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Adimurthi
S. L. Yadava

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Multiplicity Results for Semilinear Elliptic Equations in a Bounded Domain of $\mathbb{R}^2$ Involving Critical Exponents

ADIMURTHI - S.L. YADAVA

1. - Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ be a C^1-function with $f(x,-t) = -f(x,t)$. Consider the following problem

$$(1.1) \qquad \begin{aligned} -\Delta u &= f(x,u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

When $n \geq 4$ and $f(x,t) = |t|^{\frac{4}{n-2}}t + \lambda t$, Brézis-Nirenberg [8] proved that (1.1) admits a non-trivial positive solution, provided $0 < f'(0) < \lambda_1(\Omega)$. Here $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of $-\Delta$. In this context, consider the following natural questions.

(Q_1) If $0 < f'(0) < \lambda_1(\Omega)$, can one get a solution of (1.1) which changes sign?

(Q_2) If $f'(0) \geq \lambda_1(\Omega)$, does (1.1) admit a non-trivial solution?

Question (Q_1) was discussed by Atkinson-Brézis-Peletier [6] and Cerami-Solimini-Struwe [10]. In [10] it has been shown that, when $n \geq 6$, problem (1.1) admits a solution which changes sign. Using this, they also proved that, when $n \geq 7$ and Ω is a ball, (1.1) admits infinitely many radial solutions which change sign. In [6] (see also Adimurthi-Yadava [2]) it has been shown that, when $n = 3, 4, 5, 6$, (1.1) does not admit any radial solution which changes sign in a ball of sufficiently small radius.

Question (Q_2) was discussed by Capozzi-Fortunato-Palmieri [9] and they proved that if $f'(0) > 0$, then (1.1) always admits a non-trivial solution.

When Ω is a ball and $n \geq 4$, Fortunato-Jannelli [11] have proved that, for $f'(0) > 0$, (1.1) admits infinitely many solutions. In view of the results of [6], solutions obtained in [11] in a ball need not be radial.

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Let $n = 2$ and $f(x,t) = h(x,t)\ \exp(bt^2)$ be a function of critical growth on Ω. Adimurthi [1] proved that (1.1) admits a non-trivial positive solution, provided $\lim\limits_{t\to\infty}\ \inf\limits_{x\in\overline{\Omega}}\ h(x,t)t = \infty$ and $\sup\limits_{x\in\overline{\Omega}}\ f'(x,0) < \lambda_1(\Omega)$. In this paper, we discuss questions (Q_1) and (Q_2) when $n = 2$ and $f(x,t) = h(x,t)\ \exp(bt^2)$ is a function of critical growth. In this case, in order to get results similar to higher dimensions, the striking phenomenon is that the dimensional restriction is reflected in the restriction of growth of h. We prove the following main results.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $0 < \lambda_1(\Omega) < \lambda_2(\Omega) < \ldots$ be the eigenvalues of the following problem

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

Let $f(x,t) = h(x,t)\ \exp(bt^2)$ be a function of critical growth on Ω (see definition 2.1). Consider the following problem

$$\begin{aligned} -\Delta u &= h(x,u)\ \exp(bu^2) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

We have

THEOREM 1.1. *Let f satisfy*

(1) *for some positive integer k*

$$\lambda_k(\Omega) \le \inf_{x\in\overline{\Omega}}\ f'(x,0) \le \sup_{x\in\overline{\Omega}}\ f'(x,0) < \lambda_{k+1}(\Omega); \tag{1.4}$$

(2) *there exist $\mu > 0$, $\tau > 0$ such that*

$$\inf_{x\in\overline{\Omega}}\ h(x,t)t \ge e^{\mu t}, \qquad \textit{for all } t \ge \tau. \tag{1.5}$$

Then (1.3) *admits a non-trivial solution.*

THEOREM 1.2. *Suppose that*

(1) $\sup\limits_{x\in\overline{\Omega}}\ f'(x,0) < \lambda_1(\Omega)$;

(2) *given any $N > 0$, there exists $t_N > 0$ such that*

$$\inf_{x\in\overline{\Omega}}\ h(x,t)t \ge e^{Nt}, \qquad \textit{for all } t \ge t_N. \tag{1.6}$$

Then (1.3) *has a non-trivial solution which changes sign in Ω.*

THEOREM 1.3. *Let $\Omega = B(0,R) = \{x \in \mathbb{R}^2;\ |x| < R\}$ and f satisfy the conditions* (1) *and* (2) *of Theorem* 1.2. *Further, assume $f(x,t) = f(|x|,t)$. Then* (1.3) *has infinitely many radial solutions which change sign.*

REMARK 1.4. In Theorem 1.3, condition (1.6) is optimal in order to get a radial solution which changes sign. If we take $f(t) = t\ \exp(t^2 + |t|^\beta)$, $0 \leq \beta \leq 1$, then it has been shown by the authors in [4] that (1.3) does not admits any radial solution which changes sign in a disc of sufficiently small radius.

If we drop the radial requirement of the solution in Theorem 1.3, then we have a stronger result.

THEOREM 1.5. *Let Ω be a ball or rectangle and $f = h(t)\exp(bt^2)$ satisfy $\lim\limits_{t\to\infty} h(t)t = \infty$. Then* (1.3) *has infinitely many solutions.*

2. - Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. In view of the Moser-Trudinger imbedding, the following notion of functions of critical growth is introduced in [1].

DEFINITION 2.1. Let $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a C^1-function and $b > 0$. The function $f(x,t) = h(x,t)\ \exp(bt^2)$ is said to be a function of critical growth on Ω if it satisfies the following:

There exists a constant $M > 0$ such that, for every $\varepsilon > 0$ and for all $(x,t) \in \overline{\Omega} \times (0,\infty)$,

(H_1) $f(x,0) = 0,\ f(x,t) > 0,\ f(x,-t) = -f(x,t);$

(H_2) $f'(x,t) > \dfrac{f(x,t)}{t}$

where $f'(x,t) = \dfrac{\partial f}{\partial t}\ (x,t);$

(H_3) $F(x,t) \leq M(1 + f(x,t)),$

where $F(x,t) = \int\limits_0^t f(x,s)\mathrm{d}s;$

(H_4) $\lim\limits_{t\to\infty} \sup\limits_{x\in\overline{\Omega}} h(x,t)\exp(-\varepsilon t^2) = 0,\ \lim\limits_{t\to\infty} \inf\limits_{x\in\overline{\Omega}} h(x,t)\ \exp(\varepsilon t^2) = \infty.$

Let $H_0^1(\Omega)$ be the usual Sobolev space. For $u \in H_0^1(\Omega)$ and $p \geq 1$, we denote

$$\|u\|^2 = \int\limits_\Omega |\nabla u|^2 \mathrm{d}x, \qquad |u|_p^p = \int\limits_\Omega |u|^p \mathrm{d}x,$$

$$|u|_\infty = \operatorname*{ess.sup}_\Omega |u|,$$

$$u^+ = \max\ (u,0),\ u^- = \max\ (-u,0).$$

Let f be a function of critical growth on Ω. Define

$$\partial B(\Omega, f) = \left\{ u \in H_0^1(\Omega)\backslash\{0\};\ \|u\|^2 = \int_\Omega f(x,u)u \ \mathrm{d}x \right\},$$

$$\partial B_1(\Omega, f) = \left\{ u \in \partial B(\Omega, f);\ u^\pm \in \partial B(\Omega, f) \right\},$$

$$J(u) = \frac{1}{2}\ \|u\|^2 - \int_\Omega F(x,u)\mathrm{d}x,$$

$$I(u) = \frac{1}{2} \int_\Omega f(x,u)u\mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x,$$

$$\frac{a(\Omega,f)^2}{2} = \inf_{\partial B(\Omega,f)} J$$

$$\frac{a_1(\Omega,f)^2}{2} = \inf_{\partial B_1(\Omega,f)} J.$$

We need the following results from [1]

THEOREM 2.1. *Let $f(x,t) = h(x,t)\ \exp(bt^2)$ be a function of critical growth on Ω. Then*

(1) $J : H_0^1(\Omega) \to \mathbb{R}$ *satisfies the Palais-Smale condition on* $\left(-\infty,\ \frac{2\pi}{b}\right)$;

(2) *Further, assume that* $\lim\limits_{t\to\infty}\ \inf\limits_{x\in\overline{\Omega}} h(x,t)t = \infty,\ \sup\limits_{x\in\overline{\Omega}} f'(x,0) < \lambda_1(\Omega)$. *Then*

(a) $0 < a(\Omega,f)^2 < \frac{4\pi}{b}$,

(b) *There exists* $u_0 \in \partial B(\Omega, f),\ u_0 \geq 0$ *such that*

$$J(u_0) = \frac{a(\Omega,f)^2}{2}$$

and u_0 *is a solution of* (1.3).

LEMMA 2.2. *Let f be a function of critical growth on Ω and $\{u_n\}$, $\{v_n\}$ be bounded sequences in $H_0^1(\Omega)$ converging weakly and for almost all x in Ω to u, v respectively. Then*

(i) *If* $\overline{\lim\limits_{n\to\infty}}\ \|u_n\|^2 < \frac{4\pi}{b}$, *then for every non-negative integer k,*

$$\lim_{n\to\infty} \int_\Omega \frac{f(x,u_n)}{u_n}\ v_n^k\mathrm{d}x = \int_\Omega \frac{f(x,u)}{u}\ v^k\mathrm{d}x;$$

(ii) *If* $\sup\limits_n \int\limits_\Omega f(x,u_n)u_n\mathrm{d}x < \infty$, *then*

$$\lim_{n\to\infty} \int_\Omega F(x,u_n)\mathrm{d}x = \int_\Omega F(x,u)\mathrm{d}x;$$

(iii) *Suppose*

(a) $\lim_{n\to\infty} J(u_n) \le \frac{2\pi}{b}, \ \sup_n \int_\Omega f(x,u_n)u_n\mathrm{d}x < \infty,$

(b) $u \not\equiv 0$ *and* $\|u\|^2 \ge \int_\Omega f(x,u)u \ \mathrm{d}x,$

then

$$\lim_{n\to\infty} \int_\Omega f(x,u_n)u_n\mathrm{d}x = \int_\Omega f(x,u)u \ \mathrm{d}x;$$

(iv) $I(u) \ge 0$ *for all* u *and* $I(u) = 0$ iff $u \equiv 0$. *Further, there exists a constant* $K > 0$ *such that*

$$\int_\Omega f(x,u)u \ \mathrm{d}x \le K(1 + I(u)).$$

(For the proof, see lemmas from 3.1 to 3.4 and the Main Theorem in [1]).

We also need the following abstract result of Bartolo-Benci-Fortunato [7].

THEOREM 2.3. *Let* E *be a Hilbert space and* $T \in C^1(E,\mathbb{R})$ *be even and* $T(0) = 0$. *Let* T *satisfy the following*:

1. *There exists* $\eta > 0$ *such that* T *satisfies the Palais-Smale condition on* $(0,\eta)$*;*
2. *There exist two closed subspaces* V_1 *and* V_2 *of* E *and positive constants* ρ, δ, η', *with* $0 < \delta < \eta' < \eta$, *such that*

$$T(u) \le \eta', \quad \textit{for all } u \textit{ in } V_1,$$

$$T(u) \ge \delta, \quad \textit{for all } u \textit{ in } V_2 \textit{ with } \|u\| = \rho,$$

$$\dim\ V_1 < \infty, \ \operatorname{codim}\ V_2 < \infty \quad \textit{and}$$

$$\dim\ V_1 > \ \operatorname{codim}\ V_2.$$

Then there exist at least dim V_1 – Codim V_2 *pairs of critical points of* T *with values in* $[\delta, \eta']$.

Finally, we end this section with the definition of Moser functions.

Let $x_0 \in \Omega$ and $L > 0$ such that $B(x_0, L)$ is contained in Ω. For $0 < \ell < L$, define

$$m_\ell(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} \left(\log \frac{L}{\ell}\right)^{1/2} & , \ 0 \le |x - x_0| \le \ell \\ \dfrac{\log\left(\frac{L}{|x-x_0|}\right)}{\left[\log \frac{L}{\ell}\right]^{1/2}} & , \ \ell \le |x - x_0| \le L \\ 0 & , \ |x - x_0| > L. \end{cases}$$

Then $m_\ell \in H_0^1(\Omega)$ and $\|m_\ell\| = 1$.

3. - Proof of theorems (1.1) and (1.2)

The proof of these theorems mainly depends on the following lemma whose proof will be given at the end of the section.

MAIN LEMMA 3.1. *Let $f(x,t) = h(x,t)\exp(bt^2)$ be a function of critical growth on Ω and V be a finite dimensional subspace of $H_0^1(\Omega) \cap H^2(\Omega)$. Let $h_0(t) = \inf\{h(x,t);\ x \in \overline{\Omega}\}$ and $C(V) = \sup\{J(v) : v \in V\}$. Assume that one of the following holds*:

(1) $V = \{0\}$ and $\lim_{t\to\infty} h_0(t)t = \infty$;

(2) $C(V) = 0$ *and there exist* $\tau > 0$, $\mu > 0$ *such that*

$$h_0(t)t \ge e^{\mu t}, \qquad \text{for all } t \ge \tau;$$

(3) *For every* $N > 0$, *there exists* $t_N > 0$ *such that*

$$h_0(t)t \ge e^{Nt}, \qquad \text{for all } t \ge t_N.$$

Then there exists $\ell_0 > 0$ such that, for $0 < \ell < \ell_0$,

$$\sup_{\substack{v\in V\\ t\in\mathbb{R}}} J(v + tm_\ell) < C(V) + \frac{2\pi}{b}, \tag{3.1}$$

where m_ℓ is the Moser function.

Let $E_k \subset H_0^1(\Omega)$ be the eigenspace corresponding to $\lambda_k(\Omega)$ and P_k be the projection on E_k. Let

$$V_{1,k} = \bigoplus_{i=1}^{k} E_i, \qquad V_{2,k} = \bigoplus_{i=k+1}^{\infty} E_i. \tag{3.2}$$

The following lemma is proved in [3]. For the sake of completeness, we sketch the proof.

LEMMA 3.2. *Let f be a function of critical growth on Ω and, for some integer k,*

$$\lambda_k(\Omega) \leq \inf_{x\in\overline{\Omega}} f'(x,0) \leq \sup_{x\in\overline{\Omega}} f'(x,0) < \lambda_{k+1}(\Omega). \tag{3.3}$$

Then there exist $\rho > 0$, $\delta > 0$ such that

$$J(u) \leq 0, \qquad \text{for all } u \text{ in } V_{1,k} \tag{3.4}$$

$$J(u) \geq \delta, \qquad \text{for all } u \text{ in } V_{2,k} \text{ with } \|u\| = \rho. \tag{3.5}$$

PROOF. Since $\inf_{x\in\overline{\Omega}} f'(x,0) \geq \lambda_k(\Omega)$, from (H_2),

$$\frac{f(x,t)}{t} > f'(x,0),$$

for all $(x,t) \in \overline{\Omega} \times (0,\infty)$. Hence

$$F(x,t) \geq \frac{\lambda_k(\Omega)t^2}{2},$$

for all (x,t) in $\overline{\Omega} \times (0,\infty)$. Let $u \in V_{1,k}$, then

$$J(u) \leq \frac{1}{2} \sum_{i=1}^{k} \lambda_i(\Omega)|P_i u|_2^2 - \frac{\lambda_k(\Omega)}{2} |u|_2^2 \leq 0$$

which proves (3.4). To prove (3.5), let

$$d(\Omega) = \inf \left\{ \|u\|;\ u \in V_{2,k}\backslash\{0\},\ \|u\|^2 \leq \int_\Omega f(x,u)u \ \mathrm{d}x \right\}.$$

Using (i) of Lemma 2.2 and $\sup_{x\in\overline{\Omega}} f'(x,0) < \lambda_{k+1}(\Omega)$, it follows that $d(\Omega) > 0$. We prove (3.5) when $\rho = d(\Omega)/2$. Suppose (3.5) is not true, then there exists a sequence $\{u_m\}$ in $V_{2,k}$ such that

$$\|u_m\| = \frac{d(\Omega)}{2}, \qquad \lim_{m\to\infty} J(u_m) = 0, \tag{3.6}$$

$$\frac{d(\Omega)^2}{4} = \|u_m\|^2 > \int_\Omega f(x,u_m)u_m \mathrm{d}x. \tag{3.7}$$

Let $\{u_m\}$ still be a subsequence of $\{u_m\}$ which converges to u_0 weakly and for almost all x in Ω. By Fatou's lemma, (3.7) and (3.6), we obtain

$$0 \leq I(u_0) \leq \varliminf_{m\to\infty} I(u_m) \leq \lim_{m\to\infty} J(u_m) = 0.$$

Hence from (iv) of Lemma 2.2, $u_0 \equiv 0$. From (3.7) and (ii) of Lemma 2.2,

$$\lim_{m\to\infty} \int_\Omega F(x, u_m)\mathrm{d}x = 0;$$

therefore

$$\frac{d(\Omega)^2}{4} = \lim_{m\to\infty} \|u_m\|^2 = 2 \lim_{m\to\infty} \left\{ J(u_m) + \int_\Omega F(x, u_m)\mathrm{d}x \right\} = 0$$

which is a contradiction. This proves the lemma.

The following lemma has been proved in Solimini [14], Cerami-Solimini-Struwe [10]. For the sake of completeness, we give the proof.

LEMMA 3.3. *Let f be a function of critical growth on Ω and* $\sup_{x\in\overline{\Omega}} f'(x,0) <$ $\lambda_1(\Omega)$. *Then*

(1) *Let $u_0 \in \partial B_1(\Omega, f)$ be such that $J'(u_0) \not\equiv 0$ (J' denotes the derivative of J). Then*

$$J(u_0) > \inf \{J(u);\ u \in \partial B_1(\Omega, f)\}.$$

(2) *Let u_1 and u_2 be two non-negative linearly independent functions in $H_0^1(\Omega)$. Then there exist p, q in $\mathbb{R}$ such that $pu_1 + qu_2 \in \partial B_1(\Omega, f)$.*

PROOF. (1) For $p, q, t \in \mathbb{R},\ v \in H_0^1(\Omega)$, define

$$\eta(t, v) = v - tJ'(v),$$

$$\sigma(p, q) = pu_0^+ - qu_0^-$$

and

$$z(t, p, q) = \eta(t, \sigma(p, q)).$$

It is easy to see that

$$\lim_{\substack{t\to 0\\ p,q\to 1}} \frac{\mathrm{d}}{\mathrm{d}t} J(z(t, p, q)) = -\|J'(u_0)\|^2.$$

Hence we can choose $\varepsilon > 0$, $\delta > 0$ such that, for all (p, q) in $U_\varepsilon = [1-\varepsilon,\ 1+\varepsilon] \times [1-\varepsilon,\ 1+\varepsilon]$ and $0 < t \leq \delta$,

$$J(z(t, p, q)) < J(z(o, p, q)) \leq J(u_0). \tag{3.8}$$

The last inequality in the above expression follows from the fact that $u_0 \in \partial B_1(\Omega, f)$. Define $L_t : U_\varepsilon \to \mathbb{R}^2$ by

$$L_t(p, q) = \left(\rho[z(t, p, q)^+] - 1,\ \rho[z(t, p, q)^-] - 1\right),$$

where

$$\rho(v) = \frac{\int_\Omega f(x,v)v \, dx}{\|v\|^2}.$$

By choosing ε and δ sufficiently small, we obtain, for $0 < t_0 \leq \delta$ and (p,q) on ∂U_ε,

$$< L_{t_0}(p,q), \ \nu(p,q) > \geq 0,$$

where $\nu(p,q)$ denotes the unit outward normal to ∂U_ε. Hence, by Miranda's theorem [12], there exists (p_0, q_0) in U_ε such that $L_{t_0}(p_0,q_0) = 0$. This implies $z(t_0,p_0,q_0) \in \partial B_1(\Omega, f)$ and hence, from (3.8), we have

$$J(z(t_0,p_0,q_0)) < J(u_0)$$

which proves (1).

(2) Let $f_1(x,t) = f(x,t) - f'(x,0)t$ and, for v in $H_0^1(\Omega)$, define

$$\rho_1(v) = \begin{cases} \dfrac{\int_\Omega f_1(x,v)v \, dx}{\|v\|^2 - \int_\Omega f'(x,0)v^2 dx} & \text{if } v \neq 0 \\ 0 & \text{if } v = 0 \end{cases} \tag{3.9}$$

and, for $0 \leq s \leq 1$, $v_s = (1-s)u_1 - su_2$. Then by the superlinearity of f, it follows that

$$\lim_{\gamma \to \infty} \sup_{s \in [0,1]} \rho_1(\gamma \ v_s) = \infty.$$

Hence we can choose a $\gamma_0 > 0$ such that, for all $s \in [0,1]$,

$$\rho_1(\gamma_0 v_s) \geq 2. \tag{3.10}$$

Define $K = (K_1, K_2) : [0,1] \times [0,1] \to \mathbb{R}^2$ by

$$K_1(s,t) = \rho_1(\gamma_0 t v_s^-) - \rho_1(\gamma_0 t v_s^+)$$

$$K_2(s,t) = \rho_1(\gamma_0 t v_s^-) + \rho_1(\gamma_0 t v_s^+) - 2.$$

Then, for (s,t) on the boundary of $[0,1] \times [0,1]$, it follows from (3.10) that

$$< K(s,t), \ \nu(s,t) > \geq 0,$$

where $\nu(s,t)$ denotes the unit outward normal. Hence, from Miranda's theorem [12], there exists $(s_0,t_0) \in (0,1) \times (0,1)$ such that $K(s_0,t_0) = 0$. This combined with (3.9) implies $\gamma_0 t_0 v_{s_0} \in \partial B_1(\Omega, f)$ and this proves (2).

PROOF OF THEOREM 1.1. Let k be the integer such that (1.4) holds. Let $V_{1,k}$, $V_{2,k}$ be subspaces of $H_0^1(\Omega)$ given by (3.2). By Lemma 3.2, there exists $\delta > 0$, $\rho > 0$ such that

$$J(u) \leq 0, \qquad \text{for all } u \text{ in } V_{1,k}, \tag{3.11}$$

$$J(u) \geq \delta, \qquad \text{for all } u \text{ in } V_{2,k} \text{ with } \|u\| = \rho. \tag{3.12}$$

From (3.11) and (2) of Lemma 3.1, there exists $\ell_0 > 0$ such that

$$\sup_{\substack{v \in V_{1,k} \\ t \in \mathbb{R}}} J(v + tm_{\ell_0}) < \frac{2\pi}{b}.$$

From (1) of Theorem 2.1, J satisfies the Palais-Smale condition on $\left(0, \frac{2\pi}{b}\right)$. Now the proof follows by taking $E = H_0^1(\Omega)$,

$$V_1 = V_{1,k} \oplus \mathbb{R}\ m_{\ell_0},\ V_2 = V_{2,k},\ \eta = \frac{2\pi}{b},\ \eta' = \sup_{v \in V_1} J(v),\ T = J$$

in the Theorem 2.3, with ρ and δ given in (3.12).

PROOF OF THEOREM 1.2. In view of (1) of Lemma 3.3, it is sufficient to show that the infimum of J is achieved on $\partial B_1(\Omega, f)$. To show this, we first prove:

CLAIM 1. $0 < \frac{a_1(\Omega,f)^2}{2} < \frac{a(\Omega,f)^2}{2} + \frac{2\pi}{b}$.

By definition, $a_1(\Omega, f) \geq a(\Omega, f)$. From Theorem 2.1, let $u_0 \in \partial B(\Omega, f)$ be such that

$$\sup_{\alpha \in \mathbb{R}} J(\alpha u_0) = J(u_0) =. \frac{a(\Omega, f)^2}{2} > 0, \tag{3.13}$$

therefore $a_1(\Omega, f) > 0$. From (2) of Lemma 3.3, for any $\ell > 0$,

$$\frac{a_1(\Omega, f)^2}{2} \leq \sup_{p,q \in \mathbb{R}} J(pu_0 + q\ m_\ell), \tag{3.14}$$

where m_ℓ is the Moser function. From (3.13) and by taking $V = \{pu_0,\ p \in \mathbb{R}\}$ in (3) of Lemma 3.1, there exists $\ell_0 > 0$ such that, for $0 < \ell < \ell_0$,

$$\sup_{p,q \in \mathbb{R}} J(pu_0 + q\ m_\ell) < \frac{a(\Omega, f)^2}{2} + \frac{2\pi}{b}. \tag{3.15}$$

Now Claim 1 follows from (3.14) and (3.15).

Let $\{u_m\}$ be in $\partial B_1(\Omega, f)$ such that

$$\lim_{m \to \infty} J(u_m) = \frac{a_1(\Omega, f)^2}{2}.$$

Since $J = I$ on $\partial B_1(\Omega, f)$, hence from (iv) of Lemma 2.2, we obtain

$$\sup_m \|u_m\| < \infty, \qquad \sup_m \int_\Omega f(x, u_m)u_m \mathrm{d}x < \infty. \tag{3.16}$$

Therefore we can extract a subsequence of $\{u_m\}$ such that

$$u_m^\pm \to u_0^\pm \qquad \text{weakly and for almost all } x \text{ in } \Omega.$$

From (3.16) and (ii) of Lemma 2.2, we get

$$\lim_{m\to\infty} \int_\Omega F(x, u_m^\pm)\mathrm{d}x = \int_\Omega F(x, u_0^\pm)\mathrm{d}x. \tag{3.17}$$

From Claim 1, we can choose $\varepsilon > 0$, $m_0 > 0$ such that, for all $m \geq m_0$,

$$a_1(\Omega, f)^2 \leq 2\ J(u_m) \leq a(\Omega, f)^2 + \frac{4\pi}{b} - \varepsilon;$$

this, together with $J(u_m^\pm) \geq \frac{a(\Omega, f)^2}{2}$, gives

$$J(u_m^\pm) \leq \frac{2\pi}{b} - \frac{\varepsilon}{2}. \tag{3.18}$$

CLAIM 2. $u_0^\pm \not\equiv 0$ and $\|u_0^\pm\|^2 \leq \int_\Omega f(x, u_0^\pm)u_0^\pm\ \mathrm{d}x$.

We shall prove this for u_0^+. A similar proof holds for u_0^-. Suppose $u_0^+ \equiv 0$. Then, from (3.17) and (3.18), we have

$$\overline{\lim_{m\to\infty}}\ \|u_m^+\|^2 = 2 \lim_{m\to\infty} \left\{ J(u_m^+) + \int_\Omega F(x, u_m^+)\mathrm{d}x \right\} \leq \frac{4\pi}{b} - \varepsilon.$$

Therefore, from (i) of Lemma 2.2,

$$\lim_{m\to\infty} \int_\Omega f(x, u_m^+)u_m^+ \mathrm{d}x = 0. \tag{3.19}$$

Since $u_m^+ \in \partial B(\Omega, f)$, therefore from (3.19) we obtain $\lim_{m\to\infty} \|u_m^+\| = 0$. This, together with $a(\Omega, f) > 0$, gives a contradiction. This proves $u_0^+ \not\equiv 0$. Now suppose

$$\|u_0^+\|^2 > \int_\Omega f(x, u_0^+)u_0^+ \mathrm{d}x. \tag{3.20}$$

Then $\{u_m^+, u_0^+\}$ satisfie all the hypotheses of (iii) of Lemma 2.2, and hence

$$\lim_{m\to\infty} \int_\Omega f(x, u_m^+)u_m^+ \mathrm{d}x = \int_\Omega f(x, u_0^+)u_0^+ \mathrm{d}x.$$

Therefore we have

$$\|u_0^+\|^2 \leq \varliminf_{m\to\infty} \|u_m^+\|^2 = \lim_{m\to\infty} \int_\Omega f(x, u_m^+)u_m^+ \mathrm{d}x = \int_\Omega f(x, u_0^+)u_0^+ \mathrm{d}x$$

which contradicts (3.20) and this proves the Claim 2.

Since $\sup_{x\in\bar\Omega} f'(x,0) < \lambda_1(\Omega)$ and

$$\|u_0^\pm\|^2 \leq \int_\Omega f(x, u_0^\pm)u_0^\pm \mathrm{d}x,$$

it is possible to choose $0 < r_1 \leq 1$, $0 < r_2 \leq 1$, such that

$$v = r_1 u_0^+ - r_2 u_0^- \in \partial B_1(\Omega, f)$$

(for the details on the existence of r_1 and r_2, we refer [1], see step 2 in the proof of Lemma 3.4). Now

$$\begin{aligned}\frac{a_1(\Omega,f)^2}{2} &\leq J(v) \leq I(v) = I(r_1u_0^+) + I(r_2u_0^-)\\ &\leq I(u_0^+) + I(u_0^-) \leq \varliminf_{m\to\infty} I(u_m)\\ &= \lim_{m\to\infty} J(u_m) = \frac{a_1(\Omega,f)^2}{2}.\end{aligned}$$

Hence $r_1 = r_2 = 1$, $u_0 \in \partial B_1(\Omega, f)$ and $J(u_0) = \frac{a_1(\Omega,f)^2}{2}$. This completes the proof of Theorem 1.2.

REMARK 3.4. The above proof also shows that J satisfies Palais-Smale condition in

$$J^{-1}\left(-\infty, \frac{a(\Omega,f)^2}{2} + \frac{2\pi}{b}\right) \cap \{\text{sufficiently small neighbourhood of } \partial B_1(\Omega, f)\}$$

PROOF OF THE MAIN LEMMA 3.1. Let $u_\ell = v_\ell + t_\ell m_\ell$ be such that $t_\ell \geq 0$ and

$$J(u_\ell) = \sup_{\substack{v\in V\\ t\in\mathbb{R}}} J(v + t\ m_\ell).$$

Since $J'(u_\ell) = 0$ on $\{v + tm_\ell,\ v \in V,\ t \in \mathbb{R}\}$, hence

$$\|u_\ell\|^2 = \int_\Omega f(x, u_\ell)u_\ell \mathrm{d}x. \tag{3.21}$$

Now suppose (3.1) is not true, then there exists a sequence $\{\ell_n\}$ such that $\ell_n \to 0$ as $n \to \infty$ and, for $v_n = v_{\ell_n},\ m_n = m_{\ell_n},\ t_n = t_{\ell_n},\ u_n = u_{\ell_n}$,

$$C(V) + \frac{2\pi}{b} \leq J(u_n). \tag{3.22}$$

Let $x_0 \in \Omega$ be the point which occurs in the definition of Moser function.

STEP 1. $\{\|v_n\|\}$ and $\{t_n\}$ are bounded.
Suppose, on the contrary, Step 1 is not true. Then either

$$\text{(i)}\quad \overline{\lim_{n\to\infty}} \frac{t_n}{\|v_n\|} > 0 \qquad \text{or} \qquad \text{(ii)}\quad \lim_{n\to\infty} \frac{t_n}{\|v_n\|} = 0.$$

In the case (i), there exist a subsequence of $\{v_n, t_n\}$ and a constant $C_0 > 0$ such that, for large n,

$$\frac{t_n}{\|v_n\|} \geq C_0 \quad \text{and} \quad t_n \to \infty \text{ as } n \to \infty. \tag{3.23}$$

Since $\|m_n\| = 1$, we have, from (3.23) and Cauchy-Schwartz inequality,

$$\|u_n\|^2 = t_n^2 + 2t_n < v_n, m_n > + \|v_n\|^2 \leq C_1 t_n^2, \tag{3.24}$$

where $C_1 = 1 + \frac{2}{C_0} + \frac{1}{C_0^2}$. Since $\left\{\frac{\|v_n\|}{t_n}\right\}$ is bounded and $v_n \in V$, therefore $\left\{\frac{|v_n|_\infty}{t_n}\right\}$ is bounded. Hence for $x \in B(x_0, \ell_n)$ and for large n,

$$\begin{aligned} u_n(x) &= v_n(x) + t_n m_n(x) \\ &= t_n m_n(x)\left(1 + \frac{v_n(x)}{t_n}\frac{1}{m_n(x)}\right) \geq \frac{1}{2}\, t_n m_n(x). \end{aligned} \tag{3.25}$$

Hence from (H_4), (3.21), (3.23), (3.24) and (3.25),

$$\begin{aligned} C_1 t_n^2 \geq \|u_n\|^2 &= \int_\Omega f(x, u_n)u_n \mathrm{d}x \\ &\geq \int_{B(x_0,\ell_n)} h_0(u_n)u_n \exp(bu_n^2)\ \mathrm{d}x \\ &\geq C_2 \exp\left(\frac{b}{8}t_n^2 m_n^2(x_0)\right)\ell_n^2 \end{aligned}$$

for some positive constant C_2. This implies that

$$C_1 \geq C_2 \exp\left\{\frac{bt_n^2}{16\pi} \log \frac{L}{\ell_n} - 2 \log \frac{1}{\ell_n} - 2 \log t_n\right\} \to \infty$$

as $\ell_n \to 0$; which is a contradiction and hence (i) cannot occur.

In case (ii), first observe that $||v_n|| \to \infty$. Let

$$z_n = \frac{v_n}{||v_n||}, \quad \varepsilon_n = \frac{t_n^2}{||v_n||^2} + \frac{2t_n}{||v_n||} < z_n, m_n > .$$

Then, by going to a subsequence if necessary and using the fact that $z_n \in V$, we can assume that

$$\lim_{n\to\infty} z_n = z_0, \quad z_0 \in V \backslash \{0\}, \quad \lim_{n\to\infty} \varepsilon_n = 0. \tag{3.26}$$

Now

$$\begin{aligned} ||u_n||^2 &= ||v_n||^2 + 2t_n < v_n, m_n > +t_n^2 \\ &= ||v_n||^2(1 + \varepsilon_n). \end{aligned}$$

Hence

$$\frac{u_n}{||u_n||} = \frac{1}{(1+\varepsilon_n)^{1/2}} \left(z_n + \frac{t_n}{||v_n||} m_n\right) \to z_0 \not\equiv 0 \tag{3.27}$$

in $H_0^1(\Omega)$.

From (3.21), (3.27) and Fatou's lemma,

$$\begin{aligned} \infty &= \int_\Omega \varliminf_{m\to\infty} \frac{f(x,u_n)}{u_n} \left(\frac{u_n}{||u_n||}\right)^2 \mathrm{d}x \\ &\leq \varliminf_{m\to\infty} \frac{1}{||u_n||^2} \int_\Omega f(x,u_n) u_n \mathrm{d}x = 1 \end{aligned}$$

which is a contradiction. This proves Step 1.

Now, for subsequences, we have

$$\begin{aligned} &\lim_{n\to\infty} v_n = v_0 \quad \text{in } V, \quad \lim_{n\to\infty} t_n = t_0 \\ &u_n \to v_0 \quad \text{weakly in } H_0^1(\Omega) \text{ and} \\ &\text{for almost all } x \text{ in } \Omega. \end{aligned} \tag{3.28}$$

From (3.21), (3.28) and by using (ii) of Lemma 2.2, we have

$$\lim_{n\to\infty} \int_\Omega F(x,u_n)\mathrm{d}x = \int_\Omega F(x,v_0)\mathrm{d}x. \tag{3.29}$$

Now letting $n \to \infty$ in (3.22) and using (3.28) and (3.29), we get

$$C(V) + \frac{2\pi}{b} \leq J(v_0) + \frac{t_0^2}{2} \leq C(V) + \frac{t_0^2}{2}. \tag{3.30}$$

STEP 2. $t_0^2 = \frac{4\pi}{b}$ and $J(v_0) = C(V)$.

From (3.30), $t_0^2 \geq \frac{4\pi}{b}$. Suppose $t_0^2 > \frac{4\pi}{b}$, then there exist $0 < \varepsilon < 1$, $n_0 > 0$ such that, for all $n \geq n_0$,

$$t_n^2 > (1+\varepsilon)\ \frac{4\pi}{b}. \tag{3.31}$$

Let

$$\begin{aligned} \varepsilon_n &= \sup_{x \in B(x_0, \ell_n)} \frac{2|v_n(x)|}{t_n m_n(x)}, \\ C &= \inf_{|t| \geq 1}\ h_0(t)t\ \exp\left(\frac{\varepsilon}{2} bt^2\right), \end{aligned} \tag{3.32}$$

then from (3.28) it follows that $\varepsilon_n \to 0$. Hence from (3.21), (3.32) and Step 1, we have

$$\begin{aligned} M = \sup_n\ \|u_n\|^2 &\geq \int_\Omega f(x, u_n)u_n \mathrm{d}x \\ &\geq \int_{B(x_0,\ell_n)} h_0(u_n)u_n\ \exp(bu_n^2)\ \mathrm{d}x \\ &\geq C \int_{B(x_0,\ell_n)} \exp\left[\left(1 - \frac{\varepsilon}{2}\right) bu_n^2\right]\ \mathrm{d}x \\ &\geq C \int_{B(x_0,\ell_n)} \exp\left[\left(1 - \frac{\varepsilon}{2}\right)(1-\varepsilon_n)bt_n^2 m_n^2(x_0)\right]\ \mathrm{d}x. \end{aligned}$$

Therefore from (3.31), for all $n \geq n_0$,

$$\begin{aligned} M &\geq C \int_{B(x_0,\ell_n)} \exp\left[\left(1 + \frac{\varepsilon}{4}\right)(1-\varepsilon_n)4\pi m_n^2(x_0)\right]\ \mathrm{d}x \\ &= C_1 \ell_n^{-2\left[\left(1+\frac{\varepsilon}{4}\right)(1-\varepsilon_n)-1\right]} \end{aligned} \tag{3.33}$$

for some positive constant C_1. Since $\varepsilon_n \to 0$, $\ell_n \to 0$ as $n \to \infty$, (3.33) gives a contradiction. Hence $t_0^2 = \frac{4\pi}{b}$ and, from (3.30), $J(v_0) = C(V)$.

STEP 3. There exist positive constants n_0 and C_0 such that, for all $n \geq n_0$,

$$\left(t_n^2 - \frac{4\pi}{b}\right) m_n^2(x_0) - \frac{1}{b}\ \varepsilon_n m_n(x_0) + \frac{1}{b}\ \log\ p_n(x_0) \leq C_0, \tag{3.34}$$

where

$$\varepsilon_n = 2bt_n \sup_{x\in\overline{\Omega}} |v_n(x)|,$$

$$p_n(x_0) = \inf\left\{th_0(t);\ t \in \left[\frac{1}{2}\ t_n m_n(x_0), 2t_n m_n(x_0)\right]\right\}.$$

Let $M = \sup_n \|u_n\|^2$. For $x \in B(x_0, \ell_n)$, it follows from (3.25) that there exists n_0 such that $u_n(x) \in \left[\frac{1}{2}\ t_n m_n(x_0), 2t_n m_n(x_0)\right]$ for all $n \geq n_0$. Now from (3.21)

$$\begin{aligned}
M \geq \|u_n\|^2 &= \int_\Omega f(x, u_n)u_n\ dx \\
&\geq \int_{B(x_0,\ell_n)} h_0(u_n)u_n \exp(bu_n^2)\ dx \\
&\geq p_n(x_0) \int_{B(x_0,\ell_n)} \exp(bu_n^2)\ dx \\
&\geq p_n(x_0) \int_{B(x_0,\ell_n)} \exp\left\{b[t_n^2 m_n^2(x_0) + 2t_n m_n(x_0)v_n(x)]\right\}\ dx \\
&\geq \pi p_n(x_0)\ \ell_n^2\ \exp\left[bt_n^2 m_n^2(x_0) - \varepsilon_n m_n(x_0)\right].
\end{aligned} \tag{3.35}$$

By definition of Moser functions

$$\ell_n = L\ \exp[-2\pi m_n^2(x_0)]. \tag{3.36}$$

Now (3.34) follows from (3.35) and (3.36).

STEP 4. There exists a $C_1 > 0$ such that, for large n,

$$\left(\log\ \frac{L}{\ell_n}\right)^{1/2} \left(\frac{4\pi}{b} - t_n^2\right) \leq C_1\ |\Delta v_n|_2. \tag{3.37}$$

Since $t \to F(x,t)$ is convex, we have, for any ξ, η real,

$$F(x,\xi) \geq F(x,\eta) + f(x,\eta)(\xi - \eta).$$

This implies

$$\int_{\Omega} F(x,u_n)u_n \mathrm{d}x \geq \int_{\Omega} F(x,v_n)\mathrm{d}x + t_n \int_{\Omega} f(x,v_n)m_n \mathrm{d}x. \tag{3.38}$$

From (3.22) and (3.38), we get

$$\begin{aligned} C(V) + \frac{2\pi}{b} &\leq J(u_n) \\ &\leq C(V) + \frac{t_n^2}{2} + t_n \left\{ < v_n, m_n > \ - \int_{\Omega} f(x,v_n)m_n \mathrm{d}x \right\}. \end{aligned} \tag{3.39}$$

From Step 1, $\{\|v_n\|\}$ is bounded and hence $\{|v_n|_\infty\}$ is bounded. Hence, for some positive constant C_2', C_2,

$$\begin{aligned} \left| \int_{\Omega} \frac{f(x,v_n)}{v_n} v_n m_n \mathrm{d}x \right| &\leq C_2' \ |v_n|_2 \ |m_n|_2 \\ &\leq C_2 |\Delta v_n|_2 \ |m_n|_2. \end{aligned} \tag{3.40}$$

Obviously $| < v_n, m_n > | \leq |\Delta v_n|_2 \ |m_n|_2$. Therefore, from (3.40) and from the fact that $\{t_n\}$ is bounded, we obtain

$$t_n \left| < v_n, m_n > \ - \int_{\Omega} f(x,v_n)m_n \mathrm{d}x \right| \leq C_3 \ |\Delta v_n|_2 \ |m_n|_2, \tag{3.41}$$

for some positive constant C_3. By definition of Moser function, it follows that

$$|m_n|_2 = O\left(\left(\log \frac{L}{\ell_n} \right)^{-\frac{1}{2}} \right). \tag{3.42}$$

Now (3.37) follows from (3.39), (3.41) and (3.42).

We discuss conditions (1), (2), (3) of the lemma separately.

(1) By hypothesis $V = \{0\}$. Therefore $C(V) = 0$, $v_m = 0$ and, from (3.22), $t_n^2 \geq \frac{4\pi}{b}$. Substituting these values in (3.34), we get

$$C_0 \geq \frac{1}{b} \ \log \ p_n(x_0) \qquad \text{for all } n \geq n_0.$$

Since $\lim\limits_{t\to\infty} h_0(t)t = \infty$, hence $p_n(x_0) \to \infty$ as $n \to \infty$. This, together with the above equation, gives a contradiction.

(2) By hypothesis $C(V) = 0$. Therefore, from Step 2, $v_n \to 0$ in V and hence $|\Delta v_n|_2 \to 0$. Since $h_0(t)t \geq e^{\mu t}$ for $t \geq \tau$ and $m_n(x_0) \to \infty$ as $n \to \infty$, by (3.34) we get, for large n,

$$\left(\frac{\mu t_n}{2b} - \frac{\varepsilon_n}{b} - \frac{C_0}{m_n(x_0)}\right) \leq \left(\frac{4\pi}{b} - t_n^2\right) m_n(x_0).$$

Now, by substituting the value of $m_n(x_0)$ in the above equation and combining this with (3.37), we get

$$\left[\frac{\mu t_n}{2b} - \frac{\varepsilon_n}{b} - \frac{\sqrt{2\pi} C_0}{\left(\log \frac{L}{\ell_n}\right)^{1/2}}\right] \leq \frac{C_1}{\sqrt{2\pi}} |\Delta v_n|_2.$$

Since $\varepsilon_n \to 0$, $t_n \to t_0 > 0$ and $|\Delta v_n|_2 \to 0$ as $n \to \infty$, from the above equation, we get $\frac{\mu t_0}{2b} \leq 0$ which is a contradiction.

(3) By hypothesis, given any $N > 0$, there exists $t_N > 0$ such that $h_0(t)t \geq e^{Nt}$ for $t \geq t_N$. Since $m_n(x_0) \to \infty$ as $n \to \infty$, by (3.34) and (3.37), we get, for large n,

$$\left[\frac{N t_n}{2b} - \frac{\varepsilon_n}{b} - \frac{\sqrt{2\pi} C_0}{\left(\log \frac{L}{\ell_n}\right)^{1/2}}\right] \leq \frac{C_1}{\sqrt{2\pi}} |\Delta v_n|_2.$$

Since ε_n, $\{|\Delta v_n|_2\}$ are bounded and $t_n \to t_0 > 0$ as $n \to \infty$, from the above equation, we get

$$\frac{N t_0}{2b} \leq C_1'$$

for some positive constant C_1'. Since N is arbitrary, we get a contradiction and this completes the proof of the lemma.

REMARK 3.5. Let $f(x,t) = h(x,t)\exp(bt^2)$ be a function of critical growth on Ω such that $\lim\limits_{t\to\infty} \inf\limits_{x\in\overline{\Omega}} h(x,t)t = \infty$. Then, from (1) of Lemma 3.1, we have for $0 < \ell < \ell_0$, for some $\ell_0 > 0$,

$$\sup_{t\in\mathbb{R}} J(tm_\ell) < \frac{2\pi}{b}$$

which implies $a(\Omega, f)^2 < \frac{4\pi}{b}$.

4. - Proof of Theorem 1.3

For $0 \leq R_1 < R_2 \leq R$, let

$$B(R_1, R_2) = \{x \in \mathbb{R}^2;\ R_1 < |x| < R_2\},$$
$$B(0, R) = \{x \in \mathbb{R}^2; |x| < R\},$$
$$H_r\ (R_1, R_2) = \{u \in H_0^1(B(R_1, R_2));\ u \text{ is radial}\}.$$

First we state a radial version of Theorems 2.1 and 1.2. Let $\Omega = B(R_1, R_2)$.

THEOREM 4.1. *Let $f(x,t) = h(x,t)\exp(bt^2)$ be a function of critical growth on $B(R_1, R_2)$. Further, assume that*

$$f(x,t) = f(|x|, t), \qquad \sup_{x \in \overline{\Omega}} f'(x, 0) < \lambda_1(B(R_1, R_2)).$$

Then

(1) *If $\lim\limits_{t \to \infty} \inf\limits_{x \in \overline{\Omega}} h(x,t)t = \infty$, then there exists at least one pair $(u_1, -u_1)$ of non-trivial solutions of* (1.3), *with $u_1 \geq 0$, and u_1 is infimum of J on $H_r(R_1, R_2) \cap \partial B(B(R_1, R_2), f)$.*

(2) *If, given any $N > 0$, there exists $t_N > 0$ such that $\inf\limits_{x \in \overline{\Omega}} h(x,t)t \geq e^{Nt}$ for $t \geq t_N$, then there exists at least one pair $\{u_2, -u_2\}$ of non-trivial solutions of* (1.3) *and u_2 is infimum of J on $H_r(R_1, R_2) \cap \partial B_1(B(R_1, R_2))$.*

For any integer $k \geq 0$, define

$$\Sigma_k = \{P;\ P \text{ is a partition of } [0, R] \text{ with } k \text{ interior points}\}.$$

Let $P = \{0 = r_0 < r_1 < \ldots < r_{k+1} = R\}$ be in Σ_k and define

$$[P] = \Bigg\{ u \in H_r(0, R);\ u(r_i) = 0,\ \Omega_i = B(r_{i-1}, r_i),\ u_i = u|_{\Omega_i},$$

$$(-1)^{i-1} u_i \geq 0,\ \|u_i\|^2 = \int_{\Omega_i} f(x, u_i) u_i \mathrm{d}x, \text{ for } i \geq 1 \Bigg\},$$

$$\partial B_k = \{[P];\ P \in \Sigma_k\}, \tag{4.1}$$

$$\frac{a_k^2}{2} = \inf\ \{J(u); u \in \partial B_k\}.$$

For $0 \leq r < s \leq R$, let us denote

$$\begin{aligned} a_0(r,s) &= a(B(r,s), f) \\ a_1(r,s) &= a_1(B(r,s), f). \end{aligned} \tag{4.2}$$

Then $a_0(r,s)$, as a function of (r,s), satisfies the following properties:

(i) Since, for every $\delta > 0$, the injection from $H_r(\delta,R)$ into $C_0(\delta,R)$ is compact, it follows that for $r,s \in [\delta,R]$ (see Lemma 3.1 of Nehari [13]):

$$(4.3) \qquad a_0(r,s) \text{ is a continuous function of } r \text{ and } s \text{ and}$$
$$a_0(r,s) \to \infty \qquad \text{as } r-s \to 0;$$

(ii) From (i) of Lemma 2.2 and (2) Theorem 2.1, it follows that

$$(4.4) \qquad a_0(0,s) \text{ is continuous on } (0,R] \text{ and } \lim_{s\to 0} a_0^2(0,s) = \frac{4\pi}{b}.$$

PROOF OF THEOREM 1.3. From (4.1) and (4.2), it follows that for any integer $k \geq 0$

$$(4.5) \qquad a_k^2 = \inf_{P\in\Sigma_k} \sum_{i=1}^{k+1} a_0(r_{i-1},r_i)^2.$$

In order to prove the Theorem, it is enough to show that, for every k, there exists a $P_0 = \{0 = r_0^0 < r_1^0 < \ldots < r_{k+1}^0 = R\}$ in Σ_k such that

$$(4.6) \qquad a_k^2 = \sum_{i=1}^{k+1} a_0(r_{i-1}^0, r_i^0)^2.$$

To see this, from (1) of Theorem 4.1, let $0 \leq v_i \in H_r(r_{i-1}^0, r_i^0)$ such that

$$(4.7) \qquad J(v_i) = \frac{1}{2}\; a_0(r_{i-1}^0, r_i^0)^2$$

and define

$$(4.8) \qquad u(x) = (-1)^{i-1} v_i(x) \qquad \text{for } x \in B(r_{i-1}^0, r_i^0).$$

Then $u \in [P_0]$ and, by applying (2) of Theorem 4.1 in $B(r_{i-1}^0, r_{i+1}^0)$ for $1 \leq i \leq k$, it follows that u is a solution of (1.3) with k interior zeros and this proves the Theorem.

Now we prove the existence of P_0 by induction on k. $k=0$ and $k=1$ follows from Theorem 4.1. Assume that infimum is achieved in (4.5) up to $k-1$, $k \geq 3$.

CLAIM 1. $a_k^2 < a_{k-1}^2 + \frac{4\pi}{b}$.

By assumption, there exist

$$P = \{0 = r_0 < r_1 < r_2 < \ldots < r_k = R\}$$

in Σ_{k-1} and a v in $[P]$ such that

$$\frac{a_{k-1}^2}{2} = J(v). \tag{4.9}$$

From (2) of Theorem 4.1, there exists some $u \in \partial B_1(B(0,r_1), f)$ such that u satisfies (1.3) in $B(0,r_1)$ and

$$J(u) = \frac{a_1(0,r_1)^2}{2}. \tag{4.10}$$

Let $0 < \tilde{r}_1 < r_1$ be such that $u(\tilde{r}_1) = 0$ and define

$$\begin{gathered} \tilde{P} = \{0 = r_0 < \tilde{r}_1 < r_1 < \ldots < r_k = R\}, \\ w(x) = \begin{cases} u(x), & x \in B(0,r_1) \\ -v(x), & x \in B(r_1 R). \end{cases} \end{gathered} \tag{4.11}$$

Then $\tilde{P} \in \Sigma_k$, $w \in [\tilde{P}]$. Hence, from (4.9), (4.10) and Claim 1 in the proof of Theorem 1.2, we have

$$\begin{aligned} \frac{a_k^2}{2} &\leq J(w) = J(v) + J(u) - J(v|_{B(0,r_1)}) \\ &\leq \frac{a_{k-1}^2}{2} + \frac{a_1(0,r_1)^2}{2} - \frac{a_0(0,r_1)^2}{2} \\ &< \frac{a_{k-1}^2}{2} + \frac{2\pi}{b}. \end{aligned}$$

This proves Claim 1.

Let $P_m = \{0 = r_0^m < r_1^m < < \ldots < r_{k+1}^m = R\}$ be a minimizing sequence of (4.5). By going to a subsequence, we assume that $r_i^m \to r_i$.

CLAIM 2. $r_1 \neq 0$.

Suppose $r_1 = 0$. From (1) of Theorem 4.1, let $0 \leq u_i^m \in H_r(r_{i-1}^m, r_i^m)$ be such that

$$J(u_i^m) = \frac{1}{2} a_0(r_{i-1}^m, r_i^m)^2.$$

Let $v^m(x) = (-1)^{i-1} u_i^m(x)$ for $x \in B(r_{i-1}^m, r_i^m)$,

$$Q_m = \{0 = r_0^m < r_2^m < \ldots r_{k+1}^m = R\}$$

and $w^m(x)$ be such that $w^m(x) = 0$ on $B(0, r_1^m)$ and $w^m(x) = -v^m(x)$ on $B(r_1^m, R)$. Then $Q_m \in \Sigma_{k-1}$ and $w^m \in [Q_m]$. Hence from (4.4)

$$\frac{a_{k-1}^2}{2} \leq J(w^m) = J(v^m) - J(u_1^m) \longrightarrow \frac{a_k^2}{2} - \frac{2\pi}{b}$$

as $m \to \infty$, which contradicts Claim 1. This proves Claim 2.

From Claim 2, $r_1 \neq 0$ and hence, from (4.3) and (4.4), we have

$$a_k^2 = \lim_{m\to\infty} \sum_{i=1}^{k+1} a_0(r_{i-1}^m, r_i^m)^2 = \sum_{i=1}^{k+1} a_0(r_{i-1}, r_i)^2.$$

This proves the Theorem 1.3.

REMARK 4.2. As in Solimini [14], in Theorem 1.3, we can prove the existence of infinitely many radial solutions of (1.3) in a ball without the condition $\sup_{x\in\overline{\Omega}} f'(x,0) < \lambda_1(\Omega)$.

5. - Proof of Theorem 1.5

The proof of this Theorem follow exactly as the argument of Fortunato-Jannelli [11]. Hence we only sketch the main steps in the proof when Ω is a rectangle.

Without loss of generality we assume $\Omega = (a,b) \times (0,\pi)$, $a, b \in \mathbb{R}$. For $x \in \Omega$, we set $x = (x', t)$, $x' \in (a,b)$, $t \in (0,\pi)$. Let $\lambda_{j,k} = \mu_j + k^2$, where μ_j be the j-th eigenvalue of $-\frac{\partial^2}{\partial x^2} : H_0^1(a,b) \to H^{-1}(a,b)$. Let $v_j(x')$ be the eigenfunction corresponding to μ_j. Let

$$e_{j,k}(x',t) = v_j(x') \sin(kt), \qquad (x',t) \in (a,b) \times (0,\pi).$$

For $m \in \mathbb{N}$, set

$$V_m = \{u \in H_0^1(\Omega);\ u_{j,k} = 0 \text{ if } k/m \notin \mathbb{N}\}, \tag{5.1}$$

where $u_{j,k}$ is the Fourier coefficient of u with respect to $e_{j,k}$. Thus if $u \in V_m$, then

$$u(x',t) = \sum_{j,k\in N} u_{j,mk} e_{j,mk}(x',t). \tag{5.2}$$

Let us denote $\Omega_m = (a,b) \times (0,\pi/m)$ and $u_m = u|_{\Omega_m}$ for $u \in H_0^1(\Omega)$. We have the following

LEMMA 5.1. *We have*

(a) *If* $u \in V_m$, *then* $u_m \in H_0^1(\Omega_m)$ *and* $J(u) = mJ(u_m)$;

(b) $J|_{V_m}$ *satisfies the Palais-Smale condition in* $\left(-\infty, \frac{2\pi}{b} m\right)$;

(c) *For every* $R > 0$, *there exist* $m \in \mathbb{N}$ *and* $\rho > 0$ *such that*

$$J(u) \geq R \qquad \text{for all } u \in V_m \text{ with } \|u\| = \rho;$$

(d) *There exists* $W \in V_m$ *such that*

$$\sup_{t \in \mathbb{R}} J(tW) < \frac{2\pi}{b} m.$$

PROOF. From the results proved earlier, (a), (b), (c) and (d) follow easily from the following observation.

Let $g : \mathbb{R} \to \mathbb{R}$ be any even function and $W \in V_m$, then

$$\int_a^b \int_0^\pi g(W) \mathrm{d}x' \, \mathrm{d}t = m \int_a^b \int_0^{\pi/m} g(W_m) \mathrm{d}x' \, \mathrm{d}t. \tag{5.3}$$

Therefore if $u \in V_m$, then from (5.3) we have

$$||u||^2 = m \, ||u_m||^2 \tag{5.4}$$

$$\int_\Omega F(u) \mathrm{d}x = m \int_{\Omega_m} F(u_m) \mathrm{d}x. \tag{5.5}$$

PROOF OF THEOREM 1.5. The proof of Theorem 1.5 follows from (b), (c), (d) of Lemma 5.1 and Mountain Pass Theorem of Ambrosetti-Rabinowitz [5].

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T.I.F.R. Centre
Post Box No. 1234
Bangalore 560 012
India