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# Metric Space Valued Functions of Bounded Variation

LUIGI AMBROSIO

## Introduction

In this paper we introduce and study the properties of the class  $BV(\Omega, E)$  of functions of bounded variation  $u : \Omega \rightarrow E$ , where  $\Omega \subset \mathbf{R}^n$  is an open set and  $(E, \delta)$  is a locally compact metric space. It is natural to require that for any Lipschitz function  $\varphi : E \rightarrow \mathbf{R}$  and any  $u \in BV(\Omega, E)$  the function  $v = \varphi(u)$  belongs to  $BV(\Omega)$ , the classical space of real functions of bounded variation. Moreover, the total variation measure  $|Du|$  has to be greater or equal than  $|Dv|$  provided the Lipschitz constant of  $\varphi$  is not greater than 1. We have thus defined  $BV(\Omega, E)$  as the class of Borel functions  $u : \Omega \rightarrow E$  such that there exists a finite measure  $\sigma$  satisfying the condition

$$(1) \quad \sigma(B) \geq |D\varphi(u)|(B) \quad \forall B \in \mathbf{B}(\Omega),$$

for any function  $\varphi : E \rightarrow \mathbf{R}$  whose Lipschitz constant is less or equal than 1. The total variation measure  $|Du|$  is the least measure which fulfils (1). It turns out that our definition is consistent with the elementary case  $\Omega = ]a, b[ \subset \mathbf{R}$ , and  $|Du|$  agrees with the essential total variation [19, 4.5.10] defined by

$$\min \left\{ \sup \left\{ \sum_{i=1}^{k-1} \delta(v(t_i), v(t_{i+1})) \mid a < t_1 \dots < t_k < b \right\} \mid v = u \text{ a.e. in } \Omega \right\}.$$

The class  $BV(\Omega, E)$  can be characterized by the properties of the one-dimensional sections, exactly as in the case  $E = \mathbf{R}$  ([12], [19]). Furthermore, many classical properties of real functions with bounded variation do not depend on the vector structure of  $\mathbf{R}$ , and continue to hold in  $BV(\Omega, E)$ . In particular, in §2 we prove the rectifiability of the approximate discontinuity set  $S_u$  and the existence of traces  $u^+$ ,  $u^-$  on the opposite sides of  $S_u$ . We also show equality

between the Radon-Nikodym derivative of  $|Du|$  with respect to Lebesgue  $n$ -dimensional measure  $\mathcal{L}^n$  and the approximate slope

$$\operatorname{ap} \lim_{y \rightarrow x} \frac{\delta(u(y), u(x))}{|y - x|}.$$

By using the same ideas of [16], [2], we introduce the class  $SBV(\Omega, E) \subset BV(\Omega, E)$  of special functions with bounded variation and we show compactness criteria for  $SBV(\Omega, E)$  and  $BV(\Omega, E)$  with respect to the almost sure convergence.

If  $u \in BV(\Omega, E)$  is a simple function (i.e., its range is a finite set), then its total variation is a measure supported in  $S_u$ , representable by

$$|Du|(B) = \int_{B \cap S_u} \delta(u^+, u^-) d\mathcal{H}^{n-1} \quad \forall B \in \mathbf{B}(\Omega).$$

In §3 we compare  $|Du|$  with the set function  $V_u$  obtained by relaxing the total variation of locally simple functions. Formally,  $V_u$  is defined by

$$(2) \quad V_u(A) = \inf \left\{ \liminf_{h \rightarrow +\infty} \int_{A \cap S_{u_h}} \delta(u_h^+, u_h^-) d\mathcal{H}^{n-1} \mid u_h \rightarrow u \right. \\ \left. \text{a.e. in } A, u_h \text{ locally simple} \right\},$$

for any open set  $A \subset \Omega$ . Unlike the case  $E = \mathbf{R}$ ,  $V_u$  may be strictly greater than  $|Du|$ . We give an example of this phenomenon for  $E = \mathbf{R}^k$ ,  $k > 1$ . Anyway, by using a sort of Poincaré inequality for  $BV(\Omega, E)$  functions, we show that  $V_u$  is a finite measure, there exists a constant  $c(n, E)$  such that

$$|Du|(B) \leq V_u(B) \leq c(n, E)|Du|(B) \quad \forall B \in \mathbf{B}(\Omega),$$

and  $V_u, |Du|$  agree on the Borel subsets of  $S_u$ . As a consequence, in case the Hausdorff one dimensional measure of  $E$  is zero, we infer the equality  $|Du| = V_u$ .

In the last section we show by an example that the class  $BV(\Omega, E)$  may naturally appear as limit of classical problems defined in Sobolev Spaces. Given a continuous function  $g : \mathbf{R}^k \rightarrow [0, +\infty[$ , we show, by using the results of §2 and §3, that the functionals

$$(3) \quad \int_{\Omega} \left[ \epsilon |\nabla u|^2 + \frac{g(u)}{\epsilon} \right] dx \quad u \in [W^{1,2}(\Omega)]^k$$

converge as  $\epsilon \rightarrow 0^+$  to the functional

$$(4) \quad |D\pi(u)|(\Omega) \quad \pi(u) \in BV(\Omega, E)$$

under general assumptions on  $g$ . The convergence takes place in a precise variational sense ( $\Gamma$ -convergence [8], [13]). The compact set  $(E, \delta)$  in (4) is the canonical quotient space of the zero set  $Z$  of  $g$ , endowed with the Riemannian distance

$$\delta(z_1, z_2) = 2 \inf \left\{ \int_0^1 g^{1/2}(\gamma) |\gamma'| dt \mid \gamma \in [C^1([0, 1])]^k, \gamma(0) = z_1, \gamma(1) = z_2 \right\},$$

and  $\pi : Z \rightarrow E$  is the projection. By definition, the arcwise connected components of  $Z$  are identified in  $E$  to single points. Under our assumptions, the limit functional in (4) can be represented by

$$(5) \quad \int_{S_{\pi(u)}} \delta(\pi(u)^+, \pi(u)^-) d\mathcal{H}^{n-1},$$

so that the only discontinuities that this functional penalizes are due to jumps of  $\pi(u)$ . The problem of the asymptotic behaviour of the functionals (3) has been studied in connection with  $\Gamma$ -convergence theory [25] and phase transitions of fluids [9], [24], [30]. Our result clarifies the nature of the functional (5) as a total variation with respect to a non euclidean distance. Unlike the other papers on this subject, we don't make any assumption on the dimension and the smoothness of the connected components of  $Z$ . Our proof relies on the general theory of local, variational functionals [13]. A key step in the proof is the original Modica-Mortola result [25].

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**1. - The approximate limits and the lattice of measures**

This paper deals with functions of bounded variations  $u : \Omega \rightarrow E$ . Here  $\Omega \subset \mathbf{R}^n$  is a fixed open set. We shall not make any regularity assumption on  $\Omega$ , but we assume for simplicity that  $\Omega$  is bounded. The set  $(E, \delta)$  is assumed to be a separable metric space such that bounded closed sets are compact. We now recall the basic notion of asymptotic limit which will be used throughout all the paper.

DEFINITION 1.1. Let  $u : \Omega \rightarrow E$  be a Borel function. We say that  $u$  is *approximately continuous* at  $x \in \Omega$  if there exists  $z \in E$  such that all the sets

$$E_\epsilon = \{y \in \Omega \mid \delta(u(y), z) > \epsilon\}$$

have 0-density at  $x$ , i.e.,

$$\lim_{\rho \rightarrow 0^+} \frac{\mathcal{L}^n(B_\rho(x) \cap E_\epsilon)}{\rho^n} = 0 \quad \forall \epsilon > 0.$$

The point  $z$  if exists is unique, is called *approximate limit* of  $u$  at  $x$ , and denoted by

$$\text{ap lim}_{y \rightarrow x} u(y), \quad \tilde{u}(x).$$

REMARK 1.2. We have the implication [31]

$$(1.1) \quad \lim_{\rho \rightarrow 0^+} \rho^{-n} \int_{B_\rho(x)} \delta(u(y), z) dy = 0 \quad \implies \quad z = \text{ap lim}_{y \rightarrow x} u(y).$$

The opposite implication is true if  $\delta$  is bounded. We denote by  $S_u$  the set of points where the approximate limit does not exist. We point out that  $S_u$  is a negligible Borel set, and  $\tilde{u}$  is a Borel function equal to  $u$  almost everywhere [3]. In case  $E = \mathbf{R}$  we also define [19] the *approximate upper limit*

$$(1.2) \quad \text{ap lim sup}_{y \rightarrow x} u(y) = \inf \left\{ t \in [-\infty, +\infty] \mid \lim_{\rho \rightarrow 0^+} \frac{\mathcal{L}^n(\{y \in B_\rho(x) \mid u(y) > t\})}{\rho^n} = 0 \right\}$$

and the *approximate lower limit*

$$(1.3) \quad \text{ap lim inf}_{y \rightarrow x} u(y) = \sup \left\{ t \in [-\infty, +\infty] \mid \lim_{\rho \rightarrow 0^+} \frac{\mathcal{L}^n(\{y \in B_\rho(x) \mid u(y) < t\})}{\rho^n} = 0 \right\}.$$

It can be easily seen that  $u$  is approximately continuous at  $x$  if and only if the approximate upper and lower limits are finite and equal. The same ideas can also be applied to define one sided limits  $u^+(x, \nu)$ ,  $u^-(x, \nu)$ , where  $\nu \in \mathbf{S}^{n-1}$  is a given direction.

DEFINITION 1.3. We say that  $z = u^+(x, \nu)$  if the sets

$$\{y \in \Omega \mid \langle y - x, \nu \rangle > 0, d(u(y), z) > \epsilon\}$$

have 0-density at  $x$  for any  $\epsilon > 0$ . Similarly, we say that  $z = u^-(x, \nu)$  if  $z = u^+(x, -\nu)$ .

REMARK 1.4. [31] The limits  $u^+, u^-$  can be seen as a generalization of the left and right limits of functions of one real variable. It can be shown that in any point  $x$  of approximate continuity the limits  $u^+(x, \nu)$  exist for any  $\nu$  and are all equal to  $\tilde{u}(x)$ . On the other hand, if  $u^+(x, \nu), u^-(x, \nu)$  exist for some  $\nu$  and are equal, then  $x$  is a point of approximate continuity. If  $u^+(x, \nu), u^-(x, \nu)$  are not equal, then the unitary vector  $\nu$  is uniquely determined up to the sign.

The following proposition shows that points of approximate continuity can be detected by using a suitably large set of test functions.

PROPOSITION 1.1. *Let  $u : \Omega \rightarrow E$  be a Borel function. Then,*

$$(i) \quad S_u \supset S_{\varphi(u)}, \quad \widetilde{\varphi(u)} = \varphi(\tilde{u}) \text{ in } \Omega \setminus S_u$$

for any continuous function  $\varphi : E \rightarrow \mathbf{R}$ . Moreover, if  $\mathcal{F}$  is any family of continuous real valued functions defined in  $E$  which separates points, and if

$$\lim_{z \rightarrow \infty} \varphi(z) = +\infty \quad \forall \varphi \in \mathcal{F},$$

then

$$(ii) \quad S_u = \bigcup_{\varphi \in \mathcal{F}} S_{\varphi(u)}.$$

PROOF. (i) is straightforward. The inclusion  $\subset$  in (ii) has been proved in [3] in case  $E$  is compact. Let  $\tilde{E} = E \cup \{\infty\}$  be the one point compactification of  $E$ , and let  $\mathcal{G} = \{\arctan(\varphi) | \varphi \in \mathcal{F}\}$ . Then,  $\mathcal{G}$  separates points of  $\tilde{E}$  and  $x \notin S_{\psi(u)}$  for all  $\psi \in \mathcal{G}$  entails the existence of  $\tilde{u}(x)$  in  $\tilde{E}$ . If  $\tilde{u}(x)$  were equal to  $\infty$ , then the approximate limit of  $\varphi(u)$  would be equal to  $+\infty$ . Hence,  $\tilde{u}(x) \in E$  and  $x \notin S_u$ . **q.e.d.**

REMARK 1.5. Also the approximate limits  $u^+, u^-$  satisfy the properties stated in Proposition 1.1. [3]. Namely,  $u^+(x, \nu)$  exists if and only if  $\varphi(u)^+(x, \nu)$  exists for any  $\varphi \in \mathcal{F}$ , and  $\varphi(u)^+(x, \nu) = \varphi(u^+(x, \nu))$ .

By using the approximate limits, it is also possible to define approximate differentials.

DEFINITION 1.6. Let  $u : \Omega \rightarrow \mathbf{R}$  be a Borel function, and let  $x \in \Omega \setminus S_u$ . We say that  $u$  is *approximately differentiable* at  $x$  if there exists a vector  $p \in \mathbf{R}^n$  such that

$$\text{ap} \lim_{y \rightarrow x} \frac{|u(y) - \tilde{u}(x) - \langle p, y - x \rangle|}{|y - x|} = 0.$$

The approximate differential will be denoted by  $\nabla u(x)$ .

REMARK 1.7. We recall (see for instance [3]) that the approximate differential if exists is unique. Moreover, the set where it exists belongs to  $\mathbf{B}(\Omega)$  and  $x \rightarrow \nabla u(x)$  is a Borel function.

Now, we recall some fundamental properties of the class  $\mathcal{M}(\Omega)$  of  $\sigma$ -additive measures  $\mu : \mathbf{B}(\Omega) \rightarrow [0, +\infty]$ . We define

$$(1.4) \quad \mu \vee \sigma(B) = \sup \left\{ \mu(B_1) + \sigma(B_2) \mid B = B_1 \cup B_2 \right\},$$

and

$$\mathcal{M} - \sup_{h \in \mathbf{N}} \mu_h(B) = \lim_{h \rightarrow +\infty} (\mu_1 \vee \dots \vee \mu_h)(B).$$

If  $\mu_h \in \mathcal{M}(\Omega)$  for all  $h$ , then the set function  $\mathcal{M} - \sup\{\mu_h \mid h \in \mathbf{N}\}$  belongs to  $\mathcal{M}(\Omega)$ . We shall also extensively make use of the following properties:

$$(1.5) \quad \begin{aligned} \mu = \mathcal{M} - \sup_{h \in \mathbf{N}} \mu_h, \quad \mu(\Omega) < +\infty &\implies \\ &\implies \frac{\mu}{\sigma} = \sup \left\{ \frac{\mu_h}{\sigma} \mid h \in \mathbf{N} \right\} \quad \sigma - \text{a.e. in } \Omega; \end{aligned}$$

$$(1.6) \quad \begin{aligned} \mu = \mathcal{M} - \sup_{h \in \mathbf{N}} \mu_h, \quad \liminf_{k \rightarrow +\infty} \sigma_k(A) \geq \mu_h(A) \quad \forall A \in \mathbf{A}(\Omega), h \in \mathbf{N} &\implies \\ &\implies \liminf_{k \rightarrow +\infty} \sigma_k(\Omega) \geq \mu(\Omega). \end{aligned}$$

The first statement is straightforward. The second one follows from the equality

$$\mu(\Omega) = \sup \left\{ \sum_{i=1}^p \mu_i(A_i) \mid A_i \in \mathbf{A}(\Omega) \text{ mutually disjoint, } p \in \mathbf{N} \right\}.$$

Let  $P = Q \times I \subset \mathbf{R}^n$  be a product space and let  $\mu$  be a measure in  $Q$ . Let  $\sigma : Q \rightarrow \mathcal{M}(I)$  be a mapping such that

$$x \rightarrow \sigma_x(B)$$

is a Borel function for any  $B \in \mathbf{B}(I)$ ; we canonically define the measure  $\int_Q \sigma_x d\mu(x) \in \mathcal{M}(P)$  by

$$(1.7) \quad \left( \int_Q \sigma_x d\mu(x) \right) (B) = \int_Q \sigma_x(\{t \mid (x, t) \in B\}) d\mu(x) \quad \forall B \in \mathbf{B}(P).$$

If  $\sigma_h : Q \rightarrow \mathcal{M}(I)$  are mappings as above, the following equality holds:

$$(1.8) \quad \mathcal{M} - \sup_{h \in \mathbf{N}} \left[ \int_Q \sigma_{hx} d\mu(x) \right] = \int_Q [\mathcal{M} - \sup_{h \in \mathbf{N}} \sigma_{hx}] d\mu(x).$$

This formula can easily be proved for the supremum of two measures  $\sigma_x, \sigma'_x$ . If  $\sigma_x(B) + \sigma'_x(B) < +\infty$  for  $\mu$ -almost every  $x \in Q$ , it suffices to take in (1.4)

$$B_1 = \{(x, t) \in B \mid \exists \rho > 0 \text{ s.t. } \sigma'_x(B \cap B_\rho(t)) = 0\} \cup \bigcup \{(x, t) \in B \mid \liminf_{\rho \rightarrow 0^+} \frac{\sigma_x(B \cap B_\rho(t))}{\sigma'_x(B \cap B_\rho(t))} \geq 1, \sigma'_x(B \cap B_\rho(t)) > 0 \forall \rho\},$$

and

$$B_2 = \{(x, t) \in B \mid \liminf_{\rho \rightarrow 0^+} \frac{\sigma_x(B \cap B_\rho(t))}{\sigma'_x(B \cap B_\rho(t))} < 1, \sigma'_x(B \cap B_\rho(t)) > 0 \forall \rho\}.$$

If  $\sigma_x(B) + \sigma'_x(B) = +\infty$  in a set of positive measure, then both sides are equal to  $+\infty$ . The formula (1.8) follows by an induction and a limiting argument.

Finally, given a vector space  $(V, |\cdot|)$  and a set function  $\mu : \mathbf{B}(\Omega) \rightarrow V$  we define its *total variation*  $|\mu|$  by

$$(1.9) \quad |\mu|(B) = \sup \left\{ \sum_{i=1}^{\infty} |\mu(B_i)| \mid B = \bigcup_{i=1}^{\infty} B_i \right\}.$$

The vector space  $\mathcal{L}_{n,k}$  of linear mappings  $L : \mathbf{R}^n \rightarrow \mathbf{R}^k$  will be endowed with the norm

$$(1.10) \quad |L| = \sup \left\{ |\langle L^*, p \rangle| \mid p \in \mathbf{R}^k, |p| \leq 1 \right\},$$

where  $L^* \in \mathcal{L}_{k,n}$  denotes the adjoint of  $L$ .

**2. - The class  $BV(\Omega, E)$**

In this section we define  $BV(\Omega, E)$  and the total variation measure  $|Du|$ , and we show that  $BV(\Omega, E)$  inherits many classical properties of real functions with bounded variation. The basic tools in all the proofs of this section are Proposition 1.1 and (1.8). We begin by remarking that it is possible to define the supremum of non countable families  $\{\sigma_i\}_{i \in I} \subset \mathcal{M}(\Omega)$  too. It is enough to set

$$M - \sup_{i \in I} \sigma_i(B) = \sup \left\{ M - \sup_{i \in J} \sigma_i(B) \mid J \subset I \text{ countable} \right\} \quad \forall B \in \mathbf{B}(\Omega).$$

Then,  $M - \sup\{\sigma_i \mid i \in I\} \in \mathcal{M}(\Omega)$  and is a finite measure if and only if there exists a finite measure  $\sigma \in \mathcal{M}(\Omega)$  such that  $\sigma_i \leq \sigma$  for any  $i \in I$ .



DEFINITION 2.1. Let  $u$  be a Borel function such that  $\delta(u, z) \in L^1(\Omega)$  for some  $z \in E$ . We say that  $u \in BV(\Omega, E)$  if the set function

$$\mathcal{M} - \sup \left\{ |D\varphi(u)| \mid \varphi \in \text{Lip}_1(E) \right\}$$

is a finite measure and we define

$$|Du| = \mathcal{M} - \sup \left\{ |D\varphi(u)| \mid \varphi \in \text{Lip}_1(E) \right\}.$$

In particular

$$(2.1) \quad \varphi(u) \in BV(\Omega, F) \quad \text{and} \quad |D\varphi(u)|(\Omega) \leq |Du|(\Omega) \quad \forall \varphi \in \text{Lip}_1(E, F)$$

REMARK 2.2. The class  $BV(\Omega, \mathbf{R}^k)$  coincides with  $[BV(\Omega)]^k$  and  $|Du|$  agrees with the total variation of the vector measure  $Du : \mathbf{B}(\Omega) \rightarrow \mathcal{L}_{n,k}$  defined in (1.9), (1.10) (this follows by using the chain rule available in [5] and [32]). In case  $\Omega = ]a, b[ \subset \mathbf{R}$  it can be easily seen that the condition  $u \in BV(\Omega, E)$  is equivalent to

$$(2.2) \quad \sup \left\{ \sum_{i=1}^{k-1} \delta(v(t_i), v(t_{i+1})) \mid a < t_1 < \dots < t_k < b \right\} < +\infty$$

for a suitable function  $v \in \mathbf{B}(\Omega; E)$  equal to  $u$  almost everywhere and  $|Du|(\Omega)$  equals the least possible number in (2.2) as  $v$  varies in the equivalence class of  $u$ . Furthermore, ([19], 2.5.16, 4.5.10) the infimum is achieved by the right and left continuous representatives  $u_+, u_-$  of  $u$  which agree outside an at most countable set. We also have

$$(2.3) \quad \delta(u_+(s), u_+(t)) \leq |Du|([s, t]), \quad \delta(u_-(s), u_-(t)) \leq |Du|([s, t])$$

whenever  $s \leq t$ . In case  $\Omega \subset \mathbf{R}$  is not connected, then  $u \in BV(\Omega, E)$  if and only if  $u \in BV(A, E)$  for any connected component  $A$  of  $\Omega$  and

$$\sum \left\{ |Du|(A) \mid A \subset \Omega \text{ connected component} \right\} < +\infty.$$

The next proposition contains equivalent definitions of the class  $BV(\Omega, E)$  which will be useful in the sequel. In particular, we show equivalence with the Cesari-Tonelli definition based on slicing. Given  $\nu \in \mathbf{S}^{n-1}$ , we denote by  $\pi_\nu \subset \mathbf{R}^n$  the hyperplane orthogonal to  $\nu$ , by  $\Omega_\nu$  the projection of  $\Omega$  on  $\pi_\nu$  and we set for any  $x \in \Omega_\nu$

$$\Omega_x = \{ t \in \mathbf{R} \mid x + t\nu \in \Omega \}, \quad u_x(t) = u(x + t\nu) \quad t \in \Omega_x.$$

PROPOSITION 2.1. *The following conditions are equivalent:*

- (i)  $u \in BV(\Omega, E)$ ;
- (ii)  $\mathcal{M} - \sup \left\{ |D\varphi(u)| \mid \varphi(u) \in \mathcal{F} \right\}$  is a finite measure;

for a countable set of functions  $\mathcal{F} \subset \text{Lip}(E)$  fulfilling the hypotheses of Proposition 1.1, and such that

$$(2.4) \quad \sup \{ |\varphi(z) - \varphi(w)| \mid \varphi \in \mathcal{F} \} = \delta(z, w) \quad \forall z, w \in E;$$

(iii) for any choice of  $\nu \in \mathbb{S}^{n-1}$  we have

$$u_x \in BV(\Omega_x, E), \quad S_{u_x} = \{t \in \Omega_x \mid x + t\nu \in S_u\},$$

$$\lim_{s \rightarrow t} u_{x^+}(s) = \lim_{s \rightarrow t} u_{x^-}(s) = \text{ap} \lim_{y \rightarrow x+t\nu} u(y) \quad \forall t \in \Omega_x \setminus S_{u_x}$$

for  $\mathcal{H}^{n-1}$ -almost every  $x \in \Omega_\nu$  and

$$\int_{\Omega_\nu} |Du_x|(\Omega_x) d\mathcal{H}^{n-1}(x) < +\infty.$$

Moreover, denoting by  $|\langle Du, \nu \rangle|$  the measure  $\int_{\Omega_\nu} |Du_x| d\mathcal{H}^{n-1}(x)$ , we have

$$(2.5) \quad |Du| = \mathcal{M} - \sup \left\{ |\langle Du, \nu \rangle| \mid \nu \in \mathbb{S}^{n-1} \right\} = \mathcal{M} - \sup \left\{ |D\varphi(u)| \mid \varphi \in \mathcal{F} \right\}.$$

PROOF. (i)  $\implies$  (ii) is trivial. We now prove (ii)  $\implies$  (iii) and the equality

$$(2.6) \quad \mathcal{M} - \sup \left\{ |\langle Du, \nu \rangle| \mid \nu \in D \right\} = \mathcal{M} - \sup \left\{ |D\varphi(u)| \mid \varphi \in \mathcal{F} \right\}$$

for any countable dense set  $D \subset \mathbb{S}^{n-1}$ . Indeed, in the case  $n = 1$  the equality

$$|Du| = \mathcal{M} - \sup \left\{ |D\varphi(u)| \mid \varphi \in \mathcal{F} \right\}$$

is a direct consequence of (2.2) and (2.4). Moreover, if  $v$  is a real function of bounded variation, it is well known that ([19], 4.5.9)

$$(2.7) \quad |Dv| = \mathcal{M} - \sup \left\{ |\langle Dv, \nu \rangle| \mid \nu \in D \right\}, \quad |\langle Dv, \nu \rangle| = \int_{\Omega_\nu} |Dv_x| d\mathcal{H}^{n-1}(x).$$

By (2.7) and (1.8) we get

$$\begin{aligned} & \mathcal{M} - \sup \left\{ |D\varphi(u)| \mid \varphi \in \mathcal{F} \right\} = \\ & = \mathcal{M} - \sup \left\{ |\langle D\varphi(u), \nu \rangle| \mid \nu \in D, \varphi \in \mathcal{F} \right\} = \\ & = \mathcal{M} - \sup \left\{ \int_{\Omega_\nu} |D\varphi(u_x)| d\mathcal{H}^{n-1}(x) \mid \nu \in D, \varphi \in \mathcal{F} \right\} = \\ & = \mathcal{M} - \sup \left\{ \int_{\Omega_\nu} |Du_x| d\mathcal{H}^{n-1}(x) \mid \nu \in D \right\}. \end{aligned}$$

By using Theorem 3.3 of [2] (see also [19]), we can find, for a given direction  $\nu$ , a  $\mathcal{H}^{n-1}$ -negligible set  $N \subset \Omega_\nu$  such that  $\varphi(u_x) \in BV(\Omega_x, E)$ ,  $S_{\varphi(u_x)} = \{t \in \Omega_x \mid x + t\nu \in S_{\varphi(u)}\}$  and

$$\lim_{s \rightarrow t} \varphi(u_{x+})(s) = \lim_{s \rightarrow t} \varphi(u_{x-})(s) = \text{ap} \lim_{y \rightarrow x+t\nu} \varphi(u)(y) \quad \forall t \in \Omega_x \setminus S_{\varphi(u_x)}$$

for any  $\varphi \in \mathcal{F}$  and any  $x \in \Omega_\nu \setminus N$ . Hence, the statements of (iii) are true for  $x \in \Omega_\nu \setminus N$  because of Proposition 1.1.

We now show the last implication (iii)  $\implies$  (i) and (2.5). By (2.7) we get

$$|D\psi(u)| \leq \sum_{i=1}^n |\langle D\psi(u), \mathbf{e}_i \rangle| \leq \sum_{i=1}^n | \langle Du, \mathbf{e}_i \rangle |$$

for any  $\psi \in \text{Lip}_1(E)$ . Hence,  $u \in BV(\Omega, E)$ . Moreover, (2.7) and (1.8) imply

$$\begin{aligned} |Du| &= \mathcal{M} - \sup \left\{ |D\psi(u)| \mid \psi \in \text{Lip}_1(E) \right\} = \\ &= \mathcal{M} - \sup \left\{ \int_{\Omega_\nu} |D\psi(u_x)| d\mathcal{H}^{n-1}(x) \mid \nu \in D, \psi \in \text{Lip}_1(E) \right\} = \\ &= \mathcal{M} - \sup \left\{ \int_{\Omega_\nu} |Du_x| d\mathcal{H}^{n-1}(x) \mid \nu \in D \right\} = \\ &= \mathcal{M} - \sup \left\{ |\langle Du, \nu \rangle| \mid \nu \in D \right\}, \end{aligned}$$

which together with (2.6) gives (2.5). **q.e.d.**

The next theorem characterizes the Radon-Nikodym derivative of  $|Du|$  with respect to  $\mathcal{L}^n$ . We denote the derivative by  $|\nabla u|$ , because for a real

functions of bounded variation it is almost everywhere equal to the norm of the approximate differential ([10], [19]). In particular, the equality

$$\operatorname{ap} \lim_{y \rightarrow x} \frac{|u(y) - u(x)|}{|y - x|} = |\nabla u|(x)$$

holds almost everywhere in  $\Omega$ . We shall prove that the same equality holds for functions  $u \in BV(\Omega, E)$ .

**THEOREM 2.2.** *Let  $\mathcal{F} \subset \operatorname{Lip}(E)$  be as in Proposition 2.1, and let*

$$|\nabla u|(x) = \sup \left\{ |\nabla \varphi(u)|(x) \mid \varphi \in \mathcal{F} \right\}.$$

Then,  $|\nabla u| \in L^1(\Omega)$ ,  $|\nabla u| = |Du|/\mathcal{L}^n$  and

$$(i) \quad \lim_{h \rightarrow 0} \frac{\delta(u_+(t+h), u_+(t))}{|h|} = \lim_{h \rightarrow 0} \frac{\delta(u_-(t+h), u_-(t))}{|h|} = |\nabla u|(t) \quad \text{a.e. in } \Omega,$$

whenever  $\Omega \subset \mathbf{R}$ ; in the general case  $n \geq 1$  we have

$$(ii) \quad \operatorname{ap} \lim_{y \rightarrow x} \frac{\delta(u(y), u(x))}{|y - x|} = |\nabla u|(x) \quad \text{a.e. in } \Omega.$$

**PROOF.** Since  $|\nabla v| = |Dv|/\mathcal{L}^n$  for real valued  $BV$  functions, by (1.5) and (2.5) we infer

$$\begin{aligned} |\nabla u|(x) &= \sup \left\{ |\nabla \varphi(u)|(x) \mid \varphi \in \mathcal{F} \right\} \\ &= \sup \left\{ \frac{|D\varphi(u)|}{\mathcal{L}^n}(x) \mid \varphi \in \mathcal{F} \right\} = \frac{|Du|}{\mathcal{L}^n}(x) \quad \text{a.e. in } \Omega. \end{aligned}$$

(i) By (2.3) and a derivation theorem for measures in the real line we get

$$\begin{aligned} \limsup_{h \rightarrow 0^+} \frac{\delta(u_+(t+h), u_+(t))}{h} &\leq \limsup_{h \rightarrow 0^+} \frac{|Du|([t, t+h])}{h} = \frac{|Du|}{\mathcal{L}^1}(x) = \\ &= |\nabla u|(x) \quad \text{a.e. in } \Omega. \end{aligned}$$

Similarly,

$$\begin{aligned} \limsup_{h \rightarrow 0^-} \frac{\delta(u_+(t+h), u_+(t))}{|h|} &\leq \limsup_{h \rightarrow 0^-} \frac{|Du|([t+h, t])}{h} = \frac{|Du|}{\mathcal{L}^1}(x) = \\ &= |\nabla u|(x) \quad \text{a.e. in } \Omega. \end{aligned}$$

Since (i) is well known for real  $BV$  functions on the real line, we have

$$\begin{aligned} \liminf_{h \rightarrow 0} \frac{\delta(u_+(t+h), u_+(t))}{|h|} &\geq \liminf_{h \rightarrow 0} \frac{|\varphi(u_+(t+h)) - \varphi(u_+(t))|}{|h|} \geq \\ &\geq |\nabla\varphi(u)|_1(t) \quad \text{a.e. in } \Omega \end{aligned}$$

for any  $\varphi \in \text{Lip}_1(E)$ . This implies the opposite inequality. The proof for  $u_-$  is analogous.

(ii) Since we know that (ii) is true for  $BV(\Omega)$  functions, we get

$$\begin{aligned} \text{ap lim inf}_{y \rightarrow x} \frac{\delta(u(y), u(x))}{|y-x|} &\geq \text{ap lim inf}_{y \rightarrow x} \frac{|\varphi(u(y)) - \varphi(u(x))|}{|y-x|} = \\ &= |\nabla\varphi(u)|_1(x) \quad \text{a.e. in } \Omega. \end{aligned}$$

The definition of  $|\nabla u|$  yields

$$\text{ap lim inf}_{y \rightarrow x} \frac{\delta(u(y), u(x))}{|y-x|} \geq |\nabla u|_1(x) \quad \text{a.e. in } \Omega.$$

Now we estimate the approximate upper limit. Let  $\mathcal{G}$  be the set of characteristic functions of the intervals  $[t, +\infty[$  with  $t \in \mathbf{Q}$ . If  $v : \Omega \rightarrow \mathbf{R}$  is a Borel function and

$$\omega_n g(\xi) \geq \limsup_{\rho \rightarrow 0^+} \rho^{-n} \int_{B_\rho(x)} g(v) dy \quad \forall g \in \mathcal{G},$$

then

$$\xi \geq \text{ap lim sup}_{y \rightarrow x} v(y).$$

Hence it will be sufficient to check the inequality

$$(2.8) \quad \omega_n g(|\nabla u|_1(x)) \geq \limsup_{\rho \rightarrow 0^+} \rho^{-n} \int_{B_\rho(x)} g\left(\frac{\delta(u(y), u(x))}{|y-x|}\right) dy$$

for a given bounded, continuous, increasing function  $g : \mathbf{R} \rightarrow [0, +\infty[$ . We set

$$\Delta(x, h) = \begin{cases} g\left(\frac{\delta(\tilde{u}(x+h), \tilde{u}(x))}{|h|}\right) & \text{if } x, x+h \in \Omega \setminus S_u, h \in \mathbf{R}^n \setminus \{0\}; \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 2.1(iii) and (i) we get

$$\limsup_{t \rightarrow 0^+} \Delta(x, th) \leq g(|\nabla u|_1(x)) \quad \text{a.e. in } \Omega$$

for any  $h \in \mathbf{R}^n$ . Since  $S_u$  is negligible, by using the Fubini-Tonelli theorem we get

$$\begin{aligned} & \int_B \left[ \limsup_{\rho \rightarrow 0^+} \rho^{-n} \int_{B_\rho(x)} g\left(\frac{\delta(u(y), u(x))}{|y-x|}\right) dy \right] dx = \\ & = \int_B \limsup_{\rho \rightarrow 0^+} \rho^{-n} \int_{B_\rho(0)} \Delta(x, h) dh dx \leq \\ & \leq \int_B \limsup_{\rho \rightarrow 0^+} \int_{B_1(0)} \Delta(x, \rho h) dh dx \leq \\ & \leq \int_B \int_{B_1(0)} \limsup_{\rho \rightarrow 0^+} \Delta(x, \rho h) dh dx \leq \omega_n \int_B g(|\nabla u|) dx, \end{aligned}$$

for any Borel set  $B \subset \Omega$ , so that the inequality (2.8) is true almost everywhere. **q.e.d.**

**THEOREM 2.3.** *Let  $u \in BV(\Omega, E)$ . Then*

(i) 
$$\mathcal{H}^{n-1}(S_u \setminus \bigcup_{i=1}^{\infty} \Gamma_i) = 0$$

for a suitable sequence of  $C^1$  hypersurfaces  $\Gamma_i$ ;

(ii) *there exists a Borel function  $\nu_u : S_u \rightarrow \mathbf{S}^{n-1}$  such that the approximate limits*

$$u^+(x, \nu_u(x)), \quad u^-(x, \nu_u(x))$$

of Definition 1.3 exist  $\mathcal{H}^{n-1}$ -almost everywhere in  $S_u$ . Moreover,

(2.9) 
$$\nu_u = \pm \nu_v \quad \mathcal{H}^{n-1} \text{-a.e. in } S_u \cap S_v$$

for any pair of functions  $u, v \in BV(\Omega, E)$ .

**PROOF.** (i) The statement is true for real functions of bounded variation ([19], 4.5.9). The general case follows by Proposition 1.1.

(ii) The equality (2.9) and the statement of (ii) are well known if  $u$  is real valued (see for instance [3] and [19]). Let  $\mathcal{F} = \{\delta(\cdot, z)\}_{z \in D}$  with  $D$  countable and dense in  $E$ . By Proposition 1.1,

$$S_u = \bigcup_{\varphi \in \mathcal{F}} S_{\varphi(u)},$$

so that we can find a Borel function  $\nu_u : S_u \rightarrow \mathbf{S}^{n-1}$  such that

$$\nu_u = \pm \nu_{\varphi(u)} \quad \mathcal{H}^{n-1} \text{-a.e. in } S_{\varphi(u)}$$

for any  $\varphi \in \mathcal{F}$ . In particular, the approximate limits  $\varphi(u)^+(x, \nu_u(x))$ ,  $\varphi(u)^-(x, \nu_u(x))$  exist  $\mathcal{H}^{n-1}$ -almost everywhere in  $S_u$  for any  $\varphi \in \mathcal{F}$ . By Proposition 1.1. and Remark 1.5 we infer the existence of the approximate limits  $u^+(x, \nu_u(x))$ ,  $u^-(x, \nu_u(x))$  for  $\mathcal{H}^{n-1}$ -almost every  $x \in S_u$ . **q.e.d.**

REMARK 2.3. [2] The total variation of real functions  $v$  of bounded variation can be represented as follows

$$|Dv|(B) = \int_B |\nabla v| dx + \int_{B \cap S_u} |v^+ - v^-| d\mathcal{H}^{n-1}(x) + |Cv|(B) \quad \forall B \in \mathbf{B}(\Omega),$$

where  $|Cv|$  is a measure supported in a negligible set, such that

$$(2.10) \quad |Cv|(B) = 0 \quad \forall B \text{ s.t. } \mathcal{H}^{n-1}(B) < +\infty.$$

By (1.5) and (2.5) we get for any  $u \in BV(\Omega, E)$  a unique measure  $|Cu| : \mathbf{B}(\Omega) \rightarrow [0, +\infty[$  such that

$$(2.11) \quad |Du|(B) = \int_B |\nabla u| dx + \int_{B \cap S_u} \delta(u^+, u^-) d\mathcal{H}^{n-1}(x) + |Cu|(B) \quad \forall B \in \mathbf{B}(\Omega)$$

In particular, (2.10) yields

$$(2.12) \quad |Du|(B) = \int_B \delta(u^+, u^-) d\mathcal{H}^{n-1} \quad \forall B \text{ s.t. } \mathcal{H}^{n-1}(B) < +\infty.$$

We say that  $u \in SBV(\Omega, E)$  if  $|Cu| = 0$  in (2.11).

REMARK 2.4. It can be easily seen that

$$u \in SBV(\Omega, E) \iff u \in BV(\Omega, E), \quad \varphi(u) \in SBV(\Omega) \quad \forall \varphi \in \text{Lip}(E).$$

In fact, the implication  $\Leftarrow$  follows by taking the supremum in  $\mathcal{M}(\Omega)$  of both sides of the equality

$$|D(\varphi(u))|(B) = \int_B |\nabla \varphi(u)| dx + \int_{B \cap S_u} |\varphi(u)^+ - \varphi(u)^-| d\mathcal{H}^{n-1}(x) \quad \varphi \in \text{Lip}_1(E).$$

Conversely, if  $u \in SBV(\Omega, E)$ , then  $|D\varphi(u)|$  is absolutely continuous with respect to  $\mathcal{L}^n$  plus the restriction of  $\mathcal{H}^{n-1}$  to  $S_u$ , and (2.10) yields  $\varphi(u) \in SBV(\Omega)$ .

A useful compactness criterion in  $SBV(\Omega)$  has been conjectured in [16] and proved in [2]. The space  $SBV(\Omega)$  has been recently successfully exploited

in [17] to show existence of minimizers of the functional

$$\int_{\Omega} |\nabla u|^2 dx + \alpha \mathcal{H}^{n-1}(K \cap \Omega) + \beta \int_{\Omega} |u - g|^2 dx, \quad K \subset \mathbf{R}^n \text{ compact,}$$

$$u \in W^{1,2}(\Omega \setminus K).$$

The functional has been suggested and studied by Mumford-Shah in [26], [27] for a variational approach to image segmentation.

**THEOREM 2.4.** (i) *Let  $(u_h) \subset BV(\Omega, E)$  be a sequence such that*

$$\sup \left\{ |Du_h|(\Omega) + \int_{\Omega} \delta(u_h, z_0) dx \mid h \in \mathbf{N} \right\} < +\infty$$

*for some  $z_0 \in E$ . Then, there exists a subsequence  $(u_{h_k})$  converging almost everywhere to  $u \in BV(\Omega, E)$  and*

$$|Du|(\Omega) \leq \liminf_{k \rightarrow +\infty} |Du_{h_k}|(\Omega).$$

(ii) *Let  $(u_h) \subset SBV(\Omega, E)$  be a sequence such that  $u_h(\Omega) \subset K$  for some compact set  $K \subset E$  independent of  $h$  and*

$$\sup \left\{ \int_{\Omega} |\nabla u_h|^p dx + \mathcal{H}^{n-1}(S_{u_h}) \mid h \in \mathbf{N} \right\} < +\infty$$

*for some  $p > 1$ . Then, there exists a subsequence  $(u_{h_k})$  converging almost everywhere to  $u \in SBV(\Omega, E)$  and*

$$\int_{\Omega} |\nabla u|^p dx \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} |\nabla u_{h_k}|^p dx,$$

$$\mathcal{H}^{n-1}(S_u) \leq \liminf_{k \rightarrow +\infty} \mathcal{H}^{n-1}(S_{u_{h_k}}).$$

**PROOF.** We denote by  $\tilde{E} = E \cup \{\infty\}$  the one point compactification of  $E$ . Let  $D \subset E$  be a countable dense set and let  $\mathcal{F} \subset \text{Lip}(E)$  be defined by

$$\mathcal{F} = \{ \delta(\cdot, z) \mid z \in D \}.$$

Since

$$|D\varphi(u_h)|(\Omega) + \int_{\Omega} |\varphi(u_h)\varphi(z_0)| dx$$

is bounded for any  $\varphi \in \mathcal{F}$  (recall (2.11)), by using a diagonal argument and Rellich's theorem we can find a subsequence  $(u_{h_k})$  such that

$$\varphi(u_{h_k}) \rightarrow u_{\varphi} \in BV(\Omega) \quad \text{a.e. in } \Omega$$



for any  $\varphi \in \mathcal{F}$ . This easily yields the almost everywhere convergence of  $u_{h_k}$  to a Borel function  $u : \Omega \rightarrow \tilde{E}$ . We extend  $g(z) = \delta(z_0, z)$  to  $\tilde{E}$  by setting  $g(\infty) = +\infty$ . Since  $g$  is lower semicontinuous in  $\tilde{E}$ , we get

$$\int_{\Omega} g(u) dx \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} g(u_{h_k}) dx < +\infty,$$

so that  $u(x) \in E$  almost everywhere. The lower semicontinuity inequality is a straightforward consequence of (1.6), (2.5) and the lower semicontinuity of the total variation of real functions of bounded variation.

(ii) We argue as in (i), by using the compactness theorem in  $SBV(\Omega)$  proved in [2]. **q.e.d.**

### 3. - Approximation by simple functions

In this section we compare  $|Du|$  with the set function  $V_u$  obtained by relaxing the total variation of locally simple functions. By using Lemma 3.2 we obtain that  $V_u$  is a finite measure if and only if  $u \in BV(\Omega, E)$ . By using a localization technique we show in Proposition 3.4(i) that  $V_u$  and  $|Du|$  agree on the Borel subsets of  $S_u$ . This, via Fleming-Rishel formula, leads to the equality  $V_u = |Du|$  in case  $\mathcal{H}^1(E) = 0$ . In this section we assume for simplicity that  $E$  is compact.

**DEFINITION 3.1.** Let  $u : \Omega \rightarrow E$  be a Borel function. We say that  $u$  is a *simple function* if there exists a finite set  $T \subset E$  such that

$$(3.1) \quad u(x) \in T \quad \text{a.e. in } \Omega.$$

We denote by  $\mathcal{S}(\Omega, E)$  the class of simple functions and by  $R(u)$  the minimal set  $T$  satisfying (3.1).

The functions  $u \in \mathcal{S}(\Omega, E)$  which belong to  $BV(\Omega, E)$  can be easily characterized. Moreover, their total variation has a simple representation.

**PROPOSITION 3.1.** Let  $u \in \mathcal{S}(\Omega, E)$ . Then,  $u \in BV(\Omega, E)$  if and only if for any  $i \in R(u)$  the set  $\{u = i\}$  has finite perimeter in  $\Omega$ . Moreover, we have

$$(3.2) \quad |Du|(B) = \frac{1}{2} \sum_{i, j \in R(u)} \delta(i, j) \mathcal{H}^{n-1}(B \cap \partial^* \{u = i\} \cap \partial^* \{u = j\})$$

$$\forall B \in \mathbf{B}(\Omega).$$

**PROOF.** Let  $u \in BV(\Omega, E) \cap \mathcal{S}(\Omega, E)$ , let  $i \in R(u)$  and let  $\varphi(z) = \delta(i, z)$ . By the Fleming-Rishel formula (see for instance [22], [23]) almost every set

$\{x \in \Omega \mid \varphi(u) < t\}$  has finite perimeter in  $\Omega$ . The above set coincides with  $\{u = i\}$  provided  $t < \delta(i, j)$  for any  $j \in R(u) \setminus \{i\}$ .

Conversely, if  $u \in \mathcal{S}(\Omega, E)$  and  $\{u = i\}$  has finite perimeter in  $\Omega$  for any  $i \in R(u)$ , we have [31]

$$|D\varphi(u)|(B) = \frac{1}{2} \sum_{i, j \in R(u)} |\varphi(i) - \varphi(j)| \mathcal{H}^{n-1}(B \cap \partial^* \{u = i\} \cap \partial^* \{u = j\})$$

$\forall \varphi \in \text{Lip}(E), B \in \mathbf{B}(\Omega)$

and (3.2) is proved by taking the supremum in  $\mathcal{M}(\Omega)$  of both sides. **q.e.d.**

Thus, simple functions  $u$  of bounded variation can be identified with partitions in sets of finite perimeter labeled by  $R(u)$ . The total variation is computed by integrating  $\delta(i, j)$  on the intersection of the essential boundaries of the level sets  $\{u = i\}, \{u = j\}$ . The factor 1/2 appears because each pair  $(i, j)$  is counted twice.

**DEFINITION 3.2.** Let  $u : \Omega \rightarrow E$  be a Borel function. We denote by  $V_u : \mathbf{A}(\Omega) \rightarrow [0, +\infty]$  the set function

$$V_u(A) = \inf \left\{ \liminf_{h \rightarrow +\infty} |Du_h|(A) \mid u_h \in BV(A, E) \cap \mathcal{S}_{\text{loc}}(A, E), \right. \\ \left. u_h \rightarrow u \text{ a.e. in } A \right\}.$$

**REMARK 3.3.** The set function

$$\tilde{V}_u(A) = \inf \left\{ \liminf_{h \rightarrow +\infty} |Du_h|(A) \mid u_h \in BV(A, E) \cap \mathcal{S}(A, E), \right. \\ \left. u_h \rightarrow u \text{ a.e. in } A \right\}$$

is greater or equal to  $V_u(A)$  and such that  $\tilde{V}_u(A) \leq V_u(A')$  whenever  $A \subset\subset A'$ . This entails that  $V_u, \tilde{V}_u$  agree on a wide class of open sets [13]. Furthermore, the argument in the end of the proof of Theorem 4.2 (see also Remark 4.1) yields that  $V_u(A), \tilde{V}_u(A)$  are equal for any set  $A \in \mathbf{A}(\Omega)$  with  $C^2$  boundary. We have chosen to study  $V_u$  instead of  $\tilde{V}_u$  because we will be able to show (Theorem 3.3) that  $V_u$  is the trace of a Borel measure. In particular, this gives the equality

$$V_u(A) = \sup_{B \subset\subset A} \tilde{V}_u(B) \quad \forall A \in \mathbf{A}(\Omega).$$

In order to compare  $|Du|$  with  $V_u$  it is important to estimate how much  $u$  differs from an ‘‘average value’’ in small domains. The classical Poincaré-

Wirtinger inequality

$$\int_{B_\rho(x)} |u(y) - u_\rho|^{n/n-1} dy \leq c [|Du|(B_\rho(x))]^{n/n-1}, \quad u_\rho = \frac{\int_{B_\rho(x)} u dy}{\omega_n \rho^n}$$

cannot be extended to  $BV(\Omega, E)$  because of the lack of convexity of  $E$ . However, we can show the following result.

LEMMA 3.2. *There exists a constant  $\xi(n, E) \leq n - 1/2$  such that*

$$(i) \quad \min \left\{ \int_Q \delta(u, z) dx \mid z \in E \right\} \leq \xi(n, E) |Q|^{1/n} |Du|(Q)$$

for any open cube  $Q \subset \mathbf{R}^n$  and  $u \in BV(Q, E)$ . Moreover, setting  $\tau = |Q|^{1/n}$ ,

$$(ii) \quad \int_Q \delta(u(x + \tau e_i), u(x)) dx \leq \tau |Du|(Q \cup R_i \cup (Q + \tau e_i)),$$

for any  $u \in BV(Q \cup R_i \cup (Q + \tau e_i))$ , where  $R_i$  is the common face of  $Q$ ,  $(Q + \tau e_i)$ .

PROOF. (i) For simplicity we assume that  $Q$  is the unit cube in  $\mathbf{R}^n$  centered at 0. We write  $Q = P \times I$ ,  $I = ] - 1/2, 1/2[$ ,  $P$  normal to  $e_1$ . We also set

$$u_t(y) = u(y + te_1), \quad u_y(t) = u(y + te_1) \quad y \in P, t \in I.$$

Our proof is by induction on  $n$ . The basic inequalities we need are the following:

$$(3.3) \quad \int_{-1/2}^{1/2} \delta(v(t), \tilde{v}(\sigma)) dt \leq \left( \frac{1}{2} + |\sigma| \right) |Dv|(I),$$

for any  $v \in BV(I, E)$ ,  $\sigma \notin S_v$ , and

$$(3.4) \quad \int_P |Du_y|(I) d\mathcal{H}^{n-1}(y) \leq |Du|(Q),$$

$$(3.5) \quad \int_{-1/2}^{1/2} |Du_t|(P) dt \leq |Du|(Q).$$

The first inequality easily follows by the Fubini-Tonelli theorem and (2.3). The second one follows by (2.5). Let us show the last one. The inequality

$$\int_{-1/2}^{1/2} |Dv_t|(\{y \in P \mid y + te_1 \in A\}) dt \leq |Dv|(A)$$

is trivially satisfied if  $v \in C^1(A)$ ,  $A \in \mathbf{A}(\Omega)$ . By taking a sequence  $v_h \rightarrow v$  such that  $|Dv_h|(A) \rightarrow |Dv|(A)$  [23], we find that the same inequality is satisfied for  $v \in BV(\Omega)$ . Hence,

$$\int_{-1/2}^{1/2} |D\varphi(u_t)|(\{y \in P \mid y + t\mathbf{e}_1 \in B\}) dt \leq |D\varphi(u)|(B) \quad \forall B \in \mathbf{B}(\Omega)$$

for any  $\varphi \in \text{Lip}_1(E)$ . Thus, (3.5) follows by (1.8) and the definition of  $|Du|$ .

By taking a sequence in (3.3)  $(\sigma_k) \subset I \setminus S_u$  such that  $\sigma_k \rightarrow 0$  we get the first step of the induction. By (3.5), we can find  $\sigma \in ]-1/2, 1/2[$  such that

$$\mathcal{H}^{n-1}(S_u \cap (P + \sigma \mathbf{e}_1)) = 0, \quad \tilde{u}(\cdot + \sigma \mathbf{e}_1) = u_\sigma(\cdot) \quad \mathcal{H}^{n-1}\text{-a.e. in } P,$$

$u_\sigma \in BV(P, E)$  and

$$|Du_\sigma|(P) \leq |Du|(Q).$$

By induction, we can find  $z \in E$  such that

$$\int_P \delta(u_\sigma, z) dx \leq \xi(n-1, E)|Du|(Q).$$

By Proposition 2.1(iii), the equality

$$\tilde{u}(y + \sigma \mathbf{e}_1) = u_{y+(\sigma)} = u_{y-(\sigma)}$$

is true  $\mathcal{H}^{n-1}$ -almost everywhere in  $P$ . By using (3.3), (3.4) and the Fubini-Tonelli theorem we get

$$\begin{aligned} \int_Q \delta(u, z) dx &\leq \int_P \int_I \delta(u(y + t\mathbf{e}_1), \tilde{u}(y + \sigma \mathbf{e}_1)) dt d\mathcal{H}^{n-1}(y) + \\ &+ \int_P \int_I \delta(\tilde{u}(y + \sigma \mathbf{e}_1), z) dt d\mathcal{H}^{n-1}(y) \leq \\ &\leq |Du|(Q) + \xi(n-1, E)|Du|(Q). \end{aligned}$$

(ii) Let us first assume  $n = 1$ ,  $Q = ]a, a + \tau[$ . By (2.3) we get

$$\begin{aligned} \int_a^{a+\tau} \delta(u_+(t + \tau), u_+(t)) dt &\leq \int_a^{a+\tau} |Du|([t, t + \tau]) dt = \\ &= \int_{]a, a+2\tau[} \left( \int_{(s-\tau) \vee a}^{s \wedge (a+\tau)} dt \right) d|Du|(s) \leq \tau |Du|([a, a + 2\tau]). \end{aligned}$$

The general case follows by slicing along the direction  $\nu = \mathbf{e}_i$  and using (2.5) of Proposition 2.1. **q.e.d.**

**THEOREM 3.3.** *Let  $u : \Omega \rightarrow E$  be a Borel function. Then, the set function  $V_u$  is the restriction to  $\mathbf{A}(\Omega)$  of a Borel measure. In addition,  $u \in BV(\Omega, E)$  if and only if  $V_u(\Omega) < +\infty$  and*

$$(3.6) \quad |Du|(A) \leq V_u(A) \leq n(\xi(n, E) + 1)|Du|(A) \quad \forall A \in \mathbf{A}(\Omega).$$

**PROOF.** We show that  $V_u$  is the restriction to  $\mathbf{A}(\Omega)$  of a Borel measure. By a well known criterion (see for instance [13]) it suffices to show the following three properties

- (i)  $V_u(A \cup B) \geq V_u(A) + V_u(B) \quad \forall A, B \in \mathbf{A}(\Omega), A \cap B = \emptyset;$
- (ii)  $V_u(A' \cup B) \leq V_u(A) + V_u(B) \quad \forall A, A', B \in \mathbf{A}(\Omega), A' \subset\subset A;$
- (iii)  $V_u(A) = \sup \left\{ V_u(B) \mid B \subset\subset A \right\} \quad \forall A \in \mathbf{A}(\Omega).$

The first condition is straightforward. The properties (ii), (iii) can be shown by joining the minimizing sequences in definition 3.2 on different open sets. The basic property we need is the following. Let  $A, A', B$  be as in (ii). Then, there exists a constant  $c(A, A')$  such that for any  $u \in BV(A, E), v \in BV(B, E)$  it is possible to find a set of finite perimeter  $S$  in  $\mathbf{R}^n$  such that  $A' \subset\subset S \subset\subset A$  and the function

$$w(x) = \begin{cases} u(x) & \text{if } x \in S; \\ v(x) & \text{if } x \in A \cup B \setminus S, \end{cases}$$

has total variation in  $A' \cup B$  not greater than

$$|Du|(A) + |Dv|(B) + c(A, A') \int_{A \cap B \setminus \bar{A}'} \delta(u, v) dx.$$

Applying this property to the minimizing sequences in definition 3.2 we easily get (ii). The joint property can be proved by making use of the coarea formula for Lipschitz functions: since ([19], 3.2.11)

$$\int_{A \cap B \setminus \bar{A}'} \delta(u, v) dx = \int_0^{\text{dist}(\partial A', \partial A)} \int_{\{x \in A \cap B \mid \text{dist}(x, A')=t\}} \delta(\tilde{u}, \tilde{v}) d \mathcal{H}^{n-1}(x) dt,$$

we can set

$$c(A, A') = [\text{dist}(\partial A', \partial A)]^{-1}, \quad S = \{x \in A : \text{dist}(x, A') < t\}$$

for a suitable  $t \in ]0, c(A, A')[$ . The function  $w$  defined in this way belongs to  $BV(A' \cup B)$  (see [31] and [32]).

The proof of (iii) is more delicate. It is necessary to slice  $A$  by the sets

$$A_k = \left\{ x \in A \mid \text{dist}(x, \partial A) > \frac{1}{k} \right\},$$

and to apply simultaneously the joint lemma to the triplets  $(A_k, A_{k+1}, A_{k+2})$ . Locally simple functions  $u_{h,k} \rightarrow u$  such that

$$V_u(A_k) = \lim_{h \rightarrow +\infty} |Du_{h,k}|(A_k)$$

can thus be joined, yielding a sequence  $u_h \rightarrow u$  of locally simple functions such that

$$\sup_{k \in \mathbf{N}} V_u(A_k) = \lim_{h \rightarrow +\infty} |Du_h|(A).$$

A detailed description of this procedure can be found in Theorem 5.2 of [7].

The inequality  $V_u(A) \geq |Du|(A)$  is a trivial consequence of the lower semicontinuity of the total variation (Theorem 2.4). In order to show the opposite inequality, let us first assume that  $A = Q$  is a unit cube, and let us partition it in the canonical way by open cubes  $Q_j$ ,  $1 \leq j \leq h^n$  with sides of length  $1/h$ . By Lemma 3.2, we can find  $z_j \in E$  such that

$$\int_{Q_j} \delta(u, z_j) dx \leq \frac{\xi(n, E)}{h} |Du|(Q_j) \quad \forall j \in \{1, \dots, h^n\}.$$

We set

$$u_h = \begin{cases} z_j & \text{if } x \in Q_j \text{ for some } j; \\ z_1 & \text{otherwise.} \end{cases}$$

By Proposition 3.1 the functions  $u_h$  are simple and with bounded variation. Moreover,

$$\int_Q \delta(u, u_h) dx \leq \frac{\xi(n, E)}{h} |Du|(Q),$$

so that we can assume up to subsequences that  $u_h \rightarrow u$  almost everywhere as  $h \rightarrow +\infty$ . We say that  $i \sim j$  if  $Q_i, Q_j$  have a common face, and we denote it by  $R_{ij}$ . We also denote by  $\tau_{ij} \in \mathbf{S}^{n-1}$  the vector normal to  $R_{ij}$  pointing to  $Q_i$ .

By Lemma 3.2(ii) we get

$$\begin{aligned} \frac{1}{h^n} \delta(z_i, z_j) &\leq \int_{Q_i} \delta(u, z_i) dx + \int_{Q_i} \delta(u, z_j) dx \leq \frac{\xi(n, E)}{h} |Du|(Q_i) + \\ &+ \int_{Q_j} \left[ \delta\left(u\left(x + \frac{\tau_{ij}}{h}\right), u(x)\right) + \delta(u(x), z_j) \right] dx \leq \\ &\leq \frac{\xi(n, E)}{h} |Du|(Q_i) + \frac{1}{h} |Du|(Q_i \cup R_{ij} \cup Q_j) + \\ &+ \int_{Q_j} \delta(u, z_j) dx \leq \frac{\xi(n, E) + 1}{h} |Du|(Q_i \cup R_{ij} \cup Q_j). \end{aligned}$$

We can now estimate the total variation of the functions  $u_h$ . By Proposition 3.1 we get

$$\begin{aligned} |Du_h|(Q) &\leq \frac{1}{2h^{n-1}} \sum_{i \sim j} \delta(z_i, z_j) \leq \frac{\xi(n, E) + 1}{2} \sum_{i \sim j} |Du|(Q_i \cup R_{ij} \cup Q_j) \leq \\ &\leq n(\xi(n, E) + 1) |Du|(Q), \end{aligned}$$

so that, by letting  $h \rightarrow +\infty$  we find

$$V_u(Q) \leq n(\xi(n, E) + 1) |Du|(Q).$$

The same argument can be repeated for any cube  $Q \subset \Omega$ , so that, since  $V_u$  is a measure, (3.6) follows. **q.e.d.**

**REMARK 3.4.** It would be interesting to know what is the optimal constant in (3.6). It is easy to see that for  $n = 1$  the optimal constant is 1 (i.e.,  $|Du| = V_u$ ). Moreover, by using the Fleming-Rishel coarea formula it can be shown that this happens also in case  $E = \mathbf{R}$ . We conjecture that  $V_u \leq n|Du|$  for any  $E, n$  and  $u \in BV(\Omega, E)$ . In general, however,  $V_u$  may differ from  $|Du|$ , as the following example shows.

**EXAMPLE 3.5.** Let  $E = \mathbf{R}^k$ ,  $\Omega = B_1(0)$ . Let  $\Theta : \mathcal{L}_{n,k} \rightarrow [0, +\infty[$  be the greatest norm such that

$$\Theta(a \otimes b) = |a||b| \quad \forall a \in \mathbf{R}^n, b \in \mathbf{R}^k.$$

Then,

$$(3.7) \quad V_u(A) = \int_A \Theta\left(\frac{Du}{|Du|}\right) d|Du| \quad \forall A \in \mathbf{A}(\Omega).$$

The function  $\Theta(L)$  is equal to the infimum of all the sums

$$\sum_{i=1}^p |a_i||b_i|$$

corresponding to the decompositions

$$L = \sum_{i=1}^p a_i \otimes b_i.$$

In the particular case  $n = k$  we find  $\Theta(Id) = n > |Id| = 1$ , where  $Id$  is the identity matrix. We give only a sketch of the proof of (3.7), because we do not need this result here. The inequality  $\geq$  in (3.7) directly follows by a semicontinuity theorem [28]. By a well known approximation argument (see for instance [28], [23]) it is enough to show (3.7) for any continuously differentiable function  $u$ . By using the optimal approximating functions given component by component by the Fleming-Rishel formula we get

$$(3.8) \quad V_u(A) \leq \sum_{i=1}^k |Du^{(i)}|(A) \quad \forall A \in \mathbf{A}(\Omega).$$

Moreover, by changing the variables in  $\mathbf{R}^k$  with orthogonal linear mappings  $B$ , and remarking that  $V_u = V_{uB}$ , we find that (3.8) yields

$$V_u(A) \leq \int_A \Lambda \left( \frac{Du}{|Du|} \right) d|Du| \quad \forall A \in \mathbf{A}(\Omega),$$

where

$$\Lambda(L) = \inf \left\{ \sum_{i=1}^k |(LB)^{(i)}| \mid B \in O(k) \right\}$$

and  $(LB)^{(i)}$  is the  $i$ -th row of  $LB$ . Since  $\Lambda(a \otimes b) = |a||b|$ , we achieve the inequality  $\leq$  in (3.7) by showing that  $\Theta$  is the quasi-convex envelope of  $\Lambda$  and by using a relaxation theorem of Acerbi-Fusco [1].

Now we show that the measures  $|Du|$  and  $V_u$  have the same restrictions to  $S_u$  for any  $u \in BV(\Omega, E)$ . The basic idea is that for  $\mathcal{H}^{n-1}$ -almost every  $x \in S_u$  we can asymptotically compare  $u$  with the function jumping between  $u^+(x)$ ,  $u^-(x)$  along a set tangent to  $S_u$  in  $x$ .

PROPOSITION 3.4. *Let  $u \in BV(\Omega, E)$ . Then,*

$$(i) \quad |Du|(B \cap S_u) = V_u(B \cap S_u) \quad \forall B \in \mathbf{B}(\Omega);$$



(ii) if  $\mathcal{H}^1(E) = 0$ , then

$$|Du|(B) = V_u(B) = \int_{B \cap S_u} \delta(u^+, u^-) d\mathcal{H}^{n-1}(x) \quad \forall B \in \mathbf{B}(\Omega).$$

The same is true if  $\mathcal{H}^1(\tilde{u}(\Omega \setminus S_u)) = 0$ .

PROOF. (i) By the Fleming-Rishel formula and (3.6),  $|Du|$  and  $V_u$  both vanish on  $\mathcal{H}^{n-1}$ -negligible sets. Moreover,  $S_u$  can be almost covered with compact subsets of  $C^1$  hypersurfaces. Hence, by the Egoroff theorem, it is enough to show the equality  $|Du|(K) = V_u(K)$  for all compact sets  $K \subset S_u$  such that

$$(3.9) \quad u^+, u^-, \nu_u \quad \text{are continuous in } K;$$

$$(3.10) \quad \begin{aligned} \rho^{-n} \int_{B_\rho^+(x, \nu_u(x))} \delta(u(y), u^+(x)) dy &\rightarrow 0, \\ \rho^{-n} \int_{B_\rho^-(x, \nu_u(x))} \delta(u(y), u^-(x)) dy &\rightarrow 0, \end{aligned}$$

uniformly for  $x \in K$ ,

$$(3.11) \quad K \subset \Gamma, \quad \Gamma \text{ } C^1 \text{ surface.}$$

By Besicovitch's theorem on differentiation of measures, it suffices to show that

$$(3.12) \quad \liminf_{\sigma \rightarrow 0^+} \frac{V_u(K \cap B_\sigma(x_0))}{|Du|(K \cap B_\sigma(x_0))} \leq 1 \quad \forall x_0 \in K.$$

Let  $x_0 \in K$  be a fixed point, let  $M$  be the maximum of  $\delta$ , and let  $\tau_0 = \min\{\delta(u^+(x), u^-(x)) \mid x \in K\}$ . By the Fleming-Rishel formula, the set  $\{x \in \Omega \mid \delta(u(x), u^+(x_0)) < \tau\}$  has finite perimeter in  $\Omega$  for almost every  $\tau > 0$ . We choose  $\tau < 1 \wedge \tau_0$  with this property and we define

$$w(x) = \begin{cases} u^+(x_0) & \text{if } \delta(u(x), u^+(x_0)) < \tau; \\ u^-(x_0) & \text{otherwise.} \end{cases}$$

The function  $w$  is simple, and  $w \in BV(\Omega, E)$ . Now we compare  $u$  with  $w$  in small neighbourhoods of  $x_0$ . Let  $\sigma$  such that

$$|u^+(x) - u^+(x_0)| < \tau^2, \quad |u^-(x) - u^-(x_0)| < \tau^2, \quad \forall x \in B_\sigma(x_0) \cap K.$$

Then,  $K \cap B_\sigma(x_0) \subset S_w$  and  $w^+(x) = u^+(x_0)$ ,  $w^-(x) = u^-(x_0)$  for any  $x \in K \cap B_\sigma(x_0)$ . Moreover,

$$\begin{aligned} \rho^{-n} \int_{B_\rho^+(x, \nu_u(x))} \delta(u(y), w(y)) dy &\leq \frac{M}{\tau} \rho^{-n} \int_{B_\rho^+(x, \nu_u(x))} \delta(u(y), u^+(x_0)) dy \leq \\ &\leq \frac{M}{\tau} \rho^{-n} \int_{B_\rho^+(x, \nu_u(x))} \delta(u(y), u^+(x)) dy + \frac{M\tau\omega_n}{2}. \end{aligned}$$

Similarly,

$$\rho^{-n} \int_{B_\rho^-(x, \nu_u(x))} \delta(w(y), u(y)) dy \leq \frac{M}{\tau} \rho^{-n} \int_{B_\rho^-(x, \nu_u(x))} \delta(u(y), u^-(x)) dy + \frac{M\tau\omega_n}{2}.$$

By (3.10) we infer

$$\rho^{-n} \int_{B_\rho(x)} \delta(w(y), u(y)) dy \leq M\tau\omega_n + \omega(x, \rho)$$

with  $\omega(x, \rho) \rightarrow 0$  as  $\rho \downarrow 0$ , uniformly for  $x \in K$ . By using a suitable covering of the set

$$I_\rho(K) = \{x \in \mathbf{R}^n : \text{dist}(x, K) < \rho\},$$

(see for instance [7], Proposition 4.4), we get

$$\begin{aligned} \limsup_{\rho \rightarrow 0^+} \frac{1}{\rho} \int_{I_\rho(K) \cap B_\sigma(x_0)} \delta(u(x), w(x)) dx &\leq \\ &\leq M c(n) \tau \limsup_{\rho \rightarrow 0^+} \frac{\mathcal{L}^n(I_\rho(K) \cap B_{\sigma+\rho}(x_0))}{\rho}, \end{aligned}$$

for some constant  $c(n)$  depending only on  $n$ . On the other hand, by (3.11) we get ([19], 3.2.39)

$$\limsup_{\rho \rightarrow 0^+} \frac{\mathcal{L}^n(I_\rho(K) \cap B_{\sigma+\rho}(x_0))}{2\rho} \leq \mathcal{H}^{n-1}(K \cap \overline{B}_\sigma(x_0)).$$

The coarea formula, applied to the Lipschitz function  $\rho \wedge \text{dist}(x, K)$ , yields ([19], 3.2.11)

$$\frac{1}{\rho} \int_{I_\rho(K) \cap B_\sigma(x_0)} \delta(u, w) dx = \frac{1}{\rho} \int_0^\rho \int_{\{x \in B_\sigma(x_0) | \text{dist}(x, K) = s\}} \delta(\tilde{u}, \tilde{w}) d \mathcal{H}^{n-1}(x) ds.$$

Hence, we can find a sequence  $\rho_h \downarrow 0$  such that

$$\begin{aligned} \mathcal{H}^{n-1}((S_u \cup S_w) \cap \{x \in B_\sigma(x_0) \mid \text{dist}(x, K) = \rho_h\}) &= 0 \quad \forall h \in \mathbf{N}, \\ \mathcal{H}^{n-1}(\{x \in B_\sigma(x_0) \mid \text{dist}(x, K) = \rho_h\}) &< +\infty, \end{aligned}$$

and

$$\limsup_{h \rightarrow +\infty} \int_{\{x \in B_\sigma(x_0) \mid \text{dist}(x, K) = \rho_h\}} \delta(\tilde{u}, \tilde{w}) d\mathcal{H}^{n-1}(x) \leq 2Mc(n)\tau \mathcal{H}^{n-1}(K \cap \overline{B}_\sigma(x_0)).$$

We set

$$u_h(x) = \begin{cases} w(x) & \text{if } \text{dist}(x, K) < \rho_h; \\ u(x) & \text{otherwise.} \end{cases}$$

The functions  $u_h$  belong to  $BV(B_\sigma(x_0), E)$  and converge to  $u$  almost everywhere. Let  $K_h$  be the  $\rho_h$  open neighbourhood of  $K$ ; by the locality of  $V_u$  and (2.12) we infer

$$\begin{aligned} V_{u_h}(B_\sigma(x_0)) &= V_{u_h}(B_\sigma(x_0) \setminus \overline{K}_h) + V_{u_h}(B_\sigma(x_0) \cap \partial K_h) + V_{u_h}(B_\sigma(x_0) \cap K_h) \leq \\ &\leq V_u(B_\sigma(x_0) \setminus K) + n(\xi(n, E) + 1) \int_{\partial K_h} \delta(\tilde{u}, \tilde{w}) d\mathcal{H}^{n-1}(x) + V_w(B_\sigma(x_0) \cap K_h). \end{aligned}$$

By letting  $h \rightarrow +\infty$  and using the lower semicontinuity of  $V_u$  we find

$$\begin{aligned} V_u(B_\sigma(x_0)) &\leq \int_{K \cap \overline{B}_\sigma(x_0)} \delta(u^+(x_0), u^-(x_0)) d\mathcal{H}^{n-1}(x) + \\ &+ M\eta\tau \mathcal{H}^{n-1}(K \cap \overline{B}_\sigma(x_0)) + V_u(B_\sigma(x_0) \setminus K), \end{aligned}$$

where  $\eta = 2n(1 + \xi(n, E))c(n)$ . Finally, by using (2.12) and letting  $\sigma \rightarrow 0$  we get

$$\liminf_{\sigma \rightarrow 0^+} \frac{V_u(B_\sigma(x_0) \cap K)}{|Du|(B_\sigma(x_0) \cap K)} \leq \left(1 + \frac{M\eta\tau}{\tau_0}\right).$$

By letting  $\tau \downarrow 0$ , (3.12) and the thesis follow.

(ii) It is sufficient to show that  $|D\varphi(u)|(\Omega \setminus S_u) = 0$  for any  $\varphi \in \text{Lip}(E)$ . Indeed, let  $v = \varphi(u) \in BV(\Omega)$ ; by the Fleming-Rishel formula we get

$$(3.13) \quad |Dv|(\Omega \setminus S_u) = \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}(\partial^* \{x \in \Omega \mid v(x) > t\} \setminus S_u) dt.$$

If  $x \notin S_u$  belongs to the essential boundary of  $\{x \in \Omega \mid v(x) > t\}$ , necessarily  $t = \tilde{v}(x) \in \varphi(E)$ . Hence the integrand in (3.13) vanishes for any  $t \in \mathbf{R} \setminus \varphi(E)$ .

On the other hand, since  $\mathcal{H}^1(E) = 0$  and  $\varphi$  is a Lipschitz function,  $\varphi(E)$  is negligible in  $\mathbf{R}$ . **q.e.d.**

REMARK 3.6. The condition  $\mathcal{H}^1(E) = 0$  is equivalent to requiring for any  $\epsilon, \delta > 0$  the existence of a countable cover of  $E$  by balls  $B(x_i, \rho_i)$  such that

$$\sum_{i=1}^{\infty} \rho_i < \delta, \quad \rho_i < \epsilon \quad \forall i \in \mathbf{N}.$$

The compact sets  $E$  such that  $\mathcal{H}^1(E) = 0$  are totally disconnected. This easily follows by considering the function  $\varphi(y) = \delta(x, y)$  whose range is a negligible compact set.

#### 4. - A singular perturbation problem

In this section we investigate the asymptotic behaviour of the solutions of variational problems

$$(P_\epsilon) \quad \min \left\{ \int_{\Omega} \left[ \epsilon |\nabla u|^2 + \frac{g(u)}{\epsilon} \right] dx + \int_{\Omega} \psi(x, u) dx \mid u \in [W^{1,2}(\Omega)]^k \right\}$$

as  $\epsilon \rightarrow 0$ . Here  $g, \psi$  are non negative functions,  $g(u)$  is continuous and  $\psi(x, u)$  is measurable in  $x$  and continuous in  $u$ . We assume that the set  $Z = \{z \in \mathbf{R}^k \mid g(z) = 0\}$  is compact in  $\mathbf{R}^k$ ,  $\psi$  is bounded, and

$$(H_1) \quad c|z|^p - C \leq g(z) \leq C(1 + |z|^p) \quad \forall z \in \mathbf{R}^k$$

for some constants  $c, C > 0, p \geq 2$ . Throughout this section it will be convenient for us to endow  $\mathbf{R}^k$  with the degenerate distance

$$(4.1) \quad \delta(z_1, z_2) = 2 \inf \left\{ \int_0^1 g^{1/2}(\gamma) |\gamma'| dt \mid \gamma \in [C^1([0, 1])]^k, \right. \\ \left. \gamma(0) = z_1, \gamma(1) = z_2 \right\}.$$

Modica-Mortola first considered in [25] the scalar case  $k = 1$  and countable sets  $Z$ . Their result has been extended in [9] by Baldo to the vector case, under the assumption of a finite set  $Z$ . The result is that limit points of solutions of problems  $P_\epsilon$  are solutions of

$$\min \left\{ \int_{S_u} \delta(u^+, u^-) d\mathcal{H}^{n-1}(x) + \int_{\Omega} \psi(x, u) dx \mid u \in BV(\Omega, Z) \right\}.$$

In [30] it has been considered the case of a zero set consisting of two disjoint  $C^1$  loops  $\Gamma_1, \Gamma_2$ . The solutions converge to

$$\min \left\{ \delta(\Gamma_1, \Gamma_2) \mathcal{H}^{n-1}(\partial^* \{x \mid u(x) \in \Gamma_1\}) + \int_{\Omega} \hat{\psi}(x, u) dx \mid u : \Omega \rightarrow \Gamma_1 \cup \Gamma_2 \text{ Borel} \right\},$$

where

$$(4.2) \quad \hat{\psi}(x, u) = \inf \left\{ \psi(x, v) \mid v \in Z, \delta(u, v) = 0 \right\} \quad \forall u \in Z.$$

We remark that  $\delta(z_1, z_2) = 0$  if  $z_1, z_2$  belong to the same arcwise connected component of  $Z$ . In the limit problem the sets  $\Gamma_1, \Gamma_2$  are identified to single points and no cost is paid for discontinuities of  $u$  in the level sets  $\{u \in \Gamma_1\}, \{u \in \Gamma_2\}$ . Hence, it is natural to introduce the canonical quotient space  $F$  of  $(\mathbf{R}^k, \delta)$ . We denote by  $\delta$  also the distance in  $F$ , and by  $\pi : \mathbf{R}^k \rightarrow F$  the projection onto  $F$ . The space  $E = \pi(Z)$  is a compact subset of  $F$ . Now we pull back  $BV(\Omega, F)$  and the total variation, by setting

$$BV(\Omega, \mathbf{R}^k) = \{u : \Omega \rightarrow \mathbf{R}^k \mid u \text{ Borel, } \pi(u) \in BV(\Omega, F)\},$$

and

$$|Du|(B) = |D\pi(u)|(B) \quad \forall B \in \mathbf{B}(\Omega), u \in BV(\Omega, \mathbf{R}^k).$$

Since on compact sets  $\delta$  can be estimated from above with some constant times the euclidean distance in  $\mathbf{R}^k$ , we get (recall also Remark 2.1)

$$BV(r, \mathbf{R}^k) \supset [BV(r)]^k \cap [L^\infty(r)]^k.$$

We also point out that, by definition, the total variation in  $BV(\Omega, \mathbf{R}^k)$  is lower semicontinuous with respect to  $\delta$ -convergence almost everywhere, and the following compactness theorem holds:

PROPOSITION 4.1. *Let  $(u_h) \subset BV(\Omega, \mathbf{R}^k)$  be a sequence such that*

$$\sup \left\{ |Du_h|(\Omega) + \int_{\Omega} \delta(u_h, 0) dx \mid h \in \mathbf{N} \right\} < +\infty.$$

*Then, there exists a subsequence  $u_{h_k}$  and  $u \in BV(\Omega, \mathbf{R}^k)$  such that  $\delta(u_{h_k}, u) \rightarrow 0$  almost everywhere in  $\Omega$ .*

We shall make two basic assumptions on assumptions on  $Z, g$ . The first one

$$(H_2) \quad \mathcal{H}^1(E) = 0,$$

is useful to us in order to have equality of  $|Du|$  and  $V_u$  in  $BV(\Omega, E)$ ; the second one

$$(H_3) \quad \min \left\{ |u - w| \mid \delta(w, v) = 0 \right\} \leq \omega(\delta(u, v)) \quad \forall u, v \in \mathbf{R}^k$$

allows a comparison between convergence with respect to  $\delta$  and convergence with respect to the standard distance. The function  $\omega(t)$  in  $(H_3)$  is required to converge to 0 as  $t \downarrow 0$ . By  $(H_3)$  we get

$$(4.3) \quad \delta(u_h, u) \rightarrow 0 \quad \implies \quad \exists v_h \text{ such that } \delta(u_h, v_h) = 0, \quad |v_h - u| \rightarrow 0.$$

Our result, which contains as a particular case all others, is that the solutions of  $\mathcal{P}_\epsilon$  have as limit points solutions of

$$(P) \quad \min \left\{ |Du|(\Omega) + \int_{\Omega} \hat{\psi}(x, u) dx \mid u \in BV(\Omega, Z) \right\}.$$

By proposition 4.1, problem  $(P)$  has at least one solution, because (4.3) ensures the lower semicontinuity of  $\hat{\psi}(x, \cdot)$  with respect to  $\delta$ . We shall also prove that for any  $u \in BV(\Omega, Z)$  the total variation  $|Du|$  is representable by

$$|Du|(B) = \int_{B \cap S_{\pi(u)}} \delta(\pi(u)^+, \pi(u)^-) d\mathcal{H}^{n-1}(x) \quad \forall B \in \mathbf{B}(\Omega).$$

The theorem is proved by using the standard tools of  $\Gamma$ -convergence. In particular, as in §3 it will be convenient for us to consider functionals  $F(u, A)$  depending on the domain of integration too. We recall (see for instance [8], [13], [18]) that by definition a sequence of functions  $f_h : X \rightarrow [-\infty, +\infty]$   $\Gamma(X)$ -converges to  $f : X \rightarrow [-\infty, +\infty]$  if

$$(4.4) \quad \liminf_{h \rightarrow +\infty} f_h(x_h) \geq f(x)$$

for any  $x \in X$  and any sequence  $x_h \rightarrow x$ , and it is possible to find a sequence  $x_h \rightarrow x$  such that

$$(4.5) \quad \limsup_{h \rightarrow +\infty} f_h(x_h) \leq f(x).$$

The basic properties of  $\Gamma$ -convergence are uniqueness of the limit, compactness, stability under continuous perturbations. Furthermore, equicoercivity assumptions ensure convergence of minimizers to minimizers. Now we recall the basic definitions of the theory of variational functionals [13]. We denote by  $\mathcal{G}$  the class of functionals  $F : L^2(\Omega) \times \mathbf{A}(\Omega) \rightarrow [0, +\infty]$  satisfying *locality*, i.e.,

$$F(u, A) = F(v, A) \quad \text{whenever } u = v \text{ a.e. in } A,$$

lower semicontinuity, i.e.,

$$F(u, A) \leq \liminf_{h \rightarrow +\infty} F(u_h, A) \quad \text{whenever } u_h \rightarrow u \text{ in } L^2(\Omega),$$

and  $F(u, \cdot)$  is the restriction to  $\mathbf{A}(\Omega)$  of a Borel measure for any  $u \in L^2(\Omega)$ . In  $\mathcal{G}$  it is possible to define  $\Gamma$ -convergence for “almost every” open set. We say that  $\mathcal{R} \subset \mathbf{A}(\Omega)$  is a *rich* class of open sets if the set  $\{t \mid A_t \notin \mathcal{R}\}$  is at most countable for all families  $(A_t) \subset \mathbf{A}(\Omega)$  with  $A_s \subset \subset A_t$  for  $s < t$ . Then, we say that  $F_h \Gamma^*(L^2(\Omega))$ -converges to  $F$  if the class of open sets  $A \in \mathbf{A}(\Omega)$  such that  $F_h(\cdot, A) \Gamma(L^2(\Omega))$ -converges to  $F(\cdot, A)$  is rich. We set

$$F_\epsilon(u, A) = \begin{cases} \int_A [\epsilon |\nabla u|^2 + \frac{g(u)}{\epsilon}] dx & \text{if } u \in W^{1,2}(A); \\ +\infty & \text{if } u \in L^2(\Omega) \setminus W^{1,2}(A), \end{cases}$$

and

$$F_\infty(u, A) = \begin{cases} |Du|(A) & \text{if } u \in BV(A, Z); \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(A, Z). \end{cases}$$

The functionals  $F_\epsilon$  and  $F_\infty$  belong to  $\mathcal{G}$ . Our result is the following:

**THEOREM 4.2.** *Assume  $(H_1)$ ,  $(H_2)$ ,  $(H_3)$ , and let  $(\epsilon_h) \downarrow 0$  be a sequence. Then,*

$$F_{\epsilon_h}(u, A) \Gamma^*(L^2(\Omega)) - \text{converges to } F_\infty(u, A).$$

*Furthermore, the class of open sets for which  $F_{\epsilon_h}(\cdot, A) \Gamma(L^2(\Omega))$ -converges to  $F_\infty(\cdot, A)$  contains all star shaped domains and all  $C^2$  domains. If  $F_{\epsilon_h}(\cdot, \Omega) \Gamma(L^2(\Omega))$ -converges to  $F_\infty(\cdot, \Omega)$ , then any sequence  $u_{\epsilon_h}$  of solutions of  $\mathcal{P}_{\epsilon_h}$  is bounded in  $L^p(\Omega)$ , compact with respect to  $\delta$ -convergence almost everywhere, and any limit point is a solution of  $\mathcal{P}$ .*

**PROOF OF THEOREM 4.2.** We rename  $F_{\epsilon_h}$  by  $F_h$ . By a compactness theorem for  $\Gamma^*$  convergence in  $\mathcal{G}$ , we can assume with no loss of generality that  $F_h \Gamma^*(L^2(\Omega))$ -converges to a functional  $F \in \mathcal{G}$ . This is stated in the appendix of [13] in the scalar case  $k = 1$ ; the same proof works in the vector case. We need only to show that  $F = F_\infty$ . We begin with the proof of  $\geq$ . Let  $\mathcal{R}$  be the class of open sets for which  $\Gamma(L^2(\Omega))$ -convergence holds. Let  $A \in \mathcal{R}$  be an open set, and let  $(u_h) \subset [W^{1,2}(A)]^k$  be a sequence converging in  $L^2(A)$  to a Borel function  $u : \Omega \rightarrow \mathbf{R}^k$ . We can assume that the inferior limit of  $F_h(u_h, A)$  is finite, the inequality being trivial if this does not happen. Since  $\epsilon_h \downarrow 0$ , it follows that  $u \in Z$  almost everywhere. The inequality  $F(u, A) \geq F_\infty(u, A)$  then follows by the lower semicontinuity of the total variation in  $BV(\Omega, \mathbf{R}^k)$  and the following proposition.

PROPOSITION 4.3. *Let  $u \in [W^{1,2}(A)]^k$ , and assume that  $F_h(u, A) < +\infty$ . Then,  $u \in BV(A, \mathbf{R}^k)$  and*

$$(4.6) \quad |Du|(A) \leq 2 \int_A g^{1/2}(u) |\nabla u| dx \leq F_h(u, A).$$

PROOF. By using  $(H_1)$  and a truncation argument, it can be easily seen that it is not restrictive to assume  $u$  bounded. Hence, we can find a constant  $K$  such that  $\delta(x, y) \leq K|x - y|$  as  $x, y$  vary in the range of  $u$ . In particular, the total variations with respect to  $\delta$  and the euclidean distance can be compared, yielding  $|Du|(B) \leq K \int_B |\nabla u| dx$  for any Borel set  $B$ . We infer that  $|Du|$  is absolutely continuous with respect to  $\mathcal{L}^n$ , and by Theorem 2.2 we get

$$|Du|(A) = \int_A \operatorname{ap} \lim_{y \rightarrow x} \frac{\delta(u(y), u(x))}{|y - x|} dx.$$

Since

$$\limsup_{w \rightarrow z} \frac{\delta(w, z)}{|w - z|} \leq 2g^{1/2}(z),$$

we obtain

$$\begin{aligned} |Du|(A) &= \int_A \operatorname{ap} \limsup_{y \rightarrow x} \frac{\delta(u(y), u(x))}{|u(y) - u(x)|} \frac{|u(y) - u(x)|}{|y - x|} dx \\ &\leq \int_A 2g^{1/2}(u) |\nabla u| dx \leq F_h(u, A). \quad \mathbf{q.e.d.} \end{aligned}$$

Since  $F(u, A) \geq F_\infty(u, A)$  for all  $u$  and all  $A \in \mathcal{R}$ , and since  $F, F_\infty$  are measures, the inequality  $F \geq F_\infty$  follows.

The proof of inequality  $F \leq F_\infty$  is divided in three steps. In the first step we show the inequality  $F(u, Q) \leq F_\infty(u, Q)$  for all cubes  $Q$  and all functions  $u$  constant on a partition in a finite number of cubes. In the second step we prove the estimate  $F(u, A) \leq n(\xi(n, E) + 1)F_\infty(u, A)$  for any  $u \in BV(\Omega, Z)$ ,  $A \in \mathbf{A}(\Omega)$ . In the third step the proof is completed by showing that

$$V_u(A) = F(u, A) = F_\infty(u, A) = \int_{A \cap S_{\pi(u)}} \delta(n(u)^+, \pi(u)^-) d\mathcal{H}^{n-1}(x) \quad \forall A \in \mathbf{A}(\Omega)$$

for any  $u \in BV(\Omega, Z)$ .

STEP 1. We show the inequality  $F(u, Q) \leq |Du|(Q)$  for any cube  $Q$  and any function  $u$  constant on a canonical partition of  $Q$  in a finite number of cubes. Since we are dealing with measures, it will be sufficient to show it for any cube  $Q \in \mathcal{R}$ . Let us first consider the simplest case, that is, a simple function  $u$  jumping between  $z_1, z_2$  along an hyperplane  $S_u$  normal to  $e_i$  for



some  $i \in \{1, \dots, n\}$ . Let us assume, to fix the ideas,  $S_u = \{x \in \mathbf{R}^n \mid \langle x, e_1 \rangle = 0\}$  and  $Q$  centered at the origin. Let  $\gamma : [0, L] \rightarrow \mathbf{R}^k$  be any  $C^1$  arc connecting  $z_1$  and  $z_2$ , parametrized by arc length. Let  $\alpha \in ]0, 1/2[$ , let us consider solutions  $\eta_h > 0$  of the differential equations

$$\eta' = \frac{g^{1/2}(\gamma(\eta)) + \epsilon_h^\alpha}{\epsilon_h}, \quad \eta(0) = 0$$

and let  $\theta_h$  be such that  $\eta_h(\theta_h) = L$ ; since

$$\theta_h = \int_0^L \frac{\epsilon_h}{g^{1/2}(\gamma(t)) + \epsilon_h^\alpha} dt \leq L\epsilon_h^{1-\alpha}$$

the sequence  $\theta_h$  converges to 0. The functions

$$(4.7) \quad u_h(x_1, \dots, x_n) = \begin{cases} z_1 & \text{if } x_1 \leq -\theta_h/2; \\ \gamma(\eta_h(x_1 + \theta_h/2)) & \text{if } -\theta_h/2 \leq x_1 \leq \theta_h/2; \\ z_2 & \text{if } x_1 \geq \theta_h/2, \end{cases}$$

converge to  $u$  almost everywhere. Let us estimate  $F_h(u_h, Q)$ .

$$\begin{aligned} \int_Q \left[ \epsilon_h |\nabla u_h|^2 + \frac{g(u_h)}{\epsilon_h} \right] dx &= \mathcal{H}^{n-1}(Q \cap S_u) \int_0^{\theta_h} \left[ \epsilon_h |\eta_h'|^2 + \frac{g(\gamma(\eta_h))}{\epsilon_h} \right] dt = \\ &= 2 \mathcal{H}^{n-1}(Q \cap S_u) \int_0^{\theta_h} g^{1/2}(\gamma(\eta_h)) |\eta_h'| dt + \theta_h \epsilon_h^{2\alpha-1} \mathcal{H}^{n-1}(Q \cap S_u). \end{aligned}$$

By letting  $h \rightarrow +\infty$  we get

$$F(u, Q) \leq 2 \mathcal{H}^{n-1}(Q \cap S_u) \int_0^L g^{1/2}(\gamma(s)) ds.$$

Since  $\gamma$  is arbitrary, the inequality  $F(u, Q) \leq F_\infty(u, Q)$  follows.

The same argument can be (see figure) locally repeated for simple functions  $u$  whose level sets are a canonical partition  $Q_j, 1 \leq j \leq k^n$  of  $Q$ . In fact, let  $S \subset Q$  be the union of the  $(n - 2)$ -dimensional skeletons of  $\partial Q_j$ , let

$$\theta_h = \epsilon^{1-\alpha} \cdot \sup_{i,j} \int_0^1 |\gamma'_{ij}(s)| ds$$

and let  $\Psi_h : Q \rightarrow [0, 1]$  be a function whose Lipschitz constant is not greater than  $4/\theta_h$ , such that  $\Psi_h(x) = 1$  if  $\text{dist}(x, S) \geq 2\theta_h$  and  $\Psi_h(x) = 0$  if  $\text{dist}(x, S) \leq \theta_h$ . In addition, let  $v_h$  be  $C^1$  functions defined in

$$Q_h = \{x \in Q \mid \text{dist}(x, S) > \theta_h\},$$

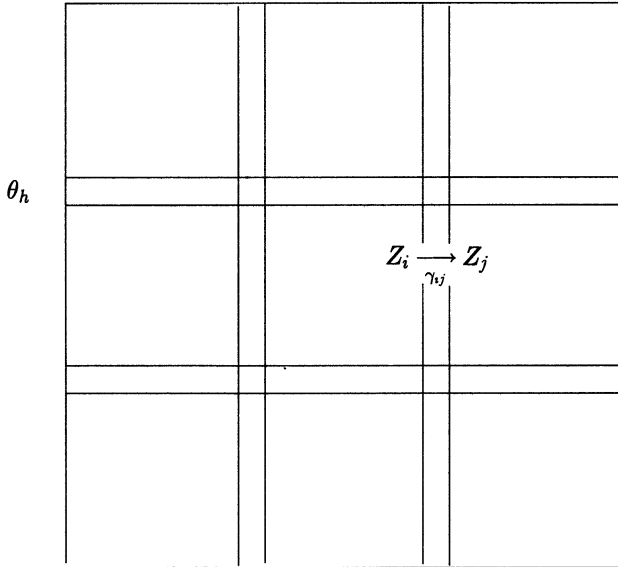
obtained by repeating the construction of Step 1, i.e., the value of  $v_h(x)$  changes from  $z_i$  to  $z_j$  along paths  $\gamma_{ij}$  in a layer centered at the common face of  $Q_i$  and  $Q_j$  whose thickness is  $\theta_h$ . The functions  $u_h = \Psi_h v_h$  converge to  $u$  almost everywhere. Since

$$\text{meas}(\{x \in Q : \text{dist}(x, S) \geq 2\theta_h\}) \leq C(n, k)\epsilon_h^{2-2\alpha}$$

and  $\alpha < 1/2$ , it can be easily seen that

$$\limsup_{h \rightarrow +\infty} \int_Q \left[ \epsilon_h |\nabla u_h|^2 + \frac{g(u_h)}{\epsilon_h} \right] dx = \limsup_{h \rightarrow +\infty} \int_{Q_h} \left[ \epsilon_h |\nabla v_h|^2 + \frac{g(v_h)}{\epsilon_h} \right] dx \leq F_\infty(u, Q) + \epsilon$$

if the paths connecting  $z_i$  to  $z_j$  are nearly optimal in (4.1).



STEP 2. Let  $u \in BV(Q, Z)$ , and let  $Q_j, 1 \leq j \leq h^n$  be a canonical partition of  $Q$ . By lemma 3.2 we get  $\eta_j \in Z$  such that  $\int_{Q_j} \delta(u, \eta_j) dx \leq \xi(n, E)/h |Du|(Q_j)$ . The function  $v_h$  whose value in  $Q_j$  is  $\eta_j$  is a good approximation of  $u$  with respect to  $\delta$  and the argument of Theorem 3.3 gives the estimates

$$\int_Q \delta(u, v_h) dx \leq \frac{\xi(n, E)}{h} |Du|(Q), \quad |Dv_h|(Q) \leq n(\xi(n, E) + 1) |Du|(Q).$$

By using  $(H_3)$  we construct  $u_h$  such that  $\pi(v_h) = \pi(u_h)$  and  $u_h$  well approximates  $u$  in  $L^2(Q, Z)$ . In fact, the Aumann's selection theorem [11] yields a Borel

function  $w_h$  such that  $\pi(w_h) = \pi(u_h)$  and

$$\int_Q |w_h - u| \, dx \leq \int_Q \omega(\delta(u, v_h)) \, dx.$$

Since  $w_h$  is a Borel function, we canonically partition any  $Q_j$  in a finite number of cubes  $Q_{j,1}, \dots, Q_{j,p}$  and find  $\xi_{j,i} \in Q_{j,i}$  such that

$$u_h(x) = \sum_{j=1}^{h^n} \sum_{i=1}^p w_h(\xi_{j,i}) \chi_{Q_{j,i}}$$

satisfies

$$\int_Q |u_h - u| \, dx \leq \int_Q \omega(\delta(u, v_h)) \, dx + \frac{1}{h}$$

if  $p$  is large enough. By construction,  $\pi(u_h) = \pi(v_h)$  and  $u_h$  is locally constant, so that step 1 gives

$$F(u_h, Q) \leq |Du_h|(Q) = |Dv_h|(Q) \leq n(\xi(n, E) + 1)|Du|(Q).$$

By letting  $h \rightarrow +\infty$  we infer the desired estimate. Since  $F, F_\infty$  are measures, the same inequality holds for any open set  $A \subset \Omega$ .

STEP 3. We first show the inequality

$$(4.8) \quad F(u, A) \leq F_\infty(u, A) \quad \forall A \in \mathbf{A}(\Omega)$$

for locally simple functions  $u$ . Let us first consider the case of a function  $u$  with only two values  $z_1, z_2$ , and let  $Q \subset \Omega$  be a cube. By the Modica-Mortola theorem ([24], [25]), for any set of finite perimeter  $B \subset Q$  and any continuous function  $f : [0, L] \rightarrow [0, +\infty[$  such that  $f(0) = f(L) = 0$ , it is possible to find a sequence of functions  $(v_h) \subset W^{1,2}(Q)$ ,  $0 \leq v_h \leq L$  such that  $v_h \rightarrow \chi_B$  and

$$\lim_{h \rightarrow +\infty} \int_Q \left[ \epsilon_h |Dv_h|^2 + \frac{f(v_h)}{\epsilon_h} \right] dx = 2 \mathcal{M}^{n-1}(\partial^* B \cap Q) \int_0^L f^{1/2}(s) \, ds.$$

We apply this result to  $B = \{u = z_1\}$ ,  $f(s) = g(\gamma(s))$ , where  $\gamma$  is a  $C^1$  path connecting  $z_1$  and  $z_1$  parametrized by arc length. Setting  $u_h = \gamma(v_h)$ , we get

$$(4.9) \quad \begin{aligned} F(u, Q) &\leq \liminf_{h \rightarrow +\infty} \int_Q \left[ \epsilon_h |Du_h|^2 + \frac{g(u_h)}{\epsilon_h} \right] dx = \\ &= \liminf_{h \rightarrow +\infty} \int_Q \left[ \epsilon_h |\nabla v_h \otimes \gamma'(v_h)|^2 + \frac{f(v_h)}{\epsilon_h} \right] dx \leq 2 \mathcal{M}^{n-1}(S_u \cap Q) \int_0^L g^{1/2}(\gamma(s)) \, ds. \end{aligned}$$

Since  $\gamma$  and  $Q$  are arbitrary, we find that (4.8) holds for simple functions with only two values. Now we need the following locality lemma.

LEMMA 4.4. *Let  $G \in \mathcal{G}$ , and assume that there exists a constant  $C \geq 0$  such that  $G(u, A) \leq C|Du|(A)$  for any  $u \in BV(\Omega, \mathbf{R}^k)$ ,  $A \in \mathbf{A}(\Omega)$ . Then,  $G(u, B) = G(v, B)$  whenever  $u, v \in BV(\Omega, \mathbf{R}^k)$ ,  $B \subset S_{\pi(u)} \cap S_{\pi(v)}$ , and (recall (2.9))*

$$\pi(u)^+(x, \nu(x)) = \pi(v)^+(x, \nu(x)) = 0, \quad \pi(u)^-(x, \nu(x)) = \pi(v)^-(x, \nu(x))$$

for  $\mathcal{H}^{n-1}$ -almost every  $x \in B$ .

The proof of the lemma is very similar to the proof of Proposition 3.4(i) (see also [7], Proposition 4.4). The basic idea is to compare  $u, v$  in small neighbourhoods of compact sets  $K \subset B$  such that

$$\lim_{\rho \rightarrow 0^+} \frac{1}{\rho} \int_{\{x \in \Omega | \text{dist}(x, K) < \rho\}} \delta(u, v) dx = 0.$$

By the estimates of step 2,  $F$  fulfils the conditions of the lemma. By using this strong locality property, we can easily see that (4.8) holds for simple functions too. Indeed, let

$$u = \sum_{i=1}^m z_i \chi_{E_i}$$

with  $E_i$  sets of finite perimeter in  $Q$ , and let

$$u_{ij}(x) = \begin{cases} z_i & \text{if } x \in E_i; \\ z_j & \text{otherwise.} \end{cases} \quad i, j \in \{1, \dots, m\}.$$

By step 2,  $F(u, \cdot)$  is a finite measure supported in  $S_u$ ; by using Proposition 3.1 and the strong locality property, we get

$$\begin{aligned} F(u, Q \cap S_u) &= \frac{1}{2} \sum_{i,j=1}^m F(u, Q \cap \partial^* \{u = i\} \cap \partial^* \{u = j\}) = \\ &= \frac{1}{2} \sum_{i,j=1}^m F(u_{ij}, Q \cap \partial^* \{u = i\} \cap \partial^* \{u = j\}) \leq \\ &\leq \frac{1}{2} \sum_{i,j=1}^m \delta(i, j) \mathcal{H}^{n-1}(Q \cap \partial^* \{u = i\} \cap \partial^* \{u = j\}) = F_\infty(u, Q), \end{aligned}$$

and (4.8) is proved for simple functions. Since  $F$  is a measure, the same inequality holds for locally simple functions too. Moreover, by Lemma 4.4 it follows that

$$\tilde{F}(v, A) = F(u, A), \quad \pi(u) = v, \quad v \in BV(\Omega, E), \quad A \in \mathbf{A}(\Omega)$$

is well defined, and (4.3) yields that  $\tilde{F}(v, A)$  is lower semicontinuous in  $v$ . By Proposition 3.4(ii) and  $(H_2)$  we can find a sequence of locally simple functions  $u_h$  such that  $\pi(u_h) \rightarrow \pi(u)$  and  $|Du_h|(A)$  converges to  $|Du|(A)$ . Hence

$$F(u, A) = \tilde{F}(\pi(u), A) \leq \liminf_{h \rightarrow +\infty} \tilde{F}(\pi(u_h), A) = \liminf_{h \rightarrow +\infty} F_\infty(u_h, A) = F_\infty(u, A),$$

and this completes the proof of inequality  $F \leq F_\infty$ .

Now we prove the last statements of the Theorem. Assume that  $A \subset \Omega$  is star shaped with respect to  $x_0$ . The inequality (4.4) follows by Proposition 4.3 as before. We need only, given  $u \in BV(A, Z)$ , to construct a sequence  $u_h \rightarrow u$  such that  $F_h(u_h, A) \rightarrow F(u, A)$ . Let

$$A_t = t(A - x_0) + x_0, \quad 0 \leq t < 1.$$

Let  $t$  be such that  $A_t \in \mathcal{R}$ , let  $u_t(x) = u((x - x_0)/t + x_0)$ , and let  $v_h(x) \rightarrow u_t(x)$  be such that  $F_h(v_h, A_t) \rightarrow F(u_t, A_t)$ . Then,  $u_h(x) = v_h(tx)$  converges to  $u$  and

$$\limsup_{h \rightarrow +\infty} F_h(u_h, A) \leq \frac{1}{t} |Du|(A).$$

Since  $t$  can be taken arbitrarily near to 1, the required sequence can be constructed by a diagonal argument.

Now we prove the convergence of minimizers to minimizers. Indeed, any sequence  $(u_{\epsilon_h})$  of minimizers is bounded in  $L^p(\Omega)$  by our assumption on  $g$ , and has equibounded total variation by Proposition 4.3. Let  $u \in BV(\Omega, Z)$  be any limit in  $(\mathbf{R}^k, \delta)$  of a subsequence of  $u_{\epsilon_h}$  (still labeled by  $u_{\epsilon_h}$  for simplicity), and let  $v \in BV(\Omega, Z)$  be any function. By Aumann's measurable selection theorem [11], we can find  $w \in BV(\Omega, Z)$  such that  $\pi(w) = \pi(v)$  and

$$\int_{\Omega} \hat{\psi}(x, v) \, dx = \int_{\Omega} \psi(x, w) \, dx.$$

By the definition of  $\Gamma$ -convergence, we can find a sequence  $w_h$  converging to  $w$  in  $L^2(\Omega, \mathbf{R}^k)$  such that

$$\lim_{h \rightarrow +\infty} F_h(w_h, \Omega) = |Dw|(\Omega) = |Dv|(\Omega).$$

Since  $u_{\epsilon_h}$  solve  $\mathcal{P}_{\epsilon_h}$ , we infer

$$\begin{aligned} |Dv|(\Omega) + \int_{\Omega} \hat{\psi}(x, v) \, dx &= \lim_{h \rightarrow +\infty} F_h(w_h, \Omega) + \int_{\Omega} \psi(x, w_h) \, dx \geq \\ &\geq \liminf_{h \rightarrow +\infty} F_h(u_{\epsilon_h}, \Omega) + \int_{\Omega} \psi(x, u_{\epsilon_h}) \, dx \geq \\ &\geq \liminf_{h \rightarrow +\infty} |Du_{\epsilon_h}|(\Omega) + \int_{\Omega} \hat{\psi}(x, u_{\epsilon_h}) \, dx \geq |Du|(\Omega) + \int_{\Omega} \hat{\psi}(x, u) \, dx. \end{aligned}$$

REMARK 4.1. By the same argument of the proof of Theorem 4.2, it can be shown that the class of open sets  $A$  for which the functionals  $F_h$   $\Gamma$ -converge to  $F_\infty$  contains all sets  $A \subset \Omega$  such that there exists a sequence of open sets  $A_h \subset \subset A$ , one to one mappings  $\varphi_h : A_h \rightarrow A$  such that  $A_h \uparrow A$  and

$$\sup \left\{ \frac{|\varphi_h(x) - \varphi_h(y)|}{|x - y|} \mid x, y \in A_h \right\} \rightarrow 1,$$

$$\inf \left\{ \frac{|\varphi_h(x) - \varphi_h(y)|}{|x - y|} \mid x, y \in A_h \right\} \rightarrow 1.$$

Any  $C^2$  domain fulfils this condition. We also remark that  $\Gamma$ -convergence of the functionals in  $\mathcal{P}_\epsilon$  does not depend on our special choice of the norm in  $\mathcal{L}_{n,k}$  (see (1.10)). In fact, any norm  $\Theta$  such that  $\Theta(a \otimes b) = |a||b|$  ensures  $\Gamma$ -convergence. The reason is that the approximating sequences have rank 1 differentials (see (4.7), (4.9)).

**List of notations**

- $S^{n-1}$  the unit sphere in  $\mathbf{R}^n$ .
- $B_\rho(x)$  the ball centered in  $x$  with radius  $\rho$ .
- $B_\rho^+(x, \nu)$  the set  $\{y \in B_\rho(x) \mid \langle y - x, \nu \rangle > 0\}$ .
- $B_\rho^-(x, \nu)$  the set  $\{y \in B_\rho(x) \mid \langle y - x, \nu \rangle < 0\}$ .
- $\mathbf{B}(\Omega)$  the Borel  $\sigma$ -algebra of  $\Omega$ .
- $\mathbf{A}(\Omega)$  the class of open subsets of  $\Omega$ .
- $\mathcal{L}^n$  the Lebesgue  $n$ -dimensional measure in  $\mathbf{R}^n$ .
- $\mathcal{H}^{n-1}$  the Hausdorff  $(n - 1)$ -dimensional measure in  $\mathbf{R}^n$ .
- $\omega_n$  the Lebesgue measure of the unit ball in  $\mathbf{R}^n$ .
- $\mathcal{L}_{n,k}$  the space of linear mappings  $L : \mathbf{R}^n \rightarrow \mathbf{R}^k$ .
- $\text{Lip}(E)$  the space of Lipschitz continuous functions  $\varphi : E \rightarrow \mathbf{R}$ .
- $\text{Lip}_1(E)$  the class of Lipschitz functions with Lipschitz constant not greater than 1.
- $\mathcal{M}(\Omega)$  the class of  $\sigma$ -additive measures  $\mu : \mathbf{B}(\Omega) \rightarrow [0, +\infty]$ .
- $a \otimes b$  the tensor product of  $a$  and  $b$ .
- $\mu/\sigma$  the Radon-Nikodym derivative of  $\mu$  with respect to  $\sigma$ .

- $e_i$  the  $i$ -th element of the canonical basis of  $\mathbf{R}^n$ .
- $\tilde{u}(x)$  the approximate limit of  $u$  at  $x$ .
- $Du$  the distributional derivative of a real  $BV$  function  $u$ .
- $S_u$  the complement of the approximate continuity set of  $u$ .
- $\nabla u$  the approximate differential of  $u$ .
- $u_+, u_-$  the right and left continuous representatives of a  $BV$  function  $u$  of one real variable.
- $u^+, u^-$  the one sided limits of a  $BV$  function  $u$ .
- $\chi_A$  the characteristic function of a set  $A$ .
- $\partial^*A$  the essential boundary of a set of finite perimeter  $A$ .

## REFERENCES

- [1] E. ACERBI - N. FUSCO, *Semicontinuity problems in the Calculus of Variations*. Arch. Rational Mech. Anal., **86**, 125-145, 1986.
- [2] L. AMBROSIO, *A compactness theorem for a special class of functions of bounded variation*. Boll. Un. Mat. It., 3-B, **7**, 857-881, 1990.
- [3] L. AMBROSIO, *Existence theory for a new class of variational problems*. To appear in Arch. Rational Mech. Anal.
- [4] L. AMBROSIO, *Variational problems in SBV*. Acta Applicandae Mathematicae, **17**, 1-40, 1989.
- [5] L. AMBROSIO - G. DAL MASO, *The chain rule for distributional derivative*. Proc. Amer. Math. Soc., **108**, 3, 691-702, 1990.
- [6] L. AMBROSIO - V.M. TORTORELLI, *Approximation of functionals depending on jumps by elliptic functionals via  $\Gamma$ -convergence*. To appear in: "Communications On Pure and Applied Mathematics".
- [7] L. AMBROSIO - S. MORTOLA - V.M. TORTORELLI, *Functionals with linear growth defined on vector valued  $BV$  functions*. To appear in "Ann. Inst. H. Poincarè".
- [8] H. ATTOUCH, *Variational convergence for functions and operators*. Pitman, Boston, 1984.
- [9] S. BALDO, *Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids*. To appear in: "Ann. Inst. H. Poincarè".
- [10] A.P. CALDERON - A. ZYGMUND, *On the differentiability of functions which are of bounded variation in Tonelli's sense*. Revista Union Mat. Arg., **20**, 102-121, 1960.
- [11] C. CASTAING - M. VALADIER, *Convex analysis and measurable multifunctions*. Lecture Notes in Math., **590**, 1977.

- [12] L. CESARI, *Sulle funzioni a variazione limitata*. Ann. Scuola Norm. Sup. Pisa, Ser. 2, Vol. 5, 1936.
- [13] G. DAL MASO - L. MODICA, *A general theory of variational functionals*. "Topics in Functional Analysis 1980-81", Scuola Normale Superiore, Pisa, 1981.
- [14] E. DE GIORGI, *Su una teoria generale della misura  $(r-1)$ -dimensionale in uno spazio a  $r$  dimensioni*. Ann. Mat. Pura Appl., **36**, 191-213, 1954.
- [15] E. DE GIORGI, *Nuovi teoremi relativi alle misure  $(r-1)$ -dimensionali in uno spazio a  $r$  dimensioni*. Ricerche Mat., **4**, 95-113, 1955.
- [16] E. DE GIORGI - L. AMBROSIO, *Un nuovo tipo di funzionale del Calcolo delle Variazioni*. Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur., 2-VIII, **82**, 1989.
- [17] E. DE GIORGI - M. CARRIERO - A. LEACI, *Existence theorem for a minimum problem with free discontinuity set*. Arch. Rational Mech. Anal., **108**, 3, 193-218, 1989.
- [18] E. DE GIORGI - T. FRANZONI, *Su un tipo di convergenza variazionale*. Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur., (8) **58**, 842-850, 1975.
- [19] H. FEDERER, *Geometric Measure Theory*. Springer Verlag, Berlin, 1969.
- [20] H. FEDERER, *A note on Gauss-Green theorem*. Proc. Amer. Mat. Soc., **9**, 447-451, 1958.
- [21] H. FEDERER, *Colloquium lectures on Geometric Measure Theory*. Bull. Amer. Math. Soc., **84**, 3, 291-338, 1978.
- [22] W.H. FLEMING - R. RISHEL, *An integral formula for total gradient variation*. Arch. Math., **11**, 218-222, 1960.
- [23] E. GIUSTI, *Minimal Surfaces and Functions of Bounded Variation*. Birkäuser, Boston, 1984.
- [24] L. MODICA, *The gradient theory of phase transitions and the minimal interface criterion*. Arch. Rational Mech. Anal., **98**, 2, 123-142, 1987.
- [25] L. MODICA - S. MORTOLA, *Un esempio di  $\Gamma$ -convergenza*. Boll. Un. Mat. Ital., **5** **14-B**, 285-299, 1977.
- [26] D. MUMFORD - J. SHAH, *Boundary detection by minimizing functionals*. Proc. of the IEEE conference on computer vision and pattern recognition, San Francisco, 1985.
- [27] D. MUMFORD - J. SHAH, *Optimal approximation by piecewise smooth functions and associated variational problems*. Comm. on Pure and Appl. Math., **17**, 4, 577-685, 1989.
- [28] Y.G. RESHETNYAK, *Weak convergence of completely additive vector functions on a set*. Siberian Math. J., **9**, 1039-1045, 1968 (translation of Sibirsk Mat. Z., **9**, 1386-1394, 1968).
- [29] E. STEIN, *Singular integrals and differentiability properties of functions*. Princeton University Press, 1970.
- [30] P. STERNBERG, *The effect of a singular perturbation on Nonconvex Variational Problems*. Arch. Rational Mech. Anal., **101**, 209-260, 1988.
- [31] A.I. VOL'PERT - S.I. HUHJAEV, *Analysis in classes of discontinuous functions and equations of mathematical physics*. Martinus Nijhoff Publisher, Dordrecht, 1985.



- [32] A.I. VOL'PERT, *The spaces  $BV$  and quasilinear equations*. Math. USSR. Sb., **17**, 225-267, 1967.
- [33] W.P. ZIEMER, *Weakly differentiable functions*. Springer Verlag, Berlin, 1989.

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