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Harmonic Mappings into Manifolds with Boundary

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1. - Introduction

In this paper we shall concentrate our attention on harmonic maps

$$u : M \to C$$

between Riemannian manifolds. Our main interest is referred to target manifolds C with boundary.

Our assumptions on the manifolds M and C will be listed in the next section. To fix ideas, we can assume that M is a compact Riemannian manifold without boundary, and that C is a compact Riemannian submanifold of the Euclidean space $\mathbf{R}^k$.

In the "smooth" case (namely, when ∂C is empty or it is strictly convex), the theory of harmonic maps has been developed by many Authors. The two Reports by J. Eells and L. Lemaire (*see* [7] and [8]) are an exhaustive survey on the results achieved in this context. The case when C has strictly convex boundary was considered in [12].

In contrast with the smooth case, not much is known when the target manifold C has boundary. In case $\dim M = 1$, which corresponds to the case of *closed geodesics* in C, some multiplicity results were proved by Marino and Scolozzi in [16] and by Canino in [2]. A multiplicity result can be found in [15] and in [18] in case M is the unit two-sphere and C is a three dimensional manifold (namely, $C = \mathbf{R}^3 \backslash \Omega$, where Ω is a bounded open set in $\mathbf{R}^3$). Finally, some regularity results for energy minimizing harmonic maps are available in the literature; we refer to the more recent papers by Duzaar [6] and Fuchs [11] (see also the *References* there in).

As there are only a few papers on the "non-smooth" case containing specialized results, and since a complete treatment of this subject is not available in the literature, we start with some preliminary remarks. With respect to the smooth case, we have to modify the definition of weak harmonic map, since the boundary of the target manifold C acts as a *unilateral* constraint. From

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this viewpoint, the notions and the tools of non-smooth Analysis look to be quite natural for a general treatment of the problem under consideration. In particular, our definition of harmonic map makes use of some notions which were introduced by De Giorgi, Marino and Tosques in [5].

A weak harmonic map is by definition a "stationary point from below" for the energy integral $E(u)$ (see Section 2) on the class

$$H^1(M,C) = \{u \in H^1(M,\mathbf{R}^k) | u(x) \in C \text{ for a.e. } x \in C\},$$

that is, a map u of class $H^1(M,C)$ is said to be harmonic if

$$\liminf_{v \to u, v \in H^1(M,C)} \frac{E(v) - E(u)}{||v - u||} \geq 0. \tag{1.1}$$

The above definition extends the usual definition of harmonic map in case C has empty boundary. From (1.1) we can see for example that every local minimum for the energy integral on the constraint $H^1(M,C)$ is a weak harmonic map.

In Section 2 we investigate the general properties of solutions to (1.1) in order to obtain a geometrical characterization of weak harmonic maps. Denoting by Δ_M the Laplace-Beltrami operator on M, we prove that a map u is weakly harmonic if and only if $\Delta_M u$ is normal to the manifold C in some weak (distributional) sense.

The possibility to look at harmonic maps both from a variational and a geometrical point of view is a useful tool in several circumstances. As a first application we compute the Euler- Lagrange equations for the energy integral on the constraint $H^1(M,C)$ (see Section 3), that is we characterize weak harmonic maps as to be distributional solutions to a system of elliptic partial differential equations. The result we achieve here extends a theorem proved by Duzaar in [6] for energy minimizing weak harmonic maps.

In Section 4 we attack a problem which was first proposed by Eells and Sampson in their celebrated paper [9]: given a smooth map $u_0 : M \to C$, can u_0 be deformed into a harmonic map $u_\infty : M \to C$?

In [9] it is proved that this problem has a positive answer in case C has empty boundary and non positive sectional curvature. Their result is obtained by proving that the evolution problem

$$\partial_t u - \Delta_M u + A(u)(Du, Du) = 0 \quad \text{on } \mathbf{R}_+ \times M \tag{1.2}$$

$$u(0,\cdot) = u_0 \quad \text{on } M \tag{1.3}$$

has a global regular solution $u : \mathbf{R}_+ \times M \to C$ and $u(\cdot, t)$ subconverges to a smooth harmonic map $u_\infty : M \to C$ as $t \to \infty$. Here $A(u) : T_uC \times T_uC \to T_u^\perp C$ is the second fundamental form of the embedding of C into $\mathbf{R}^k$ at u. The restriction on the curvature of C is in general necessary (compare with [10]) to prevent the phenomenon of "separation of spheres". In the general case we may investigate the existence of *weak solutions* to (1.2) - (1.3). The more complete results in this context are due to Struwe [20] for the case $\dim M = 2$ and to Chen-Struwe [4] for the higher dimensional case.

We follow here the approximation argument used in [3] and in [4] in order to extend the existence and partial regularity results by Chen and Struwe to the case of target manifolds with boundary. The main Theorem is stated in Section 4.

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2. - Weak harmonic maps

2.1 - *Notation and preliminaries*

Let $k \geq 1$ be an integer. If u, v are two points in the Euclidean space $\mathbf{R}^k$ we denote by $u \cdot v$ their scalar product. We shall denote by $|u| = (u \cdot u)^{1/2}$ the norm in $\mathbf{R}^k$. If E_1, E_2 are two subsets of $\mathbf{R}^k$, we define

$$d(E_1, E_2) = \inf_{u \in E_1, v \in E_2} |u - v|.$$

We denote by χ_E the characteristic function of a set E in $\mathbf{R}^k$, that is, $\chi_E(u) = 1$ if $u \in E$, $\chi_E(u) = 0$ otherwise in $\mathbf{R}^k$.

Let (M, g) be a Riemannian manifold of finite dimension m. We assume that M is compact and with empty boundary. We shall use standard notation for the spaces $L^p(M, \mathbf{R}^k)$, $W^{1,p}(M, \mathbf{R}^k)$. In case $p = 2$ we simply write H^1 instead of $W^{1,2}$, and we denote by $\|\cdot\|$ the norm in H^1. Let $H^{-1}(M, \mathbf{R}^k)$ be the dual space of $H^1(M, \mathbf{R}^k)$. We shall denote by $\langle \cdot, \cdot \rangle$ the duality product on $H^{-1} \times H^1$. If C is any subset of $\mathbf{R}^k$ we set

$$H^1(M, C) = \{u \in H^1(M, \mathbf{R}^k) \mid u(x) \in C \text{ for a.e. } x \in M\}.$$

Let u be a map in $H^1(M, \mathbf{R}^k)$. In local coordinates $x_1, \dots, x_m$ on M we define

$$e(u) = \frac{1}{2}\, g^{\alpha,\beta}\, \frac{\partial u}{\partial x_\alpha} \cdot \frac{\partial u}{\partial x_\beta}$$

where $(g^{\alpha,\beta})$ is the inverse matrix of $(g_{\alpha,\beta})$, and

$$dM = \sqrt{\det g}\; dx.$$

The *energy* of the map u is by definition

$$E(u) = \int_M e(u)\; dM.$$

Let C be a closed subset of $\mathbf{R}^k$. We shall say that a map $u \in H^1(M,C)$ is *weakly harmonic* if u is stationary from below for the energy integral on $H^1(M,C)$, that is:

$$(2.1) \qquad u \text{ is harmonic } \Leftrightarrow \liminf_{v \to u, v \in H^1(M,C)} \frac{E(v) - E(u)}{\|v - u\|} \geq 0.$$

Definition (2.1) can be rewritten in an equivalent way by using the concept of subdifferential arising from non-smooth Analysis: setting

$$\partial^- E(u) = \{\alpha \in H^{-1}(M, \mathbf{R}^k) \mid \liminf_{v \to u, v \in H^1(M,C)} \frac{E(v) - E(u) - \langle \alpha, v - u \rangle}{\|v - u\|} \geq 0\},$$

we see that a map $u \in H^1(M,C)$ is weakly harmonic iff

$$(2.2) \qquad 0 \in \partial^- E(u).$$

In this paper we shall restrict our attention to the case when C is a smooth manifold with boundary. More precisely, we assume that C is the closure of an open subset of a supporting Riemannian manifold S which is isometrically embedded into $\mathbf{R}^k$. We require that S is (topologically) closed, and

$$(2.3) \qquad d(C, \partial S) > 0.$$

The manifolds ∂C, S are supposed to be smooth (of class C^3). We also need some bound on the geometry of C. Let us denote by Π_S and by $\Pi_{\partial C}$ the nearest point projections on S, ∂C respectively. The projection $\Pi_{\partial C}$ is of class C^2 in a neighborhood of ∂C and by assumption (2.3) the projection Π_S is of class C^2 in a uniform neighborhood of C. We shall assume:

(2.4) *there exists a uniform open neighborhood $\mathcal{U}_C$ of C in $\mathbf{R}^k$ such that the map $\Pi_S : \mathcal{U}_C \to S$ has bounded first and second order derivatives.*

(2.5) *there exists a uniform open neighborhood $\mathcal{U}_{\partial C}$ of ∂C in $\mathbf{R}^k$ such that the map $\Pi_{\partial C} : \mathcal{U}_{\partial C} \to \partial C$ has bounded first and second order derivatives.*

The above hypotheses are satisfied in some interesting cases, as for example when C is a compact submanifold (with boundary) of $\mathbf{R}^k$ or when C is the complement of an open and bounded set in $\mathbf{R}^k$ with smooth boundary.

We denote by T_uS, $T_u\partial C$ the tangent spaces to S and to ∂C respectively. We denote by A the second fundamental form of the embedding of S into $\mathbf{R}^k$ and by b the second fundamental form of ∂C relative to S. By assumptions (2.4) and (2.5) the maps A and b are continuous on $\mathcal{U}_C \cap S$ and ∂C respectively, and

$$\sup_{u \in \mathcal{U}_C \cap S} |A(u)| < \infty, \qquad \sup_{u \in \partial C} |b(u)| < \infty.$$

We denote by $\omega(\cdot)$ an inner normal vectorfield to C relative to S with $\omega \in C^1(\partial C, S^{k-1})$ and with bounded first order derivatives (compare with assumption (2.5)).

For u in a suitably small neighborhood of C (also denoted by $\mathcal{U}_C$), we set

$$\Pi_C(u) = \begin{cases} \Pi_S(u) & \text{if } \Pi_S(u) \in C \\ \Pi_{\partial C}(u) & \text{otherwise.} \end{cases}$$

Notice that Π_C is the nearest point projection of $\mathcal{U}_C$ into C. By the assumptions (2.4) and (2.5), it results that $\Pi_C : \mathcal{U}_C \to C$ is (globally) Lipschitz continuous on $\mathcal{U}_C$. Consequently, by standard results on Sobolev spaces (see for example [17]) we have that

$$\Pi_C \circ w \in H^1(M, C) \text{ for every } w \in H^1(M, \mathcal{U}_C)$$

and moreover, in local coordinates $x_1, \dots, x_m$ on M, we get

$$\frac{\partial}{\partial x_\alpha}(\Pi_C \circ w) = d\Pi_S(w) \cdot \frac{\partial w}{\partial x_\alpha} \text{ a.e. on } \{x \mid \Pi_C(w(x)) = \Pi_S(w(x))\}$$

and

$$\frac{\partial}{\partial x_\alpha}(\Pi_C \circ w) = d\Pi_{\partial C}(w) \cdot \frac{\partial w}{\partial x_\alpha} \text{ a.e. on } \{x \mid \Pi_C(w(x)) = \Pi_{\partial C}(w(x))\}$$

(see for example [13], Lemma A.4).

2.2 - *Harmonic mappings and normal vectorfields*

Let u be a point in C. The *tangent cone* to C at u is defined by:

$$T_uC = \begin{cases} T_uS & \text{if } u \in C \setminus \partial C \\ \{\tau \in T_uS \mid \tau \cdot \omega(u) \geq 0\} & \text{if } u \in \partial C. \end{cases}$$

In Lemma A.1 (Appendix A) we shall give some equivalent definitions of tangent cone. In particular, it turns out that T_uC coincides with the contingent cone introduced by Bouligand at the beginning of this century, and it coincides with Clarke's tangent cone (see for example [1]).

The *normal cone* to C at $u \in C$ is by definition the negative polar cone to T_uC, that is

$$N_uC = \{\sigma \in \mathbf{R}^k \mid \sigma \cdot \tau \leq 0 \ \forall \ \tau \in T_uC\}.$$

It is straightforward to show that

$$N_uC = \begin{cases} T_u^\perp S & u \in C \setminus \partial C \\ \{\sigma \in T_u^\perp \partial C \mid \sigma \cdot \omega(u) \leq 0\} & \text{if } u \in \partial C. \end{cases}$$

Let u be a map of class $H^1(M,C)$. We shall say that a map $\tau \in H^1(M,\mathbf{R}^k)$ is an H^1 *tangential vectorfield* (TVF) to C along u if

$$\tau(x) \in T_{u(x)}C \quad \text{for a.e. } x \in M.$$

Let σ be a map in the dual space $H^{-1}(M,\mathbf{R}^k)$ of $H^1(M,\mathbf{R}^k)$. We shall say that σ is an H^{-1} *normal vectorfield* (NVF) to C along u if

$$\langle \sigma, \tau \rangle \leq 0 \text{ for every TVF to } C \text{ along } u. \tag{2.6}$$

We point out that the appellative "H^{-1} *normal vectorfield*" gives only an intuitive description of a distribution $\sigma \in H^{-1}(M,\mathbf{R}^k)$ satisfying (2.6). Our definition is justified by the fact that in case σ is represented by an L^2 function on M, condition (2.6) is equivalent to:

$$\sigma(x) \in N_{u(x)}C \text{ for a.e. } x \in M.$$

Normal vectorfields have a variational characterization. For $u \in H^1(M,\mathbf{R}^k)$ we set

$$I_C(u) = \begin{cases} 0 \text{ if } u \in H^1(M,C) \\ +\infty \text{ otherwise.} \end{cases}$$

The functional I_C is the incatrix function of the constraint $H^1(M,C)$ in the space $H^1(M,R^k)$. Let u be a map in $H^1(M,C)$; the subdifferential of I_C at the point $u \in C$ is defined by:

$$\partial^- I_C(u) = \{\sigma \in H^{-1}(M,\mathbf{R}^k) \mid \limsup_{v \to u, v \in H^1(M,C)} \langle \sigma, \frac{v-u}{\|v-u\|} \rangle \leq 0\}.$$

It is not difficult to show that

$$\partial^- E(u) = -\Delta_M u + \partial^- I_C(u) \text{ for every } u \in H^1(M,C),$$

and hence we get that u is weakly harmonic iff

$$\Delta_M u \in \partial^- I_C(u). \tag{2.7}$$

In Appendix A (see Proposition A.5), we shall prove that

$$\partial^- I_C(u) = \{\sigma \in H^{-1}(M,\mathbf{R}^k) \mid \sigma \text{ is a NVF to } C \text{ along } u\}$$

and hence from (2.7) we obtain the following result.

COROLLARY 2.1. *Let $u \in H^1(M,C)$. Then u is weakly harmonic if and only if $\Delta_M u$ is an H^{-1} normal vectorfield to C along u.*

This result will be the fundamental step for the computation of the Euler-Lagrange equations for the energy integral on $H^1(M, C)$, and it will be used also in the choice of the approximating evolution equations for harmonic maps.

REMARK 2.2. Assume that the manifold C has empty boundary. Then the projection Π_C is smooth in a uniform neighborhood of C, and hence for every $\phi \in C^\infty(M, \mathbf{R}^k)$ the map $t \to \Pi_C(u + t\phi)$ (for $|t|$ small) is a differentiable curve in $H^1(M, C)$. Classically, a map $u \in H^1(M, C)$ is said to be harmonic iff

$$\frac{d}{dt} E(\Pi_C(u + t\phi))_{|t=0} = 0 \quad \forall \phi \in C^\infty(M, \mathbf{R}^k) \tag{2.8}$$

(see for example [8] and [19]). On the other hand, it can be easily proved that (2.8) is equivalent to:

$$\langle -\Delta_M u, \psi \rangle = 0 \ \forall \psi \in H^1(M, \mathbf{R}^k), \ \ \psi \in T_u C.$$

The same arguments which lead to the proof of Corollary 2.1 can be used in order to show that (2.8) is equivalent to

$$\lim_{v \to u, v \in H^1(M,C)} \frac{E(v) - E(u)}{\|v - u\|} = 0$$

and to: $0 \in \partial^- E(u)$. □

REMARK 2.3. Let us briefly consider the Dirichlet's problem for weak harmonic maps in case ∂M is not empty. In this case, we denote by $H^{-1}(M, \mathbf{R}^k)$ the space of continuous linear forms on $H^1_0(M, \mathbf{R}^k)$. For every fixed function $g \in H^1(M, C)$, we denote by $H^1_g(M, C)$ the space of maps $u \in H^1(M, C)$ such that $u - g$ vanishes on ∂M. A map $u \in H^1_g(M, C)$ has to be said weakly harmonic iff u is stationary from below for the energy integral on $H^1_g(M, C)$, that is:

$$\liminf_{v \to u, v \in H^1_g(M,C)} \frac{E(v) - E(u)}{\|v - u\|} \geq 0.$$

This definition can be rewritten in terms of subdifferentials, and a result similar to Corollary 2.1 can be proved. □

3. - The Euler-Lagrange equations

Weak harmonic maps $u : M \to C$ are solutions to the differential inclusion

$$0 \in \partial^- E(u), \tag{3.1}$$

which by Corollary 2.1 is equivalent to

(3.2) $\Delta_M u$ is an H^{-1} normal vectorfield to C along u.

The equivalence between (3.1) and (3.2) allows us to compute the Euler-Lagrange equations for the energy integral on $H^1(M,C)$. In other words, under the assumptions on C in Section 2, we can characterize harmonic maps as to be solutions of a differential *equation.*

THEOREM 3.1. *Let $u \in H^1(M,C)$. Then u is a weak harmonic map if and only if there exists a Radon measure λ on M with*

$$0 \leq d\lambda \leq \chi_{u^{-1}(\partial C)}(-b(u)(Du,Du))dM \tag{3.3}$$

and such that

$$\langle -\Delta_M u, \Phi\rangle + \int_M A(u)(Du,Du)\cdot\Phi dM = \int_{u^{-1}(\partial C)} \omega(u)\cdot\Phi d\lambda \tag{3.4}$$

$$\forall \Phi \in L^\infty \cap H^1(M,\mathbf{R}^k).$$

As in Duzaar [6] (compare with Corollary 2.9), it can be proved that for a sufficiently regular harmonic map u it results

$$d\lambda = \chi_{u^{-1}(\partial C)}(-b(u)(Du,Du))dM.$$

Let us notice that (3.3) implies

$$-b(u)(Du,Du) \geq 0 \text{ a.e. on } u^{-1}(\partial C).$$

As it was already observed by Duzaar in [6], this inequality can be seen as a concavity condition on C in points where the image of u "essentially" touches the boundary of C.

Equation (3.4) is equivalent to:

$$\langle -\Delta_M u, \phi\rangle = \int_{u^{-1}(\partial C)} \omega(u)\cdot\phi d\lambda \tag{3.5}$$

$$\text{for every } \phi \in L^\infty \cap H^1(M,\mathbf{R}^k) \text{ with } \phi \in T_uS.$$

The proof of this equivalence is based on the following facts. We first recall that $A(u)(\tau,\tau') \in T_u^\perp S$ for every $u \in C$ and for every $\tau,\tau' \in T_uS$, and that $\omega(u) \in T_uS$ for every $u \in \partial C$. Finally, one has to use the fact that $d\Pi_S(u(\cdot))\Phi(\cdot) \in H^1(M,\mathbf{R}^k)$ for every $\Phi \in L^\infty \cap H^1(M,\mathbf{R}^k)$, and $d\Pi_S(u(x))\Phi(x)$ is tangent to S at $u(x)$ for almost every $x \in M$.

PROOF OF THEOREM 3.1. Let $u \in H^1(M,C)$ be a solution to (3.4) - (3.3), and let $\tau \in H^1(M,\mathbf{R}^k)$ be a TVF to C along u. For every real number $R > 0$

we define

$$\tau_R(x) = \begin{cases} \tau(x) & \text{if } |\tau(x)| \leq R \\ \dfrac{\tau(x)}{|\tau(x)|} R & \text{otherwise.} \end{cases} \tag{3.6}$$

Direct computations and Lebesgue's Theorem easily show that $\tau_R \to \tau$ in H^1 as $R \to +\infty$. Since $\tau_R \in L^\infty \cap H^1$ and $\tau_R(x) \in T_{u(x)}C \subseteq T_{u(x)}S$ for a.e. $x \in M$, we can use τ_R as a test function in (3.5) (which is equivalent to (3.4)) to get

$$\langle -\Delta_M u, \tau_R \rangle = \int_{u^{-1}(\partial C)} \omega(u) \cdot \tau_R d\lambda.$$

Since λ is a positive measure by (3.3), from the definition of tangent cone to C we infer that $\langle \Delta_M u, \tau_R \rangle \leq 0$ for every $R > 0$. Passing to the limit as $R \to \infty$ we finally get $\langle \Delta_M u, \tau \rangle \leq 0$, and since τ is an arbitrary tangential vectorfield to C along u, this implies that $\Delta_M u$ solves (3.2), i.e. u is weakly harmonic.

Corollary 2.1 plays a crucial role also in the proof of the converse. Our arguments are reminiscent of those used by Duzaar for the proof of Theorem 2.4 in [6].

Let $u \in H^1(M, C)$ be a weak harmonic map. Assume that $\phi \in H^1(M, \mathbf{R}^k)$ is a tangential vectorfield to S along u with $\phi = 0$ almost everywhere on $u^{-1}(\partial C)$. Then it is clear that both ϕ and $-\phi$ are tangent to C along u (since $\pm\phi \cdot \omega(u) = 0$ on $u^{-1}(\partial C)$), and hence Corollary 2.1 gives us $\pm\langle \Delta_M u, \phi \rangle \leq 0$, that is

$$\langle \Delta_M u, \phi \rangle = 0 \tag{3.7}$$

$$\text{for every } \phi \in H^1(M, \mathbf{R}^k) : \phi \in T_u S \text{ and } \phi = 0 \text{ on } u^{-1}(\partial C).$$

In the following we shall prove the existence of a bounded vectorfield $\tau \in C^1(S \cap \mathcal{U}_C, \mathbf{R}^k)$ having bounded derivatives, and such that

$$\tau(z) \in T_z S \quad \text{for every } z \in S \cap \mathcal{U}_C, \tag{3.8}$$

$$\tau(z) = \omega(z) \quad \text{for every } z \in \partial C. \tag{3.9}$$

Notice that (3.8) and (3.9) imply

$$\tau(z) \in T_z C \quad \text{for every } z \in C.$$

This remark, jointly with well-known composition theorems in Sobolev spaces (see for example [17]) guarantees that $\tau \circ u$ is an $L^\infty \cap H^1(M, \mathbf{R}^k)$ tangential vectorfield to C along u. Since u is weakly harmonic, by Corollary 2.1 we infer that

$$\langle -\Delta_M u, \eta\tau(u) \rangle \geq 0 \ \forall\ \eta \in C^\infty(M),\ \eta \geq 0.$$

Consequently, there exists a unique positive Radon measure λ on M such that

$$\langle -\Delta_M u, \eta\tau(u)\rangle = \int_M \eta \, d\lambda \;\; \forall \eta \in C^\infty(M). \tag{3.10}$$

Let us notice that (3.7) implies

$$\operatorname{supp} \lambda \subseteq u^{-1}(\partial C).$$

Moreover, using again (3.7), it is easy to prove that the measure λ is independent on the choice of the vectorfield τ, that is, it only depends on the map u and on the manifolds S and ∂C.

In order to prove the estimate:

$$d\lambda \leq \chi_{u^{-1}(\partial C)}(-b(u)(Du, Du))dM,$$

we argue as in [6] (proof of Lemma 2.2). We fix a map $\theta \in C^\infty(\mathbf{R})$ with $0 \leq \theta(s) \leq 1, \theta'(s) \leq 0, \theta(s) = 1$ if $s \leq \frac{1}{2}$ and $\theta(s) = 0$ if $s \geq 1$, and we set

$$\rho(z) = \begin{cases} d(x, \partial C) & \text{if } z \in C \\ -d(z, \partial C) & \text{if } z \in S \backslash C. \end{cases}$$

Since ∂C is smooth, the map ρ is smooth in a neighborhood of ∂C. Moreover, by assumption (2.5), it is bounded and it has bounded derivatives in a uniform neighborhood of ∂C, and

$$\nabla \rho(z) = \omega(z) \text{ for } z \in \partial C. \tag{3.11}$$

Here $\nabla \rho$ denotes the gradient of ρ as a map defined on S, that is $\nabla\rho(z) \in T_z S$ and $\nabla \rho(z) \cdot \tau = d\rho(z)\tau$ for every $\tau \in T_z S$. For every $\varepsilon > 0$ small enough we set

$$\tau_\varepsilon(z) = \begin{cases} \theta\left(\dfrac{\rho(z)}{\varepsilon}\right) \nabla\rho(z) \text{ if } \rho(z) < \varepsilon \\ 0 \text{ otherwise.} \end{cases}$$

Notice that for ε small enough the map τ_ε is well defined in a neighborhood of ∂C in S, it is of class $L^\infty \cap C^1$, it has bounded derivatives and it satisfies (3.8). In addition, it satisfies (3.9) by (3.11). Consequently, we can take $\tau = \tau_\varepsilon$ in (3.10) to get

$$\langle -\Delta_M u, \eta\tau_\varepsilon(u)\rangle = \int_M \eta d\lambda \;\; \forall \varepsilon > 0, \;\; \forall \eta \in C^\infty(M).$$

We recall that the Radon measure λ does not depend on ε. For a positive test function η with support in a coordinate chart for M we get

$$(3.12)\qquad \int_M \eta d\lambda = \int_M g^{\alpha,\beta} \frac{\partial u}{\partial x_\alpha} \cdot \frac{\partial}{\partial x_\beta} [\eta \theta(\frac{\rho(u)}{\varepsilon}) \nabla \rho(u)] dM \leq \int_M \theta(\frac{\rho(u)}{\varepsilon}) g^{\alpha,\beta} \frac{\partial u}{\partial x_\alpha} \cdot \frac{\partial}{\partial x_\beta} [\eta \nabla \rho(u)] dM.$$

Here we have used the fact that $\eta\theta'(\frac{\rho(u)}{\varepsilon}) \leq 0$ on M, which gives:

$$\int_M \eta g^{\alpha,\beta} [\frac{\partial u}{\partial x_\alpha} \cdot \nabla \rho(u))] \frac{\partial}{\partial x_\beta} [\theta(\frac{\rho(u)}{\varepsilon})] dM = \frac{1}{\varepsilon} \int_M \eta \theta'(\frac{\rho(u)}{\varepsilon}) g^{\alpha,\beta} \frac{\partial}{\partial x_\alpha} (\rho \circ u) \cdot \frac{\partial}{\partial x_\beta} (\rho \circ u) dM \leq 0.$$

We observe that as $\varepsilon \to 0$,

$$(3.13)\qquad \theta(\frac{\rho(u(\cdot))}{\varepsilon}) \rightharpoonup \chi_{u^{-1}(\partial C)} \text{ weakly* in } L^\infty(M)$$

and

$$\frac{\partial}{\partial x_\beta} [\eta \nabla \rho(u)] = \eta d\omega(u) \frac{\partial u}{\partial x_\beta} + \frac{\partial \eta}{\partial x_\beta} \omega(u) \text{ a.e. on } u^{-1}(\partial C)$$

by (3.11). On the other hand, we can use:

$$\frac{\partial u}{\partial x_\alpha} \in T_u \partial C \text{ a.e. on } u^{-1}(\partial C)$$

to get $\omega(u) \cdot \frac{\partial}{\partial x_\alpha} u = 0$ a.e. on $u^{-1}(\partial C)$, and thus we obtain

$$(3.14)\qquad \frac{\partial u}{\partial x_\alpha} \cdot \frac{\partial}{\partial x_\beta} [\eta \nabla \rho(u)] = \eta \frac{\partial u}{\partial x_\alpha} \cdot (d\omega(u) \frac{\partial u}{\partial x_\beta}) + \frac{\partial \eta}{\partial x_\beta} \frac{\partial u}{\partial x_\alpha} \cdot \omega(u) = \eta(-b(u)(\frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta})) \text{ a.e. on } u^{-1}(\partial C)$$

by definition of the second fundamental form b. Finally, (3.12), (3.13) and (3.14) yield in the limit

$$\int_M \eta d\lambda \leq \int_{u^{-1}(\partial C)} \eta(-b(u)(Du, Du)) dM \ \forall \ \eta \in C^\infty(M), \eta \geq 0$$

which completes the proof of (3.3).

Using (3.3) and a density argument, from (3.10) we easily obtain

$$\langle -\Delta_M u, \eta\tau(u)\rangle = \int_M \eta d\lambda \ \forall \eta \in L^\infty \cap H^1(M). \tag{3.15}$$

Let us now fix a tangential vectorfield ϕ to S along u with $\phi \in L^\infty \cap H^1(M, \mathbf{R}^k)$. Since $\tau(u)\cdot\phi \in L^\infty \cap H^1(M)$, from (3.15) we get

$$\langle -\Delta_M u, (\tau(u)\cdot\phi)\tau(u)\rangle = \int_{u^{-1}(\partial C)} \tau(u)\cdot\phi d\lambda = \int_{u^{-1}(\partial C)} \omega(u)\cdot\phi d\lambda.$$

On the other hand it is clear that both the maps $\pm[\phi - (\tau(u)\cdot\phi)\tau(u)]$ are tangential vectorfields to C along u (because they are orthogonal to $\tau(u) = \omega(u)$ on $u^{-1}(\partial C)$), and hence $\pm\langle -\Delta_M u, \phi - (\tau(u)\cdot\phi)\tau(u)\rangle \geq 0$ since u is weakly harmonic. This shows that

$$\langle -\Delta_M u, \phi\rangle = \langle -\Delta_M u, (\tau(u)\cdot\phi)\tau(u)\rangle = \int_{u^{-1}(\partial C)} \omega(u)\cdot\phi d\lambda,$$

and concludes the proof of Theorem 3.1. □

4. - On the heat flow method

Let $u_0 : M \to C$ be a map of class $W^{2,p}(M, \mathbf{R}^k)$ for every $p \in [1, +\infty)$. In this section we study the evolution problem

$$-\partial_t u(t,x) \in \partial^- E(u(t,x)) \text{ on } \mathbf{R}_+ \times M \tag{4.1}$$

$$u(0,\cdot) = u_0 \text{ on } M, \tag{4.2}$$

where $\partial^- E(u)$ is the subdifferential of the energy integral on the constraint $H^1(M, C)$ (*see* Section 2.1).

In order to introduce a suitable notion of weak solution to the differential inclusion (4.1), we observe that it is formally equivalent to:

$$-(\partial_t u - \Delta_M u) \in \partial^- I_C(u),$$

(*see* Section 2.2). Our notion of solution to (4.1) is based on the geometrical characterization of $\partial^- I_C(u)$ given by Proposition A.5.

DEFINITION 4.1. *A map $u : \mathbf{R}_+ \times M \to C$ is said to be a global weak solution to* (4.1)-(4.2) *if the following conditions are satisfied*:

$$\partial_t u \in L^2(\mathbf{R}_+ \times M), \ u, \ \nabla u \in L^\infty(\mathbf{R}_+, L^2(M)), \tag{4.3}$$

(4.4) $$u(0,\cdot) = u_0 \text{ in the sense of traces,}$$

(4.5) $$\int_{\mathbf{R}_+\times M} (\partial_t u \cdot \phi + g^{\alpha,\beta} \frac{\partial u}{\partial x_\alpha} \cdot \frac{\partial \phi}{\partial x_\beta}) dM dt \geq 0$$

for every testing function ϕ with compact support in $\mathbf{R}_+ \times M$, and such that

$$\phi \in H^1(\mathbf{R}_+ \times M, \mathbf{R}^k), \ \ \phi(t,x) \in T_{u(t,x)}C \ a.e. \ on \ \mathbf{R}_+ \times M.$$

REMARK 4.2. As in Theorem 3.1, we can characterize global weak solutions to (4.1) as solutions to a differential equation. Let $u : \mathbf{R}_+ \times M \to C$ be a map satisfying (4.3). Then u is a global weak solution to (4.1) if and only if there exists a Borel measure λ on $\mathbf{R}_+ \times M$, with

$$0 \leq d\lambda \leq \chi_{u^{-1}(\partial C)}(-b(u)(Du, Du)) dM dt,$$

and such that

$$\int_{\mathbf{R}_+\times M} (\partial_t u \cdot \Phi + g^{\alpha,\beta} \frac{\partial u}{\partial x_\alpha} \cdot \frac{\partial \Phi}{\partial x_\beta} + A(u)(Du, Du) \cdot \Phi) dM dt$$
$$= \int_{u^{-1}(\partial C)} \omega(u) \cdot \Phi d\lambda, \ \ \forall \Phi \in L^\infty \cap H^1(\mathbf{R}_+ \times M, \mathbf{R}^k) \text{ with compact support.}$$

We shall adopt the techniques in [4] to cover the case $\partial C \neq \emptyset$. In [4] the existence and the partial regularity of weak solutions to (4.1)-(4.2) are obtained by the penalty approximation which is used in [3] for the case $C = S^n$, and $\epsilon-$regularity, the fundamental step which can be found in [21].

Similarly to the case $\partial C = \emptyset$, what we essentially do is to approximate the differential inclusion (4.1) by a sequence of parabolic differential equations.

In order to define the approximating problems we fix a smooth, nondecreasing function η such that $\eta(s) = s$ for $s \leq \delta^2$ and $\eta(s) = 2\delta^2$ for $s \geq 4\delta^2$. By our assumptions on C, the function $d(\cdot, C)^2$ has Lipschitz continuous first order derivatives in a neighborhood of C. Then, if δ is small enough, the function $u \to \eta(d(u,C)^2)$ is of class $C^{1,1}(\mathbf{R}^k)$ and

(4.6) $$\frac{d}{du} \frac{1}{2} \eta(d(u,C)^2) = \eta'(d(u,C)^2)(u - \Pi_C u)$$

if $d(u,C) < 2\delta$. For $k = 1, 2, \ldots$, we define the functional

$$E_k(u) = \int_M e_k(u) dM,$$

where

(4.7) $$e_k(u) = \frac{1}{2}(g^{\alpha,\beta} \frac{\partial u}{\partial x_\alpha} \cdot \frac{\partial u}{\partial x_\beta} + k\eta(d(u,C)^2)),$$

and we consider the heat flow

$$\partial_t u - \Delta_M u + k \frac{d}{du}\frac{1}{2}\eta(d(u,C)^2) = 0, \text{ on } \mathbf{R}_+ \times M \tag{4.8}$$

with initial data (4.2).

By using Galerkin's method, for every k we obtain the existence of a solution u^k to (4.8)-(4.2) satisfying

$$\partial_t u^k \in L^2(\mathbf{R}_+ \times M), \ \nabla u^k \in L^\infty(\mathbf{R}_+, L^2(M)),$$
$$\eta(d(u^k, C)^2) \in L^\infty(\mathbf{R}_+, L^1(M)).$$

From the last inclusion we get also

$$\frac{d}{du}\frac{1}{2}\eta(d(u^k,C)^2) \in L^\infty(\mathbf{R}_+, L^2(M)),$$

and hence, by Sobolev embedding theorem, and using the a-priori estimates for linear parabolic equations (see [14], Chapter 3), we infer that

$$u^k, \nabla^2 u^k, \partial_t u^k \in L^p_{\text{loc}}(\mathbf{R}_+ \times M), \tag{4.9}$$

for any $p \in [1, +\infty)$.

In order to pass to the limit as $k \to \infty$, we first point out the following "energy estimate" (*see* [4], Lemma 2.1).

LEMMA 4.3. *Let $u_0 \in H^1(M, C)$. Then*

$$\int_{\mathbf{R}_+ \times M} |\partial_t u^k|^2 dM dt + \sup_{t>0} E_k(u^k(t,\cdot)) \le E_k(u_0) = E(u_0).$$

From Lemma 4.3 we infer that there exist a subsequence of $(u^k)_k$ (still denoted by u^k), and a measurable function u defined a.e. on $\mathbf{R}_+ \times M$, such that

$$\nabla u^k \to \nabla u \text{ weakly} * \text{ in } L^\infty(\mathbf{R}_+, L^2(M)),$$
$$\partial_t u^k \to \partial_t u \text{ weakly in } L^2(\mathbf{R}_+ \times M), \ u^k \to u \text{ weakly in } H^1_{\text{loc}}(\mathbf{R}_+ \times M).$$

Hence, we first get that $u^k \to u$ a.e. on $\mathbf{R}_+ \times M$ and u satisfies (4.3) and the initial conditions (4.2). In addition, we get that

$$d(u^k, C) \to 0 \text{ in } L^2_{\text{loc}}(\mathbf{R}_+ \times M),$$

and thus we can deduce

$$u \in C \text{ a.e. on } \mathbf{R}_+ \times M.$$

The choice of the approximating evolution equations is based on the following two Propositions, which will lead to our main Theorem.

PROPOSITION 4.4. *Let Q be an open subset of $\mathbf{R}_+ \times M$. If for a subsequence $(u^k)_k$ we have that*

$$e_k(u^k) \text{ is bounded in } L^\infty_{\text{loc}}(Q), \tag{4.10}$$

$$kd(u^k, C) \text{ is bounded in } L^2_{\text{loc}}(Q), \tag{4.11}$$

then

$$u, \nabla^2 u, \partial_t u \in L^2_{\text{loc}}(Q), \tag{4.12}$$

$$-(\partial_t - \Delta_M)u(t,x) \in N_{u(t,x)}C \text{ for a.e. } (t,x) \in Q. \tag{4.13}$$

PROOF. From (4.10) we know that there exists a subsequence u^k such that as $k \to \infty$

$$u^k \to u \text{ in } C^0_{\text{loc}}(Q), \;\; \nabla u^k \to \nabla u \text{ weakly} * \text{ in } L^\infty_{\text{loc}}(Q). \tag{4.14}$$

Moreover, from (4.6), (4.8) and (4.11) we find that $(\partial_t - \Delta_M)u^k$ is bounded in $L^2_{\text{loc}}(Q)$, and hence also $\partial_t u^k$ and $\nabla^2 u^k$ are. Therefore

$$(\partial_t - \Delta_M)u^k \to (\partial_t - \Delta_M)u \text{ weakly in } L^2_{\text{loc}}(Q) \tag{4.15}$$

and u satisfies (4.12).

In order to show (4.13), we consider again the vectorfields $\tau_\epsilon \in C^1(S, \mathbf{R}^k)$ already defined in the proof of Theorem 3.1. We recall here that the vectorfields τ_ϵ have the following properties:

$$\tau_\epsilon(v) \in T_v S \text{ for } v \in S, \;\; \tau_\epsilon(v) = \omega(v) \text{ for } v \in \partial C, \tag{4.16}$$

$$\|\tau_\epsilon\|_{L^\infty(S)} \le c,$$

where $c > 0$ is a constant depending only on C, and

$$\tau_\epsilon(v) \to \chi_{\partial C}(v)\omega(v) \text{ pointwise on } S \tag{4.17}$$

as $\epsilon \to 0$. Let us fix any open set $Q' \subset\subset Q$ and any map $\phi \in L^2(Q', \mathbf{R}^k)$ with $\phi(t,x) \in T_{u(t,x)}C$ for a.e. $(t,x) \in Q'$. In particular, we have that $\phi \in T_u S$ and

$$\phi \cdot \omega(u) \ge 0 \text{ a.e. on } Q' \cap u^{-1}(\partial C). \tag{4.18}$$

For every real number s we set $s^- := \min(s, 0)$ and we define

$$\phi_k^\epsilon = d\Pi_S(\Pi_C u^k)\phi - (d\Pi_S(\Pi_C u^k)\phi \cdot \tau_\epsilon(\Pi_C u^k))^- \tau_\epsilon(\Pi_C u^k).$$

Since Π_S has bounded derivatives in $\mathcal{U}_C$, from (4.16) we get that for sufficiently large k, $\phi_k^\epsilon \in L^2(Q', \mathbf{R}^k)$ and $|\phi_k^\epsilon| \le c|\phi|$ where the constant c depends only on C. Moreover, from (4.14), the continuity of τ_ϵ and Lebesgue's theorem we have that as $k \to \infty$

$$\phi_k^\epsilon \to \phi - (\phi \cdot \tau_\epsilon(u))^- \tau_\epsilon(u) \text{ in } L^2(Q', \mathbf{R}^k).$$

Here we have used the fact that $d\Pi_S(u)\phi = \phi$ a.e. on Q', since $\phi \in T_u S$ a.e. on Q'. In addition, from (4.14), (4.16) and (4.17) we get that as $\epsilon \to 0$,

$$\phi - (\phi \cdot \tau_\epsilon(u))^- \tau_\epsilon(u) \to \phi - \chi_{u^{-1}(\partial C)}(\phi \cdot \omega(u))^- \omega(u) = \phi \text{ in } L^2(Q', \mathbf{R}^k).$$

by (4.18). Combining with (4.15) we get

$$\lim_{\epsilon \to 0} \lim_{k \to \infty} \int_{Q'} (\partial_t - \Delta_M) u^k \cdot \phi_k^\epsilon dM dt = \int_{Q'} (\partial_t - \Delta_M) u \cdot \phi dM dt. \tag{4.19}$$

We notice that for every ϵ, k it results $\phi_k^\epsilon \in T_{\Pi_C u^k} C$ a.e. on Q', and that $-(\partial_t - \Delta_M) u^k \in N_{\Pi_C u^k} C$ a.e. on $\mathbf{R}_+ \times M$ by (4.8) and Lemma A.2. Hence, we infer that $(\partial_t - \Delta_M) u^k \cdot \phi_k^\epsilon \ge 0$ a.e. on Q', which together with (4.19) implies

$$\int_{Q'} (\partial_t - \Delta_M) u \cdot \phi dM dt \ge 0. \tag{4.20}$$

Since ϕ is tangent to C along u and ϕ is arbitrary, from (4.20) we easily get the conclusion of the proof of Proposition 4.4. □

We introduce now the parabolic distance on $\mathbf{R}_+ \times M$ by

$$\delta((t,x),(s,y)) = \max\{\sqrt{|t-s|}, |x-y|_g\},$$

where $|x-y|_g$ denotes the distance between x and y with respect to metric g on M. As in [4], Section 3, we have the following result.

PROPOSITION 4.5. *Let $u : \mathbf{R}_+ \times M \to C$ be a map satisfying* (4.3). *If there exists a closed subset $\Sigma \subset \mathbf{R}_+ \times M$, with locally finite m-dimensional Hausdorff measure with respect to the parabolic metric δ, and such that*

$$u, \nabla^2 u, \partial_t u \in L^2_{\text{loc}}((\mathbf{R}_+ \times M) \backslash \Sigma), \tag{4.21}$$

$$-(\partial_t - \Delta_M) u \in N_u C \textit{ a.e. on } \mathbf{R}_+ \times M, \tag{4.22}$$

then u is a global weak solution to (4.1).

PROOF. The proof is similar to the last part of the proof of Theorem 3.1 in [4]. Let $Q \subset\subset \mathbf{R}_+ \times M$ be any fixed domain. Given $R > 0$, let $\{P_j = P_{R_j}(z_j)\}_{j \in J}$

be a covering of $\Sigma \cap Q$ by parabolic cylinders $P_{R_j}(z_j)$ with $z_j \in \Sigma \cap Q$, $R_j \leq R$ and such that $\sum_{j \in J} R_j^m \leq 2H^m(\Sigma \cap Q; \delta)$, where $H^m(A; \delta)$ denotes the m-dimensional Hausdorff measure of A with respect to metric δ. Now we choose a cut-off function $\theta \in C_0^\infty(\mathbf{R}_+ \times M, [0,1])$ with support in $P_2(0)$ and such that $\theta = 1$ on $P_1(0)$, and we set

$$\theta_j(t,x) = \theta(\frac{t-t_j}{R_j^2}, \frac{x-x_j}{R_j}).$$

Let ϕ be any testing function satisfying the conditions in (4.5) and with support in Q. From (4.21) and (4.22) we have

$$0 \leq \int_Q (\partial_t u - \Delta_M u) \cdot \phi \inf_{j \in J}(1-\theta_j) dM dt = \tag{4.23}$$

$$\int_Q (\partial_t u \cdot \phi + g^{\alpha,\beta} \frac{\partial u}{\partial x_\alpha} \cdot \frac{\partial \phi}{\partial x_\beta}) \inf_{j \in J}(1-\theta_j) dM dt +$$

$$\int_Q g^{\alpha,\beta} \frac{\partial u}{\partial x_\alpha} \cdot \phi \frac{\partial}{\partial x_\beta} \{\inf_{j \in J}(1-\theta_j)\} dM dt.$$

As in [4] it can be proved that the third integral in (4.23) goes to zero as $R \to 0$. Noticing that $\inf_{j \in J}(1-\theta_j) \to 1$ a.e. on Q, we obtain (4.5), and Proposition 4.5 is proved. □

Now we can prove our main Theorem.

THEOREM 4.6. *Let $u_0 : M \to C$ be a map of class $W^{2,p}(M, \mathbf{R}^k)$ for every $p \in [1, +\infty)$. Then there exists a global weak solution $u : M \to C$ to the evolution problem (4.1)-(4.2) with $E(u(t,\cdot)) \leq E(u_0)$ and $u, \nabla^2 u, \partial_t u \in L^p_{\mathrm{loc}}((\mathbf{R}_+ \times M)\backslash\Sigma)$ for any $p \in [1,\infty)$, where the singular set Σ is closed and it has locally finite m-dimensional Hausdorff measure with respect to the parabolic metric δ.*

Moreover, as $t \to \infty$ suitably, a sequence $u(t,\cdot)$ converges weakly in $H^1(M, \mathbf{R}^k)$ to a harmonic map $u_\infty : M \to C$ with energy $E(u_\infty) \leq E(u_0)$. The map u_∞ is of class $W^{2,p}_{\mathrm{loc}}(M\backslash\Sigma_\infty)$ for any $p \in [1,\infty)$, where Σ_∞ has finite $(m-2)$-dimensional Hausdorff measure.

PROOF. As in [21], Theorem 6.1, we define

$$\Sigma = \bigcap_{0 < R \leq \min\{\epsilon_0, \sqrt{t_0}/2\}} \{z_0 \in \mathbf{R}_+ \times M \,|\, \liminf_{k \to \infty} \int_{T_R(z_0)} e_k(u^k) G_{z_0} \phi^2 dM dt \geq \epsilon_0\}$$

where the set $T_R(z_0)$ and the maps G_{z_0} and ϕ are defined in the Appendix B, and ϵ_0 is a small positive number which will be determined in Lemma B.1. The proof that Σ is closed and it has locally finite m-dimensional measure can be obtained as in [21], proof of Theorem 6.1. One needs only to replace the energy $e(u^k)$ in [21] by the penalized energy $e_k(u^k)$, and the monotonicity formula proved in [21] for the case $M = \mathbf{R}^m$, by the estimates given by Lemma B.1.

For every point $z_0 \notin \Sigma$ there exists a radius $R_0 \leq \min\{\epsilon_0, \sqrt{t_0}/2\}$ such that for a subsequence u^k it results

$$\int_{T_{R_0}(z_0)} e_k(u^k) G_{z_0} \phi^2 dM dt \leq \epsilon_0.$$

By Lemma B.3 we have that there exists an open neighborhood Q of z_0 such that the sequence $e_k(u^k)$ is bounded in $L^\infty_{\text{loc}}(Q)$. Hence, also $kd(u^k, C)$ is bounded in $L^2_{\text{loc}}(Q)$ by Lemma B.2, and thus we can apply Proposition 4.4 to get that (4.12) and (4.13) hold in Q. Since z_0 is an arbitrary point in $\mathbf{R}_+ \times M \backslash \Sigma$, this shows that the assumptions of Proposition 4.5 are satisfied, and hence u is a global weak solution to (4.1)-(4.2) in the sense of Definition 4.1. In addition, we can use Lemma B.3 to prove that

$$\nabla u \in L^\infty_{\text{loc}}(\mathbf{R}_+ \times M \backslash \Sigma), \tag{4.24}$$

and hence by (4.21), Remark 4.2 and the parabolic regularity theory we get $\partial_t u, u, \nabla^2 u \in L^p_{\text{loc}}(\mathbf{R}_+ \times M \backslash \Sigma)$.

This concludes the proof of the first part of the Theorem. The second part can be obtained as in [4], proof of Theorem 1.5. We omit the details of the proof. □

REMARK 4.7. It would be of interest to investigate whether $H^{2,\infty}$ -regularity can be achieved. For scalar variational inequalities such a result is due to Frehse (Boll. Unione Mat. Ital. 6 (1972), 312-315).

APPENDIX A

In this section we prove some results about tangential and normal vectorfields. We start with some preliminary remarks on tangent and normal cones to a manifold with boundary.

LEMMA A.1. *Let u be a point in C and $\tau \in \mathbf{R}^k$. Then the following statements are equivalent*:

(i) $$\tau \in T_u C;$$

(ii) $$\lim_{\epsilon \to 0^+} \frac{d(u + \epsilon\tau, C)}{\epsilon} = 0;$$

(iii) *For every sequence $\epsilon_n \to 0^+$, there exists a sequence $\tau_n \to \tau$ such that $u + \epsilon_n \tau_n \in C$;*

(iv) *For every sequence $\epsilon_n \to 0^+$ we have*

$$\frac{\Pi_C(u+\epsilon_n\tau)-u}{\epsilon_n} \to \tau;$$

(v) *There exist sequences $v_n \in C$ with $v_n \to u$, and $\epsilon_n \to 0^+$ such that*

$$\frac{v_n - u}{\epsilon_n} \to \tau.$$

PROOF. If C is replaced by S (and $u \in S\backslash\partial S$), or by ∂C (and $u \in \partial C$), the equivalence among (i) to (v) is obvious. For the same reason, the conclusion of Lemma A.1 is easily proved in case $u \in C\backslash\partial C$. Let us consider now the general case $u \in C$. The equivalence between (ii) and (iii), and the implication (iv) $\Longrightarrow$ (v) are trivial. Now we prove that (i) $\Longrightarrow$ (iii). Assume that $\tau \neq 0$. From $\tau \in T_uC \subseteq T_uS$ we get that for every sequence $\epsilon_n \to 0^+$ there exists a sequence $\tau_n \to \tau$ such that $v_n := u+\epsilon_n\tau_n \in S$. If for a subsequence we have that $v_n \in S\backslash C$ (otherwise, (iii) is proved), then we first get that $u \in \partial C$. Since $\omega(u)$ is the inner unit normal vector to ∂C at u relative to S, and since $v_n = u+\epsilon_n\tau_n \in S\backslash C$, we have that

$$0 \geq \lim_n \sup \omega(u) \cdot \frac{v_n - u}{|v_n - u|} = \frac{1}{|\tau|}\,\omega(u)\cdot\tau.$$

On the other hand, we know that that $\tau\cdot\omega(u) \geq 0$ since $\tau \in T_uC$. This shows that $\tau\cdot\omega(u) = 0$ which, together with $\tau \in T_uS$, implies that $\tau \in T_u\partial C$. It follows that there exists a sequence τ_n' such that $\tau_n' \to \tau$ and $u+\epsilon_n\tau_n' \in \partial C \subseteq C$, and (iii) is proved.

To prove that (iii) $\Rightarrow$ (iv), fix a sequence $\epsilon_n \to 0^+$ and set $\sigma_n = \frac{1}{\epsilon_n}(\Pi_C(u+\epsilon_n\tau)-u)$. By (iii) there exists a sequence $\tau_n \to \tau$ with $u+\epsilon_n\tau_n \in C$. Thus,

$$\begin{aligned}\lim_{n\to\infty}\sup|\sigma_n - \tau| &= \lim_{n\to\infty}\sup \frac{1}{\epsilon_n}\,|\Pi_C(u+\epsilon_n\tau) - u - \epsilon_n\tau + \epsilon_n\tau_n\epsilon_n\tau_n| \\ &\leq \lim_{n\to\infty}\sup\{\frac{1}{\epsilon_n}|\Pi_C(u+\epsilon_n\tau) - \Pi_C(u+\epsilon_n\tau_n)| + |\tau_n - \tau|\} \\ &\leq \lim_{n\to\infty}\sup(L+1)|\tau_n - \tau| = 0\end{aligned}$$

where L is the Lipschitz constant of Π_C. This shows that $\sigma_n \to \tau$ and proves the implication.

For the implication (v) $\Rightarrow$ (i), fix two sequences $v_n \in C$, $v_n \to u$ and $\epsilon \to 0^+$, such that $\tau_n := \frac{1}{\epsilon_n}(v_n - u) \to \tau$. This implies that $\tau \in T_uS$, since

$$v_n = u + \epsilon_n\tau_n = \Pi_S(u+\epsilon_n\tau_n) = u + \epsilon_n\int_0^1 d\Pi_S(u+s\epsilon_n\tau_n)\tau_n ds$$

and hence

$$\tau = \lim_{n\to\infty} \frac{v_n - u}{\epsilon_n} = \lim_{n\to\infty} \int_0^1 d\Pi_S(u + s\epsilon_n\tau_n)\tau_n ds = d\Pi_S(u)\tau \in T_uS.$$

It remains only to prove that $\omega(u) \cdot \tau \geq 0$ if $u \in \partial C$. Since $v_n \to u \in \partial C$ and $v_n \in C$, we have

$$0 \leq \lim_n \inf \omega(u) \cdot \frac{v_n - u}{|v_n - u|} = \frac{1}{|\tau|}\omega(u) \cdot \tau,$$

and the Lemma is proved. □

LEMMA A.2. *For every u in a suitably small neighborhood of C we have*

$$u - \Pi_C u \in N_{\Pi_C u} C.$$

PROOF. Fix any point u such that $\Pi_C u$ is defined. For any fixed vector $\tau \in T_{\Pi_C u}C$, by (iii) of Lemma A.1 we can find a sequence $\sigma_n \to \tau$ with $\Pi_C u + \frac{1}{n}\sigma_n \in C$. Since

$$|u - \Pi_C u|^2 = d(u, C)^2 \leq |u - (\Pi_C u + \frac{1}{n}\sigma_n)|^2,$$

we have $\frac{1}{n}|\sigma_n|^2 \geq 2(u - \Pi_C u) \cdot \sigma_n$. Passing to the limit we finally get that $(u - \Pi_C u) \cdot \tau \leq 0$, and since τ is an arbitrary tangent vector to C at $\Pi_C u$, the Lemma is proved. □

LEMMA A.3. *Let $u \in H^1(M, C)$ be a given map. Then for every sequence $v_n \in H^1(M, C)$ with $v_n \not\equiv u$ and $v_n \to u$ in H^1, there exist a subsequence of v_n (also denoted by v_n) and a* TVF *τ to C along u such that*

$$\frac{v_n - u}{||v_n - u||} \to \tau \textit{ weakly in } H^1(M, \mathbf{R}^k).$$

PROOF. Setting

$$\tau_n = \frac{v_n - u}{||v_n - u||},$$

we have $v_n \to u$ a.e. on M, $\tau_n \to \tau$ a.e. on M, and $\tau_n \to \tau$ weakly in $H^1(M, \mathbf{R}^k)$, for a map $\tau \in H^1(M, \mathbf{R}^k)$ (up to subsequences). Setting $\epsilon_n = ||v_n - u||_{H^1}$, $\epsilon_n \to 0^+$, the equivalence between (v) and (i) in Lemma A.1 leads to $\tau(x) \in T_{u(x)}C$ a.e. $x \in M$, i.e. τ is a tangential vectorfield along u. □

LEMMA A.4. *Let $u \in H^1(M, C)$ and $\tau \in H^1(M, \mathbf{R}^k) \cap L^\infty$ be a non-zero tangential vectorfield to C along u. Let $\epsilon_n \to 0^+$ be a given sequence. Then for* *n large,* $\Pi_C(u+\epsilon_n\tau)\in H^1(M,C)$ *and* $\Pi_C(u+\epsilon_n\tau)\neq u$. *In addition,*

(i) $$\frac{\Pi_C(u+\epsilon_n\tau)-u}{\epsilon_n}\to\tau \text{ in } H^1(M,\mathbf{R}^k);$$

(ii) $$\frac{\Pi_C(u+\epsilon_n\tau)-u}{\|\Pi_C(u+\epsilon_n\tau)-u\|}\to\frac{\tau}{\|\tau\|} \text{ in } H^1(M,\mathbf{R}^k).$$

PROOF. Noticing that

$$\sup_{x\in M} d(u(x)+\epsilon_n\tau(x)),C)\leq\epsilon_n\|\tau\|_{L^\infty}\to 0,$$

from assumptions (2.4) and (2.5) we have that $\Pi_C(u+\epsilon_n\tau)$ is well defined and belongs to $H^1(M,C)$ for n large enough. The fact that $\Pi_C(u+\epsilon_n\tau)\neq u$ easily follows from (iv) of Lemma A.1 and from the assumption $\tau\neq 0$. The statement (ii) trivially follows from (i). To prove (i) we define

$$v_n=\frac{\Pi_C(u+\epsilon_n\tau)-u}{\epsilon_n}.$$

Using the implication (i) $\Rightarrow$ (iv) in Lemma A.1, the Lipschitz continuity of Π_C and Lebesgue's Theorem we get that $v_n\to\tau$ in $L^2(M,\mathbf{R}^k)$. In order to complete the proof of (ii) we extend $\Pi_{\partial C}$ to a C^1 map $\hat\Pi_{\partial C}$ on $\mathbf{R}^k$ with bounded derivatives, and we set

$$w_n=\frac{\Pi_S(u+\epsilon_n\tau)-u}{\epsilon_n},\quad z_n=\frac{\hat\Pi_{\partial C}(u+\epsilon_n\tau)-u}{\epsilon_n},$$
$$A_n=\{x\in M|v_n(x)=w_n(x)\}.$$

Notice that $\Pi_C(u+\epsilon_n\tau)=\hat\Pi_{\partial C}(u+\epsilon_n\tau)=\Pi_{\partial C}(u+\epsilon_n\tau)$ a.e. on $M\backslash A_n$. Since Π_S is smooth, in local coordinates on M we have

$$\frac{\partial w_n}{\partial x_\alpha}=\int_0^1[d^2\Pi_S(u+\theta\epsilon_n\tau)(\frac{\partial u}{\partial x_\alpha}+\epsilon_n\frac{\partial\tau}{\partial x_\alpha})\tau$$
$$+d\Pi_S(u+\theta\epsilon_n\tau)\frac{\partial\tau}{\partial x_\alpha})]d\theta,\text{ a.e. on } M.$$

Since we have supposed that $\tau\in L^\infty(M,\mathbf{R}^k)$, using Lebesgue's Theorem and the assumption (2.4) on S we get that for every coordinate $\alpha=1,\ldots,m$,

(A.1) $$\frac{\partial w_n}{\partial x_\alpha}\to\frac{\partial\tau}{\partial x_\alpha}\text{ in } L^2(M,\mathbf{R}^k)\text{ and a.e. on } M.$$

Here we have used

$$\frac{\partial\tau}{\partial x_\alpha}=d^2\Pi_S(u)\frac{\partial u}{\partial x_\alpha}\tau+d\Pi_S(u)\frac{\partial\tau}{\partial x_\alpha}\text{ a.e. on } M$$

which can be obtained by differentiating the identity $d\Pi_S(u)\tau = \tau$. Similarly, we have

$$\frac{\partial z_n}{\partial x_\alpha} \to d^2\hat{\Pi}_{\partial C}(u)\frac{\partial u}{\partial x_\alpha}\tau + d\hat{\Pi}_{\partial C}(u)\frac{\partial \tau}{\partial x_\alpha}\tau \text{ in } L^2(M, \mathbf{R}^k)$$

and

$$\frac{\partial z_n}{\partial x_\alpha} \to \frac{\partial \tau}{\partial x_\alpha} \text{ a.e. on } \{x \in u^{-1}(\partial C) | \tau(x) \in T_{u(x)}\partial C\}. \tag{A.2}$$

By the definition of set A_n and by Lemma A.4 in [13] we get

$$\frac{\partial v_n}{\partial x_\alpha} = \begin{cases} \dfrac{\partial w_n}{\partial x_\alpha} & \text{a.e. on } A_n \\ \dfrac{\partial z_n}{\partial x_\alpha} & \text{a.e. on } M \backslash A_n, \end{cases} \tag{A.3}$$

and hence

$$\left|\frac{\partial v_n}{\partial x_\alpha}\right| \leq \left|\frac{\partial w_n}{\partial x_\alpha}\right| + \left|\frac{\partial z_n}{\partial x_\alpha}\right|, \text{ a.e. on } M. \tag{A.4}$$

Since the right side of (A.4) converges in $L^2(M, \mathbf{R}^k)$, in order to get $v_n \to \tau$ in H^1 it suffices to prove that

$$\frac{\partial v_n}{\partial x_\alpha} \to \frac{\partial \tau}{\partial x_\alpha} \text{ a.e. on } M. \tag{A.5}$$

Notice that for a.e. $x \in M$ there exists a subsequence $v_{n_k}(x)$ such that for all n_k, either $x \in A_{n_k}$ and hence $\frac{\partial}{\partial x_\alpha} v_{n_k}(x) = \frac{\partial}{\partial x_\alpha} w_{n_k}(x)$, or $x \in M \backslash A_{n_k}$ and hence $\frac{\partial}{\partial x_\alpha} v_{n_k}(x) = \frac{\partial}{\partial x_\alpha} z_{n_k}(x)$ by (A.3). In the first case, from (A.1) we get

$$\frac{\partial}{\partial x_\alpha} v_{n_k}(x) \to \frac{\partial}{\partial x_\alpha}\tau(x). \tag{A.6}$$

If we are in the second case, from $\Pi_C(u(x)+\epsilon_{n_k}\tau(x)) \in \partial C$ we first get that $u(x) \in \partial C$. Moreover, from implication (i) $\Rightarrow$ (iv) in Lemma A.1 we deduce that

$$\begin{aligned} \tau(x) &= \lim_{k\to\infty} \frac{\Pi_C(u(x) + \epsilon_{n_k}\tau(x)) - u(x)}{\epsilon_{n_k}} \\ &= \lim_{k\to\infty} \frac{\Pi_{\partial C}(u(x) + \epsilon_{n_k}\tau(x)) - u(x)}{\epsilon_{n_k}} = d\Pi_{\partial C}(u(x))\tau(x), \end{aligned}$$

that is $\tau(x) \in T_{u(x)}\partial C$. Then from (A.2) and (A.3) we get again

$$\frac{\partial}{\partial x_\alpha} v_{n_k}(x) = \frac{\partial}{\partial x_\alpha} z_{n_k}(x) \to \frac{\partial}{\partial x_\alpha}\tau(x). \tag{A.7}$$

Finally, since the limit of $\frac{\partial}{\partial x_\alpha} v_{n_k}(x)$ does not depend on the subsequence v_{n_k} and the point x is arbitrarily, we conclude (A.5) from (A.6) and (A.7). The proof of Lemma A.4 is complete. □

The next proposition gives a variational characterization of H^{-1} normal vectorfields. We recall that for $u \in H^1(M, C)$ the subdifferential of the indicatrix function I_C at u is defined by

$$\partial^- I_C(u) = \{\sigma \in H^{-1}(M, \mathbf{R}^k) | \limsup_{v \to u, v \in H^1(M,C)} \langle \sigma, \frac{v-u}{\|v-u\|} \rangle \le 0\}.$$

PROPOSITION A.5. *Let $u \in H^1(M, C)$ be a given map. Then*

$$\partial^- I_C(u) = \{\sigma \in H^{-1}(M, \mathbf{R}^k) | \sigma \text{ is a NVF to } C \text{ along } u\}.$$

PROOF. For any $\sigma \in H^{-1}(M, \mathbf{R}^k)$ we can find a sequence $v_n \in H^1(M, C)$ such that $v_n \neq u$ for every n, $v_n \to u$ in $H^1(M, \mathbf{R}^k)$ and

$$l_\sigma := \lim_{n \to \infty} \langle \sigma, \frac{v_n - u}{\|v_n - u\|} \rangle = \limsup_{v \to u, v \in H^1(M,C)} \langle \sigma, \frac{v-u}{\|v-u\|} \rangle.$$

By Lemma A.3 there exists a subsequence of v_n and a TVF τ_σ to C such that

$$\frac{v_n - u}{\|v_n - u\|} \to \tau_\sigma \text{ weakly in } H^1(M, \mathbf{R}^k).$$

This shows that there exists a tangential vectorfield $\tau_\sigma \in H^1(M, \mathbf{R}^k)$ with $l_\sigma = \langle \sigma, \tau \rangle$. In particular, if σ is a NVF to C along u we immediately get $l_\sigma \le 0$, i.e. $\sigma \in \partial^- I_C(u)$.

Conversely, if τ is any non-zero TVF to C along u with $\tau \in L^\infty \cap H^1(M, \mathbf{R}^k)$, then by Lemma A.4 (ii), there exists a sequence v_n in $H^1(M, C)$ such that

$$\frac{v_n - u}{\|v_n - u\|} \to \frac{\tau}{\|\tau\|} \text{ in } H^1(M, \mathbf{R}^k).$$

It follows that for a fixed $\sigma \in \partial^- I_C(u)$ we have

$$0 \ge \lim_{n \to \infty} \langle \sigma, \frac{v_n - u}{\|v_n - u\|} \rangle = \frac{1}{\|\tau\|} \langle \sigma, \tau \rangle$$

and hence $\langle \sigma, \tau \rangle \le 0$. Now, if τ is any H^1-tangential vectorfield, we can define a sequence of tangential vectorfields $\tau_R \in L^\infty \cap H^1(M, \mathbf{R}^k)$ as in (3.6), with $\tau_R \to \tau$ in H^1 as $R \to \infty$. Since $\langle \sigma, \tau_R \rangle \le 0$ for every $R > 0$, we get in the limit that $\langle \sigma, \tau \rangle \le 0$. Since τ is an arbitrary TVF to C along u, we have proved that σ is a NVF to C along u, and Proposition A.5 is completely proved. □

APPENDIX B

In this section we shall give some Lemmas for the solution u^k to the approximating equation (4.8).

We fix a positive constant ρ smaller than the injectivity radius of M. For any point $p \in M$, the geodesic ball $B_\rho(p)$ is defined and diffeomorphic to Euclidean ball $B_\rho(0) \subset \mathbf{R}^m$ via the exponential map. In normal coordinates $\{x_\alpha\}$ on $B_\rho(p)$, the metric g on M is represented by a matrix $(g_{i,j})_{1\le i,j\le m}$ with $g(0) = id$. We may restrict u^k to any such coordinate neighborhood, and regard u^k as a map on $\mathbf{R}_+ \times B_\rho(0)$ satisfying (4.8).

We fix a cut-off function $\phi \in C_0^\infty(B_\rho(0))$ with $0 \le \phi \le 1$ and such that $\phi = 1$ in a neighborhood of 0. Given any $t_0 > 0$, for every radius $R \le \min\{\rho, \sqrt{t_0}/2\}$, we define the functions $\Phi(R) = \Phi_{t_0}(R)$ and $\Psi(R) = \Psi_{t_0}(R)$ by

$$\Phi(R) = R^2 \int_{S_R(z_0)} e_k(u^k) G_{z_0} \phi^2 dM, \quad \Psi(R) = \int_{T_R(z_0)} e_k(u^k) G_{z_0} \phi^2 dM,$$

where $z_0 = (t_0, 0)$ and

$$S_R(z_0) = \{z = (t,x) | t = t_0 - R^2\},$$

$$T_R(z_0) = \{z = (t,x) | t_0 - 4R^2 < t < t_0 - R^2\},$$

$$G_{z_0}(t,x) = \frac{1}{4\pi(t_0 - t)^{m/2}} \exp(-\frac{|x|^2}{4(t_0 - t)}), \quad t < t_0,$$

$$e_k(u^k) = \frac{1}{2} g^{\alpha,\beta} \frac{\partial u^k}{\partial x_\alpha} \cdot \frac{\partial u^k}{\partial x_\beta} + \frac{k}{2} \eta(d(u^k, C)^2).$$

We have the following monotonicity inequalities, which can be proved as in [4], proof of Lemma 4.2.

LEMMA B.1. *There exists a constant $c > 0$ depending only on M and C, such that for $0 < R_1 \le R_2 \le \min\{\rho, \sqrt{t_0}/2\}$, it results*

$$\Phi(R_1) \le \exp(c(R_2 - R_1))\Phi(R_2) + cE(u_0)(R_2 - R_1),$$

$$\Psi(R_1) \le \exp(c(R_2 - R_1))\Psi(R_2) + cE(u_0)(R_2 - R_1).$$

Similarly to the case $\partial C = \emptyset$ we have the following result.

LEMMA B.2. *Let $u^k : \mathbf{R}_+ \times B_\rho(0)$ be a solution to* (4.8), *satisfying* (4.9). *Then $\partial_t e_k(u^k)$, $\Delta_M e_k(u^k) \in L^2_{\mathrm{loc}}(\mathbf{R}_+ \times B_\rho(0))$ and for sufficiently large k we have that*

$$(\partial_t - \Delta_M) e_k(u^k) + \frac{k^2}{2} d(u^k, C)^2 \le c e_k(u^k)(1 + e_k(u^k))$$

almost everywhere on $\mathbf{R}_+ \times B_\rho(0)$, where the constant $c > 0$ depends only on M and C.

PROOF. For simplicity of notation, we write u instead of u^k and we set $Q = \mathbf{R}_+ \times B_\rho(0)$. From the equation (4.8) we infer that

$$(\partial_t - \Delta_M)\frac{\partial u}{\partial x_\alpha} = F_\alpha, \qquad \alpha = 1, \dots, m$$

in the distributional sense, where

$$F_\alpha = k\frac{\partial}{\partial x_\alpha}\{\eta'(d(u,C)^2)(u - \Pi_C U)\}.$$

Since $\Pi_C : \mathcal{U}_C \to \mathcal{C}$ is globally Lipschitz continuous, from (4.9) it easily follows that $F_\alpha \in L^p(Q)$ for every $p \in [1, \infty)$. Hence, by local estimates for linear parabolic equations we get that $\nabla_x^3 u, \partial_t \nabla u, \nabla u \in L^p_{\text{loc}}(Q)$ for every $p \in [1, \infty)$, and hence the first part of the Lemma follows immediately.

To conclude the proof, we first compare (4.6) with the equation (4.8) in order to get that the following equalities hold a.e. on Q:

$$\begin{aligned}\frac{k}{2}(\partial_t - \Delta_M)\eta(d(u,C)^2) = &-\frac{k^2}{4}\eta'(d(u,C)^2)^2\Big|\frac{d}{du}(d(u,C)^2)\Big|^2 \\ &-\frac{k}{2}g^{\alpha,\beta}\frac{\partial}{\partial x_\beta}\Big\{\eta'(d(u,C)^2)\frac{d}{du}(d(u,C)^2)\Big\}\cdot\frac{\partial u}{\partial x_\alpha},\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}(\partial_t - \Delta_M)g^{\alpha,\beta}\frac{\partial u}{\partial x_\alpha}\cdot\frac{\partial u}{\partial x_\beta} = &-\frac{k}{2}g^{\alpha,\beta}\frac{\partial}{\partial x_\beta}\Big\{\eta'(d(u,C)^2)\frac{d}{du}(d(u,C)^2)\Big\}\cdot\frac{\partial u}{\partial x_\alpha} \\ &-|\nabla^2 u|^2 + R_k,\end{aligned}$$

where the error term R_k is easily estimated by

$$\text{(B.1)} \qquad |R_k| \le \frac{1}{2}|\nabla^2 u|^2 + ce_k(u) \text{ a.e. on } Q,$$

for some constant c which depends only on M and C. Notice that for $d(u,C) < 2\delta$,

$$\Big|\frac{d}{du}(d(u,C)^2)\Big|^2 = 4d(u,C)^2,$$

and hence, for sufficiently large k, it results that

$$\text{(B.2)} \qquad (\partial_t - \Delta_M)e_k(u) + |\nabla^2 u|^2 + k^2\eta'(d(u,C)^2)^2 d(u,C)^2 = I + R_k$$

a.e. on Q, where

$$I = -kg^{\alpha,\beta}\frac{\partial}{\partial x_\beta}\{\eta'(d(u,C)^2)\frac{d}{du}(d(u,C)^2)\}\cdot\frac{\partial u}{\partial x_\alpha}.$$

Setting

$$A = \{(t,x) \in Q,\ \Pi_S u \in C\},$$

by well-known theorems in Sobolev spaces (*see* for example [13], Lemma A.4), we easily get that

$$\eta'(d(u,C)^2)\frac{d}{du}(d(u,C)^2) = \eta'(d(u,S)^2)\frac{d}{du}(d(u,S)^2) \text{ a.e. on } A,$$

$$\eta'(d(u,C)^2)\frac{d}{du}(d(u,C)^2) = \eta'(d(u,\partial C)^2)\frac{d}{du}(d(u,\partial C)^2) \text{ a.e. on } Q\backslash A.$$

Using this remark we see that almost everywhere on A we have

$$\begin{aligned} I &= -kg^{\alpha,\beta}\frac{\partial}{\partial x_\beta}\{\eta'(d(u,S)^2)\frac{d}{du}(d(u,S)^2)\} \cdot \frac{\partial u}{\partial x_\alpha} \\ &= -2kg^{\alpha,\beta}\{\eta''(d(u,S)^2)d(u,S)^2\frac{\partial u}{\partial x_\alpha}\cdot\frac{\partial u}{\partial x_\beta} \\ &+ \eta'(d(u,S)^2)\frac{\partial}{\partial x_\beta}\, d(u,S)\cdot\frac{\partial}{\partial x_\alpha}\, d(u,S) \\ &+ \eta'(d(u,S)^2)d(u,S)\frac{d^2}{du^2}\, d(u,S)\frac{\partial u}{\partial x_\alpha}\cdot\frac{\partial u}{\partial x_\beta}\} \le \frac{k^2}{2}\, d(u,C)^2 + ce_k(u)^2 \end{aligned}$$

where $c > 0$ is a positive constant depending only on M and S.

By the same way we have $I \le k^2/2\ d(u,C)^2 + ce_k(u)^2$ a.e. on $Q\backslash A$, and hence we can conclude

$$I \le \frac{k^2}{2}\, d(u,C)^2 + ce_k(u)^2, \qquad \text{a.e. on } Q, \tag{B.3}$$

where $c > 0$ is a positive constant depending only on M and C. Inserting (B.3) and (B.3) into (B.2), we immediately conclude the proof of Lemma B.2. □

With minor modifications with respect to [4], and using Lemma B.2, we get the following result, which is a fundamental step in the proof of the main Theorem. We omit the details of the proof.

LEMMA B.3. *There exists a constant* $0 < \epsilon_0 < \rho$, *depending only on* M *and* C, *such that if for some* $0 < R \le \min(\epsilon_0, \sqrt{t_0}/2)$,

$$\Psi(R) = \Psi_{t_0}(R) \le \epsilon_0$$

is satisfied, then

$$e_k(u^k(t,x)) \le c(\delta R)^{-2} \text{ if } |t - t_0| < (\delta R)^2 \text{ and } |x| < \delta R,$$

with constants $c > 0$ *depending only on* M *and* C, *and* $\delta > 0$ *possibly depending in addition on* $E(u_0)$ *and* R.

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