hand, if $\mathbf{x}_0$ is any point of $\mathcal{B}$, letting $R'\to+\infty$ in (2.6), we have

$$\int_{(P\setminus P_{R''})\cap\mathcal{B}_R(\mathbf{x}_0)} (\eta(\mathbf{u}))(\mathbf{x},t)dv \le \int_{(P\setminus P_{R''_t})\cap\mathcal{B}_{R_0}(\mathbf{x}_0)} (\eta(\mathbf{u}))(\mathbf{x},0)dv, \tag{3.4}$$

which, for a fixed instant $t > 0$, implies that, for any nonempty neighbourhood $\mathcal{B}_R''$ of Γ, any perturbation initially concentrated on Γ is identically zero outside $\mathcal{B}_R''$ at t. In essence, since t is arbitrary, this result tells us that the perturbation cannot leave Γ.

4. - Qualitative properties of motions which are singular at a curve Γ

Throughout this Section, we assume that $\mathbf{A}$ satisfies hypothesis (b) and the hyperbolicity condition. We first show, by means of counter-examples, that System (2.1)-(2.3)-(2.4) admits in general infinitely many solutions in the class $\mathcal{S}_\Gamma$. Then, from (2.7) we deduce that, if the initial perturbation is identically zero outside a neighbourhood of Γ, then the resulting perturbation identically vanishes outside a neighbourhood of Γ at each instant t (*local domain of influence theorem*). Subsequently, we obtain a *thermodynamical domain of influence theorem* for motions belonging to a properly defined subclass of $\mathcal{S}_\Gamma$, and prove that in such subclass of $\mathcal{S}_\Gamma$ the Work and Energy Theorem holds in a "generalized" form.

4.1. - *Some counter-examples to uniqueness*

The most interesting feature which follows from assumption (b) is the loss of uniqueness of solutions to the boundary-initial value problem (2.1)-(2.3)-(2.4). Indeed, by extending the counter-examples given in [9], we show that the Cauchy problem associated with System (2.1) has infinitely many solutions corresponding to the same assigned body forces and initial values, at least when Γ is assumed to be a straight line. To this end, because of the linearity of the equations, it will be sufficient to show that System (2.1) has infinitely many solutions corresponding to zero body forces and initial values.

Assume that the body $\mathcal{B}$ occupies the whole space and is *isotropic* with Lamé moduli λ and μ such that $\lambda = 0$, $\mu > 0$. Furthermore, let Γ be the x^3-axis. Then, in the cylindrical coordinate system $(\delta = \delta(\mathbf{x}), \vartheta, x^3)$, if we look for solutions $\mathbf{u} \equiv (u_\delta = 0,\ u_\vartheta = 0,\ u_3 = u(\delta, t))$, then the Cauchy problem associated with System (2.1)-(2.4) reduces to

$$(4.1)\qquad \begin{aligned} &\delta\rho\ddot{u} = \frac{\partial}{\partial\delta}\left(\delta\mu\frac{\partial u}{\partial\delta}\right), \qquad \forall(\delta,t)\in(0,+\infty)\times(0,+\infty),\\ &u(\delta,0)=0, \qquad \dot{u}(\delta,0)=0, \qquad \forall\delta\in[0,+\infty). \end{aligned}$$

We append to System (4.1) the "boundary" condition

$$u(0,t) = u^*(t), \qquad \forall t \geq 0,$$

with the *compatibility conditions* $u^*(0) = \dot{u}^*(0) = 0$.

Assume now that condition (b) is satisfied with

$$\rho(\delta) = \delta^{-1} q'(\delta), \qquad \mu(\delta) = [\delta q'(\delta)]^{-1}, \qquad \forall \delta > 0$$

where all the dimensional constants are taken for simplicity equal to 1. Then, in order to solve System (4.1) by the standard method of characteristic curves, let us write Equation $(4.1)_1$ in the form

$$\begin{aligned} &\dot{u} - [q'(\delta)]^{-1} \frac{\partial u}{\partial \delta} = v \\ &\dot{v} + [q'(\delta)]^{-1} \frac{\partial v}{\partial \delta} = 0. \end{aligned} \tag{4.2}$$

Introducing the auxiliary variables

$$\begin{aligned} \xi &= q(\delta) - t \\ \tau &= q(\delta) + t, \end{aligned}$$

System (4.2) reads

$$\begin{aligned} &\frac{\partial u}{\partial \xi} = v \\ &\frac{\partial v}{\partial \tau} = 0, \end{aligned}$$

whence $v = v(\xi)$ and

$$u(\xi, \tau) = \int_0^{\xi} v(\varsigma) \mathrm{d}\varsigma + u_0(\tau)$$

or, setting $\int_0^{\xi} v(\varsigma) \mathrm{d}\varsigma = w(\xi)$,

$$u(\delta, t) = u_0(q(\delta) + t) + w(q(\delta) - t),$$

subject to conditions

$$\begin{aligned} &u_0(q(\delta)) + w(q(\delta)) = 0, \\ &u_0'(q(\delta)) - w'(q(\delta)) = 0, \end{aligned}$$

whence

$$u_0(q(\delta)) - w(q(\delta)) = K.$$

As a consequence, $u_0(q(\delta) + t) = K/2$ and $w(q(\delta) - t) = K/2$, so that

$$u(\delta, t) = 0, \qquad \forall \delta > 0, \ \forall t \in (0, q(\delta)). \tag{4.3}$$

In order to determine the solution $u(\delta, t)$ for $t \geq q(\delta)$, we must observe that

$$u_0(t) + w(-t) = \hat{u}(t),$$

so that

$$w(q(\delta) - t) = \hat{u}(t - q(\delta)) - \frac{K}{2}.$$

Finally, we have

$$u(\delta, t) = \hat{u}(t - q(\delta)), \qquad \forall \delta > 0,\ \forall t \geq q(\delta). \tag{4.4}$$

Therefore, the solution to System (4.1), expressed by (4.3)-(4.4), *does not* in general identically vanish on $(0, +\infty) \times (0, +\infty)$, and its support is the interior of the hyperparaboloid of equation $t = q(\delta)$. This region can be viewed as the join on $t \geq 0$ of the *domains of influence* of the "datum" $\hat{u}(t)$: but this "datum" is *fictitious*, and has been introduced only in order to give the solution in an explicit form. Its prescription is in general uncorrect from both the mathematical and the physical viewpoint, since the Cauchy data are completely expressed by $(4.1)_2$ and, on the other hand, we cannot be able to measure the values of the solution over one-dimensional subsets of $\mathbb{R}^3$. It is then quite natural to look for a criterion to single out *the* physically meaningful solution among the infinitely many fields expressed by (4.3)-(4.4). This will be carried out in the sequel (Sub-section 4.3), by following a method based upon the *entropy principle* and introduced in [10, 11].

At the moment, we want to point out that

i) when the data ρ and μ are assumed to be regular and positive on $\mathbb{R}^3$, then the nontrivial solutions to System (4.1) cannot be continuously differentiable on the whole of $(0, +\infty) \times (0, +\infty)$;

ii) when (b) holds, then it is possible to find smooth nontrivial solutions to System (4.1).

In order to prove the first statement, it is sufficient to note that, for any $t \geq q(\delta)$,

$$\frac{\partial u}{\partial x^i}(\mathbf{x}, t) = -\hat{u}'(t - q(\delta))\,\frac{q'(\delta)x^i}{\delta} \qquad (i = 1, 2) \tag{4.5}$$

so that, if $\lim_{\delta \to 0} q'(\delta) = \ell \neq 0$, as it happens when the data are regular, then

$$\lim_{x^1 \to 0^+} \frac{\partial u}{\partial x^1}(\mathbf{x}, t) = -\hat{u}'(t - q(0))\ell \qquad (x^2 = 0)$$

and

$$\lim_{x^1 \to 0^-} \frac{\partial u}{\partial x^1}(\mathbf{x}, t) = \hat{u}'(t - q(0))\ell \qquad (x^2 = 0).$$

In particular, if we confine ourselves to consider only smooth solutions, then we conclude that System (4.1) admits only the trivial one.

As far as ii) is concerned, we note that, by virtue of (b), (4.5) implies that ∇u is continuous on Γ.

Of course, for any twice continuously differentiable $\hat{u}$, since

$$\frac{\partial^2 u}{\partial x^i \partial x^j} = \hat{u}''(t - q(\delta)) \left(\frac{q'(\delta)}{\delta}\right)^2 x^i x^j + \hat{u}'(t - q(\delta)) \frac{q''(\delta)}{\delta} x^i x^j - \hat{u}'(t - q(\delta)) \frac{q'(\delta)}{\delta^3} x^i x^j + \hat{u}'(t - q(\delta)) \frac{q'(\delta)}{\delta} \delta^{ij},$$

if $q'(\delta) = o(\delta)$ and $q''(\delta)\delta = o(1)$, then the solution u is certainly twice continuously differentiable on $(0, +\infty) \times (0, +\infty)$.

In particular, if we choose

$$q'(\delta) = \exp\left(-\frac{1}{\delta}\right),$$

and $\hat{u}$ of class C^∞, then we are sure that the corresponding solution $\mathbf{u}$ is in turn of class C^∞ on $(0, +\infty) \times (0, +\infty)$.

4.2. - *A local domain of influence theorem*

Assume that

$$\int_{\mathcal{B} \cap P^c_{R''}} (\eta(\mathbf{u}))(\mathbf{x}, 0) \mathrm{d}v < +\infty, \qquad \forall R'' > 0,$$

and

$$\forall t \geq 0, \ \forall R'' > 0, \quad \begin{cases} \displaystyle\int_{\mathcal{B} \cap P^c_{R''_t}} \rho^{-1} \mathbf{b}^2 \mathrm{d}v < +\infty \\ \displaystyle\int_{\partial\mathcal{B} \cap P^c_{R''_t}} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla \mathbf{u}]\mathbf{n} \ \mathrm{d}a < +\infty \end{cases}$$

where we have set $\mathcal{B}^c_\xi = \mathcal{B} \backslash \mathcal{B}_\xi$, $\forall \xi > 0$. Then, letting $R, R' \to +\infty$, (2.7) yields

$$\begin{aligned} \int_{\mathcal{B} \cap P^c_{R''_t}} (\eta(\mathbf{u}))(\mathbf{x}, t) \mathrm{d}v \leq \exp\left(\frac{t}{t_0}\right) \Bigg\{ & \int_{\mathcal{B} \cap P^c_{R''}} (\eta(\mathbf{u}))(\mathbf{x}, 0) \mathrm{d}v \\ & + \int_0^t \exp\left(-\frac{s}{t_0}\right) \Bigg\{ \int_{\mathcal{B} \cap P^c_{R''_s}} \rho^{-1} \mathbf{b}^2 \mathrm{d}v \\ & + \int_{\partial\mathcal{B} \cap P^c_{R''_s}} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla \mathbf{u}]\mathbf{n} \ \mathrm{d}a \Bigg\} \mathrm{d}s \Bigg\}. \end{aligned} \tag{4.6}$$

Of course, (4.6) implies that the total mechanical energy $\int_{\mathcal{B}\cap P^c_{R''_t}} (\eta(\mathbf{u}))(\mathbf{x},t)dv$ is finite for any $t \geq 0$. It also immediately leads to the following *domain of influence theorem.*

THEOREM 4.1. *Let* $\mathbf{u}$ *be a solution to System* (2.1) *and let*

$$\begin{cases} \mathbf{u} = \mathbf{0}, \quad \dot{\mathbf{u}} = \mathbf{0} & \textit{on } (\mathcal{B} \cap P^c_{R''}) \times \{0\} \\ \mathbf{b} = \mathbf{0} & \textit{on } (\mathcal{B} \cap P^c_{R''_s}) \times \{s\}, \ \forall s \in (0,t) \\ \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla \mathbf{u}]\mathbf{n} = \mathbf{0} & \textit{on } (\partial\mathcal{B} \cap P_{R''_s}) \times \{s\}, \ \forall s \in (0,t). \end{cases}$$

Then

$$\mathbf{u} = \mathbf{0} \qquad \textit{on } (\mathcal{B} \cap P^c_{R''_s}) \times \{s\}, \ \forall s \in (0,t).$$

It is worth remarking that, if $\mathbf{u}$, $\dot{\mathbf{u}} = \mathbf{0}$ on $\mathcal{B}\times\{0\}$ and $\mathbf{b} \equiv \mathbf{0}$, $\dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{n} \equiv \mathbf{0}$, then Theorem 4.1 implies that

$$\mathbf{u} = \mathbf{0} \qquad \text{on } (\mathcal{B} \cap P^c_{q^{-1}(ct)}) \times \{t\}, \ \forall t \geq 0. \tag{4.7}$$

In the light of the counter-examples given in the previous Sub-section, the *local uniqueness* result expressed by relation (4.7) is the strongest possible (without assuming any condition on the solutions [9][1]). Indeed, the set $(\mathcal{B} \cap P^c_{q^{-1}(ct)}) \times \{t\}$ is just the region where the solution (4.4) turns out to be certainly nonzero when $\hat{u} \not\equiv 0$.

4.3. - *Physically meaningful motions*

In view of the results given in Sub-sections 4.2, 4.3, we are allowed to guess that a *purely mechanical* approach is not sufficient to give a *satisfactory* mathematical description of the motion of a linearly elastic body when $\mathbf{A}$ is assumed to satisfy condition (b) at Γ. Furthermore, as we shall see in Section 5, the same can be stated when $\mathbf{A}$ is assumed to violate the hyperbolicity condition.

We are then naturally led to look for a physically meaningful criterion which could enable us to single out the *effective* solution among the infinitely many ones satisfying System (2.1)-(2.3)-(2.4) in each of the classes $\mathcal{S}_\infty$ and $\mathcal{S}_\Gamma$. To this aim, following [10, 11], we appeal to the *laws of Thermodynamics*. Indeed, since we are dealing with singular motions of $\mathcal{B}$, we can expect that the singularity itself, in spite of the purely mechanical character of the processes, could give rise to some kind of "heat supply".

In the whole of this Sub-section we shall proceed in a purely formal way, by assuming that *all the integrals we are working with are finite*. We shall

[1] Uniqueness theorems with suitable summability conditions on the solutions are given in [13-14].

subsequently specify the conditions on data and motions assuring the finiteness of the integrals.

As is well known, for any $\mathcal{K} \subseteq \mathcal{B}$, the *first law of Thermodynamics* takes the form

$$\begin{aligned}\int_{\mathcal{K}} (\eta(\mathbf{u}) + \rho\vartheta_0\varsigma)(\mathbf{x}, t)\mathrm{d}v &= \int_{\mathcal{K}} (\eta(\mathbf{u}) + \rho\vartheta_0\varsigma)(\mathbf{x}, 0)\mathrm{d}v \\ &+ \int_0^t \left\{ \int_{\mathcal{K}} \dot{\mathbf{u}} \cdot \mathbf{b}\ \mathrm{d}v + \int_{\partial\mathcal{K}} (\mathbb{C}[\nabla\mathbf{u}]\dot{\mathbf{u}} + \mathbf{h}) \cdot \mathbf{n}\ \mathrm{d}a \right\} \mathrm{d}\tau \\ &+ H_{\mathcal{K}}(t), \quad \forall t \geq 0;\end{aligned} \tag{4.8}$$

and, denoting by $Z_{\mathcal{K}}(t)$ the *entropy production* in $\mathcal{K}$, associated with the motion $\mathbf{u}$ of $\mathcal{B}$ in the time interval $[0, t]$, the *second law of Thermodynamics* is written as

$$\begin{aligned}Z_{\mathcal{K}}(t) &= \int_{\mathcal{K}} (\rho\varsigma)(\mathbf{x}, t)\mathrm{d}v - \int_{\mathcal{K}} (\rho\varsigma)(\mathbf{x}, 0)\mathrm{d}v \\ &- \vartheta_0^{-1} \left\{ H_{\mathcal{K}}(t) + \int_0^t \mathrm{d}\tau \int_{\partial\mathcal{K}} \mathbf{h} \cdot \mathbf{n}\ \mathrm{d}a \right\} \geq 0, \qquad \forall t \geq 0,\end{aligned} \tag{4.9}$$

where $\mathbf{h}$, ϑ_0 and ς respectively denote the *heat flux* per unit surface area, the (*uniform*) *absolute temperature* and the *entropy* per unit mass.

A simple comparison between (4.8) and (4.9) shows that both the laws of Thermodynamics imply that

$$\begin{aligned}H_{\mathcal{K}}(t) \equiv &\int_{\mathcal{K}} (\eta(\mathbf{u}))(\mathbf{x}, 0)\mathrm{d}v - \int_{\mathcal{K}} (\eta(\mathbf{u}))(\mathbf{x}, t)\mathrm{d}v \\ &+ \int_0^t \left\{ \int_{\mathcal{K}} \dot{\mathbf{u}} \cdot \mathbf{b}\ \mathrm{d}v + \int_{\partial\mathcal{K}} (\mathbb{C}[\nabla\mathbf{u}]\dot{\mathbf{u}}) \cdot \mathbf{n}\ \mathrm{d}a \right\} \mathrm{d}\tau \geq 0,\end{aligned} \tag{4.10}$$

where the functional $H_{\mathcal{K}}(t)$ is called *singularity heat supply*.

REMARK 4.1. Condition (4.10) has an important physical meaning. Indeed, let us write the balance of total mechanical energy in $\mathcal{K} = \mathcal{B}_{\omega,R}(\mathbf{x}_0) \cap P^c_{R'}$:

$$\begin{aligned}
\int_{\mathcal{K}} (\eta(\mathbf{u}))(\mathbf{x}, t)\mathrm{d}v = \int_{\mathcal{K}} (\eta(\mathbf{u}))(\mathbf{x}, 0)\mathrm{d}v \\
+ \int_0^t \Bigg\{ \int_{\mathcal{K}} \dot{\mathbf{u}} \cdot \mathbf{b}\ \mathrm{d}v + \int_{\partial\mathcal{B}_\omega \cap (S_R(\mathbf{x}_0) \cap P^c_{R'})} (\mathbb{C}[\nabla\mathbf{u}]\dot{\mathbf{u}}) \cdot \mathbf{n}\ \mathrm{d}a \\
+ \int_{\mathcal{B}_\omega \cap (\partial S_R(\mathbf{x}_0) \cap P^c_{R'})} (\mathbb{C}[\nabla\mathbf{u}]\dot{\mathbf{u}}) \cdot \mathbf{e}^0_r \mathrm{d}a \\
+ \int_{\mathcal{B}_\omega \cap (S_R(\mathbf{x}_0) \cap \partial P^c_{R'})} (\mathbb{C}[\nabla\mathbf{u}]\dot{\mathbf{u}}) \cdot \mathbf{e}_\Gamma \mathrm{d}a \Bigg\} \mathrm{d}s.
\end{aligned} \tag{4.11}$$

If $\Gamma = \emptyset$ and $\mathbf{A}$ verifies the hyperbolicity condition, then any motion of $\mathcal{B}$ verifies the classical Work and Energy Theorem, so that

$$\lim_{R \to +\infty} \int_0^t \mathrm{d}\tau \int_{\mathcal{B} \cap \partial S_R(\mathbf{x}_0)} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{e}^0_r \mathrm{d}a = 0 \tag{4.12}$$

and, as a consequence, the process *is not dissipative*. Then, letting $R' \to 0$ and $R \to +\infty$ in (4.11), we deduce the obvious fact that $H_{\mathcal{B}}(t) = H(t) \equiv 0$.

As it will be clear from the counter-examples to uniqueness that will be given in the next Section, if $\Gamma = \emptyset$, but $\mathbf{A}$ *does not* satisfy the hyperbolicity condition, then the limit (4.12) is in general different from zero, and equals $-H(t)$: *i.e.*, in such a case, the singularity heat supply is equal to the work made at infinity by the "contact" external forces.

Likewise, letting first $R \to +\infty$, then $R' \to 0$ and $\omega \to 0$ in (4.11), we realize that, when $\Gamma \neq \emptyset$ and $\mathbf{A}$ satisfies condition (b), then the limit

$$H(t) = -\lim_{R' \to 0} \left(\lim_{R \to +\infty} \int_0^t \mathrm{d}\tau \int_{\partial P^c_{R'} \cap \mathcal{B}_R(\mathbf{x}_0)} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_\Gamma \mathrm{d}a \right) \neq 0$$

expresses the work made by the stress at the points of the singularity curve Γ.

In more general terms, since the restriction to $\mathcal{K} \subseteq \mathcal{B}$ of a motion $\mathbf{u}$ of $\mathcal{B}$ is a motion of $\mathcal{K}$, the same can be stated for any $\mathcal{K} \subseteq \mathcal{B}$, so that

$$H_{\mathcal{K}}(t) = -\lim_{R' \to 0} \int_0^t \mathrm{d}\tau \int_{\partial P^c_{R'} \cap \mathcal{K}} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_\Gamma \mathrm{d}a \neq 0$$

expresses the work made by the stress at the points of the singularity curve $\Gamma \cap \mathcal{K}$.

In conclusion, when a motion of $\mathcal{K} \subseteq \mathcal{B}$ is singular, either in the sense of Definition 2.2 or according to Definition 2.4, then the total mechanical energy of $\mathcal{K}$ is not conserved, and the energy released in the process is identified with $H_{\mathcal{K}}(t)$. In both cases, we find that the motions of $\mathcal{K}$ satisfy a "generalized" version of the Work and Energy Theorem, which leads us to the conclusion that

$$\vartheta_0 Z_{\mathcal{K}}(t) = H_{\mathcal{K}}(t).$$

REMARK 4.2. According to the physical content of the mathematical notion of "motion", it is quite reasonable to single out the "effective" – or "physically meaningful" – motions of $\mathcal{B}$, as the ones that satisfy the *entropy inequality* (4.9). As a consequence, the conclusions of Remark 4.1 allow us to give the following

DEFINITION 4.1. A motion $\mathbf{u}$ of $\mathcal{B}$, belonging to $\mathcal{S}_\infty \cup \mathcal{S}_\Gamma$, is said to be *physically meaningful* if and only if, for every $\mathcal{K} \subseteq \mathcal{B}$, its restriction to $\mathcal{K}$ satisfies inequality (4.10).

We shall denote by $\mathcal{F}$ the family of all physically meaningful motions of $\mathcal{B}$.

Our next step will be to verify that the nonzero solutions to System (4.1) do not satisfy inequality (4.9). Indeed, in this case, since $\mathbb{C}[\nabla \mathbf{u}]\mathbf{e}_\Gamma = \mu(\partial u/\partial \delta)\mathbf{e}_3$ and $\mu(\delta) = [\delta q'(\delta)]^{-1}$,

$$\begin{aligned}
&\int_0^t \mathrm{d}\tau \int_{\mathcal{B}_{R'}(\mathbf{x}_0)\cap\partial P_R} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla \mathbf{u}]\mathbf{e}_\Gamma \mathrm{d}a \\
&= \int_0^t \mathrm{d}\tau \int_a^b \mathrm{d}x^3 \int_0^{2\pi} \mu(R) q'(R) \hat{u}'^2(\tau - q(R)) R \ \mathrm{d}\vartheta \\
&= 2\pi t(b-a)\hat{u}'^2(t_{\mathrm{ave}} - q(R)),
\end{aligned}$$

where $a = x_0^3 - \sqrt{R'^2 - R^2}$, $b = x_0^3 + \sqrt{R'^2 - R^2}$, and $R' \gg R$. As a consequence, letting $R \to 0$ yields

$$H(t) = -4\pi R' t \hat{u}'^2(t_{\mathrm{ave}}) < 0.$$

Thus, we conclude that the *only* solution to System (4.1), endowed with a real physical meaning, is the *rest.*

4.4. - *Thermodynamical Domain of Influence and Work and Energy Theorems*

In order to present our next result, it is now necessary to state the conditions on motions assuring the finiteness of the integrals we shall work with.

In the class $\mathcal{S}ca_\Gamma$ — and under the assumption that $\mathbf{A}$ satisfies the hyperbolicity condition — they are the following:

I) $$\int_{B_R(\mathbf{x}_0)} (\eta(\mathbf{u}))(\mathbf{x},0)\mathrm{d}v < +\infty, \quad \forall \mathbf{x}_0 \in \Gamma, \quad \forall R > 0;$$

II) $$\int_{B_R(\mathbf{x}_0)} \rho^{-1}\mathbf{b}^2\mathrm{d}v < +\infty, \quad \forall \mathbf{x}_0 \in \Gamma, \quad \forall R > 0, \quad \forall t \geq 0;$$

III) $$\left| \int_{\partial\mathcal{B}\cap S_R(\mathbf{x}_0)} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{n}\ \mathrm{d}a \right| < +\infty, \quad \forall \mathbf{x}_0 \in \Gamma, \quad \forall R > 0, \quad \forall t \geq 0;$$

IV) $\forall \mathbf{x}_0 \in \Gamma$, $\forall R > 0$, $\exists M(R) > 0 : R'|\dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_\Gamma| \leq M(R)$, on $B_R(\mathbf{x}_0) \cap \partial P^c_{R'}$, $\forall R' > 0$.

We denote by $\mathcal{I}$ the class of solutions to System (2.1)-(4.10) that satisfy conditions I)-IV). In other words, $\mathcal{I} \subset \mathcal{F}$.

We are now in a position to show that the solutions to System (2.1)-(4.9) that belong to $\mathcal{F}$, enjoy the "propagation property" expressed by a Domain of Influence Theorem analogous to the one proved in Sub-section 3.1.

THEOREM 4.2. (Domain of Influence Theorem). *Let* $\mathbf{u} \in \mathcal{I}$ *be a solution to System* (2.1)-(4.10), *and assume*

$$H_{\mathcal{B}_{\omega,R}} = -\lim_{R'\to 0}\int_0^t \mathrm{d}\tau \int_{\mathcal{B}_{\omega,R}\cap\partial P_{R'}} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_\Gamma\mathrm{d}a, \quad \forall R > 0, \quad \forall \omega \geq 0.$$

Then
$$\mathbf{u} = \mathbf{0} \qquad on\ \{(\mathcal{B}\backslash\Gamma)\backslash\overline{D}_\varphi(t)\} \times [0,t].$$

PROOF. Set $\mathcal{D} = \mathcal{B}_\omega \cap P^c_{R'}$, and $f = w(\lambda_1)$ in (2.5), where λ_1 is as in the proof of Theorem 2.1. Then, using (2.10)-(2.11), we have

$$\begin{aligned}\int_{\mathcal{D}} (f\eta(\mathbf{u}))(\mathbf{x},t)\mathrm{d}v \leq &\int_{\mathcal{D}} (f\eta(\mathbf{u}))(\mathbf{x},0)\mathrm{d}v \\ &+ \int_0^t \mathrm{d}s \left\{ t_0^{-1} \int_{\mathcal{D}} (f\eta(\mathbf{u}))\mathrm{d}v + t_0 \int_{\mathcal{D}} f\rho^{-1}\mathbf{b}^2\mathrm{d}v \right. \\ &+ \int_{\partial\mathcal{B}_\omega\cap P^c_{R'}} f|\dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{n}|\mathrm{d}a \\ &\left. - \int_{\mathcal{B}_\omega\cap\partial P_{R'}} f\dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_\Gamma\mathrm{d}a \right\}.\end{aligned}$$

Hence, by first applying Grönwall's lemma, then letting $\sigma \to 0$, it follows that

$$\tag{4.13}
\begin{aligned}
\int_{\mathcal{B}_{\omega,R}(\mathbf{x}_0)\cap P^c_{R'}} (\eta(\mathbf{u}))(\mathbf{x},t)\mathrm{d}v \le \exp\left(\frac{t}{t_0}\right)\Bigg\{ &\int_{\mathcal{B}_{\omega,R^*_t}(\mathbf{x}_0)\cap P^c_{R'}} (\eta(\mathbf{u}))(\mathbf{x},0)\mathrm{d}v \\
&+\int_0^t \exp\left(-\frac{s}{t_0}\right)\Bigg\{t_0 \int_{\mathcal{B}_{\omega,R^*_{t-s}}(\mathbf{x}_0)\cap P^c_{R'}} \rho^{-1}\mathbf{b}^2\mathrm{d}v \\
&+ \int_{\partial\mathcal{B}_\omega\cap S_{R^*_{t-s}}(\mathbf{x}_0)\cap P^c_{R'}} \dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{n}\ \mathrm{d}a \\
&+ \int_{\mathcal{B}_{\omega,R^*_{t-s}}(\mathbf{x}_0)\cap\partial P^c_{R'}} \dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_\Gamma\mathrm{d}a\Bigg\}\mathrm{d}s\Bigg\}.
\end{aligned}$$

Hence, in the limit $R' \to 0$, we have

$$\tag{4.14}
\begin{aligned}
\int_{\mathcal{B}_{\omega,R}(\mathbf{x}_0)} (\eta(\mathbf{u}))(\mathbf{x},t)\mathrm{d}v \le \exp\left(\frac{t}{t_0}\right)\Bigg\{ &\int_{\mathcal{B}_{\omega,R^*_t}(\mathbf{x}_0)} (\eta(\mathbf{u}))(\mathbf{x},0)\mathrm{d}v \\
&+\int_0^t \exp\left(-\frac{s}{t_0}\right)\Bigg\{t_0 \int_{\mathcal{B}_{\omega,R^*_{t-s}}(\mathbf{x}_0)} \rho^{-1}\mathbf{b}^2\mathrm{d}v \\
&+ \int_{\partial\mathcal{B}_\omega\cap S_{R^*_{t-s}}(\mathbf{x}_0)} \dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{n}\ \mathrm{d}a\Bigg\}\mathrm{d}s \\
&- \int_0^t \exp\left(-\frac{s}{t_0}\right) H_{\mathcal{B}_{\omega,R^*_{t-s}}}(s)\mathrm{d}s\Bigg\}.
\end{aligned}$$

Since $\mathbf{u}$ satisfies (4.10), (4.14) yields the *domain of dependence inequality*

$$\tag{4.15}
\begin{aligned}
\int_{\mathcal{B}_{\omega,R}(\mathbf{x}_0)} (\eta(\mathbf{u}))(\mathbf{x},t)\mathrm{d}v \le \exp\left(\frac{t}{t_0}\right)\Bigg\{ &\int_{\mathcal{B}_{\omega,R^*_t}(\mathbf{x}_0)} (\eta(\mathbf{u}))(\mathbf{x},0)\mathrm{d}v \\
&+\int_0^t \exp\left(-\frac{s}{t_0}\right)\Bigg\{t_0 \int_{\mathcal{B}_{\omega,R^*_{t-s}}(\mathbf{x}_0)} \rho^{-1}\mathbf{b}^2\mathrm{d}v \\
&+ \int_{\partial\mathcal{B}_\omega\cap S_{R^*_{t-s}}(\mathbf{x}_0)} \dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{n}\ \mathrm{d}a\Bigg\}\mathrm{d}s\Bigg\}.
\end{aligned}$$

From now on, thanks to (4.15), we may follow step by step the proof of Theorem 3.1 to get the desired result. □

An immediate consequence of the above result is the following

THEOREM 4.3 (Uniqueness Theorem). *System* (2.1)-(2.3)-(2.4)-(4.9) *has at most one solution* $\mathbf{u} \in \mathcal{F}$.

We may now extend the classical work and energy theorem [1] to elastic solutions of the class $\mathcal{F}$ satisfying the entropy criterion (4.9).

THEOREM 4.4 (Extended work and energy theorem). *Let* $\mathbf{u} \in \mathcal{I}$ *and let* Γ *be bounded. If*

$$\begin{aligned} &\int_B \eta(\mathbf{u})(\mathbf{x},0)\mathrm{d}v < +\infty, \\ &\int_B (\rho^{-1}\mathbf{b}^2)(\mathbf{x},t)\mathrm{d}v < +\infty, \quad \forall t \geq 0, \\ &\int_{\partial B} (\dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{n})(\mathbf{x},t)\mathrm{d}a < +\infty, \quad \forall t \geq 0, \end{aligned} \tag{4.16}$$

then, $\forall t > 0$,

$$\begin{aligned} \int_B (\eta(\mathbf{u}))(\mathbf{x},t)\mathrm{d}v = &\int_B (\eta(\mathbf{u}))(\mathbf{x},0)\mathrm{d}v \\ &+ \int_0^t \left\{ \int_B \dot{\mathbf{u}} \cdot \mathbf{b}\ \mathrm{d}v + \int_{\partial B} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{n}\ \mathrm{d}a \right\} \mathrm{d}s \\ &+ \lim_{R' \to 0} \int_{B \cap \partial P^c_{R'}} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_\Gamma \mathrm{d}a, \end{aligned} \tag{4.17}$$

i.e.

$$H(t) = -\lim_{R' \to 0} \int_{B \cap \partial P^c_{R'}} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_\Gamma \mathrm{d}a.$$

PROOF. Choose $f = w(\sigma^{-1}(\varphi(R) - \varphi(r(\mathbf{x}))))$ and $D = B_\omega \cap P^c_{R'}$ in (2.5). Then

$$\begin{aligned} \int_D (f\eta(\mathbf{u}))(\mathbf{x},t)\mathrm{d}v = &\int_D (f\eta(\mathbf{u}))(\mathbf{x},0)\mathrm{d}v \\ &+ \int_0^t \left\{ \int_D f\dot{\mathbf{u}} \cdot \mathbf{b}\ \mathrm{d}v + \int_{\partial B_\omega \cap P^c_{R'}} f\dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{n}\ \mathrm{d}a \right. \end{aligned} \tag{4.18}$$

$$(4.18)\qquad + \int_{\mathcal{B}_\omega \cap \partial P^c_{R'}} f\dot{\mathbf{u}} \cdot \mathbb{C}[\nabla \mathbf{u}]\mathbf{e}_\Gamma \mathrm{d}a + \sigma^{-1} \int_{\mathcal{D}} w'\varphi'\dot{\mathbf{u}} \cdot \mathbb{C}[\nabla \mathbf{u}]\mathbf{e}_r \mathrm{d}v \Big\} \mathrm{d}s.$$

Now, thanks to I)-IV), we are allowed to take the limit $R' \to 0$ in (4.13), to get

$$\int_{\mathcal{B}_{\omega,R}(\mathbf{x}_0)} (\eta(\mathbf{u}))(\mathbf{x},t)\mathrm{d}v \leq \exp\left(\frac{t}{t_0}\right) \Big\{ \int_{\mathcal{B}_{\omega,R^*_t}(\mathbf{x}_0)} (\eta(\mathbf{u}))(\mathbf{x},0)\mathrm{d}v + \int_0^t \exp\left(-\frac{s}{t_0}\right) \Big\{ t_0 \int_{\mathcal{B}_{\omega,R^*_{t-s}}(\mathbf{x}_0)} \rho^{-1}\mathbf{b}^2 \mathrm{d}v + \int_{\partial\mathcal{B}_\omega \cap S_{R^*_{t-s}}(\mathbf{x}_0)} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla \mathbf{u}]\mathbf{n}\ \mathrm{d}a + \int_{\mathcal{B}_{\omega,R^*_{t-s}}(\mathbf{x}_0)} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla \mathbf{u}]\mathbf{e}_\Gamma \mathrm{d}a \Big\} \mathrm{d}s \Big\}.$$

Now, since Γ is bounded, a suitable positive $\overline{R}$ certainly exists such that $P_{R'} \subset S_{\overline{R}}$. As a consequence, by virtue of (4.16), we are allowed to take the limit $R \to +\infty$ in (4.13), to obtain

$$\int_{\mathcal{B}_\omega} (\eta(\mathbf{u}))(\mathbf{x},t)\mathrm{d}v < +\infty, \qquad \forall t \geq 0,$$

and since

$$|w'\varphi'\dot{\mathbf{u}} \cdot \mathbb{C}[\nabla \mathbf{u}]\mathbf{e}_r| \leq \frac{w'}{2} \{c\nabla \mathbf{u} \cdot \mathbb{C}[\nabla \mathbf{u}] + c^{-1}\varphi'^2 \rho \dot{\mathbf{u}} \cdot \mathbf{A}(\mathbf{x},\mathbf{e}_r)\dot{\mathbf{u}} \leq Wc\eta(\mathbf{u}),$$

where $W = \sup_{\mathbb{R}} w'$, we have

$$\lim_{R\to+\infty} \left| \int_0^t \mathrm{d}s \int_{\mathcal{D}} w'\varphi'\dot{\mathbf{u}} \cdot \mathbb{C}[\nabla \mathbf{u}]\mathbf{e}_r \mathrm{d}v \right| \leq Wc \int_0^t \mathrm{d}s \int_{\mathcal{B}} \eta(\mathbf{u})\mathrm{d}v < +\infty.$$

Therefore, letting $R \to +\infty$, then applying Lebesgue's dominated convergence theorem, and finally letting $\sigma \to +\infty$, $R' \to 0$, and $\omega \to 0$, we obtain (4.17). □

REMARK 4.3. If (4.16) hold, then relation (4.17) retains its validity even when $\mathbf{u}$ is *not* assumed to belong to $\mathcal{I}$, but only to satisfy condition IV).

REMARK 4.4. These last two results suggest the possibility of explicitly taking into account the singularity heat supply in the principles of Thermodynamics, by rewriting conditions (4.8)-(4.9) on the motions of $\mathcal{B}$ in the following form:

$$\int_{\mathcal{B}} (\eta(\mathbf{u}) + \rho\vartheta_0\varsigma)(\mathbf{x},t)\mathrm{d}v = \int_{\mathcal{B}} (\eta(\mathbf{u}) + \rho\vartheta_0\varsigma)(\mathbf{x},0)\mathrm{d}v + \int_0^t \left\{ \int_{\mathcal{B}} \dot{\mathbf{u}}\cdot\mathbf{b}\,\mathrm{d}v + \int_{\partial\mathcal{B}} (\mathbb{C}[\nabla\mathbf{u}]\dot{\mathbf{u}} + \mathbf{h})\cdot\mathbf{n}\,\mathrm{d}a \right\} - \lim_{R'\to 0} \int_{\mathcal{B}\cap\partial P^c_{R'}} \dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_r\mathrm{d}a, \qquad \forall t\geq 0,$$

and

$$Z_{\mathcal{B}}(t) = \int_{\mathcal{B}} (\rho\varsigma)(\mathbf{x},t)\mathrm{d}v - \int_{\mathcal{B}} (\rho\varsigma)(\mathbf{x},0)\mathrm{d}v + \vartheta_0^{-1}\left\{ \lim_{R'\to 0} \int_{\mathcal{B}\cap\partial P^c_{R'}} \dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_r\mathrm{d}a - \int_0^t \mathrm{d}\tau \int_{\partial\mathcal{B}} \mathbf{h}\cdot\mathbf{n}\,\mathrm{d}a \right\} \geq 0, \qquad \forall t\geq 0$$

when $\Gamma \neq \emptyset$ and $\mathbf{A}$ satisfies the hyperbolicity condition.

In the next Section, we shall see how should these conditions be written when $\varphi_\infty < +\infty$.

5. - Qualitative properties of motions which are singular at infinity

We now turn our attention toward the case in which the acoustic tensor $\mathbf{A}$ of the body $\mathcal{B}$ *does not* satisfy the hyperbolicity condition. Only for the sake of simplicity, we shall assume $\Gamma = \emptyset$. The reader should be aware that the assumption $\Gamma \neq \emptyset$ would not affect our reasonings with any substantial difficulty, but would only involve far more complicated calculations.

5.1. - *Counter-examples again*

It is already known [8] that the violation of hyperbolicity condition entails the loss of uniqueness of motions. We shall now discuss this result in a more general context.

Let then $\mathcal{B} \subset \mathbb{R}^3$ be the exterior of the infinite cylindrical region having axis x^3 and radius 1. We assume again that $\mathcal{B}$ is isotropic with Lamé moduli λ and μ such that $\lambda = 0$, $\mu > 0$. Using the cylindrical coordinates $(r = r(\mathbf{x}), \vartheta, x^3)$, we assume

$$\rho(\mathbf{x}) = \rho(r) = r^{-1}\varphi'(r), \qquad \mu(\mathbf{x}) = \mu(r) = [r\varphi'(r)]^{-1}, \quad \forall r > 0.$$

If we want a solution $\mathbf{u} \equiv (u_r = 0,\ u_\vartheta = 0,\ u_3 = u(r,t))$, then System (2.1)-(2.3)$_1$-(2.4) is written in the form

$$(5.1) \qquad \begin{aligned} & r\rho\ddot{u} = \frac{\partial}{\partial r}\left(r\mu\frac{\partial u}{\partial r}\right), \qquad \forall (r,t) \in (1,+\infty) \times (0,+\infty), \\ & u(1,t) = 0, \qquad \forall t \in [0,+\infty), \\ & u(r,0) = 0, \qquad \dot{u}(r,0) = 0, \qquad \forall r \in [1,+\infty). \end{aligned}$$

We use again an additional system of *boundary* conditions, namely

$$\lim_{r\to+\infty} u(r) = u_\infty(t), \qquad \forall t \in [0,+\infty),$$

with the compatibility conditions $u_\infty(0) = \dot{u}_\infty(0) = 0$. We assume that

$$u_\infty(t) = \tilde{u}_\infty(t) + \hat{u}_\infty(t),$$

with

$$\begin{aligned} \tilde{u}_\infty(0) &= \tilde{u}_\infty(\varphi_\infty) = \tilde{u}_\infty(2\varphi_\infty) = \tilde{u}'_\infty(\varphi_\infty) \\ &= \tilde{u}'_\infty(2\varphi_\infty) = \tilde{u}''_\infty(\varphi_\infty) = \tilde{u}''_\infty(2\varphi_\infty) = 0 \end{aligned}$$

and

$$\hat{u}_\infty(t) = -\tilde{u}_\infty(t + 2\varphi_\infty), \quad \forall t \in [0,+\infty).$$

By the same standard method used in Section 4, we arrive at a solution of the form

$$(5.2) \qquad u(r,t) = u_1(t + \varphi(r)) + u_2(t - \varphi(r)),$$

subject to conditions

$$(5.3) \qquad \begin{aligned} & u_1(\varphi(r)) + u_2(-\varphi(r)) = 0, \\ & u'_1(\varphi(r)) + u'_2(-\varphi(r)) = 0, \end{aligned}$$

and, by virtue of the assumption $\varphi_\infty < +\infty$,

$$
\begin{aligned}
&u_1(\varphi(1)+t)+u_2(t-\varphi(1))=0,\\
&u_1(\varphi_\infty+t)+u_2(t-\varphi_\infty)=u_\infty(t).
\end{aligned}
\tag{5.4}
$$

According to what has been done in [12], we disregard condition $(5.4)_1$, and solve System $(5.1)_1$-(5.3)-$(5.4)_2$. Moreover, we split $(5.4)_2$ into the following two conditions:

$$
\begin{aligned}
&u_1(\varphi_\infty+t)=\hat{u}_\infty(t), \quad \forall t>0\\
&u_2(t-\varphi_\infty)=\tilde{u}_\infty(t), \quad \forall t>0.
\end{aligned}
\tag{5.5}
$$

Accordingly, we first arrive at a solution of the form

$$
\hat{u}(r,t)=\begin{cases} 0 & \text{on } \{(r,t): r\in[1,+\infty),\\ & \qquad 0\le t\le \varphi_\infty-\varphi(r)\}\\ \hat{u}_\infty(t+\varphi(r)-\varphi_\infty) & \text{on } \{(r,t): r\in[1,+\infty),\\ & \qquad t\ge \varphi_\infty-\varphi(r)\},\end{cases}
\tag{5.6}
$$

which satisfies condition $(5.5)_1$, *i.e.*

$$\lim_{r\to+\infty}\hat{u}(r,t)=\hat{u}_\infty(t).$$

Then, we find another solution to System $(5.1)_{1,2}$, which has the form

$$
\tilde{u}(r,t)=\begin{cases} 0 & \text{on } \{(r,t): r\in[1,+\infty),\\ & \qquad 0\le t\le \varphi(r)-\varphi(1)\}\\ \tilde{u}_\infty(t-\varphi(r)+\varphi_\infty) & \text{on } \{(r,t): r\in[1,+\infty),\\ & \qquad t\ge \varphi(r)-\varphi(1)\}\end{cases}
$$

and satisfies condition $(5.5)_2$, *i.e.*

$$\lim_{r\to+\infty}\tilde{u}(r,t)=\tilde{u}_\infty(t).$$

The field

$$u(r,t)=\hat{u}(r,t)+\tilde{u}(r,t)$$

may be now easily shown to be twice continuously differentiable and to be a solution to the whole System (5.1) (condition $(5.1)_2$ included).

It is now easy to show that, if $H_{\mathcal{B}_R}(t)$ denotes the singularity heat supply in $\mathcal{B}_R$ at instant t, and

$$H_{\mathcal{B}_R}(t)=-\lim_{R\to+\infty}\int_0^t d\tau\int_{\mathcal{B}\cap\partial S_R}\dot{\mathbf{u}}\cdot\mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_r da,$$

it follows

$$H_{\mathcal{B}_R}(t) < 0, \quad \forall R > 1.$$

If we put

$$m(R) = \max\{\varphi(R) - \varphi(1), \varphi_\infty - \varphi(R)\},$$

we have indeed

$$\begin{aligned}
&\int_0^t \mathrm{d}\tau \int_{\mathcal{B}\cap\partial S_R} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_r \mathrm{d}a \\
&= \int_{m(R)}^t \mathrm{d}\tau \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi \mathrm{d}\phi \int_0^{2\pi} \mu(R)\varphi'(R)\{\hat{u}'_\infty(\tau + \varphi(R) - \varphi_\infty) \\
&\qquad + \tilde{u}'_\infty(\tau - \varphi(R) + \varphi_\infty)\}^2 R^2 \mathrm{d}\vartheta \\
&= 4\pi R(t - m(R))\{\hat{u}'_\infty(t_{\text{ave}} + \varphi(R) - \varphi_\infty) \\
&\qquad + \tilde{u}'_\infty(t_{\text{ave}} - \varphi(R) + \varphi_\infty)\}^2 > 0, \quad \forall R > 1.
\end{aligned}$$

This shows that, if we are able to prove a relation analogous to (4.17), then such nonzero solutions turn out to violate the principles of Thermodynamics. As a consequence, they cannot be considered as effective motions of $\mathcal{B}$.

5.2. - *The work and energy theorem*

This final Sub-section is devoted to prove a "generalized" Work and Energy Theorem similar to the one proved in Section 4.

THEOREM 5.1 (Second extended Work and Energy Theorem). *If, for any* $t \in (0, +\infty)$,

$$\lim_{R\to+\infty} \left| \int_0^t \mathrm{d}\tau \int_{\mathcal{B}\cap\partial S_R} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_r \mathrm{d}a \right| < +\infty, \tag{5.6}$$

then

$$H(t) = -\lim_{R\to+\infty} \int_0^t \mathrm{d}\tau \int_{\mathcal{B}\cap\partial S_R} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_r \mathrm{d}a,$$

where, as usual, $H(t)$ *is the singularity heat supply in* $\mathcal{B}$ *at instant* t.

PROOF. Set $\mathcal{D} = \mathcal{B}_R$ and $f \equiv 1$ in (2.5). One has

$$\int_{\mathcal{B}_R} (\eta(\mathbf{u}))(\mathbf{x}, t)\mathrm{d}v = \int_{\mathcal{B}_R} (\eta(\mathbf{u}))(\mathbf{x}, 0)\mathrm{d}v + \int_{\mathcal{B}_R} \dot{\mathbf{u}} \cdot \mathbf{b}\ \mathrm{d}v + \int_{\partial\mathcal{B}\cap\mathcal{S}_R} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{n}\ \mathrm{d}a + \int_{\mathcal{B}\cap\partial\mathcal{S}_R} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_r \mathrm{d}a\, \mathrm{d}s,$$

that is

$$H_{\mathcal{B}_R}(t) = -\int_0^t \mathrm{d}\tau \int_{\mathcal{B}\cap\partial\mathcal{S}_R} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_r \mathrm{d}a, \tag{5.7}$$

which, by virtue of conditions (4.16)-(5.6), leads to the desired conclusion. □

REMARK 5.1. If (4.16) and (5.6) are *not* assumed to hold, then the theorem is no longer true, but, of course, (5.7) retains its validity.

REMARK 5.2. The above theorem shows that, when $\varphi_\infty < +\infty$, then the laws of Thermodynamics could be written in the form

$$\int_{\mathcal{B}} (\eta(\mathbf{u}) + \rho\vartheta_0\varsigma)(\mathbf{x}, t)\mathrm{d}v = \int_{\mathcal{B}} (\eta(\mathbf{u}) + \rho\vartheta_0\varsigma)(\mathbf{x}, 0)\mathrm{d}v + \int_0^t \left\{ \int_{\mathcal{B}} \dot{\mathbf{u}} \cdot \mathbf{b}\ \mathrm{d}v + \int_{\partial\mathcal{B}} (\mathbb{C}[\nabla\mathbf{u}]\dot{\mathbf{u}} + \mathbf{h}) \cdot \mathbf{n}\ \mathrm{d}a \right\} - \lim_{R\to 0} \int_{\mathcal{B}\cap\partial\mathcal{S}_R} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_r \mathrm{d}a, \quad \forall t \geq 0$$

and

$$Z_{\mathcal{B}}(t) = \int_{\mathcal{B}} (\rho\varsigma)(\mathbf{x}, t)\mathrm{d}v - \int_{\mathcal{B}} (\rho\varsigma)(\mathbf{x}, 0)\mathrm{d}v + \vartheta_0^{-1} \left\{ \lim_{R'\to 0} \int_{\mathcal{B}\cap\partial\mathcal{S}_R} \dot{\mathbf{u}} \cdot \mathbb{C}[\nabla\mathbf{u}]\mathbf{e}_r \mathrm{d}a - \int_0^t \mathrm{d}\tau \int_{\partial\mathcal{B}} \mathbf{h} \cdot \mathbf{n}\ \mathrm{d}a \right\} \geq 0, \qquad \forall t \geq 0,$$

in order to take explicitly into account the singularity heat supply.

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